Theoretical and Numerical Results on Modeling of Oceanic Turbulent Mixing Layers

T. Chacón Rebollo, M. Gómez Mármol, S. Rubino

Universidad de Sevilla, Dpto. de Ecuaciones Diferenciales y Análisis Numérico

BCAM Seminar (Basque Center for Applied Mathematics)

December 12, 2012

basque center for applied mathematics

Outline

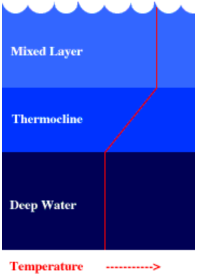
1. Introduction of the mean features of oceanic turbulent mixing layers.
2. Introduction of turbulent mixing-layer models, based on the **gradient Richardson number**.
3. Some results on the mathematical analysis of the proposed models.
4. Comparison with a more general model, based on the **Primitive Equations**.
5. Numerical tests.

Oceanic Turbulent Mixing Layers

- The right computation of the **Sea Surface Temperature** (**SST**) is needed to predict different aspects related to oceanic biosystems and global climate changes.
- This problem shows its importance especially in **tropical regions**, where the high temperatures generate well-developed surface mixing layers.
- Many physical oceanographers started to study this problem during the *80's*, formulating the so-called "standard models".
- Turbulent mixing-layer models are usually vertical first-order closure models, and they do not include pressure gradients.

Structure of Mixing Layer

- The wind-stress generates strong mixing phenomena in a layer located below the ocean surface: the **mixing layer**.
- The mixing layer is divided in two sections: the **mixed layer** and the **pycnocline**.
- Mixed layer: almost constant density.
- Pycnocline: remarkable gradient of density.



Mathematical Formulation

1. Physically, vertical fluxes strongly dominate the mixing layer.
   $\Rightarrow$ We assume velocity and density horizontally homogeneous.
2. We set:

$$\mathbf{U} = (u(z,t), v(z,t), 0), \quad \rho = \rho(z,t).$$

3. We neglect the Coriolis force, that is a good approximation for equatorial regions.
4. In tropical seas:

$$\rho = \rho(\text{Temperature}) \longrightarrow \text{ (state equation)}$$

Mathematical Formulation

The variables u, v, ρ satisfy the Reynolds-averaged equations:

$$\begin{cases} \partial_t u - a_1 \partial_{zz} u = -\partial_z \langle u' \, w' \rangle - D_1 \\ \partial_t v - a_1 \partial_{zz} v = -\partial_z \langle v' \, w' \rangle - D_2 \\ \partial_t \rho - a_2 \partial_{zz} \rho = -\partial_z \langle \rho' \, w' \rangle \end{cases} \quad (1)$$

- $(a_1, a_2) =$ laminar viscosity and diffusion
- $(D_1, D_2) = \nabla_H p =$ imposed horizontal pressure gradient

To close the problem, we use the concept of *eddy* viscosity and diffusion.

$\Rightarrow$ Application of a turbulence model based on the **gradient Richardson number**.

Richardson number-based Model

We set:

$$-\langle u'\, w' \rangle = \nu_{T1} \partial_z u, \tag{2}$$

$$-\langle v'\, w' \rangle = \nu_{T1} \partial_z v, \tag{3}$$

$$-\langle \rho'\, w' \rangle = \nu_{T2} \partial_z \rho. \tag{4}$$

The eddy coefficients ν_{T1} and ν_{T2} are expressed as functions of the **gradient Richardson number** R, defined as:

$$R = -\frac{g}{\rho_r} \frac{\partial_z \rho}{(\partial_z u)^2 + (\partial_z v)^2}. \tag{5}$$

- <u>Case $R < 0$</u> $(\partial_z \rho > 0)$:

  Unstable density configurations.

