International Conference-School Infinite-dimensional dynamics, dissipative systems, and attractors Nizhny Novgorod, Russia
July 13–17, 2015

Some recent results on tempered pullback attractors for non-autonomous variants of Navier-Stokes equations

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Outline of the talk

Motivation

Abstract results on attractors theory

- Existence of minimal pullback attractors
- Comparison of non-tempered and tempered attractors
- Comparison of pullback $\mathcal{D}_i-$attractors

Application to a 2D-Navier-Stokes model

- Pullback $\mathcal{D}$-attractors in H
- Pullback $\mathcal{D}$-attractors in V
- Tempered behaviour of the pullback attractors

Flattening property: shorter proof of asymp.compactness for V

Delay terms: "good" and "bad" ones

Navier-Stokes-Voigt

Motivation

- Non-autonomous dynamical systems
  - V. V. Chepyzhov and M. I. Vishik, Attractors of non-autonomous dynamical systems and their dimension , *J. Math. Pures Appl.* **73** (1994), 279–333.
  - V. V. Chepyzhov and M. I. Vishik, *Attractors for Equations of Mathematical Physics*, Colloquium Publications **49**, Providence, AMS, 2002.

- Random dynamical systems (unbounded time-dependent terms)
  - B. Schmalfuß, Backward cocycles and attractors of stochastic differential equations, en *International Seminar on Applied Mathematics-Nonlinear Dynamics: Attractor Approximation and Global Behaviour* (V. Reitmann, T. Redrich y N. J. Kosch, eds.), (Dresden), pp. 185–192, Technische Universität, 1992.
  - H. Crauel and F. Flandoli, Attractors for random dynamical systems, *Probab. Theory Relat. Fields* **100** (1994), 365–393.
  - H. Crauel, A. Debussche, and F. Flandoli, Random attractors, *J. Dynam. Differential Equations* **9** (1997), 307–341.
  - I. D. Chueshov, *Monotone Random Systems and Applications*, *Lecture Notes in Mathematics* **1779**, Berlin Heidelberg: Springer-Verlag, 2002.

- Deterministic non-autonomous dynamical systems with the pullback approach with fixed bounded sets

  - P. E. Kloeden and B. Schmalfuß, Nonautonomous systems, cocycle attractors and variable time-step discretization, *Numer. Algorithms*, **14** (1997) 141–152. Dynamical numerical analysis (Atlanta, GA, 1995).
  - P. E. Kloeden and B. Schmalfuß, Asymptotic behaviour of nonautonomous difference inclusions, *Systems & Control Letters*, **33** (1998), 275–280.
  - P. E. Kloeden and D. J. Stonier, Cocycle attractors in nonautonomously perturbed differential equations, *Dynam. Contin. Discrete Impuls. Systems*, **4** (1998), 211–226.
  - P. E. Kloeden, Pullback attractors in nonautonomous difference equations, *J. Difference Eqns. Applns.*, **6** (2000), 33–52.

• Deterministic non-autonomous dynamical systems with tempered universes:

- T. Caraballo, G. Łukaszewicz, and J. Real, Pullback attractors for asymptotically compact non-autonomous dynamical systems, *Nonlinear Anal.* 64 (2006), 484-498.
- T. Caraballo, G. Łukaszewicz, and J. Real, Pullback attractors for non-autonomous 2D-Navier-Stokes equations in some unbounded domains, *C. R. Math. Acad. Sci. Paris,* **342** (2006), 263–268.

⋆ Physical and mathematical questions: big-bang-bang–past, present, future; dissipative world

Abstract results on attractors theory. Existence of minimal pullback attractors

Consider given a metric space (X, d_X), and let us denote $\mathbb{R}^2_d = \{(t,\tau) \in \mathbb{R}^2 : \tau \leq t\}$.

A process on X is a mapping U such that $\mathbb{R}^2_d \times X \ni (t,\tau,x) \mapsto U(t,\tau)x \in X$ with $U(\tau,\tau)x = x$ for any $(\tau,x) \in \mathbb{R} \times X$, and $U(t,r)(U(r,\tau)x) = U(t,\tau)x$ for any $\tau \leq r \leq t$ and all $x \in X$.

Definition

A process U on X is said to be closed if for any $\tau \leq t$, and any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ and $U(t,\tau)x_n \to y \in X$, then $U(t,\tau)x = y$.

Remark

U continuous

$\Rightarrow$ *strong-weak (also known as norm-to weak)*

$\Rightarrow$ *closed*

This more relaxed concepts are useful in some situations (e.g., dyn. syst. and attractors for strong sols. for RD eqns).

$\mathcal{P}(X)$ the family of all nonempty subsets of X, and consider a family of nonempty sets $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ [not required compactness or boundedness on these sets]

Definition

U is pullback $\widehat{D}_0$-asymptotically compact if for any $t \in \mathbb{R}$ and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, the sequence $\{U(t, \tau_n)x_n\}$ is relatively compact in X.

Denote

$$\Lambda(\widehat{D}_0, t) := \bigcap_{s \le t} \overline{\bigcup_{\tau \le s} U(t,\tau) D_0(\tau)}^X \quad \forall\, t \in \mathbb{R}.$$

Proposition

U pullback $\widehat{D}_0$-asymptotically compact $\Rightarrow$ for all $t \in \mathbb{R}$, the set $\Lambda(\widehat{D}_0, t)$ given by (8) is a nonempty compact subset of X, and (attracts pullback)

$$\lim_{\tau \to -\infty} \operatorname{dist}_X(U(t,\tau)D_0(\tau), \Lambda(\widehat{D}_0, t)) = 0.$$

Let be given $\mathcal{D}$ a nonempty class of families parameterized in time $\widehat{D} = \{D(t) \ : \ t \in \mathbb{R}\} \subset \mathcal{P}(X)$. The class $\mathcal{D}$ will be called a universe in $\mathcal{P}(X)$.

Definition

It is said that $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}-$absorbing for the process U on X if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(t, \widehat{D}) \leq t$ such that

$$U(t,\tau)D(\tau) \subset D_0(t) \quad \text{for all } \tau \leq \tau_0(t,\widehat{D}).$$

Observe that in the definition above $\widehat{D}_0$ does not belong necessarily to the class $\mathcal{D}$.

Definition

U pullback $\mathcal{D}-$asymptotically compact if it is $\widehat{D}$-asymptotically compact for any $\widehat{D} \in \mathcal{D}$.

Proposition

$\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ pullback $\mathcal{D}-$absorbing for a process U on X, which is pullback $\widehat{D}_0$-asymptotically compact. Then, U is also pullback $\mathcal{D}-$asymptotically compact.

Proposition

U closed and pullback $\mathcal{D}-$asymptotically compact $\Rightarrow$ for each $\widehat{D} \in \mathcal{D}$ and any $t \in \mathbb{R}$, the set $\Lambda(\widehat{D}, t)$ is a nonempty compact subset of X, invariant for U, that attracts $\widehat{D}$ in the pullback sense, i.e.

$$\lim_{\tau \to -\infty} \mathrm{dist}_X(U(t,\tau)D(\tau), \Lambda(\widehat{D}, t)) = 0. \tag{1}$$

Moreover, it is the minimal family of closed sets satisfying (1).

Theorem

$U : \mathbb{R}^2_d \times X \to X$ closed, a universe $\mathcal{D}$ in $\mathcal{P}(X)$, and a family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ pullback $\mathcal{D}-$absorbing for U, and U pullback $\widehat{D}_0-$asymptotically compact.

Then, the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ defined by

$$\mathcal{A}_{\mathcal{D}}(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda(\widehat{D}, t)}^{X} \quad t \in \mathbb{R},$$

(a) *for any $t \in \mathbb{R}$, $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of X, and $\mathcal{A}_{\mathcal{D}}(t) \subset \Lambda(\widehat{D}_0, t)$,*

(b) *$\mathcal{A}_{\mathcal{D}}$ is pullback $\mathcal{D}-$attracting*

(c) *$\mathcal{A}_{\mathcal{D}}$ is invariant, i.e. $U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau) = \mathcal{A}_{\mathcal{D}}(t)$ for all $\tau \leq t$,*

(d) *if $\widehat{D}_0 \in \mathcal{D}$, then $\mathcal{A}_{\mathcal{D}}(t) = \Lambda(\widehat{D}_0, t) \subset \overline{D_0(t)}^{X}$, for all $t \in \mathbb{R}$.*

The family $\mathcal{A}_{\mathcal{D}}$ is minimal in the sense that if $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is a family of closed sets and $\mathcal{D}-$attracting, then $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$.

