Constructing bispectral orthogonal polynomials from the classical discrete families of Charlier, Meixner and Krawtchouk[1]

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[1] joint work with Antonio J. Durán

Outline

1. Introduction
   - Classical discrete orthogonal polynomials
   - Krall orthogonal polynomials
2. Methodology
   - $\mathcal{D}$-operators
   - Choice of polynomials
   - Identifying the measure
3. Examples
   - Charlier, Meixner and Krawtchouk polynomials

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Discrete orthogonal polynomials

A system of polynomials $(p_n)_n$ is orthogonal with respect to a discrete measure $\omega(x) = \sum_{x\in\mathcal{S}} a_x \delta_{t_x}, \mathcal{S} \subset \mathbb{N}$ if

$$\langle p_n, p_m\rangle_\omega = \sum_{x\in\mathcal{S}} a_x p_n(t_x) p_m(t_x) = \|p_n\|_\omega^2 \delta_{nm}, \quad n, m \geq 0$$

Every family of OP's $(p_n)_n$ satisfy a three-term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x), \quad n \geq 1$$

where $a_n, c_n \neq 0$, $b_n \in \mathbb{R}$ and $p_0(x) = 1, p_{-1}(x) = 0$.
Jacobi operator (tridiagonal):

$$Jp = \begin{pmatrix} b_0 & a_1 & & & \\ c_1 & b_1 & a_2 & & \\ & c_2 & b_2 & a_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix} = x \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix} = xp, \quad x \in \mathcal{S}$$

The converse result is also true (Favard's or spectral theorem).

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Classical families

If we set

$$\Delta f(x) = f(x+1) - f(x), \quad \nabla f(x) = f(x) - f(x-1)$$

the classification problem is to find discrete OP's $(p_n)_n$

Lancaster, 1941

$$\sigma(x)\Delta\nabla p_n(x) + \tau(x)\Delta p_n(x) + \lambda_n p_n(x) = 0, \quad x \in \mathcal{S} \subset \mathbb{N}$$
$$\deg\sigma \leq 2, \quad \deg\tau = 1$$

In other words, if we call the shift operator

$$\mathfrak{S}_j f(x) = f(x+j)$$

the difference equation reads

$$[\sigma(x) + \tau(x)]\mathfrak{S}_1 p_n(x) - [2\sigma(x) + \tau(x)]\mathfrak{S}_0 p_n(x)$$
$$+ \sigma(x)\mathfrak{S}_{-1} p_n(x) + \lambda_n p_n(x) = 0, \quad x \in \mathcal{S} \subset \mathbb{N}$$

Classical families

- Charlier (Poisson): $\mathcal{S} = \{0, 1, 2, \dots\}$.

$$\omega_a(x) = \sum_{x=0}^{\infty} \frac{a^x}{x!}\delta_x, \quad a > 0$$

$$ac_n^a(x+1) - (x+a)c_n^a(x) + xc_n^a(x-1) = -nc_n^a(x)$$

- Meixner (Pascal, Geometric): $\mathcal{S} = \{0, 1, 2, \dots\}$.

$$\omega_{a,c}(x) = \Gamma(c)(1-a)^c \sum_{x=0}^{\infty} \frac{(c)_x a^x}{x!}\delta_x, \quad 0 < a < 1, \quad c > 0$$

$(i)_j = i(i+1)\cdots(i+j-1)$ is the Pochhammer symbol

$$a(x+c)m_n^{a,c}(x+1) - (x + a(x+c))m_n^{a,c}(x) + xm_n^{a,c}(x-1) = n(a-1)m_n^{a,c}(x)$$

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$$\begin{aligned} a(x+c) m_n^{a,c}(x+1) &- (x + a(x+c)) m_n^{a,c}(x) \\ &+ x m_n^{a,c}(x-1) = n(a-1) m_n^{a,c}(x) \end{aligned}$$

Classical families

- Krawtchuok (Binomial, Bernoulli): $\mathcal{S} = \{0, 1, 2, \ldots N-1\}$.

