On a initial-boundary Q-Tensor problem related to Liquid Crystals

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1. Nematic Liquid Crystals
   - A simplified model by F. H. Lin
   - Some known results
2. Models with Stretching Terms
   - Nematic Liquid Crystals with Stretching Terms
   - A generic Q-tensor model
3. Some analytical results for Q-tensor models
   - Weak existence
   - Weak/strong uniqueness
   - Maximum Principle
   - Strong solution ?
   - Local weak regularity for $(\partial_t \mathbf{u}, \partial_t Q)$ and uniqueness

Complex Fluids

It is not possible to decouple microscopic and macroscopic effects.

- Fluids with elastic properties. They possesses intermediate properties between solids and liquids. Examples: liquid crystals, polymers (macromolecules), ...
- Phase-field models.
  Examples: multi-fluids (mixture of fluids), multi-phases (solidification), ...

These complex materials have practical utilities because its microstructure can be handled in order to produce good mechanical, optical or thermic properties.

Liquid Crystals

Liquid crystals (LC) are intermediate phases between solid and liquid; at the macroscopic level, they are (viscous) liquids but their molecules have a anisotropic order due to their elastic properties.

- Nematic Liquid Crystals have an orientation order.
- Smectic Liquid Crystals have also a positional order (arranged by layers).

The derivation and the analysis falls into a general energetic variational framework for complex fluids with elastic effects due to the presence of nontrivial microstructures, coupling

- Navier-Stokes equations for the velocity and pressure.
- Partial Differential Equations for the microscopic variable (called order parameter)

Figura: Types of Liquid Crystals

Nematic Liquid Crystals - Macroscopic model

Director field $\mathbf{d}(t, \mathbf{x})$, representing the average orientation of the liquid crystal molecules.
The shear stress tensor depends on elastic and viscous effects (Ericksen-Leslie theory, 1980s):

$$\sigma = \sigma^d(D, \mathbf{d}) + \lambda\, \sigma^e(\mathbf{d}),$$

where σ^d is the dissipative tensor, σ^e the elastic tensor and $\lambda > 0$ a “balance” coefficient.
Then, equations for equilibrium of forces remains as:

$$D_t \mathbf{u} + \nabla p - \nabla \cdot \sigma^d - \lambda \nabla \cdot \sigma^e = 0 \quad \text{in } Q = (0, T) \times \Omega.$$

Nematic Liquid Crystals: Microscopic Model (of Allen-Canh's type)

Starting from the Ericksen-Leslie's formulation, a penalized model is presented by [F.H. Lin]:

$$\mathcal{D}_{\mathbf{t}}\mathbf{d} + \gamma \frac{\delta \mathbf{E_e}}{\delta \mathbf{d}} = \mathbf{0}, \quad \text{in } Q,$$

where

$$\frac{\delta E_e}{\delta \mathbf{d}} = -\Delta \mathbf{d} + \nabla_{\mathbf{d}} \mathbf{F}_\epsilon(\mathbf{d})$$

is the *Euler-Lagrange* equation associated to the elastic energy functional:

$$E_e(\mathbf{d}) = \left(\frac{1}{2} \int_\Omega |\nabla \mathbf{d}|^2 + \int_\Omega \mathbf{F}_\epsilon(\mathbf{d}) \right)$$

Ginzburg-Landau's functional:

$$\mathbf{F}_\epsilon(\mathbf{d}) = \frac{1}{4\,\epsilon^2}\left(|\mathbf{d}|^2 - 1\right)^2$$

such that $\mathbf{f}_\epsilon(\mathbf{d}) = \nabla_{\mathbf{d}}(\mathbf{F}_\epsilon(\mathbf{d}))$ for every $\mathbf{d} \in \mathbb{R}^3$, hence

$$\mathbf{f}_\epsilon(\mathbf{d}) = \frac{1}{\epsilon^2}\left(|\mathbf{d}|^2 - 1\right)\,\mathbf{d},$$

where $|\mathbf{d}|$ denotes the euclidean norm in $\mathbb{R}^3$ and $\epsilon > 0$ is a penalization parameter.

