



THE \mathcal{L} -SECTIONAL CURVATURE OF S -MANIFOLDS

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ABSTRACT. We investigate \mathcal{L} -sectional curvature of S -manifolds with respect to the Riemannian connection and to certain semi-symmetric metric and non-metric connections naturally related with the structure, obtaining conditions for them to be constant and giving examples of S -manifolds in such conditions. Moreover, we calculate the scalar curvature in all the cases.

1. INTRODUCTION.

In 1963, Yano [13] introduced the notion of f -structure on a C^∞ $(2n + s)$ -dimensional manifold M , as a non-vanishing tensor field f of type $(1, 1)$ on M which satisfies $f^3 + f = 0$ and has constant rank $r = 2n$. Almost complex ($s = 0$) and almost contact ($s = 1$) are well-known examples of f -structures. The case $s = 2$ appeared in the study of hypersurfaces in almost contact manifolds [5, 8] and it motivated that, in 1970, Goldberg and Yano [9] defined globally framed f -structures (also called f .pk-structures), for which the subbundle $\ker f$ is parallelizable. Then, there exists a global frame $\{\xi_1, \dots, \xi_s\}$ for the subbundle $\ker f$ (the vector fields ξ_1, \dots, ξ_s are called the structure vector fields), with dual 1-forms η^1, \dots, η^s .

Thus, we can consider a Riemannian metric g on M , associated with a globally framed f -structure, such that $g(fX, fY) = g(X, Y) - \sum_{\alpha=1}^s \eta^\alpha(X)\eta^\alpha(Y)$, for any vector fields X, Y in M and then, the structure is called a metric f -structure. Therefore, TM splits into two complementary subbundles $\text{Im} f$ (whose differentiable distribution is usually denoted by \mathcal{L}) and $\ker f$ and, moreover, the restriction of f to $\text{Im} f$ determines a complex structure.

A wider class of globally framed f -manifolds (that is, manifolds endowed with a globally framed f -structure) was introduced in [3] by Blair according to the following definition: a metric f -structure is said to be a K -structure if the fundamental 2-form Φ , given by $\Phi(X, Y) = g(X, fY)$, for any vector fields X and Y on M , is

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closed and the normality condition holds, that is, $[f, f] + 2 \sum_{\alpha=1}^s d\eta^\alpha \otimes \xi_\alpha = 0$, where $[f, f]$ denotes the Nijenhuis torsion of f . A K -manifold is called an S -manifold if $d\eta^\alpha = \Phi$, for all $\alpha = 1, \dots, s$. If $s = 1$, an S -manifold is a Sasakian manifold. Furthermore, S -manifolds have been studied by several authors (see, for example, [4, 6, 10, 12]).

It is well known that there are not exist S -manifolds ($s \geq 2$) of constant sectional curvature and, for Sasakian manifolds, the unit sphere is the only one. This is due to the fact that $K(X, \xi_\alpha) = 1$ and $K(\xi_\alpha, \xi_\beta) = 0$, for any unit vector field $X \in \mathcal{L}$ and any $\alpha, \beta = 1 \dots, s$. For this reason, it is interesting to study the sectional curvature of planar sections spanned by vector fields of \mathcal{L} (called \mathcal{L} -sectional curvature) and to obtain conditions for this sectional curvature to be constant.

Further, in 1924 Friedmann and Schouten [7] introduced semi-symmetric linear connections on a differentiable manifold. Later, Hayden [11] defined the notion of metric connection with torsion on a Riemannian manifold. More precisely, if ∇ is a linear connection in a differentiable manifold M , the torsion tensor T of ∇ is given by $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$, for any vector fields X and Y on M . The connection ∇ is said to be symmetric if the torsion tensor T vanishes, otherwise it is said to be non-symmetric. In this case, ∇ is said to be a semi-symmetric connection if $T(X, Y) = \eta(Y)X - \eta(X)Y$, for any X, Y , where η is a 1-form on M . Moreover, if g is a Riemannian metric on M , ∇ is called a metric connection if $\nabla g = 0$, otherwise it is called non-metric. It is well known that the Riemannian connection is the unique metric and symmetric linear connection on a Riemannian manifold. Recently, S -manifolds endowed with a semi-symmetric either metric or non-metric connection naturally related with the S -structure have been studied in [1, 2].