$$\omega_{a,N}(x) = \frac{1}{(1+a)^{N-1}} \sum_{x=0}^{N-1} \binom{N-1}{x} a^x \delta_x, \quad a > 0$$

$$a(N-x-1)k_n^{a,N}(x+1) - [x + a(N-x-1)]k_n^{a,N}(x) + xk_n^{a,N}(x-1) = -n(1+a)k_n^{a,N}(x)$$

- Hahn (Hypergeometric): $\mathcal{S} = \{0, 1, 2, \ldots N\}$

$$\omega_{a,b,N}(x) = \sum_{x=0}^{N} \binom{a+x}{x} \binom{b+N-x}{N-x} \delta_x, \quad a, b > -1, \quad a, b < -N$$

$$B(x)h_n^{a,b,N}(x+1) - [B(x) + D(x)]h_n^{a,b,N}(x) + D(x)h_n^{a,b,N}(x-1) = n(n+a+b+1)h_n^{a,b,N}(x)$$

where $B(x) = (x+a+1)(x-N)$ and $D(x) = x(x-b-N-1)$.

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Krall polynomials (continuous case)

GOAL (H.L. Krall, 1939): find families of OP's $(q_n)_n$ which are also eigenfunctions of a higher-order differential operator of the form

$$D_c = \sum_{j=0}^{2m} h_j(x)\frac{d^j}{dx^j}, \quad \deg(h_j) \leq j \quad \Rightarrow \quad D_c(q_n) = \lambda_n q_n$$

A.M. Krall, Littlejohn, Koornwinder, Koekoek's, Lesky, Grünbaum, Heine, Iliev, Horozov, Zhedanov, etc (80's, 90's, 00's).

$(q_n)_n$ are typically orthogonal with respect to the measure

$$\omega(x) + \sum_{j=0}^{m-1} a_j \delta_{x_0}^{(j)}, \quad a_j \in \mathbb{R}$$

where ω is a (modified) classical weight and x_0 is an endpoint of the support of orthogonality of ω.

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Krall polynomials (discrete case)

The same question arise in the discrete setting, i.e. find families of OP's $(q_n)_n$ which are also eigenfunctions of a higher order difference operator

$$D_d = \sum_{j=r}^{s} h_j(x)\mathfrak{S}_j, \quad h_s, h_r \neq 0, \quad \Rightarrow \quad D_d(q_n) = \lambda_n q_n$$

Bavinck-van Haeringen-Koekoek, 1994: adding deltas at the endpoints of the support does not work (infinite order difference operator).

Surprisingly, it has not been until very recently (Durán, 2012) when the first examples appeared. Also $s - r = 2m$.

$(q_n)_n$ are typically orthogonal with respect to the measure

$$\omega^F(x) = \prod_{f \in F} (x - f)\, \omega(x)$$

where ω is a discrete classical weight and F is a finite set of numbers. This is also called a Christoffel transform of ω.

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Conjectures (Durán, 2012)

For a finite set F consider $r_F = \sum_{f\in F} f - \frac{n_F(n_F-1)}{2} + 1$, where $n_F = \#(F)$.

Conjecture A: Let ω_a be the Charlier weight and consider (F finite)

$$\omega_a^F = \prod_{f\in F}(x-f)\omega_a$$

The OP's $(q_n)_n$ with respect to ω_a^F are eigenfunctions of a higher-order difference operator with $-s = r = r_F$.

Conjecture B: Let $\omega_{a,c}$ be the Meixner weight and consider (F_1, F_2 finite)

$$\omega_{a,c}^{F_1,F_2} = \prod_{f\in F_1}(x+c+f)\prod_{f\in F_2}(x-f)\omega_{a,c}$$

The OP's $(q_n)_n$ with respect to $\omega_a^{F_1,F_2}$ are eigenfunctions of a higher-order difference operator with $-s = r = r_{F_1} + r_{F_2} - 1$.