Remark

Under the assumptions of Theorem 5, the family $\mathcal{A}_{\mathcal{D}}$ is called the minimal pullback $\mathcal{D}-$attractor for the process U.
If $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$, then it is the unique family of closed subsets in $\mathcal{D}$ that satisfies (b)–(c).
A sufficient condition for $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ is to have that $\widehat{D}_0 \in \mathcal{D}$, the set $D_0(t)$ is closed for all $t \in \mathbb{R}$, and the family $\mathcal{D}$ is inclusion-closed (i.e. if $\widehat{D} \in \mathcal{D}$, and $\widehat{D}' = \{D'(t) \ : \ t \in \mathbb{R}\} \subset \mathcal{P}(X)$ with $D'(t) \subset D(t)$ for all t, then $\widehat{D}' \in \mathcal{D}$).

Denote $\mathcal{D}_F^X$ the universe of fixed nonempty bounded subsets of X, i.e. the class of all families $\widehat{D}$ of the form $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of X.

For $\mathcal{D}_F^X$, the corresponding minimal pullback $\mathcal{D}_F^X$−attractor $\mathcal{A}_{\mathcal{D}_F^X}$ is the one defined by Crauel, Debussche, and Flandoli.

Corollary

Under the assumptions of Theorem 5, if the universe $\mathcal{D}$ contains the universe $\mathcal{D}_F^X$, then both attractors, $\mathcal{A}_{\mathcal{D}_F^X}$ and $\mathcal{A}_{\mathcal{D}}$, exist, and the following relation holds:

$$\mathcal{A}_{\mathcal{D}_F^X}(t) \subset \mathcal{A}_{\mathcal{D}}(t) \qquad \forall t \in \mathbb{R}.$$

Remark

Under the above assumptions, if, moreover, $\widehat{D}_0 \in \mathcal{D}$, and for some $T \in \mathbb{R}$ the set $\cup_{t \le T} D_0(t)$ is a bounded subset of X, then

$$\mathcal{A}_{\mathcal{D}_F^X}(t) = \mathcal{A}_{\mathcal{D}}(t) \qquad \forall t \le T.$$

Comparison of pullback $\mathcal{D}_i$−attractors

Theorem

Let $\{(X_i, d_{X_i})\}_{i=1,2}$ be metric spaces, $X_1 \subset X_2$ contin. injected, and for $i = 1, 2$, let $\mathcal{D}_i$ be a universe in $\mathcal{P}(X_i)$, with $\mathcal{D}_1 \subset \mathcal{D}_2$. U acts as a process in both cases, $U : \mathbb{R}^2_d \times X_i \to X_i$ for $i = 1, 2$.

$$\mathcal{A}_i(t) = \overline{\bigcup_{\widehat{D}_i \in \mathcal{D}_i} \Lambda_i(\widehat{D}_i, t)}^{X_i}, \quad i = 1, 2.$$

Then, $\mathcal{A}_1(t) \subset \mathcal{A}_2(t)$ for all $t \in \mathbb{R}$.

Suppose moreover that the two following conditions are satisfied:

(i) $\mathcal{A}_1(t)$ is a compact subset of X_1 for all $t \in \mathbb{R}$,

(ii) for any $\widehat{D}_2 \in \mathcal{D}_2$ and any $t \in \mathbb{R}$, there exist a family $\widehat{D}_1 \in \mathcal{D}_1$ and a $t^*_{\widehat{D}_1} \leq t$ (both possibly depending on t and $\widehat{D}_2$), such that U is pullback $\widehat{D}_1$-asymptotically compact, and for any $s \leq t^*_{\widehat{D}_1}$ there exists a $\tau_s \leq s$ such that

$$U(s,\tau)D_2(\tau) \subset D_1(s) \quad \text{for all } \tau \leq \tau_s.$$

Then, under all the conditions above,
$\mathcal{A}_1(t) = \mathcal{A}_2(t) \quad \text{for all } t \in \mathbb{R}.$

Remark

In the preceding theorem, if instead of assumption (ii) we consider the following condition:

(ii') *for any* $\widehat{D}_2 \in \mathcal{D}_2$ *and any sequence* $\tau_n \to -\infty$ *there exist another family* $\widehat{D}_1 \in \mathcal{D}_1$ *and another sequence* $\tau'_n \to -\infty$ *with* $\tau'_n \geq \tau_n$ *for all* n, *such that* U *is pullback* $\widehat{D}_1$*-asymptotically compact, and*

$$U(\tau'_n, \tau_n)D_2(\tau_n) \subset D_1(\tau'_n), \quad \text{for all } n, \tag{2}$$

then, with a similar proof, the equality $\mathcal{A}_2(t) = \mathcal{A}_1(t)$ *for all* $t \in \mathbb{R}$, *also holds.*

Observe that a sufficient condition for (2) is that there exists $T > 0$ *such that for any* $\widehat{D}_2 \in \mathcal{D}_2$, *there exists a* $\widehat{D}_1 \in \mathcal{D}_1$ *satisfying* $U(\tau + T, \tau)D_2(\tau) \subset D_1(\tau + T)$, *for all* $\tau \in \mathbb{R}$.

Application to a 2D-Navier-Stokes model

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f(t) \text{ in } (\tau, +\infty) \times \Omega, \\ \operatorname{div} u = 0 \text{ in } (\tau, +\infty) \times \Omega, \\ u = 0 \text{ on } (\tau, +\infty) \times \partial\Omega, \\ u(\tau, x) = u_\tau(x), \ x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is open and bounded with smooth enough $\partial\Omega$[1],
$\nu > 0$ is the kinematic viscosity,
u is the velocity field of the fluid,
p is the pressure,
u_τ is the initial velocity field, and
f the external force (time-dep.)term (Ex.: Arctic sea, control, etc)

[1] Not for the results in H but in V.

$$\mathcal{V} = \left\{ u \in (C_0^\infty(\Omega))^2 : \operatorname{div} u = 0 \right\},$$

$H =$ the closure of $\mathcal{V}$ in $(L^2(\Omega))^2$ with the norm $|\cdot|$, and inner product $(\cdot,\cdot)$, where for $u, v \in (L^2(\Omega))^2$,

$$(u, v) = \sum_{j=1}^{2} \int_\Omega u_j(x) v_j(x) \mathrm{d}x,$$

$V =$ the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^2$ with the norm $\|\cdot\|$ associated to the inner product $((\cdot,\cdot))$, where for $u, v \in (H_0^1(\Omega))^2$,

$$((u, v)) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} \mathrm{d}x.$$

Definition (Weak solution)

A weak solution is a function u that belongs to $L^2(\tau, T; V) \cap L^\infty(\tau, T; H)$ for all $T > \tau$, with $u(\tau) = u_\tau$, such that for all $v \in V$,

$$\frac{d}{dt}(u(t), v) + \nu\langle Au(t), v\rangle + b(u(t), u(t), v) = \langle f(t), v\rangle,$$

where the equation must be understood in the sense of $\mathcal{D}'(\tau, +\infty)$.

Remark

If u is a weak solution, then we deduce that for any $T > \tau$, one has $u' \in L^2(\tau, T; V')$, and so $u \in C([\tau, +\infty); H)$, whence the initial datum has full sense. Moreover, in this case the following energy equality holds for all $\tau \leq s \leq t$:

$$|u(t)|^2 + 2\nu \int_s^t \langle Au(r), u(r)\rangle dr = |u(s)|^2 + 2\int_s^t \langle f(r), u(r)\rangle dr.$$

Definition (Strong solution)

A strong solution is a weak solution u of (17) such that $u \in L^2(\tau, T; D(A)) \cap L^\infty(\tau, T; V)$ for all $T > \tau$.

Remark

If $f \in L^2_{loc}(\mathbb{R}; H)$ and u is a strong solution, then $u' \in L^2(\tau, T; H)$ for all $T > \tau$, and so $u \in C([\tau, +\infty); V)$. In this case the following energy equality holds:

$$\begin{aligned} & \|u(t)\|^2 + 2\nu \int_s^t |Au(r)|^2 \, dr + 2 \int_s^t b(u(r), u(r), Au(r)) \, dr \\ = \; & \|u(s)\|^2 + 2 \int_s^t (f(r), Au(r)) \, dr, \quad \forall \tau \le s \le t. \end{aligned}$$

Theorem (Weak and strong solutions)

$f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$ and $u_\tau \in H \Rightarrow \exists!$ weak solution $u(\cdot) = u(\cdot;\tau, u_\tau)$.

$f \in L^2_{\mathrm{loc}}(\mathbb{R}; H) \Rightarrow u \in C((\tau, T]; V) \cap L^2(\tau+\varepsilon, T; (H^2(\Omega))^2)$ for every $\varepsilon > 0$ and $T > \tau + \varepsilon$.