A simplified model [F. H. Lin]

Taking

$$\sigma^d = \nu\left(\nabla \mathbf{u} + \nabla \mathbf{u}^t\right) \qquad \text{(Stokes' law)}$$

$$\sigma^e = -(\nabla \mathbf{d})^t \nabla \mathbf{d} \quad \Rightarrow \quad \nabla \cdot \sigma^e = (\nabla \mathbf{d})^t(-\Delta \mathbf{d} + \mathbf{f}_\epsilon(\mathbf{d})) + \nabla\Big(\cdots\Big)$$

$$(NLC)\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla\widetilde{p} = -\lambda(\nabla\mathbf{d})^t\,\Delta\mathbf{d} \\ \nabla\cdot\mathbf{u} = 0 \\ \partial_t\mathbf{d} + \mathbf{u}\cdot\nabla\mathbf{d} + \gamma\left(-\Delta\mathbf{d} + \mathbf{f}_\epsilon(\mathbf{d})\right) = \mathbf{0} \\ \mathbf{u}|_{\partial\Omega} = \mathbf{0}, \quad \mathbf{d}|_{\partial\Omega} = \mathbf{d}_{\partial\Omega} \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \mathbf{d}|_{t=0} = \mathbf{d}_0 \end{cases}$$

Known results for $\mathbf{u} = \mathbf{0}$, $\mathbf{d} = \mathbf{d}_{\partial\Omega}$ on $\partial\Omega$,

[F. H. Lin & C. Liu '95]:

- Existence of global in time weak solution:
  $(\mathbf{u}, \mathbf{d}) \in L^\infty(0, T; \mathbf{L}^2(\Omega) \times \mathbf{H}^1(\Omega)) \cap L^2(0, T; \mathbf{H}^1(\Omega) \times \mathbf{H}^2(\Omega))$,
- Existence (and uniqueness) of local in time strong solution:
  $(\mathbf{u}, \mathbf{d}) \in L^\infty(0, T_*; \mathbf{H}^1(\Omega) \times \mathbf{H}^2(\Omega)) \cap L^2(0, T_*; \mathbf{H}^2(\Omega) \times \mathbf{H}^3(\Omega))$,
  for $T_* \le T$ small enough or $T_* = T$ ($\forall T > 0$) if ν large.

[F. G-G,M. A. Rodríguez-Bellido & M.A. Rojas-Medar '09]:
Regularity criteria for uniqueness and global in time regularity

[H.Wu'12]: Convergence of trajectories towards an unique equilibrium

A nematic model with Stretching

$$
(StNLC)\left\{
\begin{array}{rcl}
\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla\widetilde{p} + \lambda(\nabla\mathbf{d})^t\,\Delta\mathbf{d} & & \\
\lambda\nabla\cdot((-\Delta\mathbf{d} + \mathbf{f}_\epsilon(\mathbf{d}))\otimes\mathbf{d}) & = & 0 \\
\nabla\cdot\mathbf{u} & = & 0 \\
\partial_t\mathbf{d} + \mathbf{u}\cdot\nabla\mathbf{d} - \mathbf{d}\cdot\nabla\mathbf{u} + \gamma\,(-\Delta\mathbf{d} + \mathbf{f}_\epsilon(\mathbf{d})) & = & \mathbf{0} \\
\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \mathbf{d}|_{t=0} & = & \mathbf{d}_0
\end{array}
\right.
$$

Effects of the stretching term

- Lose of the maximum principle
- Existence of local in time strong solution only for space-periodic boundary conditions or large viscosity

Convergence of trajectories to a unique equilibrium

- [C. Liu, H. Wu & X. Xu '12]: for periodic boundary conditions
- [H. Petzeltová, E. Rocca & G. Schimperna'13]: for homogeneous Neumann boundary conditions

Free energy

Free energy operator:

$$\mathcal{E}(Q) = \int_\Omega \frac{\varepsilon}{2} |\nabla Q|^2 + F(Q)$$

where

$$F(Q) = \frac{a}{2}|Q|^2 - \frac{b}{3}(Q^2 : Q) + \frac{c}{4}|Q|^4 \quad \text{(non-convex)} \qquad (1)$$

Let $H(Q) = \dfrac{\delta \mathcal{E}(Q)}{\delta Q}$ be the variational derivative in $L^2(\Omega)$.

