Balanced Intervals of Two Sets of Points on a line or circle

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Abstract

Let n, m, k, h be positive integers such that $1 \le n \le m$, $1 \le k \le n$ and $1 \le h \le m$. Then we give a necessary and sufficient condition for every configuration with n red points and m blue points on a line or circle to have an interval containing precisely k red points and h blue points.

Key words: balanced interval, interval, two sets of points, line, circle

1. A balanced interval on a line

In this section we shall prove the following theorem.

Theorem 1 *Let n, m, k, h be integers such that $1 \le n \le m$, $1 \le k \le n$ and $1 \le h \le m$. Then for any n red points and m blue points on a line in general position (i.e., no two points lie on the same position.), there exists an interval that contains precisely k red points and h blue points if and only if*

$$\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1\right)(h-1) < m < \left(\left\lfloor \frac{n}{k-1} \right\rfloor\right)(h+1), \quad (1)$$

where the rightmost term is an infinite number when $k = 1$.

We begin with an example of our theorem. Consider a configuration consisting of 10 red points and 20 blue points on a line in general position. Then by the above theorem, we can easily show that if $k \in \{1, 2, 3, 5, 10\}$, then such a configuration has an interval containing exactly k red points and $2k$ blue points; otherwise (i.e., $k \in \{4, 6, 7, 8, 9\}$) there exist a configuration that has no such an interval (Fig. 1). We call an interval that contains given number of red points and blue points *a balanced interval.*

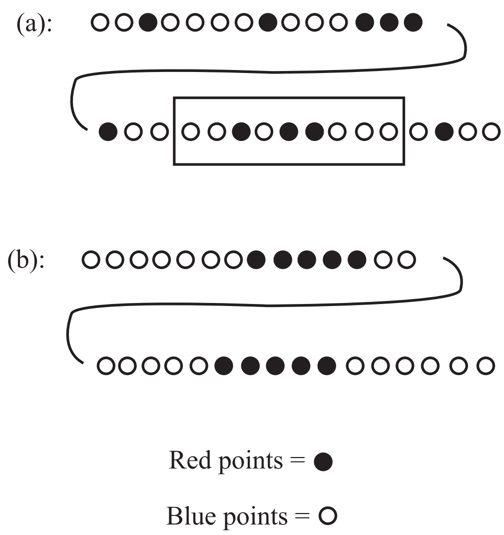

Fig. 1. (a): An interval containing 3 red points and 6 blue points; (b): A configuration that has no interval containing exactly 4 red points and 8 blue points.

Theorem 1 is an easy consequence of the following five lemmas.

For a configuration with red and blue points on a line, we denote by R and B the sets of red points and blue points, respectively. A configuration X with n red points and m blue points on the line is expressed as

$$\{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_{n+m}\},$$

where each x_i denotes a red point or a blue point ordered from left to right. The configuration X is also expressed as

$$R(1) \cup B(1) \cup \cdots \cup R(s) \cup B(s),$$

where $R(i)$ and $B(i)$ denote disjoint subsets of R and B, respectively, and some of them may be empty sets. For a set Y, we denote by $|Y|$ the cardinality of Y.

Lemma 2 *If*

$$m \leq \left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1\right)(h-1), \tag{2}$$

then there exists a configuration with n red points and m blue points that has no interval containing exactly k red points and h blue points.

PROOF. Let $t = \lfloor \frac{n}{k-1} \rfloor$. Then $n \leq (t+1)(k-1)$, and $m \geq t(h+1)$ by (4). Hence we can obtain the following configuration with n red points and m blue points:

$$R(1) \cup B(1) \cup \cdots \cup R(t+1) \cup B(t+1),$$

where $|R(i)| \leq k-1$ for every $1 \leq i \leq t+1$, $|R(1) \cup \cdots \cup R(t+1)| = n$, $|B(i)| = h+1$ for every $1 \leq i \leq t$, $|B(t+1)| = m - (h+1)t \geq 0$ and $|B(1) \cup \cdots \cup B(t+1)| = m$. Then this configuration obviously has no interval containing exactly k red points and h blue points since every interval containing k red points must include $B(j)$ for some $1 \leq j \leq t$.

Lemma 3 *If*

$$m > \left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1\right)(h-1), \tag{3}$$

then every configuration with n red points and m blue points on a line has an interval containing exactly k red points and at least h blue points.

PROOF. Let $t = \lfloor \frac{n}{k+1} \rfloor$. Let X be a configuration with n red points and m blue points. Suppose that X has no desired interval. Namely, we assume that every interval containing exactly k red points has at most $h-1$ blue points.