If $u_\tau \in V$, then $u \in C([\tau, T]; V) \cap L^2(\tau, T; (H^2(\Omega))^2)$ for every $T > \tau$, i.e. u is a strong solution.

Theorem (Weak and strong solutions)

$f \in L^2_{\rm loc}(\mathbb{R}; V')$ and $u_\tau \in H \Rightarrow \exists!$ weak solution $u(\cdot) = u(\cdot; \tau, u_\tau)$.
$f \in L^2_{\rm loc}(\mathbb{R}; H) \Rightarrow u \in C((\tau, T]; V) \cap L^2(\tau + \varepsilon, T; (H^2(\Omega))^2)$ for every $\varepsilon > 0$ and $T > \tau + \varepsilon$.
If $u_\tau \in V$, then $u \in C([\tau, T]; V) \cap L^2(\tau, T; (H^2(\Omega))^2)$ for every $T > \tau$, i.e. u is a strong solution.

Therefore, when $f \in L^2_{\rm loc}(\mathbb{R}; V')$, we can define a process $U : \mathbb{R}^2_d \times H \to H$ as

$$U(t, \tau)u_\tau = u(t; \tau, u_\tau) \quad \forall u_\tau \in H, \quad \forall \tau \leq t,$$

and if $f \in L^2_{\rm loc}(\mathbb{R}; H)$, the restriction of this process to $\mathbb{R}^2_d \times V$ is a process in V.

Pullback $\mathcal{D}$-attractors in H

Proposition (Continuity of the process)

If $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$, for any pair $(t, \tau) \in \mathbb{R}^2_d$, the map $U(t, \tau)$ is continuous from H into H.

Moreover, if $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$, then $U(t, \tau)$ is also continuous from V into V.

Pullback $\mathcal{D}$-attractors in H

Proposition (Continuity of the process)

If $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$, for any pair $(t, \tau) \in \mathbb{R}^2_d$, the map $U(t, \tau)$ is continuous from H into H.

Moreover, if $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$, then $U(t, \tau)$ is also continuous from V into V.

Lemma

Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$ and $u_\tau \in H$. Consider any $\mu \in (0, 2\nu\lambda_1)$ fixed. Then, the solution u satisfies for all $t \geq \tau$:

$$|u(t)|^2 \leq e^{-\mu(t-\tau)}|u_\tau|^2 + \frac{e^{-\mu t}}{2\nu - \mu\lambda_1^{-1}} \int_\tau^t e^{\mu s}\|f(s)\|_*^2 ds.$$

Lemma

Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$ *and* $u_\tau \in H$*. Consider any* $\mu \in (0, 2\nu\lambda_1)$ *fixed. Then, the solution* u *satisfies for all* $t \geq \tau$:

$$|u(t)|^2 \leq e^{-\mu(t-\tau)}|u_\tau|^2 + \frac{e^{-\mu t}}{2\nu - \mu\lambda_1^{-1}} \int_\tau^t e^{\mu s}\|f(s)\|_*^2 ds.$$

Definition (Universe)

We will denote by $\mathcal{D}_\mu^H$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(H)$ such that

$$\lim_{\tau \to -\infty} \left(e^{\mu\tau} \sup_{v \in D(\tau)} |v|^2 \right) = 0.$$

Remark

$\mathcal{D}_F^H \subset \mathcal{D}_\mu^H$ *and that* $\mathcal{D}_\mu^H$ *is inclusion-closed (tempered condition).*

Corollary ($\mathcal{D}_\mu^H$−absorbing family)

Assume that there exists some $\mu \in (0, 2\nu\lambda_1)$ *such that*

$$\int_{-\infty}^{0} e^{\mu s}\|f(s)\|_*^2 ds < +\infty.$$

Then, $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ *defined by* $D_0(t) = \overline{B}_H(0, R_H^{1/2}(t))$,

$$R_H(t) = 1 + \frac{e^{-\mu t}}{2\nu - \mu\lambda_1^{-1}} \int_{-\infty}^{t} e^{\mu s}\|f(s)\|_*^2 ds,$$

is pullback $\mathcal{D}_\mu^H$*−absorbing for the process* $U : \mathbb{R}_d^2 \times H \to H$ *(and therefore* $\mathcal{D}_F^H$*−absorbing too), and* $\widehat{D}_0 \in \mathcal{D}_\mu^H$.

Lemma ($\mathcal{D}_\mu^H$−asymptotic compactness)

The process U *is pullback* $\mathcal{D}_\mu^H$*−asymptotically compact.*

Corollary ($\mathcal{D}_\mu^H-$absorbing family)

Assume that there exists some $\mu \in (0, 2\nu\lambda_1)$ *such that*

$$\int_{-\infty}^{0} e^{\mu s}\|f(s)\|_*^2 ds < +\infty.$$

Then, $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\}$ *defined by* $D_0(t) = \overline{B}_H(0, R_H^{1/2}(t))$,

$$R_H(t) = 1 + \frac{e^{-\mu t}}{2\nu - \mu\lambda_1^{-1}} \int_{-\infty}^{t} e^{\mu s}\|f(s)\|_*^2 ds,$$

is pullback $\mathcal{D}_\mu^H-$*absorbing for the process* $U : \mathbb{R}_d^2 \times H \to H$ *(and therefore* $\mathcal{D}_F^H-$*absorbing too), and* $\widehat{D}_0 \in \mathcal{D}_\mu^H$.

Lemma ($\mathcal{D}_\mu^H-$asymptotic compactness)

The process U *is pullback* $\mathcal{D}_\mu^H-$*asymptotically compact.*

`Proof (energy method based on non-increasing continuous functionals) omitted, see` V `case below.`

Theorem (Pullback $\mathcal{D}_\mu^H$-attractor)

Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V')$ *satisfies for some* $\mu \in (0, 2\nu\lambda_1)$ *the above condition. Then,* $\exists$ *the minimal pullback* $\mathcal{D}_F^H$*-attractor*

$$\mathcal{A}_{\mathcal{D}_F^H} = \{\mathcal{A}_{\mathcal{D}_F^H}(t) : t \in \mathbb{R}\}$$

and the minimal pullback $\mathcal{D}_\mu^H$*-attractor*

$$\mathcal{A}_{\mathcal{D}_\mu^H} = \{\mathcal{A}_{\mathcal{D}_\mu^H}(t) : t \in \mathbb{R}\},$$

for the process U*. The family* $\mathcal{A}_{\mathcal{D}_\mu^H}$ *belongs to* $\mathcal{D}_\mu^H$*, and the following relation holds:*

$$\mathcal{A}_{\mathcal{D}_F^H}(t) \subset \mathcal{A}_{\mathcal{D}_\mu^H}(t) \subset \overline{B}_H(0, R_H^{1/2}(t)) \quad \forall t \in \mathbb{R}.$$

Remark

Useful in unbounded “Poincaré”-domains to obtain $\mathcal{A}_{\mathcal{D}_F^H}$.

Regularity: pullback $\mathcal{D}$-attractors in V

From now on we assume that $f \in L^2_{\text{loc}}(\mathbb{R}; H)$, and satisfies

$$\int_{-\infty}^{0} e^{\mu s}|f(s)|^2\, ds < +\infty, \quad \text{for some } \mu \in (0, 2\nu\lambda_1).$$

Lemma

For any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}^H_\mu$, there exists $\tau_1(\widehat{D}, t) < t - 3$, such that for any $\tau \leq \tau_1(\widehat{D}, t)$ and any $u_\tau \in D(\tau)$, it holds

$$\begin{cases} |u(r;\tau,u_\tau)|^2 \leq \rho_1(t) & \text{for all } r \in [t-3, t], \\ \|u(r;\tau,u_\tau)\|^2 \leq \rho_2(t) & \text{for all } r \in [t-2, t], \\ \int_{r-1}^{r} |Au(\theta;\tau,u_\tau)|^2 d\theta \leq \rho_3(t) & \text{for all } r \in [t-1, t], \\ \int_{r-1}^{r} |u'(\theta;\tau,u_\tau)|^2 d\theta \leq \rho_4(t) & \text{for all } r \in [t-1, t], \end{cases}$$

where

$$\rho_1(t) = 1 + \frac{e^{\mu(3-t)}}{2\nu\lambda_1 - \mu} \int_{-\infty}^{t} e^{\mu\theta} |f(\theta)|^2 \, d\theta,$$

$$\begin{aligned} \rho_2(t) \quad = \quad & \max_{r \in [t-2,t]} \left\{ \left(\frac{1}{\nu}\rho_1(r) + \left(\frac{1}{\nu^2\lambda_1} + \frac{2}{\nu} \right) \int_{r-1}^{r} |f(\theta)|^2 \, d\theta \right) \right. \\ & \left. \times \exp\left[2C^{(\nu)}\rho_1(r) \left(\frac{1}{\nu}\rho_1(r) + \frac{1}{\nu^2\lambda_1} \int_{r-1}^{r} |f(\theta)|^2 \, d\theta \right) \right] \right\}, \end{aligned}$$

$$\rho_3(t) = \frac{1}{\nu}\left(\rho_2(t) + \frac{2}{\nu}\int_{t-2}^{t} |f(\theta)|^2 \, d\theta + 2C^{(\nu)}\rho_1(t)\rho_2^2(t) \right),$$

$$\rho_4(t) = \nu\rho_2(t) + 2\int_{t-2}^{t} |f(\theta)|^2 \, d\theta + 2C_1^2\rho_2(t)\rho_3(t),$$

and $C^{(\nu)} = 27C_1^4(4\nu^3)^{-1}$.