Richardson number-based Model

IBVP for the analysis of the Mixing Layer

$$\left\{\begin{array}{l}\partial_t u - \partial_z(\nu_1 \partial_z u) = -D_1,\\ \partial_t v - \partial_z(\nu_1 \partial_z v) = -D_2,\\ \partial_t \rho - \partial_z(\nu_2 \partial_z \rho) = 0, \text{ for } t \geq 0 \text{ and } -h \leq z \leq 0,\\ u = u_b(t),\ v = v_b(t),\ \rho = \rho_b(t), \text{ for } z = -h,\\ \nu_1 \partial_z u = (\rho_a/\rho_r) V_x(t),\ \nu_1 \partial_z v = (\rho_a/\rho_r) V_y(t),\ \nu_2 \partial_z \rho = Q(t), \text{ for } z = 0,\\ u = u_0(z),\ v = v_0(z),\ \rho = \rho_0(z), \text{ for } t = 0.\end{array}\right. \tag{6}$$

$$\nu_1 = a_1 + \nu_{T1}, \quad \nu_2 = a_2 + \nu_{T2};$$

$$(V_x(t), V_y(t)) = C_D |\mathbf{U}^a(t)| \mathbf{U}^a(t), \tag{7}$$

$$\mathbf{U}^a(t) = (u_a(t), v_a(t)) = \text{air velocity}, \quad C_D (= 1.2 \cdot 10^{-3}) = \text{friction coef.} \tag{8}$$

A. C. Bennis, T. Chacón Rebollo, M. Gómez Mármol, R. Lewandowski, *Numerical modelling of algebraic closure models of oceanic turbulent mixing layers*, (2010).

Several Models for the Turbulent Coefficients

We set:

$$\nu_1 = f_1(R), \ \nu_2 = f_2(R)$$

- **Pacanowski-Philander**

$$f_1(R) = a_1 + \frac{b_1}{(1+5R)^2}, \ f_2(R) = a_2 + \frac{f_1(R)}{1+5R} \qquad (9)$$

- **Gent**

$$f_1(R) = a_1 + \frac{b_1}{(1+10R)^2}, \ f_2(R) = a_2 + \frac{b_2}{(1+5R)^3} \qquad (10)$$

- **Non-standard model**

$$f_1(R) = a_1 + \frac{b_1}{(1+5R)^2}, \ f_2(R) = a_2 + \frac{f_1(R)}{(1+5R)^2} \qquad (11)$$

Equilibrium Solutions

Steady Problem:

$$\begin{cases} \partial_z(f_1(R^e)\partial_z u^e) = D_1 \\ \partial_z(f_1(R^e)\partial_z v^e) = D_2 \\ \partial_z(f_2(R^e)\partial_z \rho^e) = 0 \end{cases} \qquad (12)$$

Integrating (12) with respect to z, we obtain:

$$\begin{cases} \partial_z u^e = \left(D_1 z + V_x^e \rho_a/\rho_r\right)/f_1(R^e) \\ \partial_z v^e = \left(D_2 z + V_y^e \rho_a/\rho_r\right)/f_1(R^e) \\ \partial_z \rho^e = Q^e/f_2(R^e) \end{cases} \qquad (13)$$

By definition of R, we deduce the implicit equation for R^e:

$$R = G(z)\frac{[f_1(R)]^2}{f_2(R)}, \qquad (14)$$

$$G(z) = -\frac{g}{\rho_r}\frac{Q^e}{\left(D_1 z + V_x^e \rho_a/\rho_r\right)^2 + \left(D_2 z + V_y^e \rho_a/\rho_r\right)^2}. \qquad (15)$$

Equilibrium Solutions

Integrating (13) with respect to z, we deduce the **equilibrium solutions**:

$$\begin{cases} u^e(z) = u_b^e + D_1 \displaystyle\int_{-h}^{z} \frac{s}{f_1(R^e(s))} ds + \frac{V_x^e \rho_a}{\rho_0} \int_{-h}^{z} \frac{1}{f_1(R^e(s))} ds \\ v^e(z) = v_b^e + D_2 \displaystyle\int_{-h}^{z} \frac{s}{f_1(R^e(s))} ds + \frac{V_y^e \rho_a}{\rho_0} \int_{-h}^{z} \frac{1}{f_1(R^e(s))} ds \\ \rho^e(z) = \rho_b^e + Q^e \displaystyle\int_{-h}^{z} \frac{1}{f_2(R^e(s))} ds \end{cases} \tag{16}$$

- $\underline{D_1 = D_2 = 0}$
  Linear profiles for velocity and density.