Velocity system

The variables describing the (QT)-model are
$(\mathbf{u}, Q, p) : (0, T) \times \Omega \to \mathbb{R}^3 \times \mathbb{R}^{3\times 3} \times \mathbb{R}$:

$$\left\{ \begin{array}{rl} D_t\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \nabla \cdot \tau(Q) + \nabla \cdot \sigma(H, Q) & \text{in } \Omega \times (0, T) \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \times (0, T) \end{array} \right.$$

where $D_t\mathbf{u} = \partial_t\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}$ is the material derivative,

$$\left\{ \begin{array}{rcl} \tau_{ij}(Q) & = & -\varepsilon \left(\partial_j Q : \partial_i Q\right) = -\varepsilon\, \partial_j Q_{kl}\, \partial_i Q_{kl}, \ \varepsilon > 0 \\ & & \text{(symmetric part)} \\ \sigma(H, Q) & = & H\,Q - Q\,H \\ & & \text{(antisymmetric part when } Q \text{ and } H \text{ are symmetric)} \end{array} \right.$$

Q-tensor system

$$\partial_t Q + (\mathbf{u}\cdot\nabla)Q - S(\nabla\mathbf{u}, Q) = -\gamma\, H(Q) \quad \text{in } \Omega\times(0,T)$$

where

$$\begin{cases} S(\nabla\mathbf{u}, Q) = \nabla\mathbf{u}\, Q^t - Q^t\,\nabla\mathbf{u}, & \text{(stretching term)} \\ H(Q) = -\varepsilon\,\Delta Q + f(Q) & \text{where} \\ f(Q) = a\,Q - \dfrac{b}{3}\left(Q^2 + QQ^t + Q^tQ\right) + c\,|Q|^2\,Q & \\ \quad \text{with } c>0,\ a,\ b\in\mathbb{R} & \\ \quad (H \text{ is a symmetric tensor if } Q \text{ is symmetric,} & \\ \quad \text{in fact } f(Q)^t = f(Q^t)) & \end{cases}$$

Previous results for an initial-value problem (in the whole $\mathbb{R}^3$) [Paicu-Zarnescu'12]

- Existence of weak solution in $(0, T)$, for each $T > 0$.
- Global strong solution in $2D$.
- Weak-Strong uniqueness

Initial and boundary conditions

Initial conditions:

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \; Q|_{t=0} = Q_0 \quad \text{in } \Omega,$$

Boundary conditions ($\Gamma = \partial\Omega$:):

- For the velocity: $\mathbf{u}|_\Gamma = \mathbf{0}$ in $(0, T)$.
- For the Q-tensor:

$$\partial_{\mathbf{n}} Q|_\Gamma = 0 \quad \text{or} \quad Q|_\Gamma = Q_\Gamma \quad \text{in } (0, T).$$

Main results for the initial-boundary QT-model

- existence of global in time weak solution (without using maximum principle).
- Modification of the model to enforce traceless and symmetry constraints for Q.
- maximum principle
- uniqueness criteria for weak solutions
- local existence (and uniqueness) of a “intermediate” regular solution

Step 1. Energy equality (Lyapunov functional)

$$\frac{d}{dt}\left(\frac{1}{2}\|\mathbf{u}\|^2_{L^2(\Omega)} + \mathcal{E}(Q)\right) + \nu\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} + \gamma\|H(Q)\|^2_{L^2(\Omega)} = 0,$$

with $\mathcal{E}(Q) = \int_\Omega \frac{\varepsilon}{2}|\nabla Q|^2 + F(Q)\,d\mathbf{x} \ngeq 0$

