Optimal Control and Partial Differential Equations

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Abstract

In this work, some type of optimal control problems with equality constraints given by Partial Differential Equations (PDE) and convex inequality constraints are considered, obtaining their corresponding first order necessary optimality conditions by means of Dubovitskii-Milyutin (DM) method.

Firstly, we consider problems with one objective functional (or scalar problems) but non-well posed equality constraints, where existence and uniqueness of state in function on control is not true (either one has existence but not uniqueness of state, or one has not existence of state for any control). In both cases, the classical Lions argument (re-writing the problem as an optimal control problem for the control without equality constraints, see for instance [14]) can not be applied.

Afterwards, we consider multiobjective problems (or vectorial problems), considering three different concepts of solution: Pareto, Nash and Stackelberg.

In all cases, an adequate abstract DM method is developed followed by an example.

1 Introduction

We consider an abstract problem of optimal control (uni-objective or multi-objective), with equality constraints given by PDE and inequality constraints (convex with nonempty interior). Our objective is to provide necessary optimality conditions, by means of DM formalism, that we will apply for some examples, where different particularities appear.

With respect to uni-objective problems, we are going to study two types of equality constraints:

- non-well posed problems, in the sense that it does not have existence of associated state to any control, as in the backward heat PDE with homogenous Dirichlet boundary condition and distributed control in all the domain
- problems with lack of uniqueness of state with respect to the control, as it is the case of the weak solutions of the three-dimensional Navier-Stokes system (or stationary Navier-Stokes with not small Reynolds number) and "partial" control, either distributed in a part of the domain, or boundary located in one (small) part of the boundary.

The DM formalism ([7]) turns out to be an operational writing of the separation of convex sets in Banach spaces, hence for instance an abstract formulation of the Lagrange multiplier theorem can be obtained. We thought that this formalism has been very little used in optimal control problems with constraints given by PDE. The first works that, to our knowledge, apply the DM formalism in this context is given by A. Papageorgiou and N. S. Papageorgiou [19]. In the works of U.Ledzewicz [13], Y. Censor [3], W. Kotarski [12] and S. Walczak [28] can be seen some extensions of this DM formalism.

Other more classic arguments and references in optimal control problems with PDE constraints are the following ones:

1. the equality constraint is re-written in function of the control and derivation respect to the control; J.L.Lions [14],
2. method of regular perturbations, respect to a scalar parameter; E. Casas [2]
3. penalty method for unstable or non-well posed problems; P.H. Rivera [24, 23]
4. lagrangian function and 3D Navier-Stokes system; M.D. Gunzburger et al. [8, 9]
5. Using abstract results of Lagrange multiplier of Ioffe-Tykhomirov type; A.V. Fursikov [5]

With respect to the vectorial or multiobjective problems, we will consider by simplicity the case of two objectives and two controls. Because it is not usual to have existence of a simultaneously (in all functionals) optimal point, several concepts of solution have been introduced in the literature. Here, we will consider three different concepts of solution (with their respective examples):

1. Nash (non-cooperative) equilibrium solutions,
2. Stackelberg (hierarchy and cooperative) solutions,
3. Pareto (strong) solutions,

Although the Pareto concept ([20]) was the first solution introduced in the literature and is based on a vectorial argument, it is often difficult to analyze it mathematically. For that reason, the other two concepts based on scalar optimization problems (associating each control with an objective functional) have been also considered ([18, 26]).

Some previous works about these concepts of solution in (vectorial) optimal control with PDE constraints are the following. The papers of J.L. Lions ([15], [16], [17]) study Pareto and Stackelberg solutions (using a Pareto solution related with a continuous of objective functionals, and not a finite number of functionals as we will consider in our work). Diaz and Lions [4] give some results for Stackelberg-Nash strategies with linear parabolic PDE constraints. Ramos, Glowinski and Periaux studied, from the mathematical and numerical point of view, the Nash equilibrium for constraints given by, either linear parabolic PDE in [21] or the Burgers equations in [22]. In all these previous papers, the classical Lions argument eliminating equality constraints re-writting in function on controls (see for instance [14]) is used.

In this work, we will use again the DM method now for vectorial optimization, in order to obtain necessary optimality conditions. In the Nash and Pareto cases, considering constraints of the NS type with partially distributed controls and objectives ([11]), whereas in the Stackelberg case, we will consider linear parabolic PDE constraints with leader criterion to lead the system near a final state (a functional of controllability type), and secondary criterion on distributed type plus the cost of the control (as in [17]).

Two important references where some extensions to the DM method applied to Pareto solutions are: Censor [3] for finite dimension constraints, and Kotarski [12] for the infinite dimension case.

2 Uniobjetive control problem

2.1 Abstract problem

Let X, Y, Z be Banach spaces (X : state, Y : control) and

- $J : X \times Y \to \mathbb{R}$ (objective functional)
- $F : D(F) \subset X \times Y \to Z$ (equality constraints),
- $\mathcal{C} \subset X \times Y$ a convex closed set, with $int\,\mathcal{C} \neq \emptyset$ (inequality constraints)

Optimal control problem:

$$(P) \qquad \min J(x,y) \quad \text{subject to} \quad F(x,y) = 0 \quad \text{and} \quad (x,y) \in \mathcal{C}$$

Several general situations

1. X, Y, Z of finite dimension for the mathematical programming,
2. $F(x,y) = 0$ is related to ordinary differential equations ODE, for Optimal control with ODE,
3. We consider $F(x,y) = 0$ related to PDE.

In this work, we consider the case of functional J and operator F are differentiable (for instance, in the Flett book [6] appears all different notions of differentiable operators between Banach spaces that we will consider in this work).