Conjecture C: Let $\omega_{a,N}$ be the Krawtchouk weight and consider (F_1, F_2 finite)

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$\mathcal{D}$-OPERATORS

Let $\mathcal{A}$ be an algebra of (differential or difference) operators and $(p_n)_n$ a family of polynomials such that there exists $D_p \in \mathcal{A}$ with $D_p(p_n) = np_n$. Given a sequence of numbers $(\varepsilon_n)_n$, let us consider the operator

$$\mathcal{D}(p_n) = \sum_{j=1}^{n} (-1)^{j+1} \varepsilon_n \cdots \varepsilon_{n-j} p_{n-j} = \varepsilon_n p_{n-1} - \varepsilon_n \varepsilon_{n-1} p_{n-2} + \cdots$$

We say that $\mathcal{D}$ is an $\mathcal{D}$-operator associated with $\mathcal{A}$ and $(p_n)_n$ if $\mathcal{D} \in \mathcal{A}$.

- Laguerre: $\varepsilon_n = -1 \Rightarrow \mathcal{D} = \frac{d}{dx}$.
- Charlier: $\varepsilon_n = 1 \Rightarrow \mathcal{D} = \nabla$.
- Meixner:
  $\varepsilon_n^1 = \frac{a}{1-a} \Rightarrow \mathcal{D}_1 = \frac{a}{1-a}\Delta, \quad \varepsilon_n^2 = \frac{1}{1-a} \Rightarrow \mathcal{D}_2 = \frac{1}{1-a}\nabla.$
- Krawtchouk:
  $\varepsilon_n^1 = \frac{1}{1-a} \Rightarrow \mathcal{D}_1 = \frac{1}{1-a}\nabla, \quad \varepsilon_n^2 = -\frac{a}{1-a} \Rightarrow \mathcal{D}_2 = -\frac{a}{1-a}\Delta.$

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$\mathcal{D}$-OPERATORS

Let $\mathcal{A}$ be an algebra of (differential or difference) operators and $(p_n)_n$ a family of polynomials such that there exists $D_p \in \mathcal{A}$ with $D_p(p_n) = np_n$. Given a sequence of numbers $(\varepsilon_n)_n$, let us consider the operator

$$\mathcal{D}(p_n) = \sum_{j=1}^{n} (-1)^{j+1} \varepsilon_n \cdots \varepsilon_{n-j} p_{n-j} = \varepsilon_n p_{n-1} - \varepsilon_n \varepsilon_{n-1} p_{n-2} + \cdots$$

We say that $\mathcal{D}$ is an $\mathcal{D}$-operator associated with $\mathcal{A}$ and $(p_n)_n$ if $\mathcal{D} \in \mathcal{A}$.

- Laguerre: $\varepsilon_n = -1 \Rightarrow \mathcal{D} = \dfrac{d}{dx}$.
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$\mathcal{D}$-operators

Theorem (Durán, 2013)

Let $\mathcal{A}$, $(p_n)_n$, $D_p(p_n) = np_n$, $(\varepsilon_n)_n$ and $\mathcal{D}$.
For an arbitrary polynomial R such that $R(n) \neq 0$, $n \geq 0$, we define a new polynomial P by

$$P(x) - P(x-1) = R(x)$$

and a sequence of polynomials $(q_n)_n$ by $q_0 = 1$ and

$$q_n = p_n + \beta_n p_{n-1}, \quad n \geq 1$$

where the numbers β_n, $n \geq 0$, are given by

$$\beta_n = \varepsilon_n \frac{R(n)}{R(n-1)}, \quad n \geq 1$$

Then there exist $D_q \in \mathcal{A}$ sucht that $D_q(q_n) = P(n)q_n$ where

$$D_q = P(D_p) + \mathcal{D}R(D_p)$$

$\mathcal{D}$-OPERATORS

GOAL: Extend the previous Theorem for the case that we consider a linear combination of $m+1$ consecutive p_n's:

$$q_n = p_n + \beta_{n,1} p_{n-1} + \beta_{n,2} p_{n-2} + \cdots + \beta_{n,m} p_{n-m}$$

Let $R_1, R_2, \ldots, R_m$ be m arbitrary polynomials and m $\mathcal{D}$-operators $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_m$ defined by the sequences $(\varepsilon_n^h)_n$, $h = 1, \ldots, m$.