Using that:

$$\begin{array}{rcl} (S(\nabla\mathbf{u}, Q), H(Q))_{L^2} & = & (\sigma(H, Q), \nabla\mathbf{u}))_{L^2} \\ (\mathbf{u}\cdot\nabla Q, H(Q))_{L^2} & = & (\nabla\cdot\tau(Q), \mathbf{u})_{L^2} \\ Q\text{-system} & & \mathbf{u}\text{-system} \end{array}$$

Step 2. Lower bound of the potential

But the term $F(Q)$ could be negative. Observe that:

$$F(Q) \geq \begin{cases} \dfrac{a}{2}|Q|^2 + \dfrac{c}{8}|Q|^4 - \alpha_1, & \text{for } \alpha_1 = \alpha_1(b,c) > 0 \text{ if } a > 0, \\ \dfrac{c}{8}|Q|^4 - \alpha_2 - \beta, & \text{for } \alpha_2 = \alpha_2(b,c),\ \beta = \beta(a,c) > 0 \\ & \text{if } a < 0, \end{cases}$$

Defining $\widetilde{F}(Q) = F(Q) + \mu$ with $\mu = \alpha_1$ if $a \geq 0$ and $\mu = \alpha_2 + \beta$ if $a < 0$, then:

$$\widetilde{F}(Q) \geq \begin{cases} \dfrac{a}{2}|Q|^2 + \dfrac{c}{8}|Q|^4 \geq 0 & \text{if } a > 0, \\ \dfrac{c}{8}|Q|^4 \geq 0 & \text{if } a < 0, \end{cases}$$

Step 3. Weak regularity

$$\frac{1}{2}\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + \frac{\varepsilon}{2}\|\nabla Q(t)\|^2_{\mathbb{L}^2(\Omega)} + \int_\Omega \widetilde{F}(Q(t)) < +\infty,$$

(here, $|\Omega| < +\infty$ is essential).
Therefore:

$$\left\{ \begin{array}{rcl} \mathbf{u} & \in & L^\infty(0,+\infty;\mathbf{L}^2(\Omega)) \cap L^2(0,+\infty;\mathbf{H}^1(\Omega)), \\ \nabla Q & \in & L^\infty(0,+\infty;\mathbb{L}^2(\Omega)) \\ \widetilde{F}(Q) & \in & L^\infty(0,+\infty;L^1(\Omega)) \\ -\varepsilon\Delta Q + f(Q) & \in & L^2(0,+\infty;\mathbb{L}^2(\Omega)) \end{array} \right.$$

Finally

$$Q \in L^\infty(0,+\infty;\mathbb{H}^1(\Omega)), \quad Q \in L^2(0,T;\mathbb{H}^2(\Omega)), \ \forall\, T > 0.$$

Modifications of the model to enforce traceless

Replace

$$H \quad \text{by} \quad \widetilde{H} = H + \alpha(Q) Id$$

with $\alpha(Q) = \frac{1}{3}\left(-a\, tr(Q) + \frac{b}{3}(tr(Q^2) + 2|Q|^2)\right)$ or

$\alpha(Q) = -\frac{1}{3} tr(f(Q))$

Idea: To eliminate the non-convex part (at least) of the trace of Q-system

Modifications of the model to enforce symmetry

Replace

$$S(\nabla \mathbf{u}, Q) \quad \text{by} \quad S(W(\mathbf{u}), Q),$$

with $W(\mathbf{u}) = (\nabla \mathbf{u} + \nabla \mathbf{u}^t)/2$
Idea: To eliminate the symmetric part of the stretching term of Q-system

Uniqueness criteria

Let $(\mathbf{u}_1, q_1, Q_1, H_c^1)$, $(\mathbf{u}_2, q_2, Q_2, H_c^2)$ be two solutions,
$(H_c)^i = -\varepsilon\,\Delta Q_i + F_c'(Q_i))$, for $F = F_c + F_e$
$\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, $q = q_1 - q_2$, $Q = Q_1 - Q_2$, $H_c = H_c^1 - H_c^2$.

