3D realization of two triangulations of a convex polygon.

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Abstract

We study the problem of construction of a convex 3-polytope whose (i) shadow boundary has \( n \) vertices and (ii) two hulls, upper and lower, are isomorphic to two given triangulations of a convex \( n \)-gon. Barnette [1] proved the existence of a convex 3-polytope in general case. We show that, in our case, a polytope can be constructed using an operation of edge creation.

Key words: triangulation, convex polytope, Steinitz theorem

1. Introduction

Let \( P \) be a convex polygon in the \( xy \)-plane with \( n \) vertices. Two triangulations of \( P \) are called distinct if the only edges they share are the edges of \( P \). Let \( T_1 \) and \( T_2 \) be two distinct triangulations of \( P \). At the First Canadian Conference on Computational Geometry Leo Guibas conjectured that it is always possible to perturb the vertices of \( P \) vertically out (i.e., by displacements parallel to the \( z \)-axis) so that the polygon \( P \) becomes a spatial polygon \( P' \) such that the convex hull of \( P' \) is a convex polyhedron consisting of two triangulated cups glued along \( P' \), and the triangulation of the upper cup (i.e., those faces oriented toward \( +z \)) is that specified as \( T_1 \), and the triangulation of the lower cup is that specified as \( T_2 \) [4].

Boris Bekster [2] disproved Guibas' conjecture by showing a counterexample, a convex hexagon with two triangulations. Marlin and Toussaint [3] considered the computational problem of deciding whether a triple \((P, T_1, T_2)\) admits a realization in \( \mathbb{R}^3 \). They reduced the problem to a linear programming problem with \( O(n^2) \) inequality constraints and \( n \) variables. The variables are \( z \)-coordinates of lifted vertices of \( P \) and the constraints correspond to vertex-face relations: the vertices must be below/above the planes passing through faces of the upper/lower cup of \( P' \). The number of constraints can be dropped to \( 2n - 6 = |T_1| + |T_2| \) by considering dihedral angles corresponding to diagonals of the triangulations [7].

Guibas conjecture is related to Steinitz's theorem [5].

Steinitz's Theorem: A graph \( G \) is isomorphic to the edge graph of a convex 3-polytope if and only if \( G \) is 3-connected and planar.

By Steinitz's theorem the graph \((P, T_1 \cup T_2)\) is the edge graph of a convex 3-polytope [3]. According to Barnette's theorem [1], every 3-polytope with a Hamiltonian circuit has realization such that the Hamiltonian circuit is a shadow boundary. This implies that Guibas' conjecture is true up to a combinatorial deformation [2]. Formally this can be stated as follows.

Theorem 1 For any two distinct triangulations \( T_1 \) and \( T_2 \) of a convex polygon \( P_2 \) in \( \mathbb{R}^2 \) with \( n \) vertices, there is a convex polytope \( P_3 \) in \( \mathbb{R}^3 \) with \( n \) vertices such that (i) the \( xy \)-shadow \( S \) of \( P_3 \) contains all its vertices, and (ii) there is an isomorphism \( \tau : P_3 \to S \) that maps the edges of \( T_1 \) (resp. \( T_2 \)) to the edges of the upper hull of \( P_3 \) (resp. the lower hull).

Barnette's proof deals with general faces (not just triangles) due to its generality. In this paper we give a different proof of Theorem 1 that uses only triangular faces of polytopes which can be turned into a more robust algorithm for finding a combinatorial realization of \((P, T_1, T_2)\) in \( \mathbb{R}^3 \).

Realization questions have been studied in com-
puter graphics and scene analysis as well. Sugihara [6] established necessary and sufficient conditions whether a line drawing in the plane can be realized in \( \mathbb{R}^3 \) by lifting.

We call a triple \((P, T_1, T_2)\) a configuration. We call a map \( \tau \) satisfying the conditions of Theorem 1 a realization.

2. Edge contraction

\[\text{Fig. 1 (a) for example where the edge } p_1p_2 \text{ is contracted. When applied for two triangulations, we want the reduced triangulations to be distinct. An edge } e \text{ of a configuration } (P, T_1, T_2) \text{ is contractible if the new triangulations } T'_1 \text{ and } T'_2 \text{ are distinct. In general, not all edges are contractible. For example, the edge } p_1p_2 \text{ in the Fig. 2 (a) is not contractible since two edges } p_1p_0 \text{ and } p_2p_6 \text{ from different triangulations coincide after the contraction of } p_1p_2.\]

\[\text{Fig. 1 (a) Edge contraction of a triangulation. (b) Edge contraction of two triangulations. The diagonals of one triangulation are solid and the diagonals of the other triangulation are dashed.}\]

\[\text{As in Barnette’s proof we use the operation of edge removal. The difference is that we will not apply it for diagonals of } P. \text{ This prevents the appearance of faces with more than three vertices. The edge contraction in a configuration is defined by identifying the edge endpoints. If applied to one triangulation of } P, \text{ it produces a triangulation, see}\]