Let $r_1, r_2, \cdots, r_n$ be the red points of X ordered from left to right. For integers $1 \leq i < j \leq n$, let $I(i,j)$ denote an open interval (r_i, r_j), and let $B(i,j)$ denote the set of blue points contained in $I(i,j)$. Furthermore, $B(-\infty, i)$ denotes the set of blue points contained in the open interval $(-\infty, r_i)$, and $B(i, \infty)$ is defined analogously. Then for any integer $1 \leq s \leq t-1$, $I(s(k+1), (s+1)(k+1))$ contains exactly k red points $\{r_j \mid s(k+1)+1 \leq j \leq (s+1)(k+1)-1\}$, and thus $|B(s(k+1), (s+1)(k+1))| \leq h-1$ by our assumption. Similarly, an open interval $(-\infty, r_{k+1})$ contains exactly k red points, and thus $|B(-\infty, k+1)| \leq h-1$. Moreover, since $n < (t+1)(k+1)$, $I(t(k+1), \infty)$ has at most k red points, and thus $B(t(k+1), \infty) \leq h-1$. Therefore

$$\begin{aligned} |B| &\leq |B(-\infty, k+1) \cup B(k+1, 2(k+1)) \cup \cdots \\ &\quad \cup B(t(k+1), \infty)| \\ &\leq (t+1)(h-1). \end{aligned}$$

This contradicts (3). Consequently the lemma is proved.

Lemma 4 *If $2 \leq k$ and*

$$m \geq \left\lfloor \frac{n}{k-1} \right\rfloor (h+1), \tag{4}$$

then there exists a configuration with n red points and m blue points on a line that has no interval containing exactly k red points and h blue points.

PROOF. Let $t = \lfloor \frac{n}{k-1} \rfloor$. Then $n \leq (t+1)(k-1)$, and $m \geq t(h+1)$ by (4). Hence we can obtain the following configuration with n red points and m blue points:

$$R(1) \cup B(1) \cup \cdots \cup R(t+1) \cup B(t+1),$$

where $|R(i)| \leq k-1$ for every $1 \leq i \leq t+1$, $|R(1) \cup \cdots \cup R(t+1)| = n$, $|B(i)| = h+1$ for every $1 \leq i \leq t$, $|B(t+1)| = m - (h+1)t \geq 0$ and $|B(1) \cup \cdots \cup B(t+1)| = m$. Then this configuration obviously has no interval containing exactly k red points and h blue points since every interval containing k red points must include $B(j)$ for some $1 \leq j \leq t$.

Lemma 5 *If $2 \leq k$ and*

$$m < \left\lfloor \frac{n}{k-1} \right\rfloor (h+1), \tag{5}$$

then every configuration with n red points and m blue points on a line has an interval containing exactly k red points and at most h blue points.

PROOF. Let $t = \lfloor \frac{n}{k-1} \rfloor$. Let X be a configuration with n red points and m blue points. Suppose that X has no desired interval. Namely, we assume that every interval containing exactly k red points has at least $h+1$ blue points.

Let $r_1, r_2, \cdots, r_n$ be the red points of X ordered from left to right. For integers $1 \leq i < j \leq n$, let $I[i,j]$, denote a closed interval $[r_i, r_j]$, and let $B'(i,j)$ denote the set of blue points contained in $I[i,j]$.

Then for any integer $0 \leq s \leq t-2$, $I[k+s(k-1), k+(s+1)(k-1)]$ contains exatly k red points $\{r_j \mid k+s(k-1) \leq j \leq k+(s+1)(k-1))\}$, and thus $|B'(k+s(k-1), k+(s+1)(k-1))| \geq h+1$ by our assumption. Similarly, we have $|B'(1,k)| \geq h+1$. Therefore

$$\begin{aligned}|B| &\geq |B'(1,k) \cup B'(k, k+(k-1)) \cup \cdots \\ &\quad \cup B'(k+(t-2)(k-1), k+(t-1)(k-1))| \\ &\geq t(h+1).\end{aligned}$$

This contradicts (5). Consequently the lemma is proved.

Lemma 6 *Consider a configuration with n red points and m blue points on a line. Suppose that there exists two intervals I and J such that both I and J contain exactly k red points respectively, I contains at most h blue points, and that J contains at least h blue points. Then there exists an interval that contains exactly k red points and h blue points.*

PROOF. If the sets of red points contained in I and J, respectively, are the same, then the lemma immediately follows. Thus we may assume that $I \cap R \neq J \cap R$, where R denote the set of n red points. Without loss of generality, we may assume that the leftmost red point of I lies to the left of J.