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \varepsilon\,\|Q\|^2_{\mathbb{H}^1(\Omega)} + \int_\Omega Q_{mn}\frac{\partial^2 F_c(R)}{\partial Q_{mn}\partial Q_{pq}} Q_{pq}\,d\mathbf{x}\right)$$
$$+\nu\|\nabla \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \gamma\|H_c\|^2_{\mathbb{L}^2(\Omega)}$$
$$\le C(t)\,\left(\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \varepsilon\,\|Q\|^2_{\mathbb{H}^1(\Omega)}\right)$$

where $C(t) \in L^1(0,T)$ under the regularity hypothesis:

$$(RH)\left\{\begin{array}{ll} \nabla \mathbf{u}_2 \in L^{\frac{2q}{2q-3}}(0,T;\mathbf{L}^q(\Omega)), & \text{for } 2 \le q \le 3 \\ \Delta Q_2 \in L^{\frac{2r}{2r-3}}(0,T;\mathbb{L}^r(\Omega)), & \text{for } 2 \le r \le 3 \quad \bullet \end{array}\right.$$

Maximum Principle

Based on:

1. $S(\nabla \mathbf{u}, Q) : Q = 0$
2. $f(Q) : Q \geq \frac{c}{2}|Q|^2 \left(|Q|^2 - \beta\right)$ for $\beta = \frac{b^2}{c^2} - \frac{2a}{c}$.

Then,

$$\partial_t \left(|Q|^2\right) + \mathbf{u} \cdot \nabla \left(|Q|^2\right) - \gamma\, \varepsilon\, \Delta \left(|Q|^2\right) + \gamma \frac{c}{2}|Q|^2 \left(|Q|^2 - \beta\right) \leq 0$$

If $\|Q_0\|_{\mathbb{L}^\infty(\Omega)} \leq \alpha$ (and $\|Q_\Gamma\|_{\mathbb{L}^\infty(\Gamma)} \leq \alpha$) with $\alpha \geq \beta$, then:

$$\|Q(t)\|_{\mathbb{L}^\infty(\Omega)} \leq \alpha \quad \forall t \geq 0.$$

Problems with the strong regularity

- Prodi's (space) estimates (taking $-\Delta \mathbf{u}$ for $\mathbf{u}$-system and $-\Delta(-\varepsilon\Delta Q + f(Q))$ for Q-system) only works for
  1. periodic-space boundary conditions for Q,
  2. large enough viscosity.
- Modified Ladyzhenskaya's (time) estimates works for Neumann and Dirichlet boundary conditions for Q.

The key point for Prodi's estimates

Due to the boundary condition for Q (non space-periodic):

$$(S(\nabla \mathbf{u}, Q), -\Delta H(Q))_{L^2} \neq (\sigma(H, Q), \nabla(-\Delta \mathbf{u}))_{L^2}$$

and

$$(\nabla \cdot \tau(Q), -\Delta \mathbf{u})_{L^2} \neq (\mathbf{u} \cdot \nabla Q, -\Delta H(Q))_{L^2}$$

because some (high nonlinear) boundary terms don't vanish.

The bad terms (only bounded for large viscosity)

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla \mathbf{u}\|^2_{L^2(\Omega)} + \|\Delta Q\|^2_{L^2(\Omega)}\right) + \nu\|A\mathbf{u}\|^2_{L^2(\Omega)} + \gamma\|\nabla(\Delta Q)\|^2_{L^2(\Omega)}$$

$$= -(\mathbf{u}\cdot\nabla\mathbf{u}, A\mathbf{u}) + (A\mathbf{u}\cdot\nabla Q, \Delta Q) + (\nabla\cdot\sigma, A\mathbf{u}) + \dots$$

$$-(\nabla(\mathbf{u}\cdot\nabla Q), \nabla(\Delta Q)) + (\nabla S(\nabla\mathbf{u}, Q), \nabla(\Delta Q))$$

$$\leq \dots + \int_\Omega |Q|\,|\nabla(\Delta Q)|\,|A\mathbf{u}|\,d\mathbf{x} \leq \dots. \qquad \bullet$$

