A quadratic distance bound on sliding between crossing-free spanning trees

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Abstract

Let S be a set of n points in the plane and let $\mathcal{T}_S$ be the set of all crossing-free spanning trees of S. We show that any two trees in $\mathcal{T}_S$ can be transformed into each other by $O(n^2)$ local and constant-size edge slide operations. No polynomial upper bound for this task has been known, but in [1] a bound of $O(n^2 \log n)$ operations was conjectured.

Key words: crossing-free spanning tree, local transformation, edge slide

1. Introduction

Let S be a set of n points in the Euclidean plane. W.l.o.g. we assume that no two points of S have the same x-coordinate, otherwise we rotate the coordinate system appropriately. A *crossing-free spanning tree* of S is a tree whose edges connect all points in S (and no others) with straight line segments that pairwise do not cross. With $\mathcal{T}_S$ we denote the set of all crossing-free spanning trees of S.

An interesting question is whether, and how fast, two members of $\mathcal{T}_S$ can be transformed into each other by means of predefined rules, often called flips. A common operation is what is called an *edge move*, which relates two trees in the set $\mathcal{T}_S$ iff they have all but one edge in common (one edge is 'flipped'). For this general setting Avis and Fukuda [2] showed that the corresponding tree graph is connected and has a diameter bounded by $2n - 4$. If we restrict the set of allowed flips to planar, length-improving edge moves then in [1] a way to transform any tree $T \in \mathcal{T}_S$ into the minimum spanning tree of S in only $O(n \log n)$ steps was given. For a more detailed discussion and some historical background see [1].

Our interest is focused on a local edge move that keeps one endpoint of the moved edge fixed and moves the other one along an adjacent tree edge. Following [3], we will call this constant-size operation an edge slide. More formally the central operation we consider is defined as follows [1]: Consider a tree $T' \in \mathcal{T}_S$. A *(planar) edge slide* on T' takes some edge $e \in T'$ and moves one of its endpoints along some edge adjacent to e in T', without generating any edge crossings. This gives a new edge f and a new tree $T'' = T' \cup \{f\} \setminus \{e\}$ such that $T'' \in \mathcal{T}_S$. An edge slide is a special kind of planar edge move: T'' is obtained by closing with f a 3-cycle C in T' and by removing e from C, in a way such that T' avoids the interior of the triangle C. Intuitively speaking, an edge slide is an edge operation as local as it can be.

In this paper we investigate the questions of how fast two crossing-free spanning trees of $\mathcal{T}_S$ can be transformed into each other by means of the edge slide operation. To this end consider the *tree graph* $\mathcal{TG}(S)$ which is an undirected graph that has $\mathcal{T}_S$ as its set of nodes. It realizes an arc between two nodes (trees) T' and T'' if and only

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if T' can be transformed into T'' by an edge slide (and vice versa). In [1] it was shown that $\mathcal{TG}(S)$ is connected. The length of a shortest path in $\mathcal{TG}(S)$ corresponds to the distance between the two respective trees. However, for the edge slide operation no polynomial upper bound on this length has been known. It was conjectured that 'if two trees are part of the same triangulation of S then they can be transformed into each other by $O(n^2)$ edge slides'. By results in [1], this would give a diameter of $O(n^2 \log n)$ for the corresponding tree graph $\mathcal{TG}(S)$. We are able to prove the following, stronger result:

Theorem 1 *Let T' and T'' be any two crossing-free spanning trees of S. Then T' can be transformed into T'' by $O(n^2)$ edge slides.*

As mentioned in [1] the edge slide operation could also prove useful in enumerating all simple polygons on a point set S via constant-size local transformations. This question is still unsettled; see e.g. Hernando et al. [5]. Our upper bound on the diameter of $\mathcal{TG}(S)$ might be useful in this respect.

2. Upper Bound Construction

Let S and $T \in \mathcal{T}_S$ be as defined in Section 1. We call a pair (e, p_j), where $e = p_i p_k$ is an edge of T and $p_i, p_j, p_k \in S$ are sorted in x-order, a *slide triangle* if the open triangle $\Delta = p_i p_j p_k$ is free of points from S and edges from T, that is, the interior of Δ is empty.

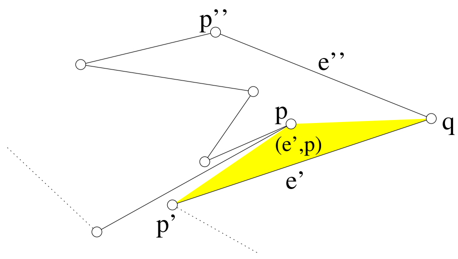

Fig. 1. A slide triangle (e', p), see Lemma 2.

