Geometric dilation of closed planar curves: A new lower bound

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Abstract

Given any simple closed curve C in the Euclidean plane, let w and D denote the minimal and the maximal caliper distances of C, correspondingly. We show that any such curve C has a geometric dilation of at least $\arcsin(\frac{w}{D}) + \sqrt{(\frac{w}{D})^2 - 1}$.

Key words: computational geometry, convex geometry, convex curves, dilation, detour, lower bound

1. Introduction

Let C be a simple closed curve C in the Euclidean plane. For any two points, p and q, on C let $\pi(p, q)$ denote the shorter of the two curve segments of C that connects p with q. Then the geometric dilation, $\delta(C)$, of C is defined as

$$\delta(C) := \sup_{p,q \in C, p \neq q} \frac{|\pi(p, q)|}{|pq|} \quad (1)$$

The computation of the geometric dilation (then called detour) was first studied in [5], where an $O(n \log n)$ approximation algorithm for polygonal chains in the plane was given. Further efficient algorithms to compute the geometric dilation of certain classes of curves and networks were presented in [1], [11], and [9].

The question of embedding a finite point set in the plane into a network with low geometric dilation was recently studied in [4]. There it has been shown that any simple closed planar curve has dilation $\delta(C) \geq \pi/2$, using Cauchy's surface area formula.

Note, that the analogue concept on graphs, where only the point set of the vertices is taken into account for computing the dilation, was extensively studied under the notion of spanners and low dilation graphs, see e.g. [7] for a survey and [3], [2] for recent results. However, there are structural differences between the two concepts, as already mentioned e.g. in [5].

In this paper we prove a powerful generalization of the lower bound from [4]. Namely, let w and D denote the width and the diameter of the convex hull of C, correspondingly, that is, the minimal and the maximal distances of a rotating caliper measuring C; see Figure 1.

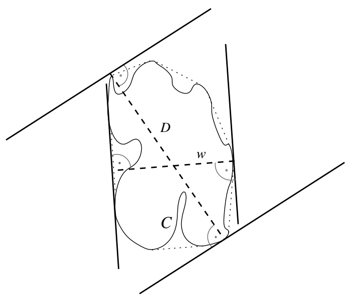

Fig. 1. Diameter D and width w of ch(C).

Then,

$$\delta(C) \geq \arcsin\left(\frac{w}{D}\right) + \sqrt{\left(\frac{w}{D}\right)^2 - 1} \quad (2)$$

holds for the geometric dilation of C. This lower bound has a minimum value of $\pi/2$ if and only if $w = D$ holds. (Note, however, that the circle is not the only closed curve satisfying $w = D$.)

The proof of formula (2) uses a well-known transformation of convex curves called the central symmetrization, see e.g. [6] and [10].

The rest of this paper is organized as follows. In Section 2 we give some necessary definitions and

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basic lemmata. Then, in Section 3 we cite the symmetrization transformation. Finally, Section 4 contains the proof of our lower bound.

2. Definitions and basic properties

Throughout this paper we consider *simple planar cycles* C, i.e. closed curves in the Euclidean plane without self-intersections. A simple cycle C is *convex* iff it always turns into the same direction, that is, iff C has a convex interior domain.

Definition 1 *(Dilation) Let C be a simple cycle, and let $p, q \in C$ be two points on C.*

- *(i) C_p^q denotes one of the two possible sub-paths of C connecting p and q, characterized by its turning direction: If one moves from p to q on C_p^q, one turns anti-clockwise ($C = C_p^q \cup C_q^p$).*
- *(ii) The* dilation *of a pair of points $(p, q) \in C \times C$ is the length of a shortest sub-path $\pi_C(p, q)$ of C connecting p and q, $|\pi_C(p, q)| = \min(|C_p^q|, |C_q^p|)$, divided by its Euclidean distance, i.e. $\delta_C(p, q) := \frac{|\pi_C(p,q)|}{|pq|}$.*
- *(iii) The* geometric dilation *of C is the supremum of the dilation values of all pairs of points of C, i.e. $\delta(C) := \sup_{p,q \in C, p \neq q} \delta_C(p, q)$.*

By continuity and compactness arguments one can show that every finite convex curve has a pair of points attaining maximum dilation.