Define the auxiliary functions $\xi_{n,i}^h$ by

$$\xi_{n,i}^h = \varepsilon_n^h \varepsilon_{n-1}^h \cdots \varepsilon_{n-i+1}^h$$

and assume that the following Casorati determinant never vanish $(n \geq 0)$

$$\Omega(n) = \begin{vmatrix} \xi_{n-1,m-1}^1 R_1(n-1) & \xi_{n-2,m-2}^1 R_1(n-2) & \cdots & R_1(n-m) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{n-1,m-1}^m R_m(n-1) & \xi_{n-2,m-2}^m R_m(n-2) & \cdots & R_m(n-m) \end{vmatrix} \neq 0$$

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Now consider the sequence of polynomials $(q_n)_n$ defined by

$$q_n(x) = \begin{vmatrix} p_n(x) & -p_{n-1}(x) & \cdots & (-1)^m p_{n-m}(x) \\ \xi^1_{n,m} R_1(n) & \xi^1_{n-1,m-1} R_1(n-1) & \cdots & R_1(n-m) \\ \vdots & \vdots & \ddots & \vdots \\ \xi^m_{n,m} R_m(n) & \xi^m_{n-1,m-1} R_m(n-1) & \cdots & R_m(n-m) \end{vmatrix}$$

Observation: q_n is a linear combination of of $m+1$ consecutive p_n's.

Define for $h = 1, \ldots, m$, the following functions

$$M_h(x) = \sum_{j=1}^{m} (-1)^{h+j} \xi^h_{x,m-j} \det\left(\xi^l_{x+j-r,m-r} R_l(x+j-r)\right)_{\left\{\begin{smallmatrix} l \neq h \\ r \neq j \end{smallmatrix}\right\}}$$

Observation: M_h are linear combinations of adjoint determinants of $\Omega(x)$.

If we assume that $\Omega(x)$ and $M_h(x)$ are polynomials in x, then $\exists\ D_q \in \mathcal{A}$ with $D_q(q_n) = P(n)q_n$ and $P(x) - P(x-1) = \Omega(x)$, where

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Choice of $R_1, R_2, \ldots, R_m$

GOAL: Make $(q_n)_n$ bispectral (we already have $D_q(q_n) = \lambda_n q_n$).

For that we have to make an appropriate choice of the arbitrary polynomials $R_1, R_2, \ldots, R_m$. This choice is based on the following recurrence formula $(h = 1, \ldots, m)$:

$$\varepsilon^h_{n+1} a_{n+1} R^h_j(n+1) - b_n R^h_j(n) + \frac{c_n}{\varepsilon^h_n} R^h_j(n-1) = (\eta_h j + \kappa_h) R^h_j(n), \quad n \in \mathbb{Z}$$

where η_h and κ_h are real numbers independent of n and j, $(a_n)_{n\in\mathbb{Z}}$, $(b_n)_{n\in\mathbb{Z}}$, $(c_n)_{n\in\mathbb{Z}}$ are the coefficients in the TTRR for the OP's $(p_n)_n$, and $(\varepsilon^h_n)_n$ defines a $\mathcal{D}$-operator for $(p_n)_n$.

