Differential properties of some families of matrix valued orthogonal polynomials and applications

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Outline

1. Scalar versus matrix orthogonality
   - Scalar case
   - Matrix case
2. New phenomena
   - Algebra of differential operators
   - Cone and convex cone of weight matrices
3. Applications
   - Quasi-birth-and-death processes
   - Quantum mechanics
   - Time-and-band limiting

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Classical families

Hermite: $\omega(t) = e^{-t^2}$, $t \in (-\infty, \infty)$:

$$H_n(t)'' - 2tH_n(t)' = -2nH_n(t)$$

Laguerre: $\omega(t) = t^{\alpha}e^{-t}$, $\alpha > -1$, $t \in (0, \infty)$:

$$tL_n^{\alpha}(t)'' + (\alpha + 1 - t)L_n^{\alpha}(t)' = -nL_n^{\alpha}(t)$$

Jacobi: $\omega(t) = t^{\alpha}(1-t)^{\beta}$, $\alpha, \beta > -1$, $t \in (0, 1)$:

$$t(1-t)P_n^{(\alpha,\beta)}(t)'' + (\alpha + 1 - (\alpha + \beta + 2)t)P_n^{(\alpha,\beta)}(t)' =$$

$$-n(n + \alpha + \beta + 1)P_n^{(\alpha,\beta)}(t)$$

Applications:

- Quantum non relativistic models (Schrödinger equation).
- Electrostatic equilibrium (with logarithmic potential).

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$$(c_2t^2 + c_1t + c_0)p_n''(t) + (d_1t + d_0)p_n'(t) = \lambda_n p_n(t)$$

$\Rightarrow$ Hermite, Laguerre and Jacobi (Bessel) polynomials

Orthonormality of $(p_n)_n$ with respect to a positive measure ω

$$\langle p_n, p_m\rangle_\omega = \int_{\mathbb{R}} p_n(t)p_m(t)d\omega(t) = \delta_{nm}, \quad n, m \geqslant 0$$

is equivalent to a three term recurrence relation

$$tp_n(t) = a_{n+1}p_{n+1}(t) + b_np_n(t) + a_np_{n-1}(t), \quad a_{n+1} \neq 0, \quad b_n \in \mathbb{R} \quad n \geqslant 0$$

Jacobi operator (tridiagonal):

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Matrix case

Matrix valued polynomials on the real line:

$$\alpha_n t^n + \alpha_{n-1} t^{n-1} + \cdots + \alpha_0, \quad \alpha_i \in \mathbb{C}^{N\times N}$$

Krein (1949): matrix valued orthogonal polynomials (MOP)

Orthogonality: weight matrix W (positive definite)

Matrix valued inner product:

$$\langle P, Q\rangle_W = \int_a^b P(t)\mathrm{d}W(t)Q^*(t), \quad P, Q \quad \text{mat. pol.}$$

- $\mathrm{d}W(t) = W(t)\mathrm{d}t + M\delta_{t_0}(t)$
- A weight matrix $W_1(t)$ reduces to scalar weights if there exists a nonsingular matrix (independent of t) T such that $W_1(t) = TW_2(t)T^*$ where $W_2(t)$ diagonal.

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- Systematic study: Asymptotics, zeros of MOP, quadrature formulae...
  Applications: scattering theory, times series and signal processing...

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$$\text{grad } F_i\leqslant i,\quad \Lambda_n \quad \text{Hermitian}$$

Equivalent to the symmetry of

$$D=\partial^2F_2(t)+\partial^1F_1(t)+\partial^1F_0(t),\quad \partial=\frac{d}{dt}$$
$$\text{con}\quad P_nD=\Lambda_nP_n$$

D es symmetric with respect to W if $\langle PD,Q\rangle_W=\langle P,QD\rangle_W$

It has not been until very recently when the first examples appeared: Grünbaum-Pacharoni-Tirao (2003) and Durán-Grünbaum (2004)

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How to get examples

- Matrix spherical functions associated to $P_n(\mathbb{C}) = \mathrm{SU}(n+1)/\mathrm{U}(n)$ Grünbaum-Pacharoni-Tirao (2003)
- Durán-Grünbaum (2004):

Symmetry equations

$$F_2 W = W F_2^*$$

$$2(F_2 W)' = F_1 W + W F_1^*$$

$$(F_2 W)'' - (F_1 W)' + F_0 W = W F_0^*$$

$$\lim_{t \to x} F_2(t) W(t) = 0 = \lim_{t \to x} (F_1(t) W(t) - W(t) F_1^*(t)), \quad \text{for} \quad x = a, b$$

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General method (Durán-Grünbaum, 2004): factorize

$$W(t) = \omega(t)T(t)T^*(t),$$

where ω is an scalar weight (Hermite, Laguerre or Jacobi) and T is a matrix function solving

$$\begin{cases} T'(t) = G(t)T(t) \\ T(t_0) = I \end{cases}$$

Examples:

If $\omega = e^{-t^2} \Rightarrow G(t) = A + 2Bt \Rightarrow \begin{cases} B = 0 \Rightarrow e^{-t^2}e^{At}e^{A^*t} \\ A = 0 \Rightarrow e^{-t^2}e^{Bt^2}e^{B^*t^2} \end{cases}$

If $\omega = t^\alpha e^{-t} \Rightarrow G(t) = A + \dfrac{B}{t} \Rightarrow \begin{cases} B = 0 \Rightarrow t^\alpha e^{-t}e^{At}e^{A^*t} \\ A = 0 \Rightarrow t^\alpha e^{-t}t^B t^B \end{cases}$

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- Moment equations (Durán-MdI)
- Matrix valued bispectral problem (Grünbaum-Tirao, 2007) $\Rightarrow$ *ad-conditions*

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Outline

Algebra of differential operators

For a fixed family $(P_n)_n$ of MOP we study the algebra over $\mathbb{C}$

$$\mathcal{D}(W) = \left\{ D = \sum_{i=0}^{k} \partial^i F_i(t) : P_n D = \Lambda_n(D) P_n, \; n = 0, 1, 2, \dots \right\}$$

Scalar case: If $\mathcal{F}$ is the second order differential operator (Hermite, Laguerre or Jacobi), then any operator $\mathcal{U}$ such that $\mathcal{U} p_n = \lambda_n p_n$

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Origin: Existence of several linearly independent second order differential operators having a fixed family of MOP as eigenfunctions

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Example

$$W_{\alpha,\nu_1,\ldots,\nu_{N-1}}(t) = t^{\alpha}e^{-t}e^{At}t^{\frac{1}{2}J}t^{\frac{1}{2}J^*}e^{A^*t},\ \alpha > -1,\ t > 0$$

$$A = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \nu_i \in \mathbb{C}\backslash\{0\},\ J = \begin{pmatrix} N-1 & 0 & \cdots & 0 & 0 \\ 0 & N-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Durán-MdI (2008)