Two main cases to study

1. Non uniqueness of state:

$$\forall\, y \in Y,\ \exists\, x \in X \ : \ F(x,y) = 0$$

but, x is not unique in general

2. No well-posedness problem:

$$\exists\, y \in Y \text{ s.t. } \quad \nexists\, x \in X \;:\; F(x,y) = 0$$

In both cases, there is not possible to write $x = x(y)$, hence the following "Lions argument" [14] can not be applied; defining $\tilde{J}(y) := J(x(y), y)$, optimality conditions are

$$\tilde{J}'(\bar{y}) = 0 \text{ (if } \mathcal{C} = X \times Y) \quad \text{or} \quad \langle \tilde{J}'(\bar{y}), y - \bar{y} \rangle \geq 0 \text{ (if } \mathcal{C} = X \times \mathcal{G})$$

Always, we will assume the so-called *Nontrivial hypothesis*:

$$\mathcal{U}_{ad} := \{(x,y) \in X \times Y \;:\; F(x,y) = 0,\; (x,y) \in \mathcal{C},\; J(x,y) < +\infty\} \neq \emptyset$$

2.2 The Dubovitskii-Milyutin method

Now, we explain the DM method in order to obtain necessary optimality systems of problem (P). Let $(\bar{x}, \bar{y}) \in D(F) \cap \mathcal{C}$, we will denote by

- $DC(\bar{x}, \bar{y})$: descent cone in $(\bar{x}, \bar{y})$ related to J
- $TC(\bar{x}, \bar{y})$: tangent cone in $(\bar{x}, \bar{y})$ related to F
- $FC(\bar{x}, \bar{y})$: feasible cone in $(\bar{x}, \bar{y})$ related to $\mathcal{C}$

(see [7] for these definitions). If C is a cone of $X \times Y$, its dual cone is defined by

$$C^* = \{f \in (X \times Y)' : \langle f, (x,y) \rangle \geq 0 \;\forall\, (x,y) \in C\}$$

Theorem 1 (Dubovitskii-Milyutin) *[6, 7] Assume that $(\bar{x}, \bar{y})$ is a (local) minimum of (P). If J, equality and inequality restrictions are "regular", in the sense that the descent, tangent and feasible cones are convex sets, then*

$$\exists\; f_0 \in DC(\bar{x}, \bar{y})^*,\;\; f_1 \in TC(\bar{x}, \bar{y})^*,\;\; f_2 \in FC(\bar{x}, \bar{y})^* \textit{ with } (f_0, f_1, f_2) \neq 0 \textit{ s.t.}$$

$$f_0 + f_1 + f_2 = 0 \quad \textit{in } (X \times Y)'.$$

Remarks: Some specifications of Theorem 1 are the followings:

- Two (or more) inequality constraints yields to the corresponding feasible cones FC_1, FC_2 which are open cones, then ([7])

$$(FC_1 \cap FC_2)^* = FC_1^* + FC_2^* \tag{1}$$

- Two equality constraints, i.e. $F = (F_1, F_2) : X \times Y \to Z_1 \times Z_2$ yields to the corresponding tangent cones TC_1, TC_2 which are closed cones, but now an equality as (1) is not true in general. Normally, an additional condition is necessary to obtain $(TC_1 \cap TC_2)^* = TC_1^* + TC_2^*$. For instance:

  1. (Walczak'84 [28]) TC_1, TC_2 have both "same sense" or "opposite sense".

2. (Ledzewicz'86 [13]) $F = (F_1, F_2) : X \times Y \to Z_1 \times Z_2$, strongly differentiable in $(\bar{x}, \bar{y})$ with
   (a) either, $Im\,(F_i) = Z_i$ $(i = 1, 2)$ and $Im\,(F)$ is closed in $Z_1 \times Z_2$,
   (b) or, $Im\,(F)$ is a closed subspace in $Z_1 \times Z_2$. Moreover, in this case,

$$TC_i^* = \{(F_i'(\bar{x}, \bar{y}))^* \lambda \ : \ \lambda \in Z_i^*\} \quad (i = 1, 2)$$

This last equality is based in the relation $N(F_i'(\bar{x}, \bar{y}))^\perp = \overline{R(F_i'(\bar{x}, \bar{y})^*)}$

Cones and dual cones ([7])

- If J is differentiable in $(\bar{x}, \bar{y})$, then $DC(\bar{x}, \bar{y}) = \{(x, y) : \ \langle J'(\bar{x}, \bar{y}), (x, y)\rangle < 0\}$. In particular, $DC(\bar{x}, \bar{y})$ is an open convex set. Furthemore,

$$DC(\bar{x}, \bar{y})^* = \{-\lambda J'(\bar{x}, \bar{y}) : \ \lambda \geq 0\}$$

- **Theorem 2 (Lyusternik)** *If F is strongly differentiable in $(\bar{x}, \bar{y})$ and $Im\, F'(\bar{x}, \bar{y}) = Z$ (regularity condition), then*

$$TC(\bar{x}, \bar{y}) = \{(x, y) \in D(F) : \ \langle F'(\bar{x}, \bar{y}), (x, y)\rangle = 0\}.$$

  In particular, $TC(\bar{x}, \bar{y})$ is a vectorial subspace of $X \times Y$. Therefore,

$$TC(\bar{x}, \bar{y})^* = \{f \in (X \times Y)' : \ \langle f, (x, y)\rangle = 0 \quad \forall\, (x, y) \in TC(\bar{x}, \bar{y})\}.$$

- Since $\mathcal{C}$ is convex in $X \times Y$ and $int\, \mathcal{C} \neq \emptyset$, then

$$FC(\bar{x}, \bar{y}) = \{\lambda(x - \bar{x}, y - \bar{y}) : \ (x, y) \in int\, \mathcal{C}, \lambda > 0\}.$$

  In particular, $FC(\bar{x}, \bar{y})$ is an open convex set in $X \times Y$. Furthemore,

$$FC(\bar{x}, \bar{y})^* = \{f \in (X \times Y)' : \ \langle f, (x - \bar{x}, y - \bar{y})\rangle \geq 0 \quad \forall\, (x, y) \in \mathcal{C}\}.$$

2.3 Example 1: Point-wise backward heat equation with initial data and total distributed control

$$X = Y = L^2(Q) \quad (Q = (0, T) \times \Omega)$$

$$J(u, v) = \frac{1}{2}\int_0^T \int_\Omega |u - u_d|^2 + \frac{N}{2}\int_0^T \int_\Omega |v - v_d|^2 \quad (N > 0, (u_d, v_d) \in X \times Y)$$

$$\begin{array}{rcl} D(F) & = & \{u : u \text{ and } u_t + \Delta u \in L^2(Q), u_{|\Sigma} = 0\} \times Y, \\ Z & = & L^2(Q) \times L^2(\Omega) \\ \mathcal{C} & = & X \times \mathcal{G} \quad (\mathcal{G} \text{ is a convex closed set in } Y) \end{array}$$

(i.e., there is not inequality constraints on state).