Modified Ladyzhenskaya's estimates

Weak estimates for $(\partial_t \mathbf{u}, \partial_t Q)$

Deriving in time $\mathbf{u}$-system and Q-system, and taking $\partial_t \mathbf{u}$ and $-\Delta(\partial_t Q)$ as test functions:

$$\frac{d}{dt}\left(\|\partial_t \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \varepsilon\,\|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)}\right) + \nu\|\partial_t \mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} + \gamma\varepsilon^2\|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)}$$
$$\leq a(t)\left(\|\partial_t \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)}\right)$$
$$+ C_{\nu,\gamma,\varepsilon}\left(\|\nabla \mathbf{u}\|^4_{\mathbf{L}^2(\Omega)} + \|H\|^4_{\mathbb{L}^2(\Omega)}\right)\left(\|\partial_t \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)}\right) \quad (2)$$

where $a \in L^1(0,T)$ (due to weak estimates).

Intermediate strong estimates for Q

Taking $\partial_t H = \partial_t(-\varepsilon \Delta Q + f(Q))$ as test function in the Q-system:

$$\frac{\gamma}{2}\frac{d}{dt}\|H\|^2_{\mathbb{L}^2(\Omega)} + \varepsilon\,\|\partial_t(\nabla Q)\|^2_{\mathbb{L}^2(\Omega)} \le C_\delta\left(1+\|Q\|_{\mathbb{H}^2(\Omega)}\right)\|\partial_t Q\|^2_{\mathbb{L}^2(\Omega)} + \delta\,\|\partial_t H\|^2_{\mathbb{L}^2(\Omega)} + C_\delta\,\|Q\|_{\mathbb{H}^2(\Omega)}\,\|\nabla \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \quad (3)$$

Intermediate strong estimates for $\mathbf{u}$

Taking $\partial_t \mathbf{u}$ as test function in the equation for $\mathbf{u}$:

$$\begin{aligned} \nu \frac{d}{dt} \|\nabla \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\partial_t \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \\ \le C\Big(\|\nabla \mathbf{u}\|^3_{\mathbf{L}^2(\Omega)} + \|Q\|_{\mathbb{H}^2(\Omega)} \|H\|^2_{\mathbb{L}^2(\Omega)} \Big) \end{aligned} \tag{4}$$

Weak-t estimates

Putting together (2)-(3)-(4), we get:

$$y'(t) + z(t) \leq \widetilde{a}(t)\, y(t) + C\, y(t)^3$$

where

$$\begin{cases} y(t) = \|\partial_t \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)} + \|\nabla \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|H\|^2_{\mathbb{L}^2(\Omega)} \\ z(t) = \|\partial_t \mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} + \|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)} \end{cases}$$

Intermediate regularity results

Thus, we obtain:

- Local in time weak solution for $(\partial_t \mathbf{u}, \partial Q)$:

$$(weak-t) \quad \begin{cases} \partial_t \mathbf{u} \in L^\infty(0,T^*;\mathbf{L}^2(\Omega)) \cap L^2(0,T^*;\mathbf{H}^1(\Omega)) \\ \partial_t Q \in L^\infty(0,T^*;\mathbb{H}^1(\Omega)) \cap L^2(0,T^*;\mathbb{H}^2(\Omega)) \\ \mathbf{u} \in L^\infty(0,T^*;\mathbf{H}^1(\Omega)) \\ Q \in L^\infty(0,T^*;\mathbb{H}^2(\Omega)) \end{cases}$$

  and uniqueness !! (because uniqueness criteria (RH) is satisfied for $q=2$ and $r=2$.)

- Under regularity hypothesis (RH), global in time weak solution for $(\partial_t \mathbf{u}, \partial_t Q)$.

