3D realization of two triangulations of a convex polygon.

Sergey Bereg [a] ,

[a] *Department of Computer Science, University of Texas at Dallas, Box 830688, Richardson, TX 75083, USA.*

Abstract

We study the problem of construction of a convex 3-polytope whose (i) shadow boundary has n vertices and (ii) two hulls, upper and lower, are isomorphic to two given triangulations of a convex n-gon. Barnette [1] proved the existence of a convex 3-polytope in general case. We show that, in our case, a polytope can be constructed using an operation of edge creation.

Key words: triangulation, convex polytope, Steinitz theorem

1. Introduction

Let P be a convex polygon in the xy-plane with n vertices. Two triangulations of P are called *distinct* if the only edges they share are the edges of P. Let T_1 and T_2 be two distinct triangulations of P. At the *First Canadian Conference on Computational Geometry* Leo Guibas conjectured that it is always possible to perturb the vertices of P vertically out (i.e., by displacements parallel to the z-axis) so that the polygon P becomes a spatial polygon P' such that the convex hull of P' is a convex polyhedron consisting ot two triangulated cups glued along P', *and* the triangulation of the upper cup (i.e., those faces oriented toward $+z$) is that specified as T_1, and the triangulation of the lower cup is that specified as T_2 [4].

Boris Bekster [2] disproved Guibas' conjecture by showing a counterexample, a convex hexagon with two triangulations. Marlin and Toussaint [3] considered the computational problem of deciding whether a triple (P, T_1, T_2) admits a realization in $\mathbb{R}^3$. They reduced the problem to a linear programming problem with $O(n^2)$ inequality constraints and n variables. The variables are z-coordinates of lifted vertices of P and the constraints correpond to vertex-face relations: the vertices must be below/above the planes passing through faces of the upper/lower cup of P'. The number of constraints can be dropped to $2n-6 = |T_1|+|T_2|$ by considering dihedral angles corresponding to diagonals of the triangulations [7].

Guibas conjecture is related to Steinitz's theorem [5].

Steinitz's Theorem: *A graph G is isomorphic to the edge graph of a convex 3-polytope if and only if G is 3-connected and planar.*

By Steinitz's theorem the graph $(P, T_1 \cup T_2)$ is the edge graph of a convex 3-polytope [3]. According to Barnette's theorem [1], every 3-polytope with a Hamiltonian circuit has realization such that the Hamiltonian circuit is a shadow boundary. This implies that Guibas' conjecture is true up to a combinatorial deformation [2]. Formally this can be stated as follows.

Theorem 1 *For any two distinct triangulations T_1 and T_2 of a convex polygon P_2 in $\mathbb{R}^2$ with n vertices, there is a convex polytope P_3 in $\mathbb{R}^3$ with n vertices such that* (i) *the xy-shadow S of P_3 contains all its vertices, and* (ii) *there is a isomorphism τ : $P_2 \to S$ that maps the edges of T_1 (resp. T_2) to the edges of the upper hull of P_3 (resp. the lower hull).*

Barnette's proof deals with general faces (not just triangles) due to its generality. In this paper we give a different proof of Theorem 1 that uses only triangular faces of polytopes which can be turned into a more robust algorithm for finding a combinatorial realization of (P, T_1, T_3) in $\mathbb{R}^3$.

Realization questions have been studied in com-

Email address: besp@utdallas.edu (Sergey Bereg).
URL: http://utdallas.edu/~sxb027100 (Sergey Bereg).

puter graphics and scene analysis as well. Sugihara [6] established necessary and sufficient conditions whether a line drawing in the plane can be realized in $\mathbb{R}^3$ by lifting.

We call a triple (P, T_1, T_2) a *configuration*. We call a map τ satisfying the conditions of Theorem 1 a *realization*.

2. Edge contraction

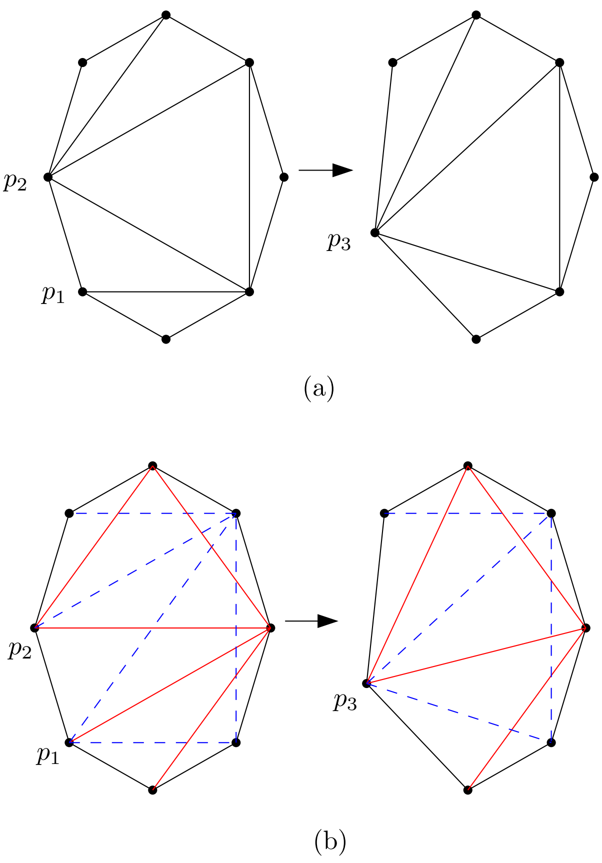


Fig. 1. (a) Edge contraction of a triangulation. (b) Edge contraction of two triangulations. The diagonals of one triangulation are solid and the diagonals of the other triangulation are dashed.