Definition 2 *(Partition Pair) Let $p \in C$ be a point on a cycle C. Then the unique* partition partner *$\hat{p}$ of p is characterized by $|\pi_C(p, \hat{p})| = |C|/2$. We say that $(p, \hat{p})$ is a* partition pair *of C.*

By continuity arguments it is easy to show that for every direction $v \in \mathbb{S}^1$ there exists a partition pair $(p, \hat{p})$, i.e. $\hat{p} - p = |\hat{p} - p|\, v$.

Definition 3 *(Breadths) Let C be a simple cycle, and let $v \in \mathbb{S}^1$ be an arbitrary direction.*

- *(i) The* v-length *of C is the maximum distance of a pair of points with direction v, i.e. $l_C(v) := \max\{|pq| \mid p, q \in C, q - p = |q - p|\, v\}$.*
- *(ii) The* v-width *(v-breadth) of C is the distance of the two supporting lines of C perpendicular to v, i.e. $w_C(v) := \max_{p \in C} p \cdot v - \min_{p \in C} p \cdot v$.*
- *(iii) The* v-partition pair distance*, $h_C(v)$, of C is the distance of the partition pair with direction v.*
- *(iv) The* diameter*, $D(C)$, of C is the maximal v-length, i.e. $D(C) := \max_{v \in \mathbb{S}^1} l_C(v)$. The* width*, $w(C)$, of C is the minimal v-length, i.e. $w(C) := \min_{v \in \mathbb{S}^1} l_C(v)$.*
- *(v) The* maximal partition pair distance *is denoted by $H(C) := \max_{v \in \mathbb{S}^1} h_C(v)$ and the* minimal partition pair distance *by $h(C) := \min_{v \in \mathbb{S}^1} h_C(v)$.*

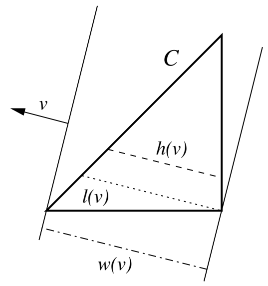

Fig. 2. Three different breadth measures.

As used in the introduction, width and diameter can also be defined using the v-width values which is proved in [8], [12] respectively:

Lemma 4 *Let C be a simple cycle, then $D(C) = \max_{v \in \mathbb{S}^1} w_C(v)$. If C is convex, then $w(C) = \min_{v \in \mathbb{S}^1} w_C(v)$.*

The next statement follows immediately from the definitions; see Figure 2.

Lemma 5 *Let C be a simple convex cycle, and let $v \in \mathbb{S}^1$ be an arbitrary direction. Then the following inequalities hold: $h_C(v) \leq l_C(v) \leq w_C(v)$.*

3. Central symmetrization

The central symmetrization (see e.g. [6], [8], [10]) is a well-known transformation which maps any convex cycle to a convex point-symmetric cycle. In our notion, the central symmetrization is based on the length values $l_C(v)$, introduced in Definition 3(i).

Amazingly, it preserves all the width values $w_C(v)$. And Cauchy's surface area formula implies that the perimeter is not changed either.

Definition 6 *Let C be a convex cycle. The* central symmetrization *of C is the cycle C' given by the parametrization $c' : \mathbb{S}^1 \to \mathbb{R}^2$, $c'(v) := \frac{l_C(v)}{2} v$.*

As depicted in Figure 3, we can construct the central symmetrization by translating all the centers of the segments of maximal length connecting pairs of points on C to the origin.

However, there is an easier and more helpful construction, described in the following lemma.