\[\text{Fig. 2. (a) The edge } (p_1, p_2) \text{ is not contractible. (b) The edge contraction for } n = 4.\]

\[\text{Lemma 2 Let } C \text{ be a configuration with } n \geq 4 \text{ vertices. There is a contractible edge of } C \text{ among the edges of the convex polygon.}\]

\[\text{Proof. If } n = 4 \text{ then every edge of the convex polygon is contractible, see Fig. 2 (b). We prove the lemma for } n \geq 5. \text{ Suppose to the contrary that there is a configuration } (P, T_1, T_2) \text{ such that all edges of } P \text{ are not contractible. Let } p_1, \ldots, p_n \text{ be}\]
the vertices of $P$ in clockwise order. The edge $p_1p_2$ is not contractible. Then there is a vertex $p_k$, $4 \leq k \leq n-1$ such that $p_1p_k$ is an edge of one triangulation, say $T_1$, and $p_fp_k$ is an edge of $T_2$, see Fig. 3 (a).

Consider an edge $p_ip_{i+1}$, $2 \leq i \leq k-1$. Since $p_ipk+1$ is not contractible, there is a vertex $p_{c(i)}$ such that $p_ip_{c(i)}$ is a diagonal of $T_j$, $j = 1$, 2 and $p_{i+1}p_{c(i)}$ is a diagonal of $T_{3-j}$, see Fig. 3 (a). We call $p_{c(i)}$ a witness since it indicates that $p_ip_{k+1}$ is not contractible. At least one vertex of $\{p_i, p_k\}$, say $p_i$ is different from $p_2$ and $p_k$. Then the edge $p_ip(i)$ does not cross one of the edges $p_1p_k$ or $p_2p_k$. Therefore $c(i)$ is an index in the range $1, \ldots, k$.

![Diagram]

Fig. 3. Lemma 2.

We call $p_{c(i)}$ a left witness if $c(i) < i$. We call $p_{c(i)}$ a right witness if $c(i) > i + 1$. Each witness is either left or right since $c(i) \neq i, i + 1$. Note that $p_{c(2)}$ is a right witness and $p_{c(k-1)}$ is a left witness. Thus there is an index $i$, $2 \leq i \leq k - 1$ such that $p_{c(i)}$ is the right index and $p_{c(i+1)}$ is the left index, see Fig. 3 (b). Then $p_ip_{c(i)}$, a diagonal of a triangulation $T_j$, intersects both diagonals $e_1 = [p_{i+1}, p_{c(i+1)}]$ and $e_2 = [p_{i+2}, p_{c(i+1)}]$. Either $e_1$ or $e_2$ is a diagonal of $T_j$. Contradiction.

3. Edge creation

We define an operation of edge creation as the reverse operation of the edge contraction. The following lemma characterizes the change of the configuration when an edge is created. We denote the sequence of indices from $i$ to $j$ in clockwise order by $\{i, i+1, \ldots, j\}$.

Lemma 3 (Edge creation) Let $C = (P, T_1, T_2)$ be a configuration with $n \geq 3$ vertices where $P = \{p_1, \ldots, p_n\}$. Suppose that an edge $e = (q_1, q_2)$ is created in place of a vertex $p_i \in P$. Let $C' = (P', T'_1, T'_2)$ be the configuration obtained by replacing a vertex $p_i$ by an edge $e = (q_1, q_2)$ in clockwise order. Then there are two edges $(p_i, p_j) \in T_1$ and $(p_j, p_k) \in T_2$ such that

- an edge $(p_i, p_j) \in T_1$, $j \in \{i+1, i+2, \ldots, j\}$ is replaced by the edge $(p_i, q_1) \in T'_1$.
- an edge $(p_i, p_j) \in T_1$, $j \in \{j, j+1, \ldots, i-1\}$ is replaced by the edge $(p_i, q_2) \in T'_1$.
- an edge $(p_j, p_k) \in T_2$, $l \in \{i+1, i+2, \ldots, p_k\}$ is replaced by the edge $(p_j, q_1) \in T'_2$.
- an edge $(p_j, p_k) \in T_2$, $l \in \{k, k+1, \ldots, i-1\}$ is replaced by the edge $(p_j, q_2) \in T'_2$.

We show that an edge can be always created.

Theorem 4 Let $C = (P, T_1, T_2)$ be a configuration with $n \geq 3$ vertices and $\mu : P \rightarrow \mathbb{R}^3$ be its realization in $\mathbb{R}^3$. Let $C' = (P', T'_1, T'_2)$ be the configuration obtained by an edge creation. There is a realization of $C'$ if there is a realization of $C$.

Theorem 1 follows from Theorem 4.

References

Fig. 4. Edge creation.