Let $(f, u_0) \in Z$ be a fixed data. For each $(u, v) \in D(F)$ one defines $F(u, v) = (\tilde{f}, \tilde{u}_0) \in Z$ as follows

$$(S) \qquad \left\{ \begin{array}{ll} u_t + \Delta u - f - v = \tilde{f} & \text{in } Q, \\ u(0) - u_0 = \tilde{u}_0 & \text{in } \Omega \end{array} \right.$$

Lemma 1 *([5, 24])*

- *$D(F)$ is a Banach space, where trace operator and integration by parts have a sense.*
- *$\exists\, D$ dense in Y such that $\forall\, v \in D,\ \exists\, u \in X$ solution of $F(u,v) = 0$.*

The previous optimal control problem has been studied in [5], where an optimality system is obtained by means of a Lagrange principle. Also optimal control problem under non-well posed constraints are studied in the works of Rivera ([24]) and Rivera and Vasconcellos ([25]) (see also [23] for unstable constraints). These works are based in the penalization method, obtaining the optimality system passing to the limit in the optimality system associated to a sequence of penalized optimal control problems.

Using DM method, it is possible to arrive (in a more systematic form) to an operational equality, which can be seen as a first optimality system. Afterwards, the difficulty will be to prove existence of the adjoint problem $(S)^*$, since $(S)^*$ will be again a non-well posed problem ([10]). Indeed, assume that $(\bar u, \bar v)$ is a (local) solution of (P), then DM formalism yields

$$f_0 + f_1 + f_2 = 0 \quad \text{with} \quad f_0 \in DC^*, f_1 \in TC^*, f_2 \in FC^*.$$

More concretely,

- $f_0 = -\lambda J'(\bar u, \bar v)$ for some $\lambda \geq 0$.
- $\langle f_1, (u,v)\rangle = 0 \quad \forall\, (u,v)$ verifying $(S)' \begin{cases} u_t + \Delta u &= v, \\ u_{|\Sigma} = 0,\ u(0) &= 0. \end{cases}$
- $f_2 \in Y^*$ such that $\langle f_2, v - \bar v\rangle \geq 0 \quad \forall\, v \in \mathcal{G}$.

One can prove that $\lambda \neq 0$ (hence $\lambda = 1$ can be taken), thanks to an absurd argument and to the density of controls v solving $(S)'$ ([10]). Then, the following optimality conditions hold:

$$\langle f_2, v\rangle = \langle J_u(\bar u, \bar v), u\rangle + \langle J_v(\bar u, \bar v), v\rangle \quad \forall\, (u,v) \text{ verifying } (S)' \tag{2}$$

$$\langle f_2, v - \bar v\rangle \geq 0 \quad \forall\, v \in \mathcal{G} \tag{3}$$

Now, the adjoint problem is introduced, in order to simplify (2)

- Step 1: Assuming that $\tilde u$ is a solution of the adjoint problem

  $$(P)^* \qquad \begin{cases} -\tilde u_t + \Delta \tilde u = J_u(\bar u, \bar v)(= \bar u - u_d) & \text{in } Q, \\ \tilde u_{|\Sigma} = 0, \qquad \tilde u(T) = 0 & \text{in } \Omega, \end{cases}$$

  then, integrating by parts in (2), the following variational formulation respect to control functions hold

  $$\langle f_2, v\rangle = (\tilde u, v)_Q + \langle J_v(\bar u, \bar v), v\rangle \quad \forall\, v \in D \text{ (dense in } Y)$$

  Then $f_2 = \tilde u + J_v(\bar u, \bar v)(= \tilde u + N(\bar v - v_d))$, hence from (3), one arrives at the so-called Euler-Lagrange conditions: $(\tilde u + N(\bar v - v_d), v - \bar v)_Q \geq 0 \quad \forall\, v \in \mathcal{G}$.

- Step 2 (Existence of solution of $(P)^*$): Since $f_2 \in L^2(Q)'$, there exists $\tilde{u} \in L^2(Q)$ such that $\langle f_2, v\rangle = (\tilde{u}, v)_Q + \langle J_v(\bar{u}, \bar{v}), v\rangle$ for any $v \in Y$. Then, from (2) one deduces $(\tilde{u}, v)_Q = \langle J_u(\bar{u}, \bar{v}), u\rangle$ for any (u, v) verifying $(S)'$. Using that $v = u_t + \Delta u$ and integrating by parts (which it is possible thanks to Lemma 2), it is not difficult to obtain ([10]) that $\tilde{u}$ is a solution of $(P)^*$

Accordingly, we arrive at the following first order necessary optimality conditions:

Theorem 3 *If $(\bar{u}, \bar{v}) \in X \times Y$ is a (local) minimum of (P), then*

- *$(\bar{u}, \bar{v})$ verify (S) with $(\tilde{f}, \tilde{u}_0) = (0, 0)$: state problem*
- *$\tilde{u}$ verifies $(P)^*$: adjoint problem*
- *$(\tilde{u} + N(\bar{v} - v_d), v - \bar{v})_Q \geq 0 \quad \forall\, v \in \mathcal{G}$: Euler-Lagrange conditions.*

Sufficient condition: Since J is strictly convex, if $(\bar{u}, \bar{v})$ verifies previous optimality conditions, then one can proves that $(\bar{u}, \bar{v})$ is the (unique) global minimum of (P).