Existence of Unsteady Solutions

Theorem

If $\mathbf{C} = (V_x, V_y, Q)(t) \in [L^2(0,T)]^3, \mathbf{U}_b = (u_b, v_b, \rho_b)(t) \in [C^0(0,T)]^3, \mathbf{U}_0 = (u_0, v_0, \rho_0)(z) \in [H^1(I)]^3$, *and these quantities are close enough to the corresponding quantities at the equilibrium, problem (6) with smoothly extended viscosities* ν_1, ν_2 *admits a unique solution* $\mathbf{U}$ *in an open neighborhood of the equilibrium* $\mathbf{U}^e$*, satisfying the estimate:*

$$||\mathbf{U} - \mathbf{U}^e||_{L^2(0,T;[H^2(I)]^3)} + ||\partial_t(\mathbf{U} - \mathbf{U}^e)||_{L^2(0,T;[L^2(I)]^3)} \leq$$

$$C(||\mathbf{C} - \mathbf{C}^e||_{L^2(0,T)} + ||\mathbf{U}_b - \mathbf{U}_b^e||_{L^\infty(0,T)} + ||\mathbf{U}_0 - \mathbf{U}^e||_{H^1(I)}) \quad (17)$$

where C *is a positive constant independent of* $\mathbf{U}$.

Sketch of the Proof: Inverse Function Theorem (Banach Spaces).

Non-linear Stability of Continuous Equilibria

Theorem

Under the hypotheses of the existence theorem, and for small enough data, the equilibrium solutions of problem (6) with smoothly extended viscosities ν_1, ν_2 are non-linearly exponentially asymptotically stable, in the sense that:

$$||\mathbf{U}'(t)||_{L^2(I)} \leq \mathrm{e}^{-\lambda t}||\mathbf{U}_0'||_{L^2(I)},$$

for some $\lambda > 0$, where $\mathbf{U}' = \mathbf{U} - \mathbf{U}^e$.

Sketch of the Proof: Standard Inequalities and Grönwall Lemma.

Conservative Finite Difference Scheme

Assume that the interval $[-h, 0]$ is divided into N subintervals of lenght $\Delta z = h/N$, with nodes $z_i = -h + i\Delta z$, $i = 0, \ldots, N$, and construct the FE space:

$$V_\Delta = \left\{ w_\Delta \in C^0([-h,0]) \text{ s.t. } w_{\Delta|_{]z_{i-1},z_i[}} \text{ is affine, } i = 1, \ldots, N;\ w_\Delta(-h) = 0 \right\}. \tag{18}$$

- **Eq. for u (Semi-Implicit Method)**: Obtain $u_\Delta \in u_b + V_\Delta$ s.t.

$$\int_{-h}^{0} \frac{u_\Delta^{n+1} - u_\Delta^n}{\Delta t} w_\Delta + \int_{-h}^{0} f_1(R_\Delta^n)\partial_z u_\Delta^{n+1} \partial_z w_\Delta = \frac{\rho_a}{\rho_r} V_x w_\Delta(0) - D_1 \int_{-h}^{0} w_\Delta. \tag{19}$$

- **FD Scheme for u**: For $i = 1, \ldots, N-1$,

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{f_1(R_{i-1/2}^n) u_{i-1}^{n+1} - \left[f_1(R_{i-1/2}^n) + f_1(R_{i+1/2}^n) \right] u_i^{n+1} + f_1(R_{i+1/2}^n) u_{i+1}^{n+1}}{(\Delta z)^2} = -D_1, \tag{20}$$

$$f_1(R_{N-1/2}^n) \frac{u_N^{n+1} - u_{N-1}^{n+1}}{\Delta z} = \frac{\rho_a}{\rho_r} V_x. \tag{21}$$