Factorization

$W_{\alpha,\nu_1,\ldots,\nu_{N-1}}(t) = t^{\alpha}e^{-t}T(t)T^*(t)$, where

$$\begin{cases} T'(t) = \frac{1}{2}\left(A + \frac{J}{t}\right)T(t), \\ T(1) = e^{A} \end{cases} \qquad \mathrm{ad}_A J = [A, J] = -A$$

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Second order differential operators

$D_1 = \partial^2 tI + \partial^1[(\alpha+1)I + J + t(A-I)] + \partial^0[(J+\alpha I)A - J]$

$$i(N-i)|\nu_{N-1}|^2 = (N-1)|\nu_i|^2 + (N-i-1)|\nu_i|^2|\nu_{N-1}|^2,\ i = 1, \ldots, N-2$$

$\Rightarrow D_2 = \partial^2 F_2 + \partial^1 F_1 + \partial^0 F_0$, where

$$F_2 = t(J - At),$$

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where $(Y)_{i+1,i} = \frac{i(N-i)}{\nu_i}$ and $(Y)_{i,j} = 0$ otherwise.

$$\prod_{i=1}^{N}\left((i-1)D_1 - D_2 + \left[\frac{(N-1)(N-i)}{|\nu_{N-1}|^2} + (i-1)(N-i)\right] I\right) = 0$$

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Algebra of differential operators

The weight matrix $W_{\alpha,a}$

$$W_{\alpha,a}(t) = t^{\alpha} e^{-t} \underbrace{\begin{pmatrix} t(1+|a|^2 t) & at \\ \bar{a}t & 1 \end{pmatrix}}_{R_a(t)}, \quad \alpha > -1, \quad t > 0$$

Rodrigues' formula

$$\mathcal{P}_{n,\alpha,a}(t) = \Phi_{n,\alpha,a}\left[t^{\alpha+n} e^{-t}(R_a(t) + X_{n,a})\right]^{(n)} R_a^{-1}(t) t^{-\alpha} e^{t}, \quad n = 1, 2, \ldots$$

$$\Phi_{n,\alpha,a} = \begin{pmatrix} 1 & -a(1+\alpha) \\ 0 & 1/\lambda_{n,a} \end{pmatrix}, \quad X_{n,a} = \begin{pmatrix} 0 & -an \\ 0 & 0 \end{pmatrix}, \quad \lambda_{n,a} = 1 + n|a|^2$$

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New linearly independent differential operators

$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
1	0	2	2	2	2	2	2	2

- Grünbaum-Pacharoni-Tirao (2003) and Castro-Grünbaum (2005): Examples of MOP satisfying first order differential equations (weight matrices reduce to scalar weights).

Basis for the second order differential operators

$$L_1 = \partial^2 \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} + \partial^1 \begin{pmatrix} \alpha+2-t & at \\ 0 & \alpha+1-t \end{pmatrix} + \partial^0 \begin{pmatrix} -\frac{1+|a|^2}{|a|^2} & (1+\alpha)a \\ 0 & -\frac{1}{|a|^2} \end{pmatrix}$$

$$L_2 = \partial^2 \begin{pmatrix} t & -2at^2 \\ 0 & -t \end{pmatrix} + \partial^1 \begin{pmatrix} \alpha+2+t & -\frac{(2+|a|^2(2\alpha+5))t}{\bar{a}} \\ \frac{2}{a} & -t-\alpha-1 \end{pmatrix} +$$

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Third order differential operators I

$$
\begin{aligned}
L_3 =&\partial^3 \begin{pmatrix} -|a|^2 t^2 & a t^2(1+|a|^2 t) \\ -\bar{a} t & |a|^2 t^2 \end{pmatrix} + \\
&\partial^2 \begin{pmatrix} -t(2+|a|^2(\alpha+5)) & a t(2\alpha+4+t(1+|a|^2(\alpha+5))) \\ -\bar{a}(\alpha+2) & t(2+|a|^2(\alpha+2)) \end{pmatrix} + \\
&\partial^1 \begin{pmatrix} t-2(\alpha+2)(1+|a|^2) & \frac{|a|^2(\alpha+1)(\alpha+2)+t(1+2|a|^2(1+|a|^2(\alpha+2))}{\bar{a}} \\ -\frac{1}{a} & 2\alpha+2-t \end{pmatrix} + \\
&\partial^0 \begin{pmatrix} 1+\alpha & -\frac{1}{\bar{a}}(1+\alpha)(|a|^2\alpha-1) \\ \frac{1}{a} & -(1+\alpha) \end{pmatrix}
\end{aligned}
$$

Third order differential operators II

$$L_4 = \partial^3 \begin{pmatrix} |a|^2 t^2 & a t^2(-1 + |a|^2 t) \\ \overline{a} t & -|a|^2 t^2 \end{pmatrix} +$$
$$\partial^2 \begin{pmatrix} |a|^2 t(\alpha + 5) & -at(-2\alpha - 4 + t(3 + |a|^2(\alpha + 5))) \\ \overline{a}(\alpha + 2) & -|a|^2 t(\alpha + 2) \end{pmatrix} +$$
$$\partial^1 \begin{pmatrix} 2|a|^2(\alpha + 2) + t & a(\alpha + 1)(\alpha + 2) - t\left(\frac{1}{\overline{a}} + 2a(2 + |a|^2)(\alpha + 2)\right) \\ -\frac{1}{a} & -t \end{pmatrix}$$
$$+ \partial^0 \begin{pmatrix} 1 + \alpha & -\frac{1}{\overline{a}}(1 + \alpha)(1 + |a|^2(\alpha + 2)) \\ \frac{1}{a} & -(1 + \alpha) \end{pmatrix}$$

Conjecture

- Even order: Order $0 = \{I\}$, Order 2i $= \{L_1^i, L_1^{i-1} L_2\}$, $i \geqslant 1$
- Odd order: Order $1 = \{0\}$, Order 4i-1 $= \{L_1^i L_3 - L_3 L_1^i, L_2^i L_3 + L_3 L_2^i\}$, $i \geqslant 0$

  Order 4i+1 $= \{L_1^i L_3 + L_3 L_1^i, L_2^i L_3 - L_3 L_2^i\}$, $i \geqslant 0$

Third order differential operators II

$$
\begin{aligned}
L_4 =& \partial^3 \begin{pmatrix} |a|^2 t^2 & a t^2(-1+|a|^2 t) \\ \bar{a} t & -|a|^2 t^2 \end{pmatrix} + \\
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& \partial^1 \begin{pmatrix} 2|a|^2(\alpha+2)+t & a(\alpha+1)(\alpha+2) - t\left(\frac{1}{\bar{a}} + 2a(2+|a|^2)(\alpha+2)\right) \\ -\frac{1}{a} & -t \end{pmatrix} \\
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A sample of relations