In this paper, we investigate \mathcal{L} -sectional curvature of S -manifolds with respect to the Riemannian connection and to the semi-symmetric metric and non-metric connections introduced in [1, 2], obtaining conditions for them to be constant and giving examples of S -manifolds in such conditions. Moreover, we calculate the scalar curvature in all the cases.

2. PRELIMINARIES ON S -MANIFOLDS.

A $(2n + s)$ - dimensional differentiable manifold M is called a *metric f -manifold* if there exist a $(1, 1)$ type tensor field f , s vector fields ξ_1, \dots, ξ_s , called *structure vector fields*, s 1-forms η^1, \dots, η^s and a Riemannian metric g on M such that

$$(2.1) \quad f^2 = -I + \sum_{\alpha=1}^s \eta^\alpha \otimes \xi_\alpha, \quad \eta^\alpha(\xi_\beta) = \delta_{\alpha\beta}, \quad f\xi_\alpha = 0, \quad \eta^\alpha \circ f = 0,$$

$$(2.2) \quad g(fX, fY) = g(X, Y) - \sum_{\alpha=1}^s \eta^\alpha(X)\eta^\alpha(Y),$$

for any $X, Y \in \mathcal{X}(M)$, $\alpha, \beta \in \{1, \dots, s\}$. In addition:

$$(2.3) \quad \eta^\alpha(X) = g(X, \xi_\alpha), \quad g(X, fY) = -g(fX, Y).$$

Then, a 2-form Φ is defined by $\Phi(X, Y) = g(X, fY)$, for any $X, Y \in \mathcal{X}(M)$, called the *fundamental 2-form*. In what follows, we denote by \mathcal{M} the distribution spanned by the structure vector fields ξ_1, \dots, ξ_s and by \mathcal{L} its orthogonal complementary

distribution. Then, $\mathcal{X}(M) = \mathcal{L} \oplus \mathcal{M}$. If $X \in \mathcal{M}$, then $fX = 0$ and if $X \in \mathcal{L}$, then $\eta^\alpha(X) = 0$, for any $\alpha \in \{1, \dots, s\}$, that is, $f^2X = -X$.

In a metric f -manifold, special local orthonormal basis of vector fields can be considered: let U be a coordinate neighborhood and E_1 a unit vector field on U orthogonal to the structure vector fields. Then, from (2.1)-(2.3), fE_1 is also a unit vector field on U orthogonal to E_1 and the structure vector fields. Next, if it is possible, let E_2 be a unit vector field on U orthogonal to E_1, fE_1 and the structure vector fields and so on. The local orthonormal basis $\{E_1, \dots, E_n, fE_1, \dots, fE_n, \xi_1, \dots, \xi_s\}$, so obtained is called an f -basis.

Moreover, a metric f -manifold is *normal* if

$$[f, f] + 2 \sum_{\alpha=1}^s d\eta^\alpha \otimes \xi_\alpha = 0,$$

where $[f, f]$ denotes the Nijenhuis tensor field associated to f . A metric f -manifold is said to be an S -manifold if it is normal and

$$\eta^1 \wedge \dots \wedge \eta^s \wedge (d\eta^\alpha)^n \neq 0 \text{ and } \Phi = d\eta^\alpha, \ 1 \leq \alpha \leq s.$$

Observe that, if $s = 1$, an S -manifold is a Sasakian manifold. For $s \geq 2$, examples of S -manifolds can be found in [3, 4, 10].