Analysis of the Numerical Scheme

Hypothesis:

$$f_1, f_2 \in W^{1,\infty}(\mathbb{R}^3) \cap C^1(\mathbb{R}^3),\ \exists\, 0 < \nu \le M \text{ s.t. } \nu \le f_1(R), f_2(R) \le M,\ \forall R \in \mathbb{R}. \tag{22}$$

Stability result:

Lemma

Under Hypothesis (22), the discrete unsteady solution $\mathbf{U}_\Delta$ (in its semi-implicit and implicit version) satisfy:

$$\|\mathbf{U}_\Delta\|_{L^\infty(L^2)} + \|\mathbf{U}_\Delta\|_{L^2(H^1)} + \|\partial_t \mathbf{U}_\Delta\|_{L^2(H^{-1})} \le C,$$

where C is a positive constant that only depends on the data (and not on Δz and Δt).

Discrete maximum principle:

Lemma

Under Hypothesis (22), if u_0, u_b and V_x are positive, and $D_1 \le 0$, then u_Δ is positive in $[-h, 0] \times [0, +\infty)$.

Analysis of Discrete Equilibria

Existence and uniqueness:

Theorem

Under Hypothesis (22), for small enough data, the steady version of the discrete problem admits a unique solution that satisfies:

$$C_1 \leq |\partial_z u_\Delta|, |\partial_z v_\Delta|, |\partial_z \rho_\Delta| \leq C_2.$$

Convergence and non-linear stability:

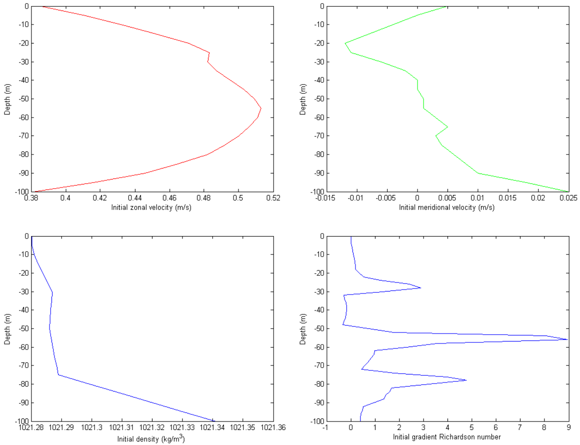
Theorem

Under Hypothesis (22), for small enough data:

$$\{(u_\Delta, v_\Delta, \rho_\Delta)\}_{\Delta z > 0} \longrightarrow (u^e, v^e, \rho^e) \text{ in } [H^1(I)]^3,$$

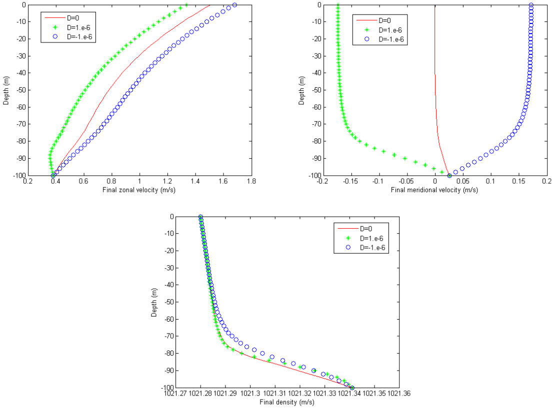
$$\limsup_{n \to +\infty} ||\mathbf{U}_\Delta^n - \Pi_\Delta \mathbf{U}^e||_{L^2(I)} \leq C \Delta z.$$