Four quadratic relations

$$L_1^2 = L_2^2 \qquad L_3^2 = -L_4^2$$
$$L_1 L_2 = L_2 L_1 \quad L_3 L_4 = -L_4 L_3$$

Four permutational relations

$$L_1 L_3 - L_2 L_4 = 0 \quad L_2 L_3 - L_1 L_4 = 0$$
$$L_3 L_2 + L_4 L_1 = 0 \quad L_3 L_1 + L_4 L_2 = 0$$

Four more quadratic relations

$$L_3 = L_1 L_4 - L_4 L_1$$
$$L_3 = L_2 L_3 + L_3 L_2 \quad L_4 = L_2 L_4 + L_4 L_2$$

Cubic relations

$$L_1 L_3^2 = L_3^2 L_1 \quad L_2 L_3^2 = L_3^2 L_2$$

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$$L_3 = L_1 L_4 - L_4 L_1 \quad L_4 = L_1 L_3 - L_3 L_1 \checkmark$$
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L_2 in terms of L_1 and L_3

$$\begin{aligned}\left[|a|^2(2+\alpha)-1\right]\left[|a|^2(\alpha-1)-1\right]L_2 &= 2|a|^2\left[|a|^2(2\alpha+1)-2\right]L_1\\ &+\left[|a|^4(\alpha^2+\alpha-5)-|a|^2(2\alpha+1)+1\right]L_1^2\\ &-2|a|^2\left[|a|^2(2\alpha+1)-2\right]L_1^3+3|a|^4L_1^4\\ &-\frac{1}{2}\left[|a|^2(2\alpha+1)-2\right]L_3^2+\frac{15}{2}|a|^2L_3^2L_1-\frac{9}{2}|a|^2L_3L_1L_3\end{aligned}$$

Conjecture

$$\mathcal{D}(W_{\alpha,a}) \quad \text{generated by} \quad \{I, L_1, L_3\}$$

Note

For the exceptional values of $\alpha = 1+\frac{1}{|a|^2}$ or $\alpha = -2+\frac{1}{|a|^2}$

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Cone and convex cone of weight matrices

Dual situation to $\mathcal{D}(W)$: given a fixed differential operator D we study:

$$\mathfrak{X}(D) = \{W : P_n^W D = \Gamma_n P_n^W, \quad n \geqslant 0, \quad \text{i.e.} \quad D \in \mathcal{D}(W)\}$$

$$\Upsilon(D) = \{W : \ D \text{ is symmetric with respect to } W \quad \text{i.e.} \quad D \in \mathcal{S}(W)\}$$

- $\Upsilon(D) \subset \mathfrak{X}(D)$
- If $\mathfrak{X}(D) \neq \emptyset$, it is a cone: $W \in \mathfrak{X}(D) \Rightarrow \alpha W \in \mathfrak{X}(D), \ \alpha > 0$
- If $\Upsilon(D) \neq \emptyset$, it is a convex cone:
  $W_1, W_2 \in \Upsilon(D) \Rightarrow \gamma W_1 + \zeta W_2 \in \Upsilon(D), \ \gamma, \zeta \geqslant 0$

$\Rightarrow$ (Monic) MOP $P_{n,\zeta/\gamma}$ with respect to $\gamma W_1 + \zeta W_2$ satisfy

$$P_{n,\zeta/\gamma} D = \Gamma_n P_{n,\zeta/\gamma}$$

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Adding a Dirac delta distribution

We look for uniparametric families of (monic) MOP $(P_{n,\gamma})_n$ such that they are eigenfunctions of a fixed second order differential operator D

$$P_{n,\gamma}D = \Gamma_n P_{n,\gamma}, \quad n = 0, 1, \dots$$

$P_{n,\gamma}$ orthogonal with respect to $W + \gamma M(t_0)\delta_{t_0}, \quad \gamma \geqslant 0$

Scalar case $(\omega + m\delta_{t_0})$

- Second order: there are NOT symmetric second order differential operators.
- Fourth order: t_0 at the endpoints of the support, which is NOT symmetric with respect to the original weight (Krall, 1941):

$$\text{Laguerre type } e^{-t} + M\delta_0$$

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Method to find examples

Theorem (Durán-MdI, 2008)

Let W be a weight matrix and $D = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0$. Assume that associated with the real point $t_0 \in \mathbb{R}$ there exists a Hermitian positive semidefinite matrix $M(t_0)$ satisfying

$$F_2(t_0)M(t_0) = 0,$$
$$F_1(t_0)M(t_0) = 0,$$
$$F_0 M(t_0) = M(t_0)F_0^*.$$

Then

$$D \text{ is symmetric with respect to } W$$
$$\Leftrightarrow$$
$$D \text{ is symmetric with respect to } W + M(t_0)\delta_{t_0}$$

Example where $t_0 \in \mathbb{R}$

$$W(t) = e^{-t^2} \begin{pmatrix} 1 + a^2 t^2 & at \\ at & 1 \end{pmatrix}, \quad t \in \mathbb{R}, \quad a \in \mathbb{R} \setminus \{0\}$$

Durán-Grünbaum (2004): weight matrix

Castro-Grünbaum (2006): Algebra of differential operators

Symmetry equations $\Rightarrow$ Expression for the 5-dimensional (real) linear space of symmetric differential operators of order at most two

Constraints:

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$t_0 = 0$

$$D = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0(t),$$

$$F_2(t) = \begin{pmatrix} 1 - at & -1 + a^2t^2 \\ -1 & 1 + at \end{pmatrix}$$

$$F_1(t) = \begin{pmatrix} -2a - 2t & 2a + 2(2 + a^2)t \\ 0 & -2t \end{pmatrix}$$

$$F_0(t) = \begin{pmatrix} -1 & 2\frac{2+a^2}{a^2} \\ \frac{4}{a^2} & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$\Rightarrow$ D is symmetric with respect to the family of weight matrices

$$\Upsilon(D) = \left\{ e^{-t^2} \begin{pmatrix} 1 + a^2t^2 & at \\ at & 1 \end{pmatrix} + \gamma \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \delta_0(t), \quad \gamma \geqslant 0 \right\} = \mathfrak{X}(D)$$