Remark

$$\lim_{t \to -\infty} e^{\mu t} \rho_1(t) = 0.$$

So $\{\overline{B}_H(0, \rho_1^{1/2}(t)) : t \in \mathbb{R}\} \in \mathcal{D}_\mu^H$.

We will denote by $\mathcal{D}_\mu^{H,V}$ the class of all families $\widehat{D}_V$ of elements of $\mathcal{P}(V)$ of the form $\widehat{D}_V = \{D(t) \cap V : t \in \mathbb{R}\}$, where $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathcal{D}_\mu^H$.
$\mathcal{D}_F^V$ the universe of families (parameterized in time but constant for all $t \in \mathbb{R}$) of nonempty fixed bounded subsets of V.
$\mathcal{D}_\mu^{H,V} \subset \mathcal{P}(V)$ is inclusion-closed, and evidently $\mathcal{D}_F^V \subset \mathcal{D}_\mu^{H,V}$.

Corollary (Absorbing in H+regularizing+tempered)

The family

$$\widehat{D}_{0,V} = \{\overline{B}_H(0, \rho_1^{1/2}(t)) \cap V : t \in \mathbb{R}\}$$

belongs to $\mathcal{D}_\mu^{H,V}$ *and satisfies that for any* $t \in \mathbb{R}$ *and any* $\widehat{D} \in \mathcal{D}_\mu^H$*, there exists a* $\tau(\widehat{D}, t) < t$ *such that*

$$U(t,\tau)D(\tau) \subset D_{0,V}(t) \quad \text{for all } \tau \leq \tau(\widehat{D}, t).$$

In particular, the family $\widehat{D}_{0,V}$ *is pullback* $\mathcal{D}_\mu^{H,V}-$*absorbing for the process* $U : \mathbb{R}_d^2 \times V \to V$.

Lemma (Asymptotic compactness in V norm)

The process $U : \mathbb{R}^2_d \times V \to V$ is pullback $\mathcal{D}^{H,V}_{\mu}-$ asymptotically compact.

Sketch of the proof:

$$\begin{cases} u^n \overset{*}{\rightharpoonup} u & \text{weak-star in } L^\infty(t-2,t;V), \\ u^n \rightharpoonup u & \text{weakly in } L^2(t-2,t;D(A)), \\ (u^n)' \rightharpoonup u' & \text{weakly in } L^2(t-2,t;H), \\ u^n \to u & \text{strongly in } L^2(t-2,t;V), \\ u^n(s) \to u(s) & \text{strongly in } V,\ a.e.\ s \in (t-2,t). \end{cases}$$

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$$u^n \to u \quad \text{strongly in} \quad C([t-2,t];H).$$

For all sequence $\{s_n\} \subset [t-2, t]$ with $s_n \to s_*$, it holds that

$$u^n(s_n) \rightharpoonup u(s_*) \quad \text{weakly in } V,$$

Claim:

$$u^n \to u \quad \text{strongly in } C([t-1, t]; V),$$

If not, $\{t_n\} \subset [t-1, t]$, $t_n \to t_* \geq t-1$

$$\|u^n(t_n) - u(t_*)\| \geq \varepsilon \quad \forall n \geq 1.$$

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$$\|u(t_*)\| \leq \liminf_{n\to\infty} \|u^n(t_n)\|.$$

for all $t-2 \leq s_1 \leq s_2 \leq t$

$$\begin{aligned} & \|u^n(s_2)\|^2 + \nu \int_{s_1}^{s_2} |Au^n(r)|^2 dr \\ \leq \; & \|u^n(s_1)\|^2 + 2C^{(\nu)} \int_{s_1}^{s_2} |u^n(r)|^2 \|u^n(r)\|^4 dr + \frac{2}{\nu} \int_{s_1}^{s_2} |f(r)|^2 dr, \end{aligned}$$

and

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In particular we can define the functions

$$\begin{aligned} J_n(s) &= \|u^n(s)\|^2 - 2C^{(\nu)} \int_{t-2}^{s} |u^n(r)|^2 \|u^n(r)\|^4 dr - \frac{2}{\nu} \int_{t-2}^{s} |f(r)|^2 dr, \\ J(s) &= \|u(s)\|^2 - 2C^{(\nu)} \int_{t-2}^{s} |u(r)|^2 \|u(r)\|^4 dr - \frac{2}{\nu} \int_{t-2}^{s} |f(r)|^2 dr. \end{aligned}$$

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J_n are non-increasing, so

$$\begin{aligned} J_n(t_n) - J(t_*) &\leq J_n(\tilde{t}_{k_\delta}) - J(t_*) \\ &\leq |J_n(\tilde{t}_{k_\delta}) - J(t_*)| \\ &\leq |J_n(\tilde{t}_{k_\delta}) - J(\tilde{t}_{k_\delta})| + |J(\tilde{t}_{k_\delta}) - J(t_*)| < \delta. \end{aligned}$$

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This yields that

$$\limsup_{n\to\infty} J_n(t_n) \leq J(t_*),$$

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This yields that

$$\limsup_{n\to\infty} J_n(t_n) \leq J(t_*),$$

and therefore,

$$\limsup_{n\to\infty} \|u^n(t_n)\| \leq \|u(t_*)\|.$$

Thus, $u^n(t_n) \to u(t_*)$ strongly in V.

Theorem

There exist the minimal pullback $\mathcal{D}_F^V$-attractor

$$\mathcal{A}_{\mathcal{D}_F^V} = \{\mathcal{A}_{\mathcal{D}_F^V}(t) : t \in \mathbb{R}\},$$

and the minimal pullback $\mathcal{D}_\mu^{H,V}$-attractor

$$\mathcal{A}_{\mathcal{D}_\mu^{H,V}} = \{\mathcal{A}_{\mathcal{D}_\mu^{H,V}}(t) : t \in \mathbb{R}\}$$

for the process $U : \mathbb{R}_d^2 \times V \to V$, and

$$\mathcal{A}_{\mathcal{D}_F^V}(t) \subset \mathcal{A}_{\mathcal{D}_F^H}(t) \subset \mathcal{A}_{\mathcal{D}_\mu^H}(t) = \mathcal{A}_{\mathcal{D}_\mu^{H,V}}(t) \quad \text{for all } t \in \mathbb{R},$$

In particular, the following pullback attraction result in V holds:

$$\lim_{\tau \to -\infty} \mathrm{dist}_V(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_\mu^H}(t)) = 0 \quad \text{for all } t \in \mathbb{R} \text{ and any } \widehat{D} \in \mathcal{D}_\mu^H.$$

Finally, if moreover f satisfies

$$\sup_{s \le 0} \left(e^{-\mu s} \int_{-\infty}^{s} e^{\mu\theta} |f(\theta)|^2 \, d\theta \right) < +\infty,$$

then (from ρ_i, $i = 1, 2$)

$$\mathcal{A}_{\mathcal{D}_F^V}(t) = \mathcal{A}_{\mathcal{D}_F^H}(t) = \mathcal{A}_{\mathcal{D}_\mu^H}(t) = \mathcal{A}_{\mathcal{D}_\mu^{H,V}}(t) \quad \text{for all } t \in \mathbb{R},$$

and for any bounded subset B of H

$$\lim_{\tau \to -\infty} \operatorname{dist}_V(U(t,\tau)B, \mathcal{A}_{\mathcal{D}_F^H}(t)) = 0 \quad \text{for all } t \in \mathbb{R}.$$

Remark (Infinitely many bigger universes)

If $f \in L^2_{\mathrm{loc}}(\mathbb{R}; H)$ *satisfies* $\int_{-\infty}^0 e^{\mu s}|f(s)|^2\,ds < +\infty$*, then*

$$\int_{-\infty}^0 e^{\sigma s}|f(s)|^2\,ds < +\infty, \quad \text{for all } \sigma \in (\mu, 2\nu\lambda_1).$$

Thus, for any $\sigma \in (\mu, 2\nu\lambda_1)$, $\exists\ \mathcal{D}^H_\sigma$*-pullback attractor,* $\mathcal{A}_{\mathcal{D}^H_\sigma}$.

