New error estimates for a viscosity-splitting scheme for the Navier-Stokes equations

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The purpose of this poster is to present some new error estimates for a time fractional-step scheme with decomposition of the viscosity, improving previous results obtained in [1], [3] and [2].

The evolution of viscous, incompressible fluid flow in a bounded domain $\Omega \subset \mathbb{R}^3$ in a time interval $[0, T]$, is governed by the unsteady, incompressible Navier-Stokes equations:

$$(P) \qquad \left\{ \begin{array}{rcll} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\,\Delta\mathbf{u} + \nabla p & = & \mathbf{f} & \text{in } \Omega(0,T), \\ \nabla\cdot\mathbf{u} & = & 0 & \text{in } \Omega(0,T), \\ \mathbf{u} & = & 0 & \text{on } \partial\Omega(0,T), \\ \mathbf{u}_{|t=0} & = & \mathbf{u}_0 & \text{in } \Omega. \end{array} \right.$$

where $\mathbf{u}(\mathbf{x}, t)$ is the fluid velocity, $p(\mathbf{x}, t)$ is the pressure, ν is the viscosity (which is assumed constant) and $\mathbf{f}$ is an external force term. Finally, ∇ is the gradient operator (tridimensional) and Δ is the Laplacian operator.

The numerical analysis for these equations has received much attention in the last decades and many numerical schemes are now available for that objective. The main (numerical) difficulties in this problem are the coupling of the momentum equation with the incompressibility condition $\nabla\cdot\mathbf{u} = 0$, and the nonlinearity of the convective terms $(\mathbf{u}\cdot\nabla)\mathbf{u}$.

Fractional step methods are becoming widely used in this context [4]. They allow to separate the effects of the two previous difficulties.

In this poster, we are going to provide new error estimates for a viscosity splitting fractional-step method, which was already introduced and studied in [1], [2] and [3]. It is a two-step scheme splitting the nonlinearity and the incompressibility of the problem into different steps (but keeping viscosity term and boundary conditions in both steps).

Basically, if we consider an uniform partition $t_m = k\,m$ of $[0, T]$, with step $k = T/N$, given $\mathbf{u}^m$ an approximation of $\mathbf{u}(t_m)$, first one computes an intermediate velocity $\mathbf{u}^{m+1/2}$ by means of a convection-diffusion problem, and afterwards $(\mathbf{u}^{m+1}, p^{m+1})$ is obtained solving a Stokes type problem. It allows to enforce the original boundary conditions of the problem in both steps, which leads to the convergence of both velocities to a continuous solution of (P) in the space $\mathbf{H}_0^1(\Omega)$, (see [1], [2]). Indeed, in [1], [2], Blasco and Codina prove the convergence of this scheme. For this purpose, they begin obtaining some a priori stability estimates for both velocities $\mathbf{u}^{m+1/2}$ and $\mathbf{u}^{m+1}$, in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$ and afterwards, they make a pass to the limit to obtain the convergence, using some compactness results to "control" the limit of the convective terms.

On the other hand, in [3] error estimates of order 1 in $l^2(\mathbf{H}^1) \cap l^\infty(\mathbf{L}^2)$ for the end-of-step velocity error $\mathbf{e}^{m+1} = \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1}$ and order 1/2 in $l^2(L^2)$ for the pressure error $e_p^{m+1} = p(t_{m+1}) - p^{m+1}$ are obtained.

The main contribution of our study is to improve the pressure error estimate in $l^\infty(\mathbf{L}^2)$ towards order 1, based on new error estimates on the discrete in time derivative of velocities. Indeed, we will improve the pressure error estimate towards order 1 in $l^2(L^2)$ (and order 1 in $l^\infty(L^2)$).

For this purpose, we will need an estimate of order 1 for the discrete time derivative of $\mathbf{e}^{m+1}$ in $l^2(\mathbf{L}^2)$, that we are going to prove in two steps. Firstly, we will obtain order 1/2 in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$ for the discrete time derivative of $\mathbf{e}^{m+1/2}$ and $\mathbf{e}^{m+1}$ and afterwards, we will improve towards order 1 in $l^\infty(\mathbf{V}') \cap l^2(\mathbf{L}^2)$, for the discrete time derivative of $\mathbf{e}^{m+1}$, that is for $\delta_t \mathbf{e}^{m+1} := (\mathbf{e}^{m+1} - \mathbf{e}^m)/k$.

On the other hand, imposing stronger regularity hypotheses, we can improve the pressure error estimate towards order 1 in $l^\infty(L^2)$. For this purpose, we will need an estimate of order 1 for the discrete time derivative of $\mathbf{e}^{m+1}$ in $l^\infty(\mathbf{L}^2)$.

Due to this improvement, this viscosity splitting scheme is fully comparable with the well-known projection scheme (with pressure correction), [4].

References

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