We shall show that we can move I to J step by step in such a way that the number of red points is a constant k and the number of blue points changes ± 1 at each step. We first remove the blue points left to the leftmost red point of I one by one, and then add the consecutive blue points lying to the right of I one by one, and denote the resulting interval by I_1 (Fig. 3). We next simultaneously remove the leftmost red point of I_1 and add the red point lying to the right of I_1, and get an interval I_2, which also contains exactly k red points and whose blue points are the same as those in I_1 (Fig. 3). By repeating this procedure, we can get an interval whose red point set is equal to that of J. Therefore, we can move I to J in the desired way. Consequently, we can find the required interval, which contains exactly k red points and h blue points.

2. A balanced interval on a circle

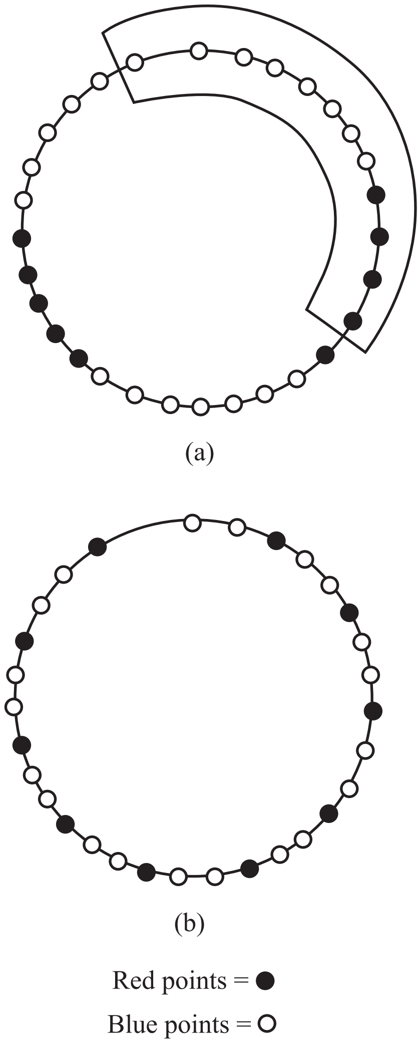

Fig. 2. (a): An interval containing 4 red points and 8 blue points; (b): A configuration that has no interval containing exactly 4 red points and 5 blue points.

In this section, we consider the following theorem, and give its example in Figure reffig:2

Theorem 7 *Let n, m, k, h be integers such that $1 \leq n \leq m$, $1 \leq k \leq n$ and $1 \leq h \leq m$. Then for any n red points and m blue points on a circle in general position (i.e., no two points lie on the same position.), there exists an interval that con-*

tains precisely k red points and h blue points if and only if

$$\frac{n}{k+1}(h-1) < m < \frac{n}{k-1}(h+1), \qquad (6)$$

where the rightmost term is an infinite number when $k = 1$.

Theorem 8 can be proved by showing similar lemmas as in the case of line. We conclude the paper by the next conjecture.

Conjecture 8 *Let n, m, k, h be integers such that $1 \leq n \leq m$, $1 \leq k \leq n$ and $1 \leq h \leq m$. Then for any n red points and m blue points in the plane in general position (i.e., no three points lie on the same.), there exists a wedge that contains precisely k red points and h blue points if and only if*

$$\frac{n}{k+2}(h-1) < m < \frac{n}{k-2}(h+1), \qquad (7)$$

where the rightmost term is an infinite number when $k = 1$.

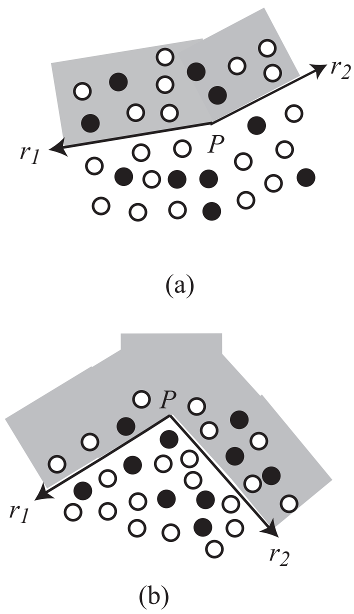

Fig. 3. Wedges containing 4 red points and 8 blue points.

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