If ∇ is a linear connection on an S -manifold and K denotes the sectional curvature associated with ∇ , the \mathcal{L} -sectional curvature $K_{\mathcal{L}}$ of ∇ is defined as $K_{\mathcal{L}}(X, Y) = K(X, Y)$, for any $X, Y \in \mathcal{L}$. The *scalar curvature* of the S -manifold with respect to ∇ is given by

$$(2.4) \quad \tau = \frac{1}{2} \sum_{i,j=1}^{2n+s} K(e_i, e_j),$$

for any local orthonormal frame $\{e_1, \dots, e_{2n+s}\}$ of tangent vector fields to M .

3. THE \mathcal{L} -SECTIONAL CURVATURE OF S -MANIFOLDS.

From now on, let M denote an S -manifold $(M, f, \xi_1, \dots, \xi_s, \eta^1, \dots, \eta^s, g)$ of dimension $2n + s$. We are going to study the sectional curvature of M with respect to different types of connections on M .

3.1. The case of the Riemannian connection. First, let ∇ denote the Riemannian connection of g . For the sectional curvature K of ∇ , in [6] it is proved that

$$(3.1) \quad K(\xi_\alpha, X) = R(\xi_\alpha, X, X, \xi_\alpha) = g(fX, fX),$$

for any $X \in \mathcal{X}(M)$ and $\alpha \in \{1, \dots, s\}$. Consequently, if $s = 1$, the unit sphere is the only Sasakian manifold of constant (sectional) curvature. If $s \geq 2$, from (3.1), we deduce that M cannot have constant sectional curvature. For this reason, it is necessary to introduce a more restrictive curvature. In general, a plane section π on a metric f -manifold M is said to be an f -section if it is determined by a unit vector X , normal to the structure vector fields and fX . The sectional curvature of π is called an f -sectional curvature. An S -manifold is said to be an S -space-form

if it has constant f -sectional curvature c and then, it is denoted by $M(c)$. The curvature tensor field R of $M(c)$ satisfies [12]:

$$\begin{aligned}
 (3.2) \quad R(X, Y, Z, W) &= \sum_{\alpha, \beta=1}^s \{g(fX, fW)\eta^\alpha(Y)\eta^\beta(Z) \\
 &\quad -g(fX, fZ)\eta^\alpha(Y)\eta^\beta(W) + g(fY, fZ)\eta^\alpha(X)\eta^\beta(W) \\
 &\quad -g(fY, fW)\eta^\alpha(X)\eta^\beta(Z)\} \\
 &\quad + \frac{c+3s}{4} \{g(fX, fW)g(fY, fZ) - g(fX, fZ)g(fY, fW)\} \\
 &\quad + \frac{c-s}{4} \{\Phi(X, W)\Phi(Y, Z) - \Phi(X, Z)\Phi(Y, W) - 2\Phi(X, Y)\Phi(Z, W)\},
 \end{aligned}$$

for any $X, Y, Z, W \in \mathcal{X}(M)$.

Therefore, if M is an S -space-form of constant f -sectional curvature c and considering an f -basis, from (3.1) and (3.2), we deduce that the scalar curvature of M with respect to the curvature tensor field of the Riemannian connection ∇ satisfies:

$$\tau = \frac{n(n-1)(c+3s)}{2} + n(c+2s).$$

Now, in view of (3.1) it is interesting to investigate the conditions for $K_{\mathcal{L}}$ to be constant. In this context, we observe that, if $n = 1$, $K_{\mathcal{L}}$ is actually the f -sectional curvature. Moreover, for $n \geq 2$, we can prove the following theorem.