Classical discrete family	$\mathcal{D}$-operators	$R_j(x)$
Charlier: c^a_n, $n \geq 0$	∇	$c^{-a}_j(-x-1)$, $j \geq 0$
Meixner: $m^{a,c}_n$, $n \geq 0$	$\frac{a}{1-a}\Delta$	$m^{1/a,2-c}_j(-x-1)$, $j \geq 0$
	$\frac{1}{1-a}\nabla$	$m^{a,2-c}_j(-x-1)$, $j \geq 0$
Krawtchouk: $k^{a,N}_n$, $n \geq 0$	$\frac{1}{1+a}\nabla$	$k^{a,-N}_j(-x-1)$, $j \geq 0$
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Identifying the measure

Given a set G of m positive integers, $G = \{g_1, \ldots, g_m\}$ we then define the sequence of polynomials $(q_n^G)_n$ by

$$q_n^G(x) = \begin{vmatrix} p_n(x) & -p_{n-1}(x) & \cdots & (-1)^m p_{n-m}(x) \\ \xi_{n,m}^1 R_{g_1}^1(n) & \xi_{n-1,m-1}^1 R_{g_1}^1(n-1) & \cdots & R_{g_1}^1(n-m) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{n,m}^m R_{g_m}^m(n) & \xi_{n-1,m-1}^m R_{g_m}^m(n-1) & \cdots & R_{g_m}^m(n-m) \end{vmatrix}$$

$(q_n^G)_n$ will be orthogonal w.r.t a Christoffel transform of ω (or several)

$$\omega^F(x) = \prod_{f \in F} (x - f)\, \omega(x)$$

How is the set G related with the set F?: G will be identified by one of the following sets:

$$I(F) = \{1, 2, \ldots, f_k\} \setminus \{f_k - f, f \in F\},$$
$$J_h(F) = \{0, 1, 2, \ldots, f_k + h - 1\} \setminus \{f - 1, f \in F\}, \quad h \geq 1$$

where $f_k = \max F$ and $k = \#(F)$.

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Outline

Charlier polynomials

Let $F \subset \mathbb{N}$ be finite and consider $G = I(F) = \{g_1, \dots, g_m\}$.
Let ω_a be the Charlier measure and $(c_n^a)_n$ its sequence of OP's. Assume that $\Omega_G(n) = \det\left(c_{g_l}^{-a}(-n-j-1)\right)_{l,j=1}^{m} \neq 0$.

If we define $(q_n)_n$ by

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then the polynomials $(q_n)_n$ are orthogonal with respect to the measure

$$\tilde{\omega}_a^F = \prod_{f \in F} (x + f_k + 1 - f)\omega_a(x + f_k + 1)$$

and they are eigenfunctions of a higher order difference operator D_q with $-s = r = \sum_{f \in F} f - \frac{n_F(n_F-1)}{2} + 1$, where $n_F = \#(F)$ and $f_k = \max F$.

Proof of Conjecture A: $\omega_a^F = a^{f_k+1}\tilde{\omega}_a^F(x - f_k - 1)$.

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and they are eigenfunctions of a higher order difference operator D_q with $-s = r = \sum_{f\in F} f - \frac{n_F(n_F-1)}{2} + 1$, where $n_F = \#(F)$ and $f_k = \max F$.

Proof of Conjecture A: $\omega_a^F = a^{f_k+1}\tilde{\omega}_a^F(x-f_k-1)$.

Charlier polynomials

Let $F \subset \mathbb{N}$ be finite and consider $G = I(F) = \{g_1, \ldots, g_m\}$.
Let ω_a be the Charlier measure and $(c_n^a)_n$ its sequence of OP's. Assume that $\Omega_G(n) = \det\left(c_{g_l}^{-a}(-n-j-1)\right)_{l,j=1}^{m} \neq 0$.
If we define $(q_n)_n$ by

$$q_n(x) = \begin{vmatrix} c_n^a(x) & -c_{n-1}^a(x) & \cdots & (-1)^m c_{n-m}^a(x) \\ c_{g_1}^{-a}(-n-1) & c_{g_1}^{-a}(-n) & \cdots & c_{g_1}^{-a}(-n+m-1) \\ \vdots & \vdots & \ddots & \vdots \\ c_{g_m}^{-a}(-n-1) & c_{g_m}^{-a}(-n) & \cdots & c_{g_m}^{-a}(-n+m-1) \end{vmatrix}$$

then the polynomials $(q_n)_n$ are orthogonal with respect to the measure

$$\tilde{\omega}_a^F = \prod_{f\in F}(x + f_k + 1 - f)\omega_a(x + f_k + 1)$$

and they are eigenfunctions of a higher order difference operator D_q with $-s = r = \sum_{f\in F} f - \frac{n_F(n_F-1)}{2} + 1$, where $n_F = \#(F)$ and $f_k = \max F$.