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Thus, for any $\sigma \in (\mu, 2\nu\lambda_1)$, $\exists$ $\mathcal{D}^H_\sigma$-pullback attractor, $\mathcal{A}_{\mathcal{D}^H_\sigma}$.

Since $\mathcal{D}^H_\mu \subset \mathcal{D}^H_\sigma$, by comparison, for any $t \in \mathbb{R}$,

$$\mathcal{A}_{\mathcal{D}^H_\mu}(t) \subset \mathcal{A}_{\mathcal{D}^H_\sigma}(t) \quad \textit{for all } \sigma \in (\mu, 2\nu\lambda_1).$$

Moreover, if f satisfies $\sup_{s\le 0}\left(e^{-\mu s}\int_{-\infty}^s e^{\mu\theta}|f(\theta)|^2\,d\theta\right) < +\infty$, then, comparing with the $\mathcal{D}^H_F$ attractor,

$$\mathcal{A}_{\mathcal{D}^H_F}(t) = \mathcal{A}_{\mathcal{D}^H_\mu}(t) = \mathcal{A}_{\mathcal{D}^H_\sigma}(t) \quad \textit{for all } t \in \mathbb{R}, \textit{ and any } \sigma \in (\mu, 2\nu\lambda_1).$$

Tempered behaviour of the pullback attractors

The pullback attractor $\mathcal{A}_{\mathcal{D}_\mu^H} \in \mathcal{D}_\mu^H$, i.e. one has that

$$\lim_{t\to-\infty}\left(e^{\mu t}\sup_{v\in\mathcal{A}_{\mathcal{D}_\mu^H}(t)}|v|^2\right)=0.$$

Proposition

$f \in L^2_{loc}(\mathbb{R}; H)$: $\sup_{s\leq 0}\left(e^{-\mu s}\int_{-\infty}^{s} e^{\mu\theta}|f(\theta)|^2\,d\theta\right) < +\infty$,
$\widehat{D} \in \mathcal{D}_\mu^H$ invariant w.r.t. U: $D(t) = U(t,\tau)D(\tau)$ for all $\tau \leq t$.
Then,

$$\lim_{t\to-\infty}\left(e^{\mu t}\sup_{v\in D(t)}\|v\|^2\right)=0.$$

Proposition (More a-priori + derivating eqn.)

$f \in W^{1,2}_{loc}(\mathbb{R}; H)$: $\int_{-\infty}^{0} e^{\mu s}|f(s)|^2\, ds < +\infty$, then for each $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\mu^H$ there exists $\tau_1(\widehat{D}, t) < t - 3$ such that

$$|AU(r,\tau)u_\tau|^2 \leq \rho_6(t) \quad \text{for all } r \in [t-1, t],\ \tau \leq \tau_1(\widehat{D}, t),\ u_\tau \in D(\tau),$$

where

$$\rho_6(t) = \frac{4}{\nu^2}(\rho_5(t) + \max_{r\in[t-1,t]} |f(r)|^2) + \frac{2C^{(\nu)}}{\nu}\rho_1(t)\rho_2(t)^2,$$

with $\rho_5(t)$ defined by

$$\rho_5(t) = \left(\rho_4(t) + \frac{1}{\nu\lambda_1}\int_{t-2}^{t} \left|f'(\theta)\right|^2 d\theta\right) \exp\left(\frac{C_1^2}{\nu}\rho_2(t)\right).$$

Proposition (Above result + estimating f)

$f \in W^{1,2}_{loc}(\mathbb{R}; H)$: $\sup_{s\le 0}\left(e^{-\mu s}\int_{-\infty}^{s} e^{\mu\theta}|f(\theta)|^2\, d\theta\right) < +\infty$,

$$\lim_{t\to-\infty}\left(e^{\mu t}\int_{t-1}^{t}|f'(\theta)|^2\, d\theta\right) = 0, \qquad \lim_{t\to-\infty}\left(e^{\mu t}|f(t)|^2\right) = 0.$$

Then, for every invariant family $\widehat{D} \in \mathcal{D}^H_\mu$:

$$\lim_{t\to-\infty}\left(e^{\mu t}\sup_{v\in D(t)}\|v\|^2_{(H^2(\Omega))^2}\right) = 0.$$

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Proof: $|f(r)| \le |f(t-1)| + \left(\int_{t-1}^{t}|f'(\theta)|^2\,d\theta\right)^{1/2} \forall r \in [t-1, t].$

Flattening property: shorter proof of asymp.compact in V

A splitting of the solutions into high and low components

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- The squeezing property:

  • C. Foias, O. Manley and R. Temam, Modelling of the interaction of small and large eddies in two-dimensional turbulent flows, *RAIRO Modl. Math. Anal. Numr.* **22** (1988), 93–118.

  • R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer, New York, 1988.

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  • C. Foias, O. P. Manley, R. Temam and Y. M. Trve, Asymptotic analysis of the Navier–Stokes equations , *Phys. D* **9** (1983), 157–188.

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- Existence of attractors
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Definition (Pullback $\widehat{D}_0$-flattening property)

U satisfies the pullback $\widehat{D}_0$-flattening property if for any $t \in \mathbb{R}$ and $\varepsilon > 0$, there exist $\tau_\varepsilon < t$, a finite dimensional subspace X_ε of X, and a mapping $P_\varepsilon : X \to X_\varepsilon$ such that

$$\bigcup_{\tau \leq \tau_\varepsilon} P_\varepsilon U(t,\tau) D_0(\tau) \text{ is bounded in } X$$

$$\|(Id_X - P_\varepsilon)U(t,\tau)u^\tau\|_X < \varepsilon \quad \text{for any } \tau \leq \tau_\varepsilon,\ u^\tau \in D_0(\tau).$$

Pullback $\widehat{D}_0$-flattening $\Rightarrow$ pullback $\widehat{D}_0$-asymptotic compact

Proposition (Flattening implies asymp.compact)

$t \in \mathbb{R}$, sequences $(t \geq)\tau_n \to -\infty$, $x_n \in D_0(\tau_n)$. Then $\{U(t, \tau_n)x_n : n \geq 1\}$ is relatively compact in X (Banach space).

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$t \in \mathbb{R}$, sequences $(t \geq)\tau_n \to -\infty$, $x_n \in D_0(\tau_n)$. Then $\{U(t,\tau_n)x_n : n \geq 1\}$ is relatively compact in X (Banach space).

Proof. Fix $k \geq 1$ (integer), $\exists P_k : X \to X_k$ (fin.dim.subspace of X)
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$\{U(t,\tau_n)x_n : n \geq 1\}$ possesses a Cauchy subseq. in X (Banach)

If $f \in L^2_{\rm loc}(\mathbb{R}; H)$ satisfies $\int_{-\infty}^0 e^{\mu s}|f(s)|^2 ds < \infty$ for some $\mu \in (0, 2\nu\lambda_1)$, then, for any $t \in \mathbb{R}$,

$$\lim_{\rho\to\infty} e^{-\rho t} \int_{-\infty}^{t} e^{\rho s}|f(s)|^2 \, ds = 0.$$

Proposition

For any $\varepsilon > 0$ and $t \in \mathbb{R}$, there exists $m = m(\varepsilon, t) \in \mathbb{N}$ such that for any $\widehat{D} \in \mathcal{D}_\mu^H$, the projection $P_m : V \to V_m := \mathrm{span}[w_1, \ldots, w_m]$ satisfies the following properties:

$$\{P_m U(t,\tau)D(\tau) : \tau \le \tau_1(\widehat{D}, t)\} \textit{ is bounded in } V,$$

and

$$\|(I - P_m)U(t,\tau)u_\tau\| < \varepsilon \quad \textit{for any } \tau \le \tau_1(\widehat{D}, t), \; u_\tau \in D(\tau),$$

`Proof:` Recall the strong estimates we had...

$$\forall t \in \mathbb{R},\ \widehat{D} \in \mathcal{D}_{\mu}^{H},\ \exists \tau_1(\widehat{D}, t) < t - 2 \text{ s. t. } \forall \tau \leq \tau_1(\widehat{D}, t),\ u_\tau \in D(\tau)$$

$$\begin{aligned} |u(r;\tau,u_\tau)|^2 &\leq R_1^2(t) \quad \forall\, r \in [t-2, t], \\ \|u(r;\tau,u_\tau)\|^2 &\leq R_2^2(t) \quad \forall\, r \in [t-1, t], \\ \nu \int_{t-1}^{t} |Au(\theta;\tau,u_\tau)|^2 \, d\theta &\leq R_3^2(t), \end{aligned}$$