The following open problems are being studied in this moment, and the corresponding results will be appear in [10]:

1. The case $int\, \mathcal{G} = \emptyset$; for instance the case of point-wise constraints for the controls:

$$\mathcal{G} = \{v \in Y \ : \ a \leq v(t, x) \leq b \quad \text{a.e } (t, x) \in Q\} \quad (a, b \in \mathbb{R}, a < b)$$

2. The case with partially distributed control, i.e. $Y = L^2(0, T; L^2(\omega))$ and in definition of J appears $\frac{N}{2}\int_0^T \int_\omega |v - v_d|^2$, for some $v_d \in L^2(0, T; L^2(\omega))$.

2.4 Example 2: Weak solutions of the $3D$ NAVIER-STOKES Equations (NS), with "partial" control (either distributed in $\omega \subset\subset \Omega$, or boundary on $\gamma \subset\subset \partial\Omega$)

$$X = L^2(Q), \quad Y = L^2(0, T; L^2(\omega)), \quad (Q = (0, T) \times \Omega)$$
$$J(u, v) = \frac{1}{2}\int_0^T \int_\Omega |u - u_d|^2 + \frac{N}{2}\int_0^T \int_\omega |v - v_d|^2 \qquad (N > 0, u_d \in X, v_d \in Y)$$
$$D(F) = (L^2(0, T; V) \cap L^\infty(0, T; H)) \times Y,$$
$$Z = H^{-1}(Q) \times H$$

where $H = \{u \in L^2(\Omega) : \nabla \cdot u = 0, u \cdot n_{|\partial\Omega} = 0\}$ and $V = H_0^1(\Omega) \cap H$ are the standard L^2 and H^1 spaces in the NS framework.

$$\mathcal{C} = X \times \mathcal{G} \quad \text{where } \mathcal{G} \text{ is a convex closed set in } Y$$

Let $(f,u_0)\in Z$ be a fixed data. For each $(u,v)\in D(F)$ one defines $F(u,v) = (\tilde f,\tilde u_0)\in Z$ as follows

$$(S)\qquad \begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega u\varphi - \int_\Omega u\cdot\nabla\varphi u + \int_\Omega \nabla u\nabla\varphi - \int_\Omega f\varphi \\ -\displaystyle\int_\omega v\varphi = \int_\Omega \tilde f\varphi \quad \forall\,\varphi\in H_0^1(Q)\ s.t.\ \nabla\cdot\varphi = 0 \\ u(0)-u_0 = \tilde u_0 \quad \text{in } \Omega \end{cases}$$

In the case of boundary control acting on $\gamma\subset\partial\Omega$, one changes $Y = L^2(0,T;L^2(\omega))$ by $Y = L^2(0,T;L^2(\gamma))$ and in the objective functional $\int_0^T\int_\omega |v-v_d|^2\,dx$ by $\int_0^T\int_\gamma |v-v_d|^2\,d\sigma$ with $v_d\in L^2(0,T;\gamma)$ a data.

Lemma 2 *(see for instance [27]) For each $v\in Y$, there exists at least a weak solution u of (S). But uniqueness of weak solution is an open problem.*

In [9], with Lagrange multiplier technique and considering state and control variables independently, a necessary optimality system of problem (P) is obtained.

Using DM method, one also can arrive at the following optimality system ([11]):

Theorem 4 *If $(\bar u,\bar v)$ is a (local) solution of (P), then there exists $\tilde u\in L^2(0,T;V)\cap L^\infty(0,T;H)$ and $\lambda>0$ (in this case could be taken $\lambda=1$) such that:*

$$(\bar u,\bar v)\ \textit{verify}\ (S),\ \ \textit{with}\ (\tilde f,\tilde u_0) = (0,0)$$

$$(P)^*\qquad \begin{cases} -\dfrac{d}{dt}\displaystyle\int_\Omega \tilde u\varphi - \int_\Omega \bar u\cdot\nabla\varphi\tilde u - \int_\Omega \tilde u\cdot\nabla\varphi\bar u + \int_\Omega \nabla\tilde u\nabla\varphi \\ = \lambda J'_u(\bar u,\bar v)(\varphi) = \lambda(\bar u - u_d,\varphi)_Q \quad \forall\,\varphi\in H_0^1(Q)\ \textit{such that}\ \nabla\cdot\varphi = 0 \\ \tilde u(T) = 0 \quad \textit{in}\ \Omega \end{cases}$$

$$(\tilde u + \lambda J'_v(\bar u,\bar v),\bar v - v)_{(0,T)\times\omega} = (\tilde u + \lambda N(\bar v - v_d),\bar v - v)_{(0,T)\times\omega} \geq 0 \quad \forall\, v\in\mathcal{G}.$$

3 Multi-objective Optimal Control

For simplicity, we consider two objectives $J=(J_1,J_2): X\times Y\to\mathbb{R}^2$ and two controls acting $y=(y_1,y_2)\in Y = Y_1\times Y_2$

Several strategies, that can be cooperative or noncooperative, can be taken in order to define a solution.

3.1 Abstract problem

Let $X, Y = Y_1\times Y_2, Z$ be Banach spaces (X : state, Y_1,Y_2 : two controls) and

- $J: X\times Y\to\mathbb{R}^2$; two objective functionals

- $F : D(F) \subset X \times Y \to Z$; equality constraints
- $\mathcal{C} \subset X \times Y_1 \times Y_2$ is a convex closed set, with $int\,\mathcal{C} \neq \emptyset$; inequality constraints.