Theorem 3.1. *Let M be a $(2n + s)$ -dimensional S -manifold with $n \geq 2$. If the \mathcal{L} -sectional curvature $K_{\mathcal{L}}$ with respect to the Riemannian connection ∇ is constant equal to c , then $c = s$. In this case, the scalar curvature of M is:*

$$\tau = ns(2n + 1).$$

Proof. It is clear that if $K_{\mathcal{L}}$ is constant equal to c , then M is an S -space-form $M(c)$. Consequently, from (3.2), we have

$$(3.3) \quad K_{\mathcal{L}}(X, Y) = \frac{c+3s}{4} + \frac{3(c-s)}{4}g(X, fY)^2,$$

for any orthonormal vector fields $X, Y \in \mathcal{L}$. Now, since $n \geq 2$, we can choose X and Y such that $g(X, fY) = 0$. Thus, from (3.3) we deduce

$$\frac{c+3s}{4} = c,$$

that is, $c = s$.

Now, considering a local orthonormal frame of tangent vector fields such that $e_{2n+\alpha} = \xi_\alpha$, for any $\alpha = 1, \dots, s$, since $K(e_i, e_j) = K_{\mathcal{L}}(e_i, e_j) = s$, $i, j = 1, \dots, 2n$, $i \neq j$, and using (3.1) and (2.4), we get the desired result for the scalar curvature. □

By using (3.2) and (3.3), we have:

Corollary 3.2. *Let $M(c)$ be an S -space-form of constant f -sectional curvature c . Then, M is of constant \mathcal{L} -sectional curvature (equal to c) if and only if $c = s$*

Example 3.3. Let us consider $\mathbf{R}^{2n+2+(s-1)}$ with coordinates

$$(x_1 \dots, x_{n+1}, y_1, \dots, y_{n+1}, z_1, \dots, z_{s-1})$$

and with its standard S -structure of constant f -sectional curvature $-3(s-1)$, given by (see [10]):

$$\xi_\alpha = 2 \frac{\partial}{\partial z_\alpha}, \quad \eta^\alpha = \frac{1}{2} \left(dz_\alpha - \sum_{i=1}^{n+1} y_i dx_i \right), \quad \alpha = 1, \dots, s-1,$$

$$g = \sum_{\alpha=1}^{s-1} \eta^\alpha \otimes \eta^\alpha + \frac{1}{4} \sum_{i=1}^{n+1} (dx_i \otimes dx_i + dy_i \otimes dy_i),$$

$$fX = \sum_{i=1}^{n+1} (Y_i \frac{\partial}{\partial x_i} - X_i \frac{\partial}{\partial y_i}) + \sum_{\alpha=1}^{s-1} \sum_{i=1}^{n+1} Y_i y_i \frac{\partial}{\partial z_\alpha},$$

where

$$X = \sum_{i=1}^{n+1} (X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}) + \sum_{\alpha=1}^{s-1} Z_\alpha \frac{\partial}{\partial z_\alpha}$$

is any vector field tangent to $\mathbf{R}^{2n+2+(s-1)}$.

Now, let $S^{2n+1}(2)$ be a $(2n+1)$ -dimensional ordinary sphere of radius 2 and $M = S^{2n+1}(2) \times \mathbf{R}^{s-1}$ a hypersurface of $\mathbf{R}^{2n+2+(s-1)}$. Let

$$\xi_s = \sum_{i=1}^{n+1} \left(-y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i} \right) - \sum_{i=1}^{n+1} \sum_{\alpha=1}^{s-1} y_i^2 \frac{\partial}{\partial z_\alpha}$$

and $\eta^s(X) = g(X, \xi_s)$, for any vector field X tangent to M . Then, if we put

$$\tilde{\xi}_\alpha = s\xi_\alpha; \quad \tilde{\eta}^\alpha = \frac{1}{s}\eta^\alpha; \quad \alpha = 1, \dots, s;$$

$$\tilde{f} = f; \quad \tilde{g} = \frac{1}{s}g + \frac{1-s}{s^2} \sum_{\alpha=1}^s \eta^\alpha \otimes \eta^\alpha,$$

it is known ([10]) that $(M, \tilde{f}, \tilde{\xi}_1, \dots, \tilde{\xi}_s, \tilde{\eta}^1, \dots, \tilde{\eta}^s, \tilde{g})$ is an S -space-form of constant f -sectional curvature $c = s$. Moreover, from (3.2), it is easy to show that the \mathcal{L} -sectional curvature $K_{\mathcal{L}}$ is also constant and equal to s .