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$$D = \partial^2 F_2(t) + \partial^1 F_1(t) + \partial^0 F_0(t),$$

$$F_2(t) = \begin{pmatrix} -\xi^{\mp}_{a,t_0} + at_0 - at & -1 - (a^2 t_0)t + a^2 t^2 \\ -1 & -\xi^{\mp}_{a,t_0} + at \end{pmatrix}$$

$$F_1(t) = \begin{pmatrix} -2a + 2\xi^{\mp}_{a,t_0} t & -2t_0 - 2a\xi^{\mp}_{a,t_0} + 2(2+a^2)t \\ 2t_0 & 2(\xi^{\mp}_{a,t_0} - at_0)t \end{pmatrix}$$

$$F_0(t) = \begin{pmatrix} \xi^{\mp}_{a,t_0} + 2\frac{t_0}{a} & 2\frac{2+a^2}{a^2} \\ \frac{4}{a^2} & -\xi^{\mp}_{a,t_0} - 2\frac{t_0}{a} \end{pmatrix}$$

$$M(t_0) = \begin{pmatrix} (\xi^{\pm}_{t_0,a})^2 & \xi^{\pm}_{t_0,a} \\ \xi^{\pm}_{t_0,a} & 1 \end{pmatrix}, \quad \xi^{\pm}_{a,t_0} = \frac{at_0 \pm \sqrt{4 + a^2 t_0^2}}{2}$$

Another example where $t_0 \in \mathbb{R}$

$$W(t) = t^{\alpha} e^{-t} \begin{pmatrix} t^2 + a^2(t-1)^2 & a(t-1) \\ a(t-1) & 1 \end{pmatrix}, \quad t > 0, \quad \alpha > -1$$

Durán-Grünbaum (2004)

$t_0 = -1, \alpha = 0, a = 1$

$$D = \partial^2 \begin{pmatrix} -\frac{\sqrt{2}(\sqrt{2}+2t)}{2} & -1+2t^2 \\ 1 & \frac{\sqrt{2}(\sqrt{2}-2t)}{2} \end{pmatrix} +$$

$$\partial^1 \begin{pmatrix} (1-\sqrt{2})(5+2\sqrt{2}-t) & -2\sqrt{2}+6t \\ -2 & (1+\sqrt{2})(t-1) \end{pmatrix} + \partial^0 \begin{pmatrix} -1+\frac{\sqrt{2}}{2} & \frac{3}{2} \\ \frac{1}{2} & 1-\frac{\sqrt{2}}{2} \end{pmatrix}$$

$$M = \begin{pmatrix} 3+2\sqrt{2} & -1-\sqrt{2} \\ -1-\sqrt{2} & 1 \end{pmatrix}$$

Another example where $t_0 \in \mathbb{R}$

$$W(t) = t^{\alpha} e^{-t} \begin{pmatrix} t^2 + a^2(t-1)^2 & a(t-1) \\ a(t-1) & 1 \end{pmatrix}, \quad t > 0, \quad \alpha > -1$$

Durán-Grünbaum (2004)

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$$M = \begin{pmatrix} 3+2\sqrt{2} & -1-\sqrt{2} \\ -1-\sqrt{2} & 1 \end{pmatrix}$$

Example where δ_0 of size $N \times N$

$$W_{\alpha,\nu_1,\dots,\nu_{N-1}}(t) = t^\alpha e^{-t} e^{At} t^{\frac{1}{2}J} t^{\frac{1}{2}J^*} e^{A^* t}, \ \alpha > -1, \ t > 0$$

$$A = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \nu_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \nu_i \in \mathbb{R}\backslash\{0\}, \ J = \begin{pmatrix} N-1 & 0 & \cdots & 0 & 0 \\ 0 & N-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Durán-MdI (2008)

Second order differential operators

$$D_1 = \partial^2 tI + \partial^1[(\alpha+1)I + J + t(A-I)] + \partial^0[(J+\alpha I)A - J]$$

$$D_2 = \partial^2 t(J - At) + \partial^1((1+\alpha)I + J)J + Y - t(J + (\alpha+2)A + Y^* - \mathrm{ad}_A Y)$$

$$+ \partial^0 \frac{N-1}{\nu_{N-1}^2}[J - (\alpha I + J)A]$$

Example where δ_0 of size $N \times N$

Symmetry equations $\Rightarrow$ Expression for the 3-dimensional (real) linear space of symmetric differential operators of order at most two

$t_0 = 0$

$$D = -(N-1)D_1 + D_2$$

$$(M)_{ij} = \left(\prod_{k=\min\{i,j\}}^{\max\{i,j\}-1} \frac{\nu_k(\alpha + N - k)}{N-k} \right) \left(\prod_{k=1}^{N-\max\{i,j\}} \frac{\nu_{N-k}(\alpha + k)}{k} \right)^2$$

$\Rightarrow$ D is symmetric with respect to the family of weight matrices

$$\Upsilon(D) = \{W_{\alpha,\nu_1,\ldots,\nu_{N-1}}(t) + \gamma M \delta_0(t), \quad \gamma \geqslant 0\} = \mathfrak{X}(D)$$

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Outline

1. Scalar versus matrix orthogonality
   - Scalar case
   - Matrix case
2. New phenomena
   - Algebra of differential operators
   - Cone and convex cone of weight matrices
3. Applications
   - Quasi-birth-and-death processes
   - Quantum mechanics
   - Time-and-band limiting

Birth-and-death processes

Transition probability matrix

$$P = \begin{pmatrix} b_0 & a_0 & & & \\ c_1 & b_1 & a_1 & & \\ & c_2 & b_2 & a_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad b_n \geqslant 0, a_n, c_n > 0, \quad a_n + b_n + c_n = 1$$

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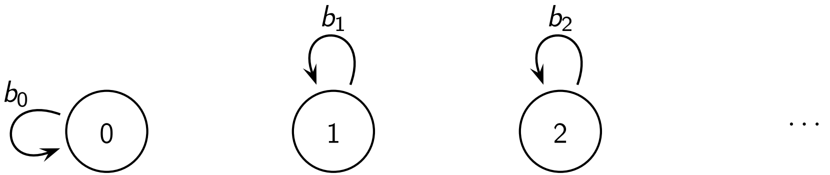

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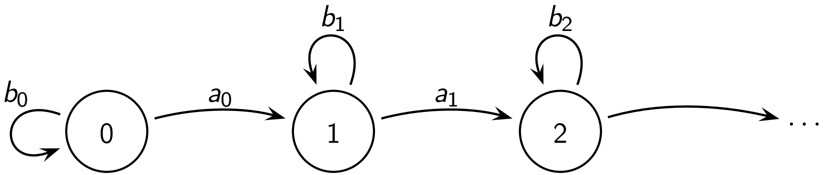

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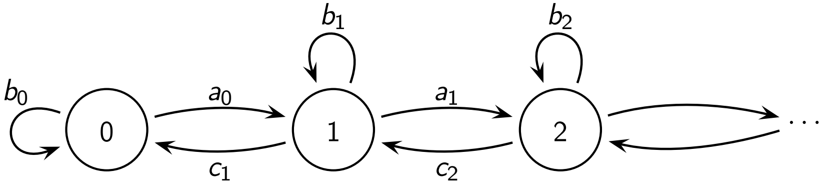