Optimal control problem:

$$(P) \qquad \min J(x, y_1, y_2) \quad \text{subject to} \quad F(x, y_1, y_2) = 0 \quad \text{and} \quad (x, y_1, y_2) \in \mathcal{C}$$

The following nontrivial hypothesis must be imposed:

$$\mathcal{U}_{ad} := \{(x, y) \in X \times Y \ : \ F(x, y) = 0, \ (x, y) \in \mathcal{C}, \ J_i(x, y) < +\infty \ (i = 1, 2)\} \neq \emptyset$$

3.2 Nash Equilibrium

The definition of Nash solution appears in the noncooperative game theory [18].

In the general case when the state x is not well defined in function of the controls, and convex restrictions are introduced for the controls $(y_1, y_2) \in \mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$ a closed convex of $Y_1 \times Y_2$, then one has the following:

Definition 1 *$(\bar{x}, \bar{y}_1, \bar{y}_2)$ is a Nash solution if*

1. *Fixed $\bar{y}_2$, then $(\bar{x}, \bar{y}_1) \in X \times Y_1$ is a solution of*

$$(P)_1 \qquad \min_{(x,y_1) \in X \times \mathcal{G}_1 : F(x,y_1,\bar{y}_2)=0} J_1(x, y_1, \bar{y}_2)$$

2. *Fixed $\bar{y}_1$, then $(\bar{x}, \bar{y}_2) \in X \times Y_2$ is a solution of*

$$(P)_2 \qquad \min_{(x,y_2) \in X \times \mathcal{G}_2 : F(x,\bar{y}_1,y_2)=0} J_2(x, \bar{y}_1, y_2)$$

If we do not assume inequality constraints, i.e. $\mathcal{C} = X \times Y_1 \times Y_2$ and assume well-posed equality constraints, i.e. state x is uniquely defined in function of controls (y_1, y_2) solving $F(x(y_1, y_2), y_1, y_2) = 0$, one has the following necessary optimality condition:

Theorem 5 *If $(\bar{x}, \bar{y}_1, \bar{y}_2) \in \mathcal{U}_{ad}$ is a Nash solution, then:*

$$\frac{\partial \widetilde{J_1}}{\partial y_1}(\bar{y}_1, \bar{y}_2) = 0 = \frac{\partial \widetilde{J_2}}{\partial y_2}(\bar{y}_1, \bar{y}_2)$$

being $\widetilde{J}_i(y_1, y_2) = J_i(x(y_1, y_2), y_1, y_2)$ $(i = 1, 2)$.

This optimality condition for a Nash solution, can be interpreted as a fixed point for the following operators:

$$R_1 : y_1 \in Y_1 \to \bar{y}_2 \in Y_2 : \frac{\partial \widetilde{J_2}}{\partial y_2}(y_1, \bar{y}_2) = 0 \to \bar{y}_1 \in Y_1 : \frac{\partial \widetilde{J_1}}{\partial y_1}(\bar{y}_1, \bar{y}_2) = 0$$

or

$$R_2 : y_2 \in Y_2 \to \bar{y}_1 \in Y_1 : \frac{\partial \widetilde{J_1}}{\partial y_1}(\bar{y}_1, y_2) = 0 \to \bar{y}_2 \in Y_2 : \frac{\partial \widetilde{J_2}}{\partial y_2}(\bar{y}_1, \bar{y}_2) = 0.$$

Also in the general case of Definition 1, the Nash solution can be interpreted as a fixed point for a multivalued operator.

3.3 Example 3: Weak solutions of NS with two partially distributed controls and convex constraints for the controls

We use the notations of example 2. In addition, let $\Omega \subset \mathbb{R}^3$ be the total domain occupied by the fluid, $\omega_{i,d} \subset \Omega$ the "observability" domains and $\omega_i \subset \Omega$ with $\omega_1 \cap \omega_2 = \emptyset$ the "control" domains.

$$X = L^2(0,T;L^2(\Omega)), \quad Y_i = L^2(0,T;L^2(\omega_i)) \quad (i=1,2)$$

$$J_i(u,v_i) = \frac{1}{2}\int_0^T \int_{\omega_{i,d}} |u-u_{i,d}|^2 + \frac{N_i}{2}\int_0^T \int_{\omega_i} |v_i|^2$$

where $N_i > 0$ and $u_{i,d} \in L^2(0,T;L^2(\omega_{i,d}))$.

$$D(F) = L^2(0,T;V) \cap L^\infty(0,T;H) \times Y_1 \times Y_2,$$

$$Z = H^{-1}(Q) \times H$$

$$\mathcal{C} = X \times \mathcal{G}_1 \times \mathcal{G}_2 \quad (\mathcal{G}_i \text{ is a convex closed set in } Y_i)$$

Let $(f,u_0) \in Z$ a fixed data. For each $(u,v_1,v_2) \in D(F)$ one defines $F(u,v_1,v_2) = (\tilde{f},\tilde{u}_0) \in Z$ as follows

$$(S) \qquad \begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega u\varphi - \int_\Omega u\cdot\nabla\varphi u + \int_\Omega \nabla u \nabla\varphi - \int_\Omega f\varphi \\ -\displaystyle\int_{\omega_{1,d}} v_1\varphi - \int_{\omega_{2,d}} v_2\varphi = \int_\Omega \tilde{f}\varphi \quad \forall\,\varphi \in H_0^1(Q) \ s.t.\ \nabla\cdot\varphi = 0 \\ u(0) - u_0 = \tilde{u}_0 \quad \text{in } \Omega \end{cases}$$

Taking into account the optimality system in the uni-objective problem (see Example 2 in a previous Section), the following optimality system can be obtained [11]:

Theorem 6 *If $(\bar{u},\bar{v}_1,\bar{v}_2)$ is a Nash solution, then there exists $\lambda_1 \geq 0, \lambda_2 \geq 0$ with $(\lambda_1,\lambda_2) \neq 0$ and $\tilde{u}_1,\tilde{u}_2 \in L^2(0,T;V)\cap L^\infty(0,T;H)$ such that*

$$(\bar{u},\bar{v}_1,\bar{v}_2) \text{ verify } (S), \text{ with } (\tilde{f},\tilde{u}_0) = (0,0)$$

$$(P)_1^* \quad \begin{cases} -\dfrac{d}{dt}\displaystyle\int_\Omega \tilde{u}_1\varphi - \int_\Omega \bar{u}\cdot\nabla\varphi\tilde{u}_1 - \int_\Omega \tilde{u}_1\cdot\nabla\varphi\bar{u} + \int_\Omega \nabla\tilde{u}_1\nabla\varphi \\ = \lambda_1 (J_1)'_u(\bar{u},\bar{v}_1)(\varphi) = \lambda_1(\bar{u}-u_{1,d},\varphi)_{(0,T)\times\omega_{1,d}} \quad \forall\,\varphi \in H_0^1(Q) \ s.t.\ \nabla\cdot\varphi = 0 \\ \tilde{u}_1(T) = 0 \quad in\ \Omega \end{cases}$$

$$(\tilde{u}_1+\lambda_1(J_1)'_{v_1}(\bar{u},\bar{v}_1),\bar{v}_1-v_1)_{(0,T)\times\omega_1} = (\tilde{u}_1+\lambda_1 N_1(\bar{v}_1-v_{1,d}),\bar{v}_1-v_1)_{(0,T)\times\omega_1} \geq 0 \quad \forall\,v_1 \in \mathcal{G}_1.$$

$$(P)_2^* \quad \begin{cases} -\dfrac{d}{dt}\displaystyle\int_\Omega \tilde{u}_2\varphi - \int_\Omega \bar{u}\cdot\nabla\varphi\tilde{u}_2 - \int_\Omega \tilde{u}_2\cdot\nabla\varphi\bar{u} + \int_\Omega \nabla\tilde{u}_2\nabla\varphi \\ = \lambda_1 (J_2)'_u(\bar{u},\bar{v}_2)(\varphi) = \lambda_2(\bar{u}-u_{2,d},\varphi)_{(0,T)\times\omega_{2,d}} \quad \forall\,\varphi \in H_0^1(Q) \ s.t.\ \nabla\cdot\varphi = 0 \\ \tilde{u}_2(T) = 0 \quad in\ \Omega \end{cases}$$

$$(\tilde{u}_2+\lambda_1(J_2)'_{v_2}(\bar{u},\bar{v}_2),\bar{v}_2-v_2)_{(0,T)\times\omega_2} = (\tilde{u}_2+\lambda_2 N_2(\bar{v}_2-v_{2,d}),\bar{v}_2-v_2)_{(0,T)\times\omega_2} \geq 0 \quad \forall\,v_2 \in \mathcal{G}_2.$$

Some results about existence of this optimality system can be obtained ([11]), imposing $\mathcal{G}_1$ and $\mathcal{G}_2$ bounded in Y_1 and Y_2 respectively.

3.4 Stackelberg (hierarchical-cooperative) solution

Assume two controls $y = (y_1, y_2)$, where y_1 is the "leader" and y_2 the "follower", the equality restrictions $F : D(F) \subset X \times Y_1 \times Y_2 \to Z$ and two objective functionals $J = (J_1, J_2) : X \times Y_1 \times Y_2 \to \mathbb{R}^2$, being J_1 the main objective and J_2 the secondary one. Inequality constraints are not considered, i.e. $\mathcal{C} = X \times Y_1 \times Y_2$.

First of all, assume that state x is well defined in function of (y_1, y_2) solving $F(x(y_1, y_2), y_1, y_2) = 0$. Then, definition of a Stackelberg solution follows two steps:

1. (Follower step) Given $y_1 \in Y_1$, we assume that there exists an (unique) solution $y_2 = S(J_2, y_1)$ of the (uniobjective) optimization problem related to J_2:

$$\min_{y_2} J_2(x(y_1, y_2), y_1, y_2)$$

2. (Leader step) To find y_1 as a solution of the (uniobjective) optimization problem related to J_1 and the previous solution $y_2 = S(J_2, y_1)$:

$$\min_{y_1} J_1(x(y_1, S(y_1)), y_1, S(y_1))$$

Then, if the state x is not well defined in function of the controls, one can define the Stackelberg solutions as follows:

Definition 2 *$(\bar{x}, \bar{y}_1, \bar{y}_2)$ is a (global) Stackelberg solution if $(\bar{x}, \bar{y}_2) \in S(J_2, \bar{y}_1)$ and*

$$J_1(\bar{x}, \bar{y}_1, \bar{y}_2) \le J_1(x, y_1, y_2) \quad \forall\, y_1 \quad \forall\, (x, y_2) \in S(J_2, y_1),$$

where $S(J_2, y_1)$ is the set of solutions of the minimum problem

$$\min_{(x,y_2):F(x,y_1,y_2)=0} J_2(x, y_1, y_2)$$

3.5 Example 4: strong solution of a parabolic linear PDE ([17])

Let Ω be the total domain, with $\omega_1 \subset \Omega$ the leader "control" domain and $\omega_2 \subset \Omega$ the follower one.

$$X = C^0([0, T]; L^2(\Omega)), \quad Y_i = L^2(0, T; L^2(\omega_i)) \quad (i = 1, 2)$$

$$J_1(u, v_1) = \psi_K(u(T)) + \frac{N_1}{2} \int_0^T \int_{\omega_1} |v_1|^2$$

where $\psi_K(g) = \begin{cases} 0 & \text{if } g \in K \\ +\infty & \text{otherwise} \end{cases}$, $K = \overline{\mathcal{B}}_{L^2(\Omega)}(u_T; \varepsilon)$, with $u_T \in L^2(\Omega)$ and $\varepsilon > 0$.

$$J_2(u, v_2) = \frac{1}{2} \int_0^T \int_\Omega |u - u_d|^2 + \frac{N_2}{2} \int_0^T \int_{\omega_2} |v_2|^2 \quad (u_d \in L^2(Q))$$

$$D(F) = \{u \in L^2(0, T; H^2(\Omega)) \cap C^0([0, T]; H^1(\Omega)) : \; u_t \in L^2(0, T; L^2(\Omega))\} \times Y_1 \times Y_2,$$

$$Z = L^2(Q) \times L^2(\Omega)$$

$$\mathcal{C} = X \times \mathcal{G}_1 \times \mathcal{G}_2 \quad (\mathcal{G}_i \text{ is a convex closed set in } Y_i)$$

Notice that J_1 is an objective of controllability type plus a part for the cost of control, whereas J_2 is a classical distributed objective.