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\end{aligned}
$$

where

$$
\begin{aligned}
R_1^2(t) &= 1 + e^{-\mu(t-2)}(2\nu\lambda_1 - \mu)^{-1} \int_{-\infty}^{t} e^{\mu\theta} |f(\theta)|^2\, d\theta,\\
R_2^2(t) &= \nu^{-1} \left(R_1^2(t) + (\nu^{-1}\lambda_1^{-1} + 2) \int_{t-2}^{t} |f(\theta)|^2\, d\theta \right)\\
&\quad \times \exp\left[2\nu^{-1} C^{(\nu)} R_1^2(t) \left(R_1^2(t) + \nu^{-1}\lambda_1^{-1} \int_{t-2}^{t} |f(\theta)|^2\, d\theta \right) \right],\\
R_3^2(t) &= R_2^2(t) + 2\nu^{-1} \int_{t-1}^{t} |f(\theta)|^2\, d\theta + 2C^{(\nu)} R_1^2(t) R_2^4(t).
\end{aligned}
$$

$\{w_j\}_{j\geq 1}$ special basis $\Rightarrow P_m$ non-expansive in V
$\Rightarrow \{P_m U(t,\tau)D(\tau) : \tau \leq \tau_1(\widehat{D},t)\}$ bounded in V $\forall m \geq 1$.

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$q_m(r) = u(r) - P_m u(r)$ and the second energy equality

$$\frac{1}{2}\frac{d}{dr}\|q_m(r)\|^2 + \nu|Aq_m(r)|^2 = -b(u(r), u(r), Aq_m(r)) + (f(r), Aq_m(r))$$

$$\leq \frac{\nu}{2}|Aq_m(r)|^2 + \frac{1}{\nu}|f(r)|^2 + \frac{C_1^2}{\nu}R_1(t)R_2^2(t)|Au(r)| \text{ a.e. } t-1 < r < t.$$

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$|Aq_m(r)|^2 \geq \lambda_{m+1}\|q_m(r)\|^2$, implies that (a.e. $t-1 < r < t$)

$$\frac{d}{dr}\|q_m(r)\|^2 + \nu\lambda_{m+1}\|q_m(r)\|^2 \leq 2\nu^{-1}|f(r)|^2 + 2C_1^2\nu^{-1}R_1(t)R_2^2(t)|Au(r)|$$

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\begin{aligned}
& e^{\nu\lambda_{m+1}t}\|q_m(t)\|^2 \\
\leq & e^{\nu\lambda_{m+1}(t-1)}\|q_m(t-1)\|^2 + 2\nu^{-1}\int_{t-1}^{t} e^{\nu\lambda_{m+1}r}|f(r)|^2\,dr \\
& + 2C_1^2\nu^{-1}R_1(t)R_2^2(t)\int_{t-1}^{t} e^{\nu\lambda_{m+1}r}|Au(r)|\,dr \\
\leq & e^{\nu\lambda_{m+1}(t-1)}\|u(t-1)\|^2 + 2\nu^{-1}\int_{t-1}^{t} e^{\nu\lambda_{m+1}r}|f(r)|^2\,dr \\
& + 2C_1^2\nu^{-1}R_1(t)R_2^2(t)\left(\int_{t-1}^{t} e^{2\nu\lambda_{m+1}r}\,dr\right)^{1/2}\left(\int_{t-1}^{t}|Au(r)|^2\,dr\right)^{1/2} \\
\leq & e^{\nu\lambda_{m+1}(t-1)}R_2^2(t) + 2\nu^{-1}\int_{t-1}^{t} e^{\nu\lambda_{m+1}r}|f(r)|^2\,dr \\
& + 2C_1^2\nu^{-3/2}R_1(t)R_2^2(t)R_3(t)(2\nu\lambda_{m+1})^{-1/2}e^{\nu\lambda_{m+1}t}.
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& + 2C_1^2\nu^{-1}R_1(t)R_2^2(t)\int_{t-1}^{t} e^{\nu\lambda_{m+1}r}|Au(r)|\,dr \\
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\end{aligned}
$$

Since $\lambda_m \to \infty$ as $m \to \infty$, $\exists m = m(\varepsilon, t) \in \mathbb{N}$ s.t.
$\|(I - P_m)U(t,\tau)u_\tau\| < \varepsilon\, \forall \tau \leq \tau_1(\widehat{D}, t)$, $u_\tau \in D(\tau)$.

Navier-Stokes eqns with delay terms

- T. Caraballo and J. Real, Navier-Stokes equations with delays, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **457** (2001), 2441–2453.
- T. Caraballo and J. Real, Asymptotic behaviour of two-dimensional Navier-Stokes equations with delays, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **459** (2003), 3181–3194.
- T. Caraballo and J. Real, Attractors for 2D-Navier-Stokes models with delays, *J. Differential Equations* **205** (2004), 271–297.

The functional Navier-Stokes problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f(t) + g(t, u_t) \quad \text{in} \ \ \Omega \times (\tau, \infty), \\ \mathrm{div}\ u \ = 0 \quad \text{in} \ \ \Omega \times (\tau, \infty), \\ u = 0 \quad \text{on} \ \ \partial\Omega \times (\tau, \infty), \\ u(x, \tau) = u^\tau(x), \quad x \in \Omega, \\ u(x, \tau + s) = \phi(x, s), \quad x \in \Omega, \ s \in (-h, 0), \end{cases}$$

u_t the function defined a.e. on $(-h, 0)$ by the relation $u_t(s) = u(t + s)$, a.e. $s \in (-h, 0)$.

$C_H = C([-h,0];H)$ with norm $|\varphi|_{C_H} = \max_{s\in[-h,0]} |\varphi(s)|$,
$L^2_X = L^2(-h,0;X)$ for $X = H,\ V$.

$g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfies

(I) $\forall \xi \in C_H$, $\mathbb{R} \ni t \mapsto g(t,\xi) \in (L^2(\Omega))^2$ is measurable,

(II) $g(t,0) = 0$, for all $t \in \mathbb{R}$,

(III) $\exists L_g > 0$ s.t. $\forall t \in \mathbb{R}$, ξ, $\eta \in C_H$,

$$|g(t,\xi) - g(t,\eta)| \leq L_g |\xi - \eta|_{C_H},$$

(IV) $\exists C_g > 0$ s.t. $\forall \tau \leq t$, u, $v \in C([\tau - h, t]; H)$,

$$\int_\tau^t |g(s,u_s) - g(s,v_s)|^2\, ds \leq C_g^2 \int_{\tau-h}^t |u(s) - v(s)|^2\, ds.$$

Observe that $(I)-(III)$ imply that given $T>\tau$ and $u \in C([\tau-h,T];H)$, the function $g_u:[\tau,T]\to(L^2(\Omega))^2$ defined by $g_u(t)=g(t,u_t)$ for all $t\in[\tau,T]$, is measurable and, in fact, belongs to $L^\infty(\tau,T;(L^2(\Omega))^2)$.

Observe that $(I) - (III)$ imply that given $T > \tau$ and $u \in C([\tau - h, T]; H)$, the function $g_u : [\tau, T] \to (L^2(\Omega))^2$ defined by $g_u(t) = g(t, u_t)$ for all $t \in [\tau, T]$, is measurable and, in fact, belongs to $L^\infty(\tau, T; (L^2(\Omega))^2)$.

Then, thanks to (IV), the mapping

$$\mathcal{G} : u \in C([\tau - h, T]; H) \to g_u \in L^2(\tau, T; (L^2(\Omega))^2)$$

has a unique extension to a mapping $\widetilde{\mathcal{G}}$ which is uniformly continuous from $L^2(\tau - h, T; H)$ into $L^2(\tau, T; (L^2(\Omega))^2)$. From now on, we will denote $g(t, u_t) = \widetilde{\mathcal{G}}(u)(t)$ for each $u \in L^2(\tau - h, T; H)$, and thus property (IV) will also hold for all u, $v \in L^2(\tau - h, T; H)$.