Introducing the polynomials $(q_n)_n$ by the conditions $q_{-1}(t) = 0$, $q_0(t) = 1$ and the recursion relation

$$t \begin{pmatrix} q_0(t) \\ q_1(t) \\ \vdots \end{pmatrix} = P \begin{pmatrix} q_0(t) \\ q_1(t) \\ \vdots \end{pmatrix}$$

i.e.

$$tq_n(t) = a_n q_{n+1}(t) + b_n q_n(t) + c_n q_{n-1}(t), \quad n = 0, 1, \ldots$$

there exists a unique measure $d\omega(t)$ supported in $[-1, 1]$ such that

$$\int_{-1}^{1} q_i(t) q_j(t) d\omega(t) \Big/ \int_{-1}^{1} q_j(t)^2 d\omega(t) = \delta_{ij}$$

n-step transition probability matrix:

$$\text{Prob}\{E_i \to E_j \text{ in } n \text{ steps}\} = P_{ij}^n = \sum_{k_1,k_2,\ldots,k_{n-1}} P_{ik_1}P_{k_1k_2}\cdots P_{k_{n-1}j}$$

Karlin y McGregor (1959): integral representation of P^n

Karlin-McGregor formula

$$P_{ij}^n = \int_{-1}^1 t^n q_i(t)q_j(t)\mathrm{d}\omega(t) \Big/ \int_{-1}^1 q_j(t)^2\mathrm{d}\omega(t)$$

Invariant measure or distribution

A non-null vector $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2, \ldots)$ with non-negative components

$$\boldsymbol{\pi}P = \boldsymbol{\pi}$$

$$\Rightarrow \pi_i = \frac{a_0a_1\cdots a_{i-1}}{c_1c_2\cdots c_i} = \frac{1}{\int_{-1}^1 q_i^2(t)\mathrm{d}\omega(t)} = \frac{1}{\|q_i\|^2}$$

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Quasi-birth-and-death processes

Transition probability matrix

$$P = \begin{pmatrix} B_0 & A_0 & & & \\ C_1 & B_1 & A_1 & & \\ & C_2 & B_2 & A_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad \begin{array}{l} (A_n)_{ij}, (B_n)_{ij}, (C_n)_{ij} \geqslant 0, \ \det(A_n), \det(C_n) \neq 0 \\ \displaystyle\sum_j (A_n)_{ij} + (B_n)_{ij} + (C_n)_{ij} = 1, \ i = 1, \ldots, N \end{array}$$

Particular case: pentadiagonal matrix

$$P = \left(\begin{array}{cc|cc|cc|cc|c} b_0 & a_0 & d_0 & 0 & & 0 & & & \\ c_1 & b_1 & a_1 & d_1 & & & & & \\ \hline e_2 & c_2 & b_2 & a_2 & d_2 & 0 & & 0 & \\ 0 & e_3 & c_3 & b_3 & a_3 & d_3 & & & \\ \hline & 0 & e_4 & c_4 & b_4 & a_4 & d_4 & 0 & \ddots \\ & & 0 & e_5 & c_5 & b_5 & a_5 & d_5 & \\ \hline & & & \ddots & & \ddots & & \ddots & \ddots \end{array}\right)$$

Quasi-birth-and-death processes

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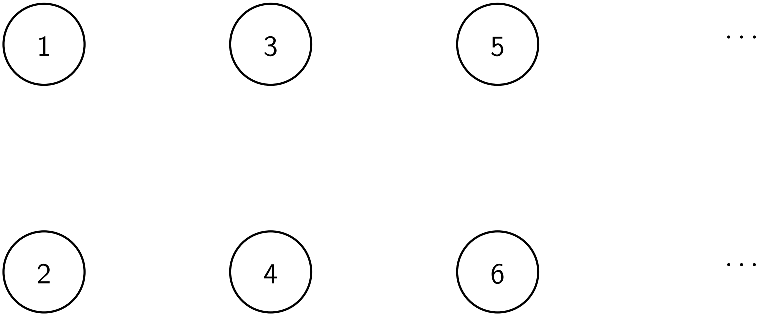
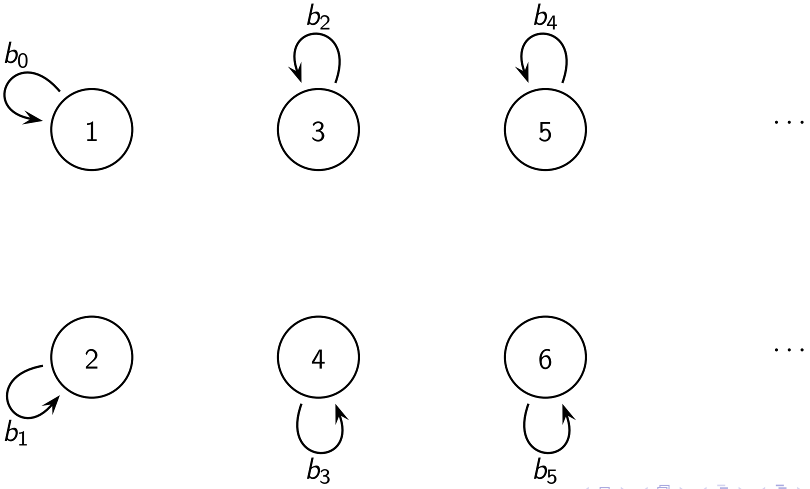
$$P = \left(\begin{array}{cc|cc|cc|cc|c} b_0 & a_0 & d_0 & 0 & \multicolumn{2}{c|}{\multirow{2}{*}{0}} & & & \\ c_1 & b_1 & a_1 & d_1 & & & & & \\ \hline e_2 & c_2 & b_2 & a_2 & d_2 & 0 & \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \\ 0 & e_3 & c_3 & b_3 & a_3 & d_3 & & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & e_4 & c_4 & b_4 & a_4 & d_4 & 0 & \ddots \\ & & 0 & e_5 & c_5 & b_5 & a_5 & d_5 & \\ \hline & & \multicolumn{2}{c|}{\ddots} & \multicolumn{2}{c|}{\ddots} & \multicolumn{2}{c|}{\ddots} & \ddots \end{array}\right)$$

Network



Network



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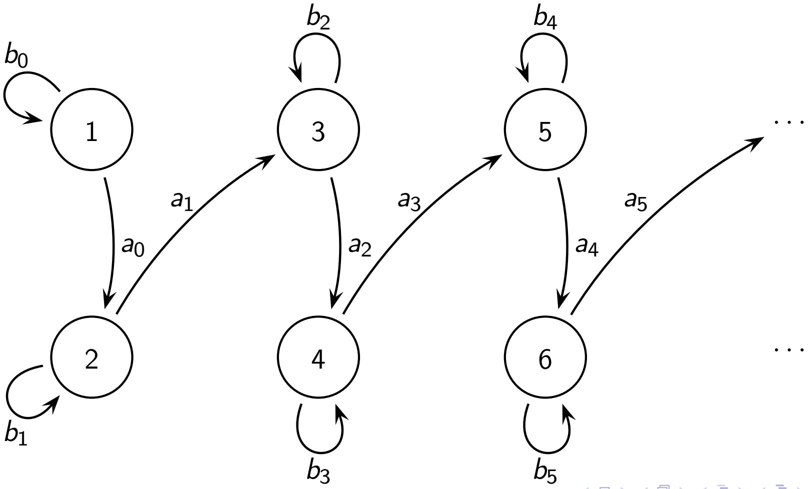