Initial Conditions with Unstable Density Profile



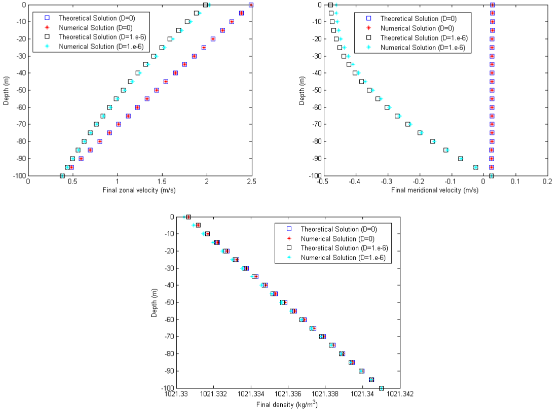
Formation of the Mixing Layer

$$T = 48h.$$



Equilibrium Solutions

$$\boxed{T = 10000h.}$$



Primitive Equations of the Ocean

- The **Primitive Equations** govern the behavior of oceanic flows for large time and horizontal spatial scales.
- Physical basic model to analyze global climate changes and oceanic biosystems.
- Reduced formulation$\rightarrow$PDEs for horizontal velocity, surface pressure and density.
  (**Boussinesq Equations** + **Hydrostatic Approximation**)

Setting of the Model

1. We consider the rigid-surface domain of the flow:
$$\omega = (0, L), \ L > 0.$$
2. We define the 2D domain of the flow:
$$\Omega = \left\{(x, z) \in \mathbb{R}^2 \text{ s.t. } x \in \omega, \ -h < z < 0\right\}.$$
3. We set:
$$\mathbf{U} = (u(x, z, t), w(x, z, t)), \ \rho = \rho(x, z, t), \ p = p(x, z, t).$$
4. We neglect the Coriolis force.

The anisotropy of the domain ($L >> h$) permits to apply the hydrostatic approximation:
$$\partial_z p = -\frac{\rho}{\rho_r} g.$$

Setting of the Model

The set of 2D Primitive Equations governing the mixing layer becomes :

$$\begin{cases} \nabla \cdot \mathbf{U} = 0, \\ \partial_t u + (\mathbf{U} \cdot \nabla) u - a_1 \Delta u + \partial_x p = -\nabla \cdot \langle \mathbf{U}' \, u' \rangle, \\ \partial_z p = -\frac{\rho}{\rho_r} g, \\ \partial_t \rho + (\mathbf{U} \cdot \nabla)\rho - a_2 \Delta \rho = -\nabla \cdot \langle \mathbf{U}' \, \rho' \rangle. \end{cases} \tag{23}$$

We set:

$$-\langle \mathbf{U}' \, u' \rangle = (\nu_h^t \, \partial_x u, \; \nu_v^t \, \partial_z u), \tag{24}$$

$$-\langle \mathbf{U}' \, \rho' \rangle = (k_h^t \, \partial_x \rho, \; k_v^t \, \partial_z \rho), \tag{25}$$

where:

(i) $\nu_h^t = (C_s \, \Delta x)^2 \, |\partial_x u|, \quad k_h^t = \frac{a_2}{a_1} \, \nu_h^t$ (Smagorinsky),

(ii) $\nu_v^t = \frac{b_1}{(1+5R)^2}, \quad k_v^t = \frac{a_1}{(1+5R)^2} + \frac{\nu_v^t}{(1+5R)^2}, \quad R = -\frac{g}{\rho_r} \frac{\partial_z \rho}{(\partial_z u)^2}$ (Richardson).

Reduced Formulation

- $\partial\Omega =$ boundary of $\Omega = \Gamma_b \cup \Gamma_s \cup \Gamma_{\ell 1} \cup \Gamma_{\ell 2}$.
- Vertical integration of the **hydrostatic pressure**:

$$\int_z^0 \partial_s p = -\frac{g}{\rho_r} \int_z^0 \rho \Rightarrow \boxed{p(x,z,t) = p_s(x,t) + \frac{g}{\rho_r} \int_z^0 \rho}$$

$$\partial_x p = \partial_x p_s + \frac{g}{\rho_r} \partial_x \int_z^0 \rho = \nabla_H(\text{Surf. Pres.}) + \text{Baroclinic Contr.}$$