Let $(f, u_0) \in Z$ a fixed data. For $(u, v_1, v_2) \in D(F)$ one defines $F(u, v_1, v_2) = (\tilde{f}, \tilde{u}_0) \in Z$ as follows

$$(S) \qquad \begin{cases} u_t + Au - f - v_1\chi\varphi_{\omega_1} - v_2\chi\varphi_{\omega_2} = \tilde{f} \quad \text{in } Q \\ u(0) - u_0 = \tilde{u}_0 \quad \text{in } \Omega, \quad u_{|\Sigma} = 0, \end{cases}$$

where A is a linear strictly elliptic operator of second order.

We can think of u as being the concentration of some chemical product in, say, a lake Ω, and think of $\omega \subset \Omega$ as the place where we can apply a control v. We have divided ω into two parts ω_1, ω_2. The main objective is to have at final time T, $u(T)$ "very close" of an optimal concentration u_T acting the leader control v_1 in $(0,T) \times \omega_1$, but we want to achieve this without, in the course of the action, going "too far" from a function u_d, and this is the work of the follower control v_2 acting in $(0,T) \times \omega_2$.

In [17] this problem is studied. First, the nontrivial condition is proved by means of an approximate controllability technique (for a linear system), using the optimality system for the follower. Finally, optimality system for the leader is obtained using Fenchel and Rockafeller's duality.

We think that DM method must also work in this ploblem.

3.6 Pareto (cooperative) solution

In the Pareto context, one only a control $y \in Y$ will be considered.

Definition 3 *$(\bar{x}, \bar{y}) \in \mathcal{U}_{ad}$ is a global (respectively local) Pareto solution if there not exists $(x, y) \in \mathcal{U}_{ad}$ (respectively $\mathcal{E}(\bar{x}, \bar{y})$) verifying:*

$$J_1(x, y) \leq J_1(\bar{x}, \bar{y}) \quad \textit{and} \quad J_2(x, y) \leq J_2(\bar{x}, \bar{y}),$$

with strictly inequality for at least one J_i.

We enounce the following result, which is an extension of the DM method to multiobjective case, proved by Censor [3] for one equality constraint and by Kotarski [12] for more equality constraints.

Theorem 7 *If $(\bar{x}, \bar{y})$ is a (local) Pareto solution, and multiobjective functionals, equality and inequality constraints are "regular", in the sense that the corresponding descent and noncreasing, tangent and feasible cones are convex sets, then there exists*

$$f_1^i \in DC(J_i)^*, f_2^i \in NC(J_i)^*, f_3^i \in TC^*, f_4^i \in FC^* \quad (i = 1, 2)$$

such that

$$f_1^1 + f_2^2 + f_3^1 + f_4^1 = 0 \quad \textit{and} \quad f_1^2 + f_2^1 + f_3^2 + f_4^2 = 0.$$

Therefore, if $DC(J_i)^* = NC(J_i)^*$ for each $i = 1, 2$, two previous operational equalities remains only in the equality:

$$f_1 + f_2 + f_3 + f_4 = 0, \quad f_1 \in DC(J_i)^*(i = 1, 2), f_3 \in TC^*, f_4 \in FC^*.$$

A sufficient condition that imply $DC(J_i)^* = NC(J_i)^*$ is that J_i is a Ponstein convex functional (see [3]). For instance, an strictly convex functional has in particular the Ponstein convexity property.

3.7 Example 5: Weak solutions of NS with two partially distributed controls and convex constraints for the controls

Example 3 will be considered. The following optimality conditions can be obtained ([11]):

Theorem 8 *If $(\bar{u}, \bar{v}_1, \bar{v}_2)$ is a (local) Pareto solution, then there exists $\lambda_1 \geq 0, \lambda_2 \geq 0$ with $(\lambda_1, \lambda_2) \neq 0$ and $\tilde{u} \in L^2(0, T; V) \cap L^\infty(0, T; H)$ such that:*

$$(\bar{u}, \bar{v}_1, \bar{v}_2) \textit{ verifying } (S), \textit{ with } (\tilde{f}, \tilde{u}_0) = (0, 0)$$

$(P)^*$

$$\begin{cases} -\dfrac{d}{dt}\displaystyle\int_\Omega \tilde{u}\varphi - \int_\Omega \bar{u} \cdot \nabla\varphi\tilde{u} - \int_\Omega \tilde{u} \cdot \nabla\varphi\bar{u} + \int_\Omega \nabla\tilde{u}\nabla\varphi \\ = \lambda_1 (J_1)'_u(\bar{u}, \bar{v}_1)(\varphi) + \lambda_2 (J_2)'_u(\bar{u}, \bar{v}_2)(\varphi) \\ = \lambda_1(\bar{u} - u_{1,d}, \varphi)_{(0,T)\times\omega_{1,d}} + \lambda_2(\bar{u} - u_{2,d}, \varphi)_{(0,T)\times\omega_{2,d}} \ \forall \varphi \in H_0^1(Q) \ s.t. \ \nabla \cdot \varphi = 0 \\ \tilde{u}_1(T) = 0 \quad in\ \Omega \end{cases}$$

$$(\tilde{u}+\lambda_1(J_1)'_{v_1}(\bar{u}, \bar{v}_1), \bar{v}_1 - v_1)_{(0,T)\times\omega_1} = (\tilde{u}+\lambda_1 N_1(\bar{v}_1 - v_{1,d}), \bar{v}_1 - v_1)_{(0,T)\times\omega_1} \geq 0 \quad \forall v_1 \in \mathcal{G}_1.$$

$$(\tilde{u}+\lambda_2(J_2)'_{v_2}(\bar{u}, \bar{v}_2), \bar{v}_2 - v_2)_{(0,T)\times\omega_2} = (\tilde{u}+\lambda_2 N_2(\bar{v}_2 - v_{2,d}), \bar{v}_2 - v_2)_{(0,T)\times\omega_2} \geq 0 \quad \forall v_2 \in \mathcal{G}_2.$$