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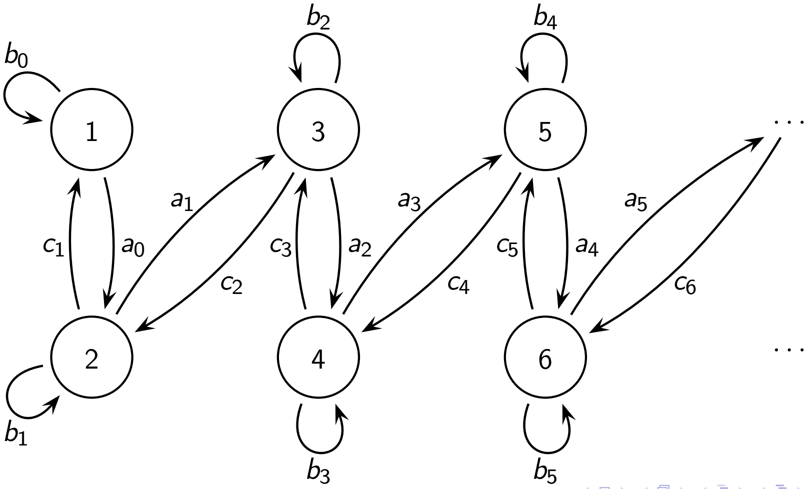

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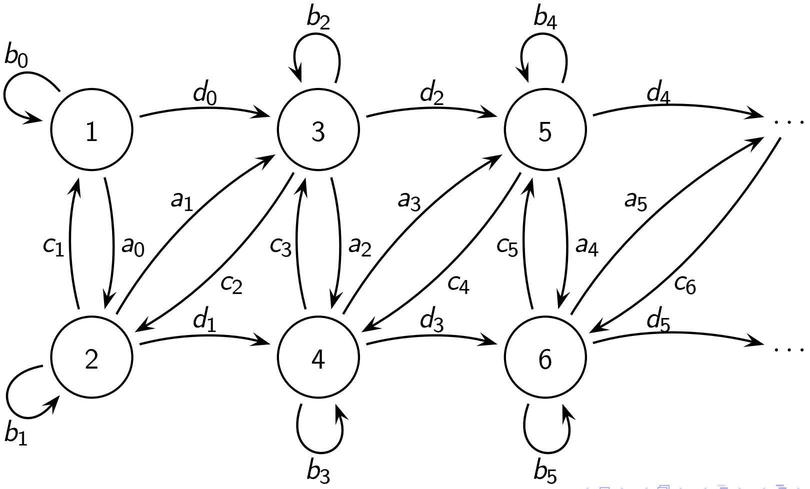
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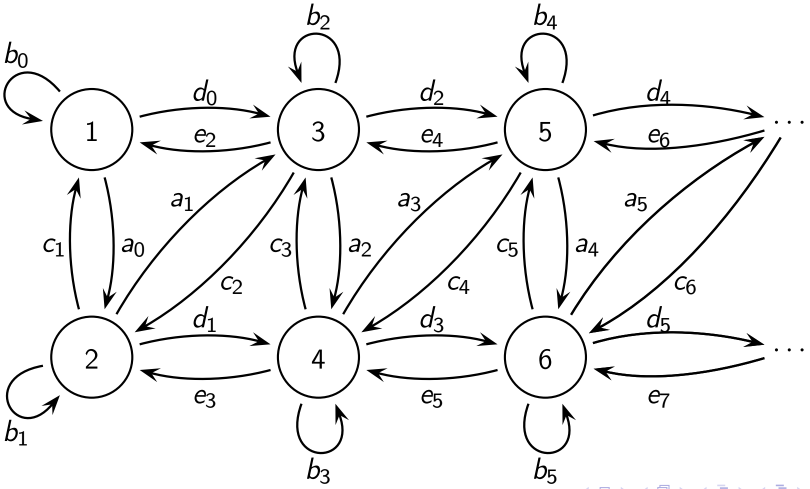
b_2
b_4
1
3
5
2
4
6
d_0
d_2
d_4
d_1
d_3
d_5
a_0
a_1
a_2
a_3
a_4
a_5
c_1
c_2
c_3
c_4
c_5
c_6
b_1
b_3
b_5
...
...

Network


b_0
b_2
b_4
d_0
d_2
d_4
1
3
5
. . .
e_2
e_4
e_6
a_1
a_3
a_5
c_1
a_0
c_3
a_2
c_5
a_4
c_2
c_4
c_6
d_1
d_3
d_5
2
4
6
. . .
e_3
e_5
e_7
b_1
b_3
b_5

MOP: Grünbaum (2007) and Dette-Reuther-Studden-Zygmunt (2007): Introducing the matrix polynomials $(Q_n)_n$ by the conditions $Q_{-1}(t) = 0$, $Q_0(t) = I$ and the recursion relation

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i.e.

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and under certain technical conditions over A_n, B_n, C_n, there exists an unique weight matrix $\mathrm{d}W(t)$ supported in $[-1, 1]$ such that

$$\left(\int_{-1}^{1} Q_i(t) \mathrm{d}W(t) Q_j^*(t) \right) \left(\int_{-1}^{1} Q_j(t) \mathrm{d}W(t) Q_j^*(t) \right)^{-1} = \delta_{ij} I$$

Karlin-McGregor formula

$$P_{ij}^n = \left(\int_{-1}^{1} t^n Q_i(t) \mathrm{d}W(t) Q_j^*(t) \right) \left(\int_{-1}^{1} Q_j(t) \mathrm{d}W(t) Q_j^*(t) \right)^{-1}$$

Invariant measure or distribution

Non-null vector with non-negative components

$$\boldsymbol{\pi} = (\boldsymbol{\pi}^0; \boldsymbol{\pi}^1; \cdots) \equiv (\pi_1^0, \pi_2^0, \ldots, \pi_N^0; \pi_1^1, \pi_2^1, \ldots, \pi_N^1; \cdots)$$

such that

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$$\Rightarrow \pi_i^j = ?$$

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The family of processes (size $N \times N$)

Conjugation

$$W(t) = T^* \widetilde{W}(t) T$$

where

$$T = \begin{pmatrix} 1 & 1 \\ 0 & -\dfrac{\alpha+\beta-k+2}{\beta-k+1} \end{pmatrix}$$