Definition

A weak solution $u \in L^2(\tau - h, T; H) \cap L^2(\tau, T; V) \cap L^\infty(\tau, T; H)$ for all $T > \tau$, with $u(\tau) = u^\tau$, $u(t) = \phi(t - \tau)$ a.e. $t \in (\tau - h, \tau)$,

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$$\frac{d}{dt}(u(t), v) + \nu\langle Au(t), v\rangle + b(u(t), u(t), v) = \langle f(t), v\rangle + (g(t, u_t), v).$$

Remark

u weak solution, then $u' \in L^2(\tau, T; V')$, *so* $u \in C([\tau, \infty); H)$.
Energy equality:

$$|u(t)|^2 + 2\nu \int_s^t \|u(r)\|^2 dr = |u(s)|^2 + 2\int_s^t \left[\langle f(r), u(r)\rangle + (g(r, u_r), u(r))\right] dr$$

for all $\tau \le s \le t$.

Definition

A strong solution is a weak solution u such that $u \in L^2(\tau, T; D(A)) \cap L^\infty(\tau, T; V)$ for all $T > \tau$.

Remark

If $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^2)$ and u is a strong solution, then $u' \in L^2(\tau, T; H)$ for all $T > \tau$, and so $u \in C([\tau, \infty); V)$.
Second energy equality:

$$\begin{aligned} & \|u(t)\|^2 + 2\nu \int_s^t |Au(r)|^2 \, dr + 2 \int_s^t b(u(r), u(r), Au(r)) \, dr \\ = \; & \|u(s)\|^2 + 2 \int_s^t (f(r) + g(r, u_r), Au(r)) \, dr \quad \forall \tau \le s \le t. \end{aligned}$$

Theorem

Let us consider $u^\tau \in H$, $\phi \in L^2_H$, $f \in L^2_{loc}(\mathbb{R}; V')$, *and* $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ *satisfying (I)–(IV).*

Then, for each $\tau \in \mathbb{R}$, *there exists a unique weak solution* u.

Moreover, if $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^2)$, *then*

(a) $u \in C([\tau+\varepsilon, T]; V) \cap L^2(\tau+\varepsilon, T; D(A))$ *for all* $T > \tau + \varepsilon > \tau$.

(b) *If* $u^\tau \in V$, u *is in fact a strong solution.*

We may consider the Banach space C_H,
and the Hilbert space $M_H^2 = H \times L_H^2$ with associated norm

$$\|(u^\tau, \phi)\|_{M_H^2}^2 = |u^\tau|^2 + \int_{-h}^{0} |\phi(s)|^2\, ds \quad \text{for } (u^\tau, \phi) \in M_H^2.$$

A fifth assumption on g and f for asymptotic estimates:

(V) Assume that $\nu\lambda_1 > C_g$, and $\exists \eta \in (0, 2(\nu\lambda_1 - C_g))$ s.t. for any $u \in L^2(\tau - h, t; H)$,

$$\begin{aligned} \int_\tau^t e^{\eta s} |g(s, u_s)|^2\, ds &\leq C_g^2 \int_{\tau-h}^{t} e^{\eta s} |u(s)|^2\, ds \quad \forall \tau \leq t, \\ \int_{-\infty}^{0} e^{\eta s} \|f(s)\|_*^2\, ds &< \infty. \end{aligned}$$

Definition

For any $\eta > 0$, we will denote by $\mathcal{D}_\eta(C_H)$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C_H)$ such that

$$\lim_{\tau \to -\infty} \left(e^{\eta\tau} \sup_{\varphi \in D(\tau)} |\varphi|^2_{C_H} \right) = 0.$$

Analogously, we will denote by $\mathcal{D}_\eta(M^2_H)$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(M^2_H)$ such that

$$\lim_{\tau \to -\infty} \left(e^{\eta\tau} \sup_{(w,\varphi) \in D(\tau)} \|(w,\varphi)\|^2_{M^2_H} \right) = 0.$$

Theorem

$f \in L^2_{loc}(\mathbb{R}; V')$ and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ satisfy (I)–(V).

Then, $\exists \, \{\mathcal{A}_{\mathcal{D}_F(C_H)}(t)\}_{t\in\mathbb{R}}$, $\{\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t)\}_{t\in\mathbb{R}}$, $\{\mathcal{A}_{\mathcal{D}_F(M^2_H)}(t)\}_{t\in\mathbb{R}}$, and $\{\mathcal{A}_{\mathcal{D}_\eta(M^2_H)}(t)\}_{t\in\mathbb{R}}$, in C_H and M^2_H respectively.

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$$\begin{aligned} \mathcal{A}_{\mathcal{D}_F(C_H)}(t) &\subset \mathcal{A}_{\mathcal{D}_\eta(C_H)}(t), \text{ and } \mathcal{A}_{\mathcal{D}_F(M^2_H)}(t) \subset \mathcal{A}_{\mathcal{D}_\eta(M^2_H)}(t) \ \forall\, t \in \mathbb{R}, \\ j(\mathcal{A}_{\mathcal{D}_F(C_H)}(t)) &\subset \mathcal{A}_{\mathcal{D}_F(M^2_H)}(t) \quad \forall\, t \in \mathbb{R}, \quad \text{and} \\ j(\mathcal{A}_{\mathcal{D}_\eta(C_H)}(t)) &= \mathcal{A}_{\mathcal{D}_\eta(M^2_H)}(t) \quad \forall\, t \in \mathbb{R}, \end{aligned}$$

[j the canonical injection of C_H into M^2_H : $j(\varphi) = (\varphi(0), \varphi)$.]

If f also satisfies $\sup_{s\le 0}\left(e^{-\eta s}\int_{-\infty}^{s} e^{\eta\theta}\|f(\theta)\|_*^2\,d\theta\right) < \infty$, the inclusions are in fact equalities.

A modification of Navier-Stokes eqns:

W. Liu, *Discrete Contin. Dyn. Syst. Ser. B* **2** (2002), 47–56.
A time-delayed term in the Burgers' equation was considered
G. Planas and E. Hernández, *Discrete Contin. Dyn. Syst. Ser. B* **21** (2008), 1245–1258.

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u(t-\rho(t))\cdot\nabla)u + \nabla p = f(t) + g(t, u_t) \text{ in } \Omega\times(\tau,\infty) \\ \operatorname{div} u = 0 \text{ in } \Omega\times(\tau,\infty), \\ u = 0 \text{ on } \partial\Omega\times(\tau,\infty), \\ u(x,\tau) = u^\tau(x) \text{ in } \Omega, \\ u(x,\tau+s) = \phi(x,s) \text{ in } \Omega\times(-h,0), \end{cases}$$

where $\Omega\subset\mathbb{R}^2$, $\tau\in\mathbb{R}$, $h>0$
u_t denotes the delay function $u_t(s) = u(t+s)$
$\rho\in C^1(\mathbb{R};[0,h])$ with $\rho'(t)\le\rho^*<1\ \forall t\in\mathbb{R}$.

Interesting features and goal:

("Small delays don't matter" ... unless in the nonlinearity)

- $u' \in L^{4/3}(V')$ even in 2D
- Lack of uniqueness and more troubles for dynamical systems: see Ball (1997), Kapustyan & Valero (2007), MR & Robinson (2003)...
- Goal here: under slightly better conditions, uniqueness, and (pullback) attractors
- Remarkable fact: special type of (tempered) universes

TRILINEAR TERM AND WEAK SOLUTION:

$$|b(u,v,w)| \leq C|u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2} \qquad \forall\, u,v,w \in V.$$

Suppose that $u^\tau \in H$, $\phi \in L^2_V$, and $f \in L^2_{loc}(\mathbb{R}; V')$.

Remark

$$|b(u(t-\rho(t)), u(t), v)| \leq \widetilde{C}\|u(t-\rho(t))\|\|u(t)\|^{1/2}|u(t)|^{1/2}\|v\|, \ \forall v \in V$$

$$1/2 + 1/4 = 3/4 \quad \Rightarrow \quad B(u(\cdot - \rho(\cdot)), u(\cdot)) \in L^{4/3}(\tau, T; V').$$

$u' \in L^{4/3}(\tau, T; V') \Rightarrow$
$u \in C([\tau, T]; V') \quad and \quad u \in C_w([\tau, T]; H)\ \forall T > \tau$
(whence initial datum $u^\tau \in H$ *meaningful).*

Existence and uniqueness:

Theorem

(Existence of weak solution by compactness method) $u^\tau \in H$, $\phi \in L^2_V$, $f \in L^2_{loc}(\mathbb{R}; V')$, *and* $g : \mathbb{R} \times C_H \to (L^2(\Omega))^2$ *satisfying assumptions (H1)–(H4). Then, there exists at least one weak solution* $u(\cdot; \tau, u^\tau, \phi)$.