Lemma 2 *Let P be a simple polygon with vertex set S, and let δP be the boundary of P with one marked edge e^*. If $\delta P \setminus \{e^*\}$ is no x-monotonous path then in the interior of P there always exists a slide triangle $(e, p) \subset P$, $e \neq e^*$.*

Proof Since $\delta P \setminus \{e^*\}$ is not x-monotonous there exists a vertex $q \in S$ with two edges from $\delta P \setminus \{e^*\}$ both emanating to the same side, i.e., both to the left or right of q. Let e' and e'', respectively, be these edges. W.l.o.g. we assume that they emanate from q to the left, e' lies below e'' and the left endpoint p' of e' lies to the left of the left endpoint $p''$of e'', see Figure 1 (all other cases are symmetric). If the open triangle $\Delta = qp'p''$ is empty we have a slide triangle (e', p''). Otherwise consider the point p which among all points of P in the interior of Δ minimizes the angle $\angle pp'q$ at p'. Note that there are no edges (partially) inside Δ that have q as an endpoint or intersect e' or e''. Therefore p provides a slide triangle (e', p). □

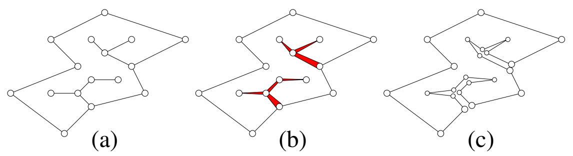

Fig. 2. Cutting a tree polygon (a) along interior edges (b) to obtain a simple polygon (c).

A *tree polygon* P is a simple polygon with interior points, each point connected to the boundary δP via a unique (simple) path such that the resulting graph is planar, see Figure 2(a). In other words, the graph without the edges of δP is a forest. We claim that we can handle this more general situation like a simple polygon: Cut along interior edges and move them apart at the cuts infinitesimally, i.e., duplicate the related vertices, see Figure 2(b) and (c). Observe that the proof of Lemma 2 still holds for this setting by considering edges e' and e'' that are neighboring in the cyclic order around q.

We call the x-monotonous path connecting all vertices of S in their x-sorted order the *canonical spanning tree* $T_c \in \mathcal{T}_S$ of S.

Theorem 3 *For a point set S and a crossing-free spanning tree $T \in \mathcal{T}_S$, $T \neq T_c$, there always exists a slide triangle (e, p), $p \in S$ and $e \in T$ such that the path $\pi \in T$ connecting p to e, say at point q, is x-monotonous. Moreover $\pi \cup pq$ is a simple polygon without interior points.*

Proof We first show that there always exists some slide triangle (e, p). The union of T and the boundary of the convex hull of S partitions S into $k \geq 1$ tree polygons P_i, $i = 1, \ldots, k$. Since T is a connected spanning tree each P_i has a unique edge which stems from the boundary of the convex hull of S. We mark these edges. From Lemma 2 and the discussion afterwards we know that we get a slide triangle inside some P_i unless for all P_i the remaining (non marked) part is x-monotonous. But in the latter case T must be x-monotonous, too, that is, $T = T_c$, a contradiction.

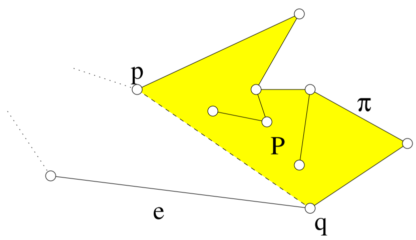

Fig. 3. A path π connecting p to q.

Let q be the (first) endpoint of e to which p is connected. If the edge pq belongs to T we are done. Thus assume that p is connected to q via a path π of length greater than 1, see Figure 3. Since (e, p) is a slide triangle the edge pq does not cross an edge of T. Thus the 'pocket' formed by π together with the edge pq and possible interior edges and points is a tree polygon P. If $\delta P \setminus pq$ is an x-monotonous path we are done. Otherwise we mark the edge pq and apply induction on P. Note that only one edge of P is marked, since T does not contain cycles. Moreover, in every induction step we obtain a smaller instance, since we get rid of at least one edge of T. □

For an edge e its weight is defined as the number of points from S which lie in the open x-interval spanned by e, that is, the number of points which lie between the endpoints of e in the x-sorted order. The weight of a tree T, denoted by $w(T)$, is the sum of the weights of its edges. Obviously T_c has weight zero and is the only tree with this property. Since each of the $n-1$ edges of T has at most weight $n-2$ the weight of a tree with n points is bounded by $(n-1)(n-2) < n^2$. A tight bound is given by the following lemma, for which we omit the proof in this extended abstract.