Grünbaum-MdI (2008)

$$\widetilde{W}(t) = t^{\alpha}(1-t)^{\beta} \begin{pmatrix} kt+\beta-k+1 & (1-t)(\beta-k+1) \\ (1-t)(\beta-k+1) & (1-t)^2(\beta-k+1) \end{pmatrix}$$

$t \in (0,1)$, $\alpha, \beta > -1$, $0 < k < \beta + 1$

Pacharoni-Tirao (2006)

We consider the family of MOP $(Q_n(t))_n$ such that

- Three term recurrence relation

$$tQ_n(t) = A_n Q_{n+1}(t) + B_n Q_n(t) + C_n Q_{n-1}(t), \quad n = 0, 1, \ldots$$

where the Jacobi matrix is stochastic

- Choosing $Q_0(t) = I$ the leading coefficient of Q_n is

$$\frac{\Gamma(\beta+2)\Gamma(\alpha+\beta+2n+2)}{\Gamma(\alpha+\beta+n+2)\Gamma(\beta+n+2)} \begin{pmatrix} \frac{k+n}{k} & -\frac{n(\alpha+\beta+2n+2)}{(\alpha+\beta+n+2)(\alpha+\beta-k+2)} \\ 0 & \frac{(n+\alpha+\beta-k+2)(\alpha+\beta+2n+2)}{(\alpha+\beta+n+2)(\alpha+\beta-k+2)} \end{pmatrix}$$

- Moreover, the corresponding norms are diagonal matrices:

$$\|Q_n\|_W^2 = \frac{\Gamma(n+\alpha+1)\Gamma(n+1)\Gamma(\beta+2)^2(n+\alpha+\beta-k+2)}{\Gamma(n+\alpha+\beta+2)\Gamma(n+\beta+2)} \times \begin{pmatrix} \frac{n+k}{k(2n+\alpha+\beta+2)} & 0 \\ 0 & \frac{(n+\alpha+1)(n+k+1)}{(\beta-k+1)(2n+\alpha+\beta+3)(n+\alpha+\beta+2)} \end{pmatrix}$$

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where the Jacobi matrix is stochastic

- Choosing $Q_0(t) = I$ the leading coefficient of Q_n is

$$\frac{\Gamma(\beta+2)\Gamma(\alpha+\beta+2n+2)}{\Gamma(\alpha+\beta+n+2)\Gamma(\beta+n+2)} \begin{pmatrix} \frac{k+n}{k} & -\frac{n(\alpha+\beta+2n+2)}{(\alpha+\beta+n+2)(\alpha+\beta-k+2)} \\ 0 & \frac{(n+\alpha+\beta-k+2)(\alpha+\beta+2n+2)}{(\alpha+\beta+n+2)(\alpha+\beta-k+2)} \end{pmatrix}$$

- Moreover, the corresponding norms are diagonal matrices:

$$\|Q_n\|_W^2 = \frac{\Gamma(n+\alpha+1)\Gamma(n+1)\Gamma(\beta+2)^2(n+\alpha+\beta-k+2)}{\Gamma(n+\alpha+\beta+2)\Gamma(n+\beta+2)} \times \begin{pmatrix} \frac{n+k}{k(2n+\alpha+\beta+2)} & 0 \\ 0 & \frac{(n+\alpha+1)(n+k+1)}{(\beta-k+1)(2n+\alpha+\beta+3)(n+\alpha+\beta+2)} \end{pmatrix}$$

Pentadiagonal Jacobi matrix

Particular case $\alpha = \beta = 0$, $k = 1/2$:

$$P = \begin{pmatrix} \frac{5}{9} & \frac{2}{9} & \frac{2}{9} & & & & & & \\ \frac{2}{9} & \frac{7}{18} & \frac{4}{45} & \frac{3}{10} & & & & & \\ \frac{5}{36} & \frac{1}{18} & \frac{107}{225} & \frac{3}{50} & \frac{27}{50} & & & & \\ & \frac{1}{6} & \frac{4}{75} & \frac{23}{50} & \frac{6}{175} & \frac{2}{7} & & & \\ & & \frac{14}{75} & \frac{2}{75} & \frac{597}{1225} & \frac{4}{147} & \frac{40}{147} & & \\ & & & \frac{1}{5} & \frac{6}{245} & \frac{47}{98} & \frac{8}{441} & \frac{5}{18} & \\ & & & & \frac{81}{392} & \frac{3}{196} & \frac{1955}{3969} & \frac{5}{324} & \frac{175}{648} \\ & & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Invariant measure

Invariant measure

The row vector

$$\boldsymbol{\pi} = (\boldsymbol{\pi}^0; \boldsymbol{\pi}^1; \cdots)$$

$$\boldsymbol{\pi}^n = \left(\frac{1}{\left(\|Q_n\|_W^2\right)_{1,1}}, \frac{1}{\left(\|Q_n\|_W^2\right)_{2,2}}, \cdots, \frac{1}{\left(\|Q_n\|_W^2\right)_{N,N}} \right), \quad n \geqslant 0$$

is an invariant measure of P

Particular case $N = 2$, $\alpha = \beta = 0$, $k = 1/2$:

$$\pi^n = \left(\frac{2(n+1)^3}{(2n+3)(2n+1)}, \frac{(n+1)(n+2)}{2n+3} \right), \quad n \geqslant 0$$

$$\pi = \left(\frac{2}{3}, \frac{2}{3}; \frac{16}{15}, \frac{6}{5}; \frac{54}{35}, \frac{12}{7}; \frac{128}{63}, \frac{20}{9}; \frac{250}{99}, \frac{30}{11}; \frac{432}{143}, \frac{42}{13}; \frac{686}{195}, \frac{56}{15}; \cdots \right)$$

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Recurrence

Theorem (Grünbaum-MdI, 2008)