Some results about existence of this optimality system can be obtained ([11]), imposing $\mathcal{G}_1$ and $\mathcal{G}_2$ bounded in Y_1 and Y_2 respectively.

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References

[1] Alexeév, V.; Fomine, S., Tykhomirov, V. – *Commande Optimale*, Mir, Moscou 1982.

[2] Casas, E. – *An Optimal Control problem governed by the evolution Navier-Stokes equations* Optimal Control of Viscous Flow (Ed. by S.S. Sritharan) Siam 1998, 79-96.

[3] Censor, Y. – *Pareto Optimality in Multiobjective Problems* Applied Math. Optim. 4, (1977) 41-59.

[4] Diaz, J. I., Lions, J. L. – *On the Approximate Controllability of Stalckelberg - Nash Strategies*, Ocean Circulation and pollution control, a mathematical and numerical investigation (Madrid 1997) 17-27, Springer 2004.

[5] Fursikov, A.V. – *Optimal control of distributed systems. Theory and applications.* American Mathematical Society, Providence, Rhode Island, 1999.

[6] Flett, T.M. – *Differential Analysis. Differentiation, differential equations and differential inequalities* Cambrigde University Press, 1980.

[7] Girsanov, I. V. – *Lectures on Mathematical Theory of Extremum Problems* Springer, New York 1972. (Lectures notes in economics and mathematical systems, 67).

[8] Gunzburger, M.D., Hou, L., Svobodny, T.P. – *Analysis and finite element approximation of optimal control problems for the stationary Navier-Stokes equations with Dirichlet controls.* Mathematical modelling and numerical analysis , Vol. 25, No. 6, (1991) 711-748.

[9] Gunzburger, M.D., Hou, L., Svobodny, T.P. – *Boundary velocity control of incompressible flow with an application to viscous drag reduction.* SIAM J. Control and Optimization, Vol. 30, No. 1, (1992) 167-181.

[10] Gayte, I., Guillén-González, F., Rojas-Medar, M.A. Work in progress.

[11] Guillén-González, F., Lopes Marques, F.P., Rojas-Medar, M.A. Work in progress.

[12] Kotarski, W. – *Characterization of Pareto Optimal Points in Problems with Multi-Equality Constraints*, Optimization 20 (1989) 93-106.

[13] Ledzewicz-Kowalewska, U. – *On some specification of the Dubovitskii-Milyutin method*. Nonlinear Analysis. Theory, Methods and Applications, Vol. 10, No. 12, (1986) 1367-1371.

[14] Lions, J. L. – *Optimal Control of Systems Governed by Partial Differential Equations*, Springer, 1970.

[15] Lions, J. L. – *Contrôle de Pareto de Systèmes Distribués. Le cas stationaire*, C.R. Acad. Sc. Paris, t. 302, Série I, n ^{0} 6, (1986) 223-227.

[16] Lions, J. L. – *Contrôle de Pareto de Systèmes Distribués. Le cas d'évolution* C.R. Acad. Sc. Paris, t. 302, Série I, n ^{0} 11, (1986) 413-417.

[17] Lions, J. L. – *Some Remarks on Stalckelberg's Optimization Mathematical Models and Methods in Applied Sciences*, Vol. 4, (1994) 477-487.

[18] Nash, J. F. – *Noncooperative games*, Annals of Mathematics, 54 (1951) 286-295.

[19] Papageorgiou, A., Papageorgiou, N.S. – *Necessary and sufficient conditions for optimality in nonlinear distributed parameter systems with variable initial state.* J. Math. Soc. Japan, Vol. 42, No. 3, (1990) 387-396.

[20] Pareto, V. – *Cours d'économie politique*, Rouge, Laussane, Switzerland, 1896.

[21] Ramos, A.M., Glowinski, R., Periaux, J. – *Nash Equilibria for the Multiobjective Control of Linear Partial Differential Equations*, Journal of Optimization Theory and Applications, Vol. 112, (2002) 457-498.

[22] Ramos, A.M., Glowinski, R., Periaux, J. – *Pointwise Control of the Burgers Equation and Related Nash Equilibria Problems: A Computational Approach,* Journal of Optimization Theory and Applications, Vol. 112, (2002) 499-516.

[23] Rivera, P.H. – *Optimal control of unstable non linear evolution systems.* Annales Faculté des Sciences Toulouse, Vol VI, (1984) 33-50.

[24] Rivera, P.H. – *On the optimal control of non-well posed linear evolution systems.* Mat. Aplic. Comp., Vol 7, 1, (1988) 55-67.

[25] Rivera, P.H., Vasconcellos, C.F. – *Optimal control for a backward parabolic system.* SIAM J. Control and Optimization, Vol. 25, No. 5, (1987) 1163-1172.

[26] Von Stalckelberg, H. – *Marktform und Gleichgewicht*, Springer, Berlin 1934.

[27] Temam, R. – *Navier-Stokes equations*, Noth-Holland, Amsterdam, Rev. Edit. 1979.

[28] Walczak, S. – *Some properties of cones in normed spaces and their application to investigating extremal problems.* Journal of optimization theory and applications, Vol. 42, No. 4, (1984) 561-582.

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