As in Barnette's proof we use the operation of edge removal. The difference is that we will not apply it for diagonals of P. This prevents the appearence of faces with more than three vertices. The *edge contraction* in a configuration is defined by idetifying the edge enpoints. If applied to one triangulation of P, it produces a triangulation, see Fig. 1 (a) for example where the edge p_1p_2 is contracted. When applied for two triangulations, we want the reduced triangulations to be distinct. An edge e of a configuration (P, T_1, T_2) is *contractible* if the new triangulations T_1' and T_2'' are distinct. In general, not all edges are contractible. For example, the edge p_1p_2 in the Fig. 2 (a) is not contractible since two edges p_1p_6 and p_2p_6 from different triangulations coincide after the contraction of p_1p_2.

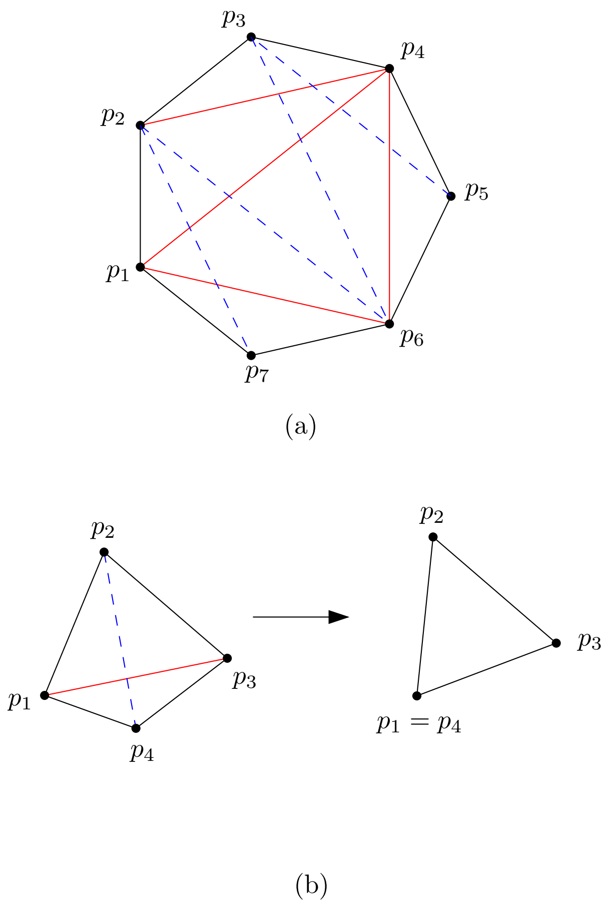
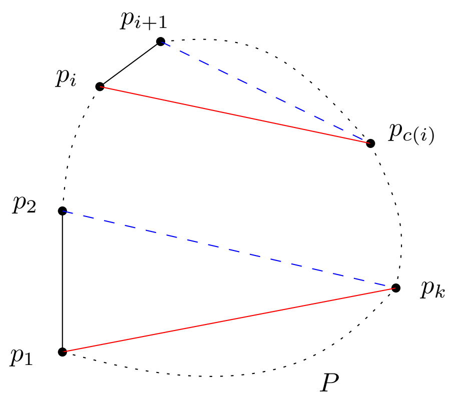


Fig. 2. (a) The edge (p_1, p_2) is not contractible. (b) The edge contraction for $n = 4$.

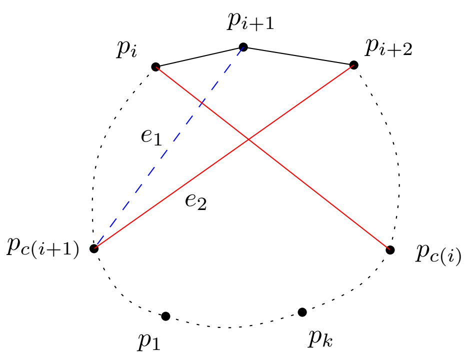
Lemma 2 *Let $\mathcal{C}$ be a configuration with $n \geq 4$ vertices. There is a contractible edge of $\mathcal{C}$ among the edges of the convex polygon.*

PROOF. If $n = 4$ then every edge of the convex polygon is contractible, see Fig. 2 (b). We prove the lemma for $n \geq 5$. Suppose to the contrary that there is a configuration (P, T_1, T_2) such that all edges of P are not contractible. Let $p_1, \ldots, p_n$ be the vertices of P in clockwise order. The edge p_1p_2 is not contractible. Then there is a vertex $p_k, 4 \leq k \leq n-1$ such that p_1p_k is an edge of one triangulation, say T_1, and p_2p_k is an edge of T_2, see Fig. 3 (a).