Intermediate regularity results

Thus, we obtain:

- Local in time weak solution for $(\partial_t \mathbf{u}, \partial Q)$:

$$(weak-t) \quad \begin{cases} \partial_t \mathbf{u} \in L^\infty(0,T^*;\mathbf{L}^2(\Omega)) \cap L^2(0,T^*;\mathbf{H}^1(\Omega)) \\ \partial_t Q \in L^\infty(0,T^*;\mathbb{H}^1(\Omega)) \cap L^2(0,T^*;\mathbb{H}^2(\Omega)) \\ \mathbf{u} \in L^\infty(0,T^*;\mathbf{H}^1(\Omega)) \cap L^2(0,T^*;\mathbf{H}^2(\Omega))\ ? \\ Q \in L^\infty(0,T^*;\mathbb{H}^2(\Omega)) \cap L^2(0,T^*;\mathbb{H}^3(\Omega))\ ? \end{cases}$$

  and uniqueness !! (because uniqueness criteria (RH) is satisfied for $q = 2$ and $r = 2$.)
- Under regularity hypothesis (RH), global in time weak solution for $(\partial_t \mathbf{u}, \partial_t Q)$.

Why long-time behavior for intermediate regularity is not clear

Using Prodi's estimates, we would have the following generic situation (**without stretching**):

$$\left.\begin{array}{ll}\text{(weak)} & \mathbf{E}'(\mathbf{t}) + \mathbf{F}(\mathbf{t}) \le \mathbf{0} \\ \text{(strong)} & \mathbf{F}'(\mathbf{t}) + \mathbf{G}(\mathbf{t}) \le \mathbf{C_2}(\mathbf{F}(\mathbf{t})^{\mathbf{3}} + \mathbf{1})\end{array}\right\} \Rightarrow \exists \lim_{t\to+\infty} \mathbf{F}(\mathbf{t}) = 0.$$

$$\left\{\begin{array}{lcl}\mathbf{E}(\mathbf{t}) & = & \|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla \mathbf{Q}\|^2_{\mathbb{L}^2(\Omega)}, \\ \mathbf{F}(\mathbf{t}) & = & \|\nabla \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{H}\|^2_{\mathbb{L}^2(\Omega)} \\ \mathbf{G}(\mathbf{t}) & = & \|A\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla \mathbf{H}\|^2_{\mathbb{L}^2(\Omega)}.\end{array}\right.$$

[B. Climent-Ezquerra, F.G-G, M.A.Rodriguez-Bellido'10]

Why long-time behavior for intermediate regularity is not clear

We have the following generic situation for $\mathbf{t} \in (\mathbf{0}, +\infty)$:

$$\text{(weak)} \quad \mathbf{E}'(\mathbf{t}) + \mathbf{F}(\mathbf{t}) \leq \mathbf{0}$$

$$\text{(strong)} \quad \widetilde{\mathbf{F}}'(\mathbf{t}) + \widetilde{\mathbf{G}}(\mathbf{t}) \leq \mathbf{C_2}(\widetilde{\mathbf{F}}(\mathbf{t})^3 + \mathbf{1})$$

$$\begin{cases} \mathbf{E}(\mathbf{t}) = \|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla \mathbf{Q}\|^2_{\mathbb{L}^2(\Omega)}, \\ \mathbf{F}(\mathbf{t}) = \|\nabla \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{H}\|^2_{\mathbb{L}^2(\Omega)} \end{cases}$$

$$\begin{cases} \widetilde{\mathbf{F}}(t) = \|\nabla \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{H}\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\partial_t \mathbf{Q}\|^2_{\mathbb{H}^1(\Omega)} \\ \widetilde{\mathbf{G}}(t) = \|\partial_t \mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} + \|\partial_t \mathbf{Q}\|^2_{\mathbb{H}^2(\Omega)}. \end{cases}$$

Final comment

This methodology can be extended to a more general Q-tensor model and Erickseen-Leslie nematic models, and could be applicable to other Diffuse-Interface models.

Some open problems

1. To design “energy-stable” numerical schemes, by using traceless and symmetry
2. Global in time intermediate regular solutions with “explicit” conditions for initial data
3. Local in time strong regularity

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Thank you very much!