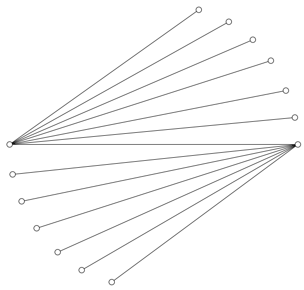

Fig. 4. A tree with maximum weight of $\frac{3n^2-10n+8}{4}$.

Lemma 4 *The weight $w(T)$ of a crossing-free spanning tree $T \in \mathcal{T}_S$ is bounded by $0 \leq w(T) \leq \lfloor \frac{3n^2-10n+8}{4} \rfloor$, $n \geq 2$, and these bounds are tight.*

Lemma 5 *Any crossing-free spanning tree $T \in \mathcal{T}_S$ can be transformed into T_c by at most $2 \cdot w(T)$ edge slides.*

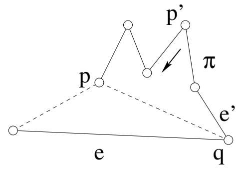

Fig. 5. A slide triangle (e, p) with x-monotonous path π connecting p to q.

Proof If $T = T_c$ the statement is obviously true, so let $T \neq T_c$. Let (e, p) be a slide triangle as provided by Theorem 3, see Figure 5. Let $k \geq 1$ be the number of edges of the x-monotonous path π connecting p to some endpoint q of e. We claim that we can reduce the weight of e by at least k by performing $2k-1$ edge slides. To this end let e' be the edge of π incident to q. Our first task is to slide e' along π to obtain the edge qp.

Assume that $k > 1$. Then π avoids the interior of the slide triangle (e, p) and thus contains at least one vertex p' pointed away from the edge qp. Since π is x-monotonous we can slide the edge of π which has p' as its left endpoint 'towards' p along the edge of π which has p' as its right endpoint. We repeat this process until we obtain the edge qp, i.e., $k = 1$. Since each edge slide reduces the length of the current path from p to q by one, we carry out exactly $k-1$ steps.

Now we can slide e along qp, reducing its weight by at least k (the vertices of π different from q). Finally we slide qp back to e' by reversing the steps of the first phase.

As long as the resulting tree is not T_c we repeat all above steps. After each iteration the weight of a single edge has been decreased by at least half of the number of the involved edge slide operations. We thus can transform T into T_c with at most $2w(T)$ edge slides. □

We are now ready to prove our main result as proposed in Section 1. We give here a more explicit statement and Theorem 1 then follows as a corollary.

Theorem 6 *For any pair* $T', T'' \in \mathcal{T}_S$ *we can transform* T' *into* T'' *by at most* $2(w(T') + w(T'')) \leq 3n^2$ *edge slides.*

Proof Lemma 5 shows that we can transform any tree $T' \in \mathcal{T}_S$ into T_c with at most $2w(T')$ edge slides. By symmetry of the edge slide operation we can use the reverse transformation for T''. Together with the upper bound $w(T'), w(T'') \leq \frac{3n^2}{4}$ from Lemma 4, the theorem follows. □

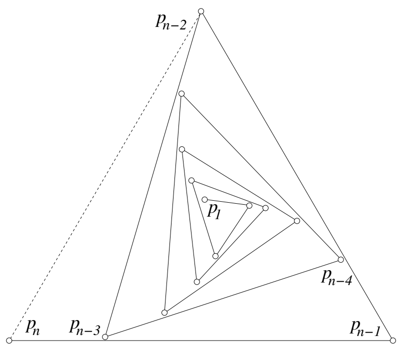

Fig. 6. To obtain the edge $p_n p_{n-2}$ requires $(n-1)(n-2)/2$ edge slides for odd $n \geq 3$.

Figure 6 shows that there are examples requiring $\Omega(n^2)$ edge slides to transform two spanning trees into each other. Thus the bound of Theorem 1 is tight. We omit the details on the lower bound construction in this extended abstract.

3. Discussion and Open Problems

One might wonder whether the slide-distance between two spanning trees which do not intersect each other is smaller than in the general case. A similar result holds for triangulations, where the flip-distance can be bounded by the number of crossing edges [4]. However, from the example in Figure 6 it follows that even for two trees differing in only one edge the slide-distance is quadratic.

Another observation is that the weight of a spanning-tree is direction-sensitive. So an obvious question is whether there always exists a 'nice' direction with sub-quadratic weight? Again a negative answer is given by the example of Figure 6, having weight $\Theta(n^2)$ for any direction of the x-axis.

So far we only obtained results on the number of necessary slide operations. On the algorithmic side we are also interested in the time complexity to compute the $O(n^2)$ slide sequence. We plan to investigate this question in the near future.

A related algorithmic question is how fast we can compute a direction to minimize the weight of a given tree. This can be done in time $O(n^2 \log n)$, but we omit the details in this extended abstract.

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