Consider an edge $p_ip_{i+1}, 2 \leq i \leq k-1$. Since p_ip_{i+1} is not contractible, there is a vertex $p_{c(i)}$ such that $p_ip_{c(i)}$ is a diagonal of $T_j, j = 1,2$ and $p_{i+1}p_{c(i)}$ is a diagonal of T_{3-j}, see Fig. 3 (a). We call $p_{c(i)}$ a *witness* since it indicates that p_ip_{i+1} is not contractible. At least one vertex of $\{p_i, p_{i+1}\}$, say p_l, is different from p_2 and p_k. Then the edge $p_lp_{c(i)}$ does not cross one of the edges p_1p_k or p_2p_k. Therefore $c(i)$ is an index in the range $1, \ldots, k$.



(a)



(b)

Fig. 3. Lemma 2.

We call $p_{c(i)}$ a *left witness* if $c(i) < i$. We call $p_{c(i)}$ a *right witness* if $c(i) > i+1$. Each witness is either left or right since $c(i) \neq i, i+1$. Note that $p_{c(2)}$ is a right witness and $p_{c(k-1)}$ is a left witness. Thus there is an index $i, 2 \leq i \leq k-2$ such that $p_{c(i)}$ is the right index and $p_{c(i+1)}$ is the left index, see Fig. 3 (b). Then $p_ip_{c(i)}$, a diagonal of a triangulation T_j, intersects both diagonals $e_1 = (p_{i+1}, p_{c(i+1)})$ and $e_2 = (p_{i+2}, p_{c(i+1)})$. Either e_1 or e_2 is a diagonal of T_j. Contradiction.

3. Edge creation

We define an operation of *edge creation* as the reverse operation of the edge contraction. The following lemma chracterizes the change of the configuration when an edge is created. We denote the sequence of indices from i to j in clockwise order by $\{i, i+1, \ldots, j\}$.

Lemma 3 (Edge creation) *Let $\mathcal{C} = (P, T_1, T_2)$ be a configuration with $n \geq 3$ vertices where $P = \{p_1, \ldots, p_n\}$. Suppose that an edge $e = (q_1, q_2)$ is created in place of a vertex $p_i \in P$. Let $\mathcal{C}' = (P', T_1', T_2')$ be the configuration obtained by replacing a vertex p_i by an edge $e = (q_1, q_2)$ in clockwise order. Then there are two edges $(p_i, p_j) \in T_1$ and $(p_i, p_k) \in T_2$ such that*

- *an edge $(p_l, p_i) \in T_1, l \in \{i+1, i+2, \ldots, j\}$ is replaced by the edge $(p_l, q_1) \in T_1'$, and*
- *an edge $(p_l, p_i) \in T_1, l \in \{j, j+1, \ldots, i-1\}$ is replaced by the edge $(p_l, q_2) \in T_1'$, and*
- *an edge $(p_l, p_i) \in T_2, l \in \{i+1, i+2, \ldots, p_k\}$ is replaced by the edge $(p_l, q_1) \in T_2'$, and*
- *an edge $(p_l, p_i) \in T_2, l \in \{k, k+1, \ldots, i-1\}$ is replaced by the edge $(p_l, q_2) \in T_2'$.*

We show that an edge can be always created.

Theorem 4 *Let $\mathcal{C} = (P, T_1, T_2)$ be a configuration with $n \geq 3$ vertices and let $\mu : P \to \mathbb{R}^3$ be its realization in $\mathbb{R}^3$. Let $\mathcal{C}' = (P', T_1', T_2')$ be the configuration obtained by an edge creation. There is a realization of $\mathcal{C}'$ if there is a realization of $\mathcal{C}$.*

Theorem 1 follows from Theorem 4.

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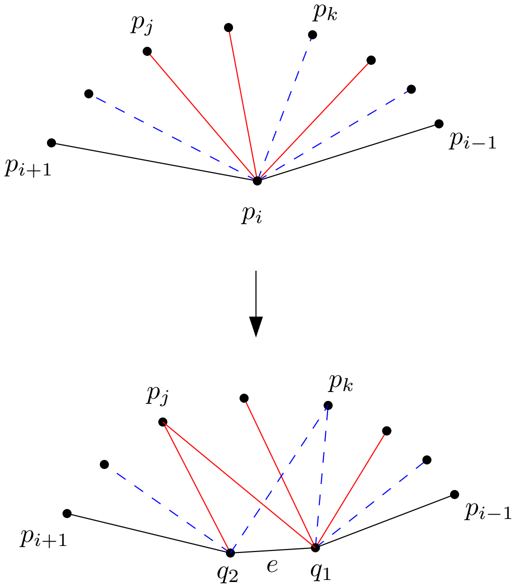

Fig. 4. Edge creation.

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