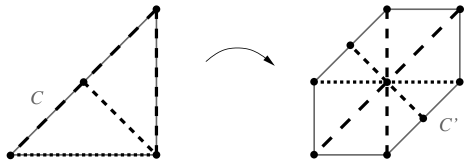

Fig. 3. The central symmetrization of an isosceles right-angled triangle.

Lemma 7 *Let $X := f(C) \cup C$ be the face bounded by C including C itself. Then define a set X' to be the* arithmetic mean *of X and $-X$; see [10]. It is the* Minkowski sum *$X \oplus -X$ scaled by $1/2$. Then, the central symmetrization C' is the boundary of this arithmetic mean X':*

$$X := f(C) \cup C$$
$$X' := \frac{1}{2}(X \oplus -X)) = \left\{ \frac{1}{2}(u-v) \,\middle|\, u, v \in X \right\}$$
$$\Rightarrow \quad C' = \partial X'$$

PROOF. The proof that this second way of constructing C' is also correct is straightforward. Let $v \in \mathbb{S}^1$ be an arbitrary direction. Define $l := \sup\{c \in \mathbb{R}^{>0} \mid cv \in X'\}$. Then due to X' being closed, lv is an element of X'. And for $k > l$ the point kv is not in X'. Thus, $lv \in \partial X'$.

It also follows that there are $p, q \in X$ such that $lv = (1/2)(q-p)$. On the other hand, the definition of l yields that $k > l$ implies there are no $p, q \in X$ satisfying $kv = (1/2)(q-p)$. Thus, $l = (1/2)l_C(v)$.

Hence, our analysis results in the following parametrization of $\partial X'$: $c'(v) = (1/2)l_C(v)v$. And this is exactly the parametrization we used to define C'.

The following lemma, stated here without proof, lists the most important properties of the central symmetrization. The fact that the width values are preserved is mentioned without proof in [6]. Gritzmann and Klee [8] prove the width-preserving and length-preserving property. The statement that the perimeter is preserved is also proved in [10].

Lemma 8 *Let C be a simple convex cycle, and let C' be its central symmetrization. Then, the cycle C' has the following properties:*

(i) Cycle C' is convex.

(ii) Cycle C' is point-symmetric with respect to the origin.

(iii) For every direction $v \in \mathbb{S}^1$, $h_{C'}(v) = l_{C'}(v) = l_C(v) \geq h_C(v)$, and $w_{C'}(v) = w_C(v)$.

(iv) Width, diameter and perimeter are preserved by central symmetrization, i.e. $w(C') = w(C)$, $D(C') = D(C)$ and $|C'| = |C|$.

Because we can show that the dilation of a convex cycle is always attained by a partition pair, it follows easily from those properties that the dilation of the transformed cycle cannot be larger then the original one:

Lemma 9 *The dilation of C' is not larger than the original dilation, i.e. $\delta(C') \leq \delta(C)$.*

4. The lower bound

To apply the transformation described in Section 3, we need the fact that the dilation of the boundary of the convex hull of any planar cycle C is at most the dilation of C itself. Due to space limitations we state this here without proof.

Theorem 10 *Let $C \subset \mathbb{R}^2$ be a simple closed curve. Let $\partial\text{ch}(C)$ denote the boundary of the convex hull of C. Then holds*

$$\delta(C) \geq \delta(\partial\text{ch}(C)).$$

Now we prove our result on the lower bound, using the central symmetrization transformation.

Theorem 11 *Let $C \subset \mathbb{R}^2$ be a simple closed curve. Let w be the width and let D be the diameter of* ch(C)*, the convex hull of C. Then the dilation of C is bounded from below by*

$$\delta(C) \geq \arcsin\left(\frac{w}{D}\right) + \sqrt{\left(\frac{D}{w}\right)^2 - 1}.$$

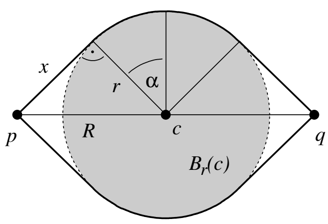

Fig. 4. The shortest cycle not intersecting the disk $B_r(c)$.