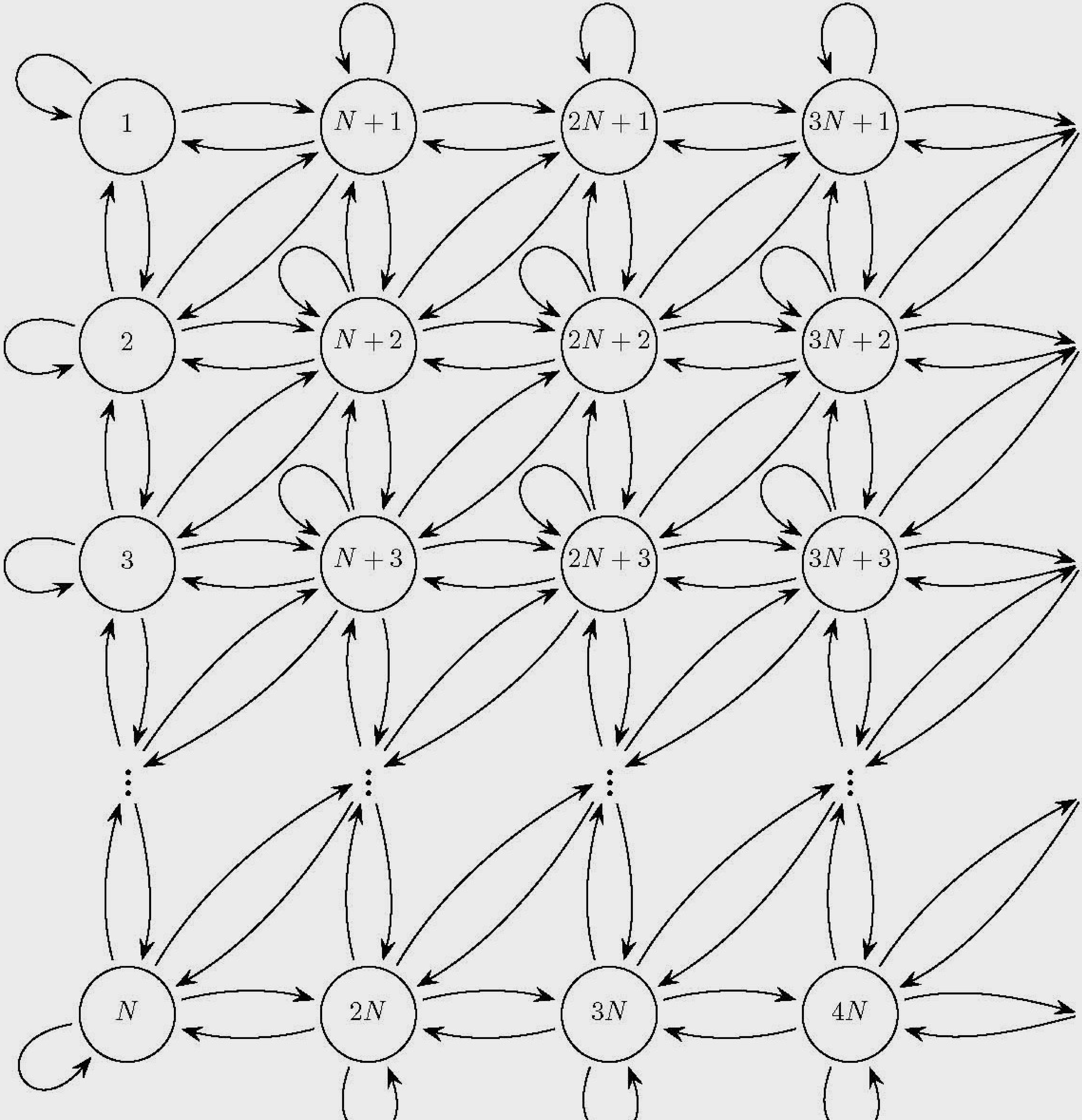
Let

$$P = \begin{pmatrix} B_0 & A_0 & & & \\ C_1 & B_1 & A_1 & & \\ & C_2 & B_2 & A_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

be the transition probability matrix. The Markov process that results from P is never positive recurrent.

If $-1 < \beta \leqslant 0$ then the process is null recurrent.

If $\beta > 0$ then the process is transient.



1
N + 1
2N + 1
3N + 1
2
N + 2
2N + 2
3N + 2
3
N + 3
2N + 3
3N + 3
N
2N
3N
4N

Quantum mechanics

Dirac's equation (central Coulomb potential)

$$T'(t) = \left(A + \frac{B}{t}\right) T(t)$$

where

$$A = \begin{pmatrix} 0 & 1+\omega \\ 1-\omega & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -a & b \\ -b & a \end{pmatrix}$$

Rose (1961)

Choosing $\omega = \pm\sqrt{a^2 - b^2}/a$ (lowest possible energy level) the solution of the Dirac's equation gives rise to a matrix weight whose MOP are eigenfunctions of certain second order differential equation

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Theorem (Durán-Grünbaum, 2006)

Consider the following instance of the Dirac's equation

$$T'(t) = \left(\widetilde{A} + \frac{\widetilde{B}}{t}\right) T(t)$$

$$\widetilde{A} = \sqrt{1 - 1/(4a)^2} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} -1/2 & a - 1/2 \\ 0 & 1/2 \end{pmatrix}$$

Then $W(t) = t^{\alpha+1} e^{-t} T(t) e^{-D_{\widetilde{A}} t} H e^{-D^*_{\widetilde{A}} t} T^*(t)$, *where* $H = e^{D_{\widetilde{A}}} T^{-1}(1) (T^{-1})^*(1) e^{D^*_{\widetilde{A}}}$ *allows for the following second order differential operator*

$$D = \partial^2 tI + \partial^1 (-tI + 2E + (\alpha + 1)I) + \partial^0 (-E + E_0)$$

$$E = \begin{pmatrix} 0 & 1/2 \\ 0 & 1 \end{pmatrix}, \quad E_0 = \frac{1 + \alpha}{5} \begin{pmatrix} -1 & 1/2 \\ -2 & 1 \end{pmatrix}$$

Time-and-band limiting

Given a full matrix M (integral operator) the computation of all its eigenvectors can be explicitly given if one finds a tridiagonal matrix S (differential operator) with simple spectrum such that

$$MS = SM$$

Classical scalar orthogonal polynomials: Grünbaum (1983)

Matrix case: Durán-Grünbaum (2005)

Example of QBD for $N = 2, \alpha = \beta = 0, k = 1/2$

$$W(t) = \begin{pmatrix} \frac{1}{2}t + \frac{1}{2} & 2t - 1 \\ 2t - 1 & \frac{9}{2}t^2 - \frac{11}{2}t + 2 \end{pmatrix}, \quad t \in [0, 1]$$

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Considering the same family $(Q_n)_n$ as before we have that

$$\|Q_n\|_W^2 = \begin{pmatrix} \frac{(2n+1)(2n+3)}{2(n+1)^3} & 0 \\ 0 & \frac{2n+3}{(n+1)(n+2)} \end{pmatrix}$$

and we can produce a family of normalized MOP $P_n = \|Q_n\|_W^{-1} Q_n$

Reproducing kernel

$$(M)_{i,j} = \int_0^{\Omega} P_i(t) W(t) P_j^*(t) \mathrm{d}t, \quad i,j = 0,1,\dots,T$$

"Band limiting": Restriction to the interval $(0,\Omega)$

"Time limiting": Restriction to the range $0,1,\dots,T$

$\Rightarrow$ There exists a block tridiagonal matrix S (pentadiagonal) such that M commutes with S

Scalar case: the vector space of all possible S's is 2-dimensional

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