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Remark

(Uniqueness improving the initial data) $u^\tau \in H$ and $\phi \in L^2_V \cap L^\infty_H$. Then

$$|b(u(t-\rho(t)), u(t), v)| \leq C|u(t-\rho(t))|^{1/2}\|u(t-\rho(t))\|^{1/2}\|v\| \times |u(t)|^{1/2}\|u(t)\|^{1/2} \Rightarrow$$

$B(u(\cdot - \rho(\cdot)), u(\cdot)) \in L^2(\tau, T; V')$ for all $T > \tau$, and so $u' \in L^2(\tau, T; V')$

$\Rightarrow$ uniqueness + energy equality

An appropriate concept of (tempered) universe

Definition

We will denote by $\mathcal{D}_\eta^{H,L^2_H}(H \times (L^2_V \cap L^\infty_H))$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(H \times (L^2_V \cap L^\infty_H))$ such that

$$\lim_{\tau \to -\infty} \left(e^{\eta\tau} \sup_{(\zeta,\varphi)\in D(\tau)} (|\zeta|^2 + \|\varphi\|^2_{L^2_H}) \right) = 0.$$

Observe that the above definition does not make the most use of the natural norm of (ζ, φ) in $H \times (L^2_V \cap L^\infty_H)$, but just in $H \times L^2_H$.

Navier-Stokes-Voigt

$\Omega \subset \mathbb{R}^3$ bounded domain with smooth (e.g., C^2) $\partial\Omega$.

$$\begin{cases} \dfrac{\partial}{\partial t}\left(u - \alpha^2 \Delta u\right) - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f(t) \text{ in } \Omega \times (\tau, \infty), \\ \operatorname{div} u = 0 \text{ in } \Omega \times (\tau, \infty), \\ u = 0 \text{ on } \partial\Omega \times (\tau, \infty), \\ u(x, \tau) = u_\tau(x), \ x \in \Omega, \end{cases}$$

a length scale parameter $\alpha > 0$, characterizing the elasticity of the fluid (in the sense that the ratio α^2/ν describes the reaction time that is required for the fluid to respond to the applied force)

Motivation NSV

-The Navier-Stokes-Voigt (NSV) model of viscoelastic incompresible fluid was introduced by Oskolkov [LOMI 1973]
-gives an approximate description of the Kelvin-Voigt fluid, [Oskolkov, 1985]
-proposed as a regularization of the 3D-Navier-Stokes with purpose of direct numerical simulations [Cao, Lunasin, Titi, 2006]
-The extra regularizing term $-\alpha^2 \Delta u_t$ changes the parabolic character of the equation, which makes it so that in 3D the problem is well-posed (forward and backward), but one does not observe any immediate smoothing of the solution
-the inviscid equation is the simplified Bardina subgrid scale model of turbulence (relation studied in [Cao, Lunasin, Titi, 2006]
-global compact attractor and estimates on fractal and Hausdorff dim by Kalantarov and Titi [LOMI, 1988; J. Nonlinear Sci. 2009]
-uniform attractors by Yue and Zhong [DCDS-B, 2011]

The autonomous equation $u + \alpha^2 Au = g$

For $g \in V'$, $\exists!$ solution u_g (Lax-Milgram)
The mapping $\mathcal{C} : u \in V \mapsto u + \alpha^2 Au \in V'$ is linear and bijective.
$\mathcal{C}^{-1}(H) = D(A)$

Definition

u is a weak solution if u belongs to $L^2(\tau, T; V)$ for all $T > \tau$, and

$$\frac{d}{dt}(u(t) + \alpha^2 Au(t)) + \nu Au(t) + B(u(t)) = f(t), \quad \text{in } \mathcal{D}'(\tau, \infty; V'),$$

$$u(\tau) = u_\tau .$$

Remark

If $u \in L^2(\tau, T; V)$ for all $T > \tau$ and satisfies the eqn, then

$$v(\cdot) = u(\cdot) + \alpha^2 Au(\cdot) \in L^2(\tau, T; V') \text{ and } v' = \frac{dv}{dt} \in L^1(\tau, T; V').$$

So, $v \in C([\tau, \infty); V')$, and $u \in C([\tau, \infty); V)$.
In particular, $u(\tau) = u_\tau$ has a sense.
Moreover, then, $v' \in L^2(\tau, T; V')$, and $u' \in L^2(\tau, T; V)$.
Thus, *u is a weak solution* **iff** *$u \in C([\tau, \infty); V)$, $u' \in L^2(\tau, T; V)$ for all $T > \tau$, and*

$$u(t)+\alpha^2 Au(t)+\int_\tau^t (\nu Au(s)+B(u(s)))\, ds = u_\tau+\alpha^2 Au_\tau+\int_\tau^t f(s)ds.$$

Lemma

If u is a weak solution, then

$$\frac{1}{2}\frac{d}{dt}(|u(t)|^2 + \alpha^2\|u(t)\|^2) + \nu\|u(t)\|^2 = \langle f(t), u(t)\rangle, \quad a.e.\ t > \tau.$$

Theorem

Let $f \in L^2_{loc}(\mathbb{R}; V')$ *be given. Then, for each* $\tau \in \mathbb{R}$ *and* $u_\tau \in V$*, there exists a unique weak solution.*
Moreover, if $f \in L^2_{loc}(\mathbb{R}; H)$ *and* $u_\tau \in D(A)$*, then*

$$u \in C([\tau, \infty); D(A)), \quad u' \in L^2(\tau, T; D(A)) \text{ for all } T > \tau,$$

and

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|^2 + \alpha^2 |Au(t)|^2) + \nu |Au(t)|^2 + (B(u(t)), Au(t)) = (f(t), Au(t)),$$

Existence of minimal pullback attractors in V norm

Lemma

Assume that $f \in L^2_{loc}(\mathbb{R}; V')$ *and* $u_\tau \in V$. *Then, for any*

$$0 < \sigma < 2\nu(\lambda_1^{-1} + \alpha^2)^{-1},$$

$$\begin{aligned} & \|u(t)\|^2 + \varepsilon\alpha^{-2} \int_\tau^t e^{\sigma(s-t)} \|u(s)\|^2 \, ds \\ \leq \; & (1 + \alpha^{-2}\lambda_1^{-1}) e^{\sigma(\tau - t)} \|u_\tau\|^2 + \alpha^{-2}\varepsilon^{-1} \int_\tau^t e^{\sigma(s-t)} \|f(s)\|_*^2 \, ds \end{aligned}$$

for all $t \geq \tau$, *where* $\varepsilon = \nu - \frac{\sigma}{2}(\lambda_1^{-1} + \alpha^2)$.

Definition

For $\sigma \in (0, 2\nu(\lambda_1^{-1} + \alpha^2)^{-1})$ s.t. $\int_{-\infty}^0 e^{\sigma s} \|f(s)\|_*^2 ds < \infty$, we will denote by $\mathcal{D}_\sigma^V$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(V)$ s.t. $\lim_{\tau \to -\infty} (e^{\sigma\tau} \sup_{v \in D(\tau)} \|v\|^2) = 0$.

Attraction in $D(A)$ norm

Lemma

Assume that $f \in L^2_{loc}(\mathbb{R}; H)$ *s.t.* $\sup_{r\le 0}\int_{r-1}^{r}\|f(s)\|_*^2 ds$. *Then, if*

$$0<\sigma<2\nu(\lambda_1^{-1}+\alpha^2)^{-1}, \quad and \quad 0<\underline{\sigma}<\sigma/3,$$

$$\|u(t)\|^2+\alpha^2|Au(t)|^2 \le e^{\sigma(\tau-t)}(\|u_\tau\|^2+\alpha^2|Au_\tau|^2)+2\varepsilon^{-1}$$
$$\times\int_\tau^t e^{\sigma(s-t)}|f(s)|^2 ds + 4C_\varepsilon C_{\underline{\sigma}}^3(\sigma-3\underline{\sigma})^{-1}\left(e^{-3\underline{\sigma}(t-\tau)}\|u_\tau\|^6+M_{t,\underline{\sigma}}^3\right)$$

for all $t\ge\tau$, *where* $M_{t,\underline{\sigma}}=\sup_{r\le t}\int_{-\infty}^{r} e^{\underline{\sigma}(s-r)}\|f(s)\|_*^2 ds$.

Definition

For any σ, $\underline{\sigma}>0$, consider the universe $\mathcal{D}_\sigma^{D(A)}\cap\mathcal{D}_{\underline{\sigma}}^{V}$ formed by $\widehat{D}=\{D(t): t\in\mathbb{R}\}\subset\mathcal{P}(D(A))$ such that

$$\lim_{\tau\to-\infty}\left(e^{\sigma\tau}\sup_{v\in D(\tau)}|Av|^2\right)=\lim_{\tau\to-\infty}\left(e^{\underline{\sigma}\tau}\sup_{v\in D(\tau)}\|v\|^2\right)=0.$$