PROOF. Because of Theorem 10 we can assume w.l.o.g. that C is convex. To show the main idea of the proof we first consider a point-symmetric cycle $\tilde{C} \subset \mathbb{R}^2$ with center-point c, see Figure 4. Then, obviously, the partition pairs are also point-symmetric with respect to c. Let (p, q) be a partition pair having maximum distance $|pq| = H(\tilde{C})$. We define $R := H(\tilde{C})/2$.

Let $B_r(c)$ be the open disc with center point c and radius $r := h(\tilde{C})/2$, that is $B_r(c) := \{b \in \mathbb{R}^2 | \, |b - c| < r\}$. Then $\tilde{C}$ cannot intersect with $B_r(c)$, otherwise there would exist a partition pair having a distance smaller than $h(\tilde{C}) = 2r$.

By using the shortest possible cyclic path connecting p and q in the Euclidean plane, not intersecting $B_r(c)$ but enclosing it, we obtain a curve $\tilde{C}_{minPer}$, as shown in Figure 4. By construction $\tilde{C}_{minPer}$ is the point-symmetric curve of smallest perimeter that has minimum partition pair distance $h(\tilde{C})$ and maximum partition pair distance $H(\tilde{C})$. Due to symmetry reasons this perimeter is $P = 4x + 4r\alpha$, with x denoting the lengths of the straight path segments from p and q to the tangent points on $B_r(c)$, correspondingly, and $r\alpha$ the lengths of the path segments on $\partial B_r(c)$.

Using Pythagoras we get $x = \sqrt{R^2 - r^2}$. And by considering the angles in the rectangular triangle, we obtain $\sin\alpha = \cos(\frac{\pi}{2} - \alpha) = \frac{r}{R}$. Because the maximum dilation of the convex cycle $\tilde{C}$ is attained by a partition pair having shortest path distance $\frac{\tilde{P}}{2} \geq \frac{P}{2}$, it holds:

$$\delta(\tilde{C}) \geq \frac{\frac{P}{2}}{2r} = \frac{4x + 4r\alpha}{4r} = \frac{\sqrt{R^2 - r^2}}{r} + \arcsin\left(\frac{r}{R}\right)$$
$$= \sqrt{\left(\frac{R}{r}\right)^2 - 1} + \arcsin\left(\frac{r}{R}\right) \qquad (3)$$

Now, let C be an arbitrary convex cycle, and let C' be its central symmetrization. Then, Lemma 9 yields that $\delta(C) \geq \delta(C')$. And by Lemma 8(iv) we know that width and diameter are preserved, i.e. $w(C') = w(C)$ and $D(C') = D(C)$. However, in a point-symmetric convex cycle v-length and v-partition pair distance are equal, implying $w(C') = h(C')$ and $D(C') = H(C')$.

Thus, if we apply formula (3) to C' keeping in mind that $r = h(C')/2 = w(C')/2 = w(C)/2 = w/2$ and $R = H(C')/2 = D(C')/2 = D(C)/2 = D/2$, we get:

$$\delta(C) \geq \sqrt{\left(\frac{D}{w}\right)^2 - 1} + \arcsin\left(\frac{w}{D}\right)$$

This lower bound equals the global lower bound of $\frac{\pi}{2}$ shown in [4] only for curves of constant width ($w = D$). Further arguments show that the inequality gets strict if C is not point-symmetric or not convex. Hence, only circles have dilation $\frac{\pi}{2}$.

Remark 12 *By replacing $l_C(v)$ by $h_C(v)$ in Definition 3 we get a new transformation, the* partition pair transformation. *Ideas analogous to the ones presented here show that*

$$\delta(C) \geq \sqrt{\left(\frac{H}{h}\right)^2 - 1} + \arcsin\left(\frac{h}{H}\right)$$

where $H = H(\mathrm{ch}(C))$ and $h = h(\mathrm{ch}(C))$ for every simple cycle C.

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