Proof of Conjecture A: $\omega_a^F = a^{f_k+1}\tilde{\omega}_a^F(x - f_k - 1)$.

Charlier polynomials: explicit example

Let $a = 1$, $F = \{1, 3\}$, $G = I(F) = \{1, 3\}$.

$\frac{q_n^G}{\Omega(n)} = c_n^1 + \beta_{n,1} c_{n-1}^1 + \beta_{n,2} c_{n-2}^1$ are orthogonal w.r.t

$$\tilde{\omega}_1^F = (x+3)(x+1)\omega_1(x+4)$$

The difference operator (of order 8) satisfying $D_q(q_n^G) = P(n) q_n^G$ is

$$D_q = P(D_1) + M_1(D_1)\nabla R_1(D_1) + M_2(D_1)\nabla R_2(D_1)$$

where

$$D_1 = -x\mathfrak{S}_{-1} + (x+1)\mathfrak{S}_0 - \mathfrak{S}_1, \quad D_1(c_n^1) = nc_n^1, \quad n \geq 0$$

$$R_1(x) = -x, \qquad R_2(x) = -\frac{1}{6}(x^3 + 3x^2 + 5x + 2)$$

$$M_1(x) = x^2 + 2x + 2, \qquad M_2(x) = -2$$

$$P(x) = -\frac{x}{12}(x^3 - 2x^2 - x - 2)$$

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Meixner polynomials

In this case have two different $\mathcal{D}$-operators. That means that we will have to consider two sets of positive integers $F_1, F_2 \subset \mathbb{N}$.

Consider $H = J_h(F_1) = \{h_1, \dots, h_{m_1}\}, K = I(F_2) = \{k_1, \dots, k_{m_2}\}$ and $m = m_1 + m_2$, $\omega_{a,c}$ the Meixner measure and $(m_n^{a,c})_n$.

If we define $(q_n)_n$ by

$$q_n(x) = \begin{vmatrix} \frac{(1-a)^m m_n^{a,c}(x)}{a^m} & \frac{-(1-a)^{m-1} m_{n-1}^{a,c}(x)}{a^{m-1}} & \cdots & (-1)^m m_{n-m}^{a,c}(x) \\ m_{h_1}^{1/a,2-c}(-n-1) & m_{h_1}^{1/a,2-c}(-n) & \cdots & m_{h_1}^{1/a,2-c}(-n+m-1) \\ \vdots & \vdots & \ddots & \vdots \\ m_{h_{m_1}}^{1/a,2-c}(-n-1) & m_{h_{m_1}}^{1/a,2-c}(-n) & \cdots & m_{h_{m_1}}^{1/a,2-c}(-n+m-1) \\ \frac{m_{k_1}^{a,2-c}(-n-1)}{a^m} & \frac{m_{k_1}^{a,2-c}(-n)}{a^{m-1}} & \cdots & m_{k_1}^{a,2-c}(-n+m-1) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{m_{k_{m_2}}^{a,2-c}(-n-1)}{a^m} & \frac{m_{k_{m_2}}^{a,2-c}(-n)}{a^{m-1}} & \cdots & m_{k_{m_2}}^{a,2-c}(-n+m-1) \end{vmatrix}$$

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Meixner polynomials

Then the polynomials $(q_n)_n$ are orthogonal with respect to the measure

$$\tilde{\omega}_{a,c}^{F_1,F_2,h} = \prod_{f\in F_1}(x+c-f)\prod_{f\in F_2}(x+f_{2,M}+1-f)\omega_{a,c-f_{1,M}-f_{2,M}-h-1}(x+f_{2,M}+1)$$

and they are eigenfunctions of a higher order difference operator D_q with

$$-s = r = \sum_{f\in F_2} f - \sum_{f\in F_1} f - \frac{n_{F_1}(n_{F_1}-1)}{2} - \frac{n_{F_2}(n_{F_2}-1)}{2} + n_{F_1}(f_{1,M}+h)+1$$

where $n_{F_i} = \#(F_i)$ and $f_{i,M} = \max F_i$, $i = 1, 2$.