3.2. The case of a semi-symmetric metric connection. In [1], a semi-symmetric metric connection on M , naturally related to the S -structure, is defined by

$$(3.4) \quad \nabla_X^* Y = \nabla_X Y + \sum_{j=1}^s \eta^j(Y) X - \sum_{j=1}^s g(X, Y) \xi_j,$$

for any $X, Y \in \mathcal{X}(M)$. For the sectional curvature K^* of ∇^* , the following theorem was proved in [1]:

Theorem 3.4. *Let M be an S -manifold. Then, the sectional curvature of ∇^* satisfies*

- (i) $K^*(X, Y) = K(X, Y) - s;$
- (ii) $K^*(X, \xi_\alpha) = K^*(\xi_\alpha, X) = 2 - s;$
- (iii) $K^*(\xi_\alpha, \xi_\beta) = K^*(\xi_\beta, \xi_\alpha) = 2 - s,$

for any orthonormal vector fields $X, Y \in \mathcal{L}$ and $\alpha, \beta \in \{1, \dots, s\}$, $\alpha \neq \beta$.

Therefore, from Theorem 3.1, if $s \neq 2$, an S -manifold cannot have constant sectional curvature with respect to the semi-symmetric metric connection defined in (3.4). For $s = 2$, $M = S^{2n+1}(2) \times \mathbf{R}$ endowed with the connection ∇^* and the S -structure given in Example 3.3 is an S -manifold of constant sectional curvature (equal to 0) with respect to ∇^* . Moreover, for any s , by using Theorem 3.1 again and (i) of Theorem 3.4, if the \mathcal{L} -sectional curvature associated with ∇^* is constant equal to c , then $c = 0$ and examples of such a situation are given in Example 3.3. In this case, the scalar curvature is given by:

$$\tau^* = \frac{(4ns + s(s - 1))(2 - s)}{2}.$$

Regarding the f -sectional curvature of ∇^* , from Theorem 4.5 in [1], we know that it is constant if and only if the f -sectional curvature associated with the Riemannian connection is constant too. In this case, if c denotes the constant f -sectional curvature of the Riemannian connection, $c - s$ is the constant f -sectional curvature of ∇^* . Furthermore, from (i) of Theorem 3.4 and (3.3) it is easy to show that

$$K_{\mathcal{L}}^*(X, Y) = \frac{c - s}{4}(1 + 3g(X, fY)^2),$$

for any orthonormal vector fields $X, Y \in \mathcal{L}$. Therefore, considering an f -basis, we deduce that the scalar curvature of a $(2n + s)$ -dimensional S -manifold of constant f -sectional curvature c with respect to ∇^* satisfies:

$$\tau^* = \frac{n(n + 1)(c - s) + (4ns + s(s - 1))(2 - s)}{2}.$$

3.3. The case of a semi-symmetric non-metric connection. In [2], a semi-symmetric non-metric connection on M , naturally related to the S -structure, is defined by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sum_{j=1}^s \eta^j(Y)X,$$

for any $X, Y \in \mathcal{X}(M)$. To consider the sectional curvature of $\tilde{\nabla}$ has no sense because $\tilde{R}(\xi_\alpha, X, X, \xi_\alpha) = 1$, while $\tilde{R}(X, \xi_\alpha, \xi_\alpha, X) = 2$, for any unit vector field $X \in \mathcal{L}$ and any $\alpha \in \{1, \dots, s\}$ (see [2] for the details). However, for the \mathcal{L} -sectional curvature $\tilde{K}_{\mathcal{L}}$, we have that $\tilde{K}_{\mathcal{L}}(X, Y) = K_{\mathcal{L}}(X, Y)$, for any orthogonal vector fields $X, Y \in \mathcal{L}$. Consequently, Theorem 3.3 and Example 3.3 can be applied here. In the case of constant \mathcal{L} -sectional curvature (equal to s) and since $\tilde{R}(\xi_\alpha, \xi_\beta, \xi_\beta, \xi_\alpha) = 1$, for any $\alpha, \beta \in \{1, \dots, s\}$, $\alpha \neq \beta$, the scalar curvature is given by:

$$\tilde{\tau} = 2ns(n + 1) + \frac{s(s - 1)}{2}.$$

Regarding the f -sectional curvature of $\tilde{\nabla}$, in [2] it is proved that it is constant if and only if the f -sectional curvature associated with the Riemannian connection is constant too. In this case, both constant are the same and the curvature tensor field of $\tilde{\nabla}$ is completely determined by c . Furthermore, since from (3.3),

$$\tilde{K}_{\mathcal{L}}(X, Y) = \frac{c + 3s}{4} + \frac{3(c - s)}{4}g(X, fY)^2,$$

for any orthonormal vector fields $X, Y \in \mathcal{L}$, considering an f -basis, we deduce that the scalar curvature of a $(2n + s)$ -dimensional S -manifold of constant f -sectional

curvature c with respect to $\tilde{\nabla}$ satisfies:

$$\tilde{\tau} = \frac{n(n+1)(c+3s) + s(s-1)}{2}.$$

REFERENCES

- [1] M. Akif Akyol, A. Turgut Vanli and L.M. Fernández, Curvature properties of a semi-symmetric metric connection defined on S -manifolds, *Ann. Polonici Math.*, Vol:107, No.1 (2013), 71-86.
- [2] M. Akif Akyol, A. Turgut Vanli and L.M. Fernández, Semi-symmetry properties of S -manifolds endowed with a semi-symmetric non metric connection, to appear in *An. Sti. Univ. "Al. I. Cuza" (Iasi)* (2015).
- [3] D.E. Blair, Geometry of manifolds with structural group $U(n) \times O(s)$, *J. Differ. Geom.*, Vol:4 (1970), 155-167.
- [4] D.E. Blair, On a generalization of the Hopf fibration, *An. Sti. Univ. "Al. I. Cuza" (Iasi)*, Vol:17 (1971), 171-177.
- [5] D.E. Blair and G.D. Ludden, Hypersurfaces in almost contact manifolds, *Tôhoku Math. J.*, Vol:21 (1969), 354-362.
- [6] J.L. Cabrerizo, L.M. Fernández and M. Fernández, The curvature tensor fields on f -manifolds with complemented frames, *An. Sti. Univ. "Al. I. Cuza" (Iasi)*, Vol:36 (1990), 151-161.
- [7] A. Friedmann and J.A. Schouten, Über die Geometrie der halbsymmetrischen Übertragung, *Math. Zeitschr.*, Vol:21 (1924), 211-223.
- [8] S.I. Goldberg and K.Yano, Globally framed f -manifolds, *Illinois J. Math.*, Vol:15 (1971), 456-474.
- [9] S.I. Goldberg and K.Yano, On normal globally framed manifolds, *Tôhoku Math. J.*, Vol:22 (1970), 362-370.
- [10] I. Hasegawa, Y. Okuyama and T. Abe, On p -th Sasakian manifolds, *J. Hokkaido Univ. of Education (Section II A)*, Vol:37 (1986), 1-16.
- [11] H.A. Hayden, Subspaces of a space with torsion, *Proc. London Math. Soc.*, Vol 34 (1932), 27-50.
- [12] M. Kobayashi and S. Tsuchiya, Invariant submanifolds of an f -manifold with complemented frames, *Kodai Math. Sem. Rep.*, Vol:24 (1972), 430-450.
- [13] K. Yano, On a structure defined by a tensor field f of type (1,1) satisfying $f^3 + f = 0$, *Tensor, N. S.*, Vol:14 (1963), 99-109.

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