Proof of Conjecture B: Write $\tilde{F}_1 = \{f_{1,M} - f + 1, f \in F_1\}$, $\tilde{c} = c + f_{1,M} + f_{2,M} + 2$ and $h = \min F_1$. In particular $J_h(\tilde{F}_1) = I(F_1)$. Therefore we have

$$\omega_{a,c}^{F_1,F_2} = (1-a)^{c-\tilde{c}}\tilde{\omega}_{a,\tilde{c}}^{\tilde{F}_1,F_2,h}(x - f_{2,M} - 1)$$

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Krawtchouk polynomials

Again we have two different $\mathcal{D}$-operators. Consider $F_1, F_2 \subset \mathbb{N}$ finite and $K = I(F_1) = \{k_1, \ldots, k_{m_2}\}$, $H = J_h(F_2) = \{h_1, \ldots, h_{m_1}\}$, $m = m_1 + m_2$, $\omega_{a,N}$ the Krawtchouk measure and $(k_n^{a,N})_n$. We assume that $f_{1,M}, f_{2,M} < N/2$ (so that $F_1 \cap \{N-1-f, f \in F_2\} = \emptyset$), where $f_{i,M} = \max F_i$.

If we define $(q_n)_n$ by

$$q_n(x) = \begin{vmatrix} (1+a)^m k_n^{a,N}(x) & -(1+a)^{m-1} k_{n-1}^{a,N}(x) & \cdots & (-1)^m k_{n-m}^{a,N}(x) \\ k_{k_1}^{a,-N}(-n-1) & k_{k_1}^{a,-N}(-n) & \cdots & k_{k_1}^{a,-N}(-n+m-1) \\ \vdots & \vdots & \ddots & \vdots \\ k_{k_{m_1}}^{a,-N}(-n-1) & k_{k_{m_1}}^{a,-N}(-n) & \cdots & k_{k_{m_1}}^{a,-N}(-n+m-1) \\ (-a)^m k_{h_1}^{1/a,-N}(-n-1) & (-a)^{m-1} k_{h_1}^{1/a,-N}(-n) & \cdots & k_{h_1}^{1/a,-N}(-n+m-1) \\ \vdots & \vdots & \ddots & \vdots \\ (-a)^m k_{h_{m_2}}^{1/a,-N}(-n-1) & (-a)^{m-1} k_{h_{m_2}}^{1/a,-N}(-n) & \cdots & k_{h_{m_2}}^{1/a,-N}(-n+m-1) \end{vmatrix}$$

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Then the polynomials $(q_n)_n$ are orthogonal with respect to the measure

$$\tilde{\omega}_{a,N}^{F_1,F_2,h} = \prod_{f\in F_1}(x+f_{1,M}+1-f)\prod_{f\in F_2}(N-x-1+f)\omega_{a,N+f_{1,M}+f_{2,M}+h+1}(x+f_{1,M}+1)$$

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where $n_{F_i} = \#(F_i)$ and $f_{i,M} = \max F_i$, $i = 1, 2$.

Proof of Conjecture C: Write $\tilde{F}_1 = \{f_{1,M} - f + 1, f \in F_1\}$, $\tilde{F}_2 = \{f_{1,M} + f_{2,M} - f + 2, f \in F_2\}$, $\tilde{N} = N - f_{1,M} - f_{2,M} - 2$ and $h = \min F_1$. In particular $J_h(\tilde{F}_2) = I(F_2)$. Therefore we have

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$$\omega_{a,N}^{F_1,F_2} = \tilde{\omega}_{a,\tilde{N}}^{\tilde{F}_1,\tilde{F}_2,h}(x - f_{1,M} - 1)$$