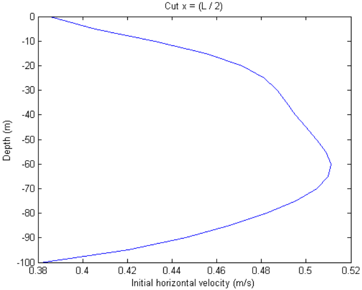
Reduced Formulation

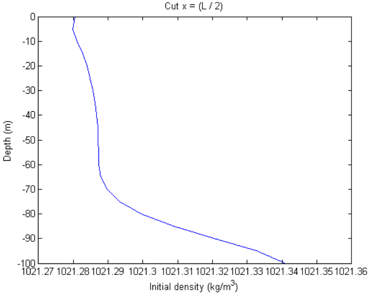
IBVP for the analysis of the Mixing Layer

$$\begin{cases} (a)\ \partial_t u + (\mathbf{U}\cdot\nabla)u - \partial_x(\nu_h\partial_x u) - \partial_z(\nu_v\partial_z u) + \partial_x p_s + (g/\rho_r)\partial_x \int_z^0 \rho = 0 & \text{in } \Omega\times]0,T[, \\ (b)\ \partial_t \rho + (\mathbf{U}\cdot\nabla)\rho - \partial_x(k_h\partial_x \rho) - \partial_z(k_v\partial_z \rho) = 0 & \text{in } \Omega\times]0,T[, \\ (c)\ \partial_z w = -\partial_x u & \text{in } \Omega\times]0,T[, \\ (d)\ u_{|\Gamma_b} = u_b,\ \rho_{|\Gamma_b} = \rho_b & \text{in } [0,T], \\ (e)\ w_{|\Gamma_b} = w_{|\Gamma_s} = 0 & \text{in } [0,T], \\ (f)\ u_{|\Gamma_{\ell 1}} = u_{|\Gamma_{\ell 2}},\ \rho_{|\Gamma_{\ell 1}} = \rho_{|\Gamma_{\ell 2}} & \text{in } [0,T], \\ (g)\ \nu_v\partial_z u_{|\Gamma_s} = (\rho_a/\rho_r)\,V_x,\ k_v\partial_z\rho_{|\Gamma_s} = Q & \text{in } [0,T], \\ (h)\ u(0) = \widehat{u}_0,\ \rho(0) = \widehat{\rho}_0 & \text{in } \Omega. \end{cases} \tag{26}$$

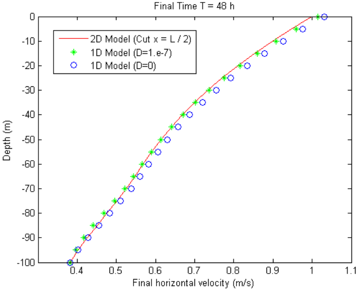
1. Spatial discretization: Galerkin FEM.
2. Temporal discretization: Semi-Implicit Euler Scheme.

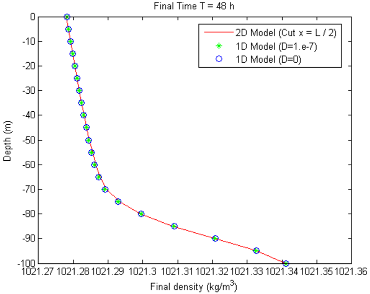
Initial Velocity and Density Profiles



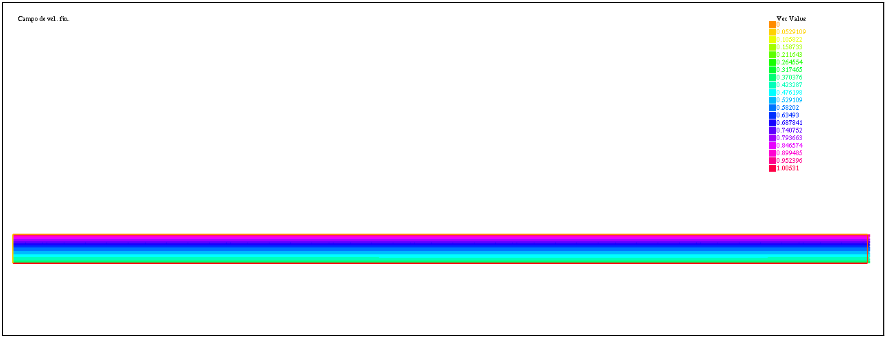


Comparison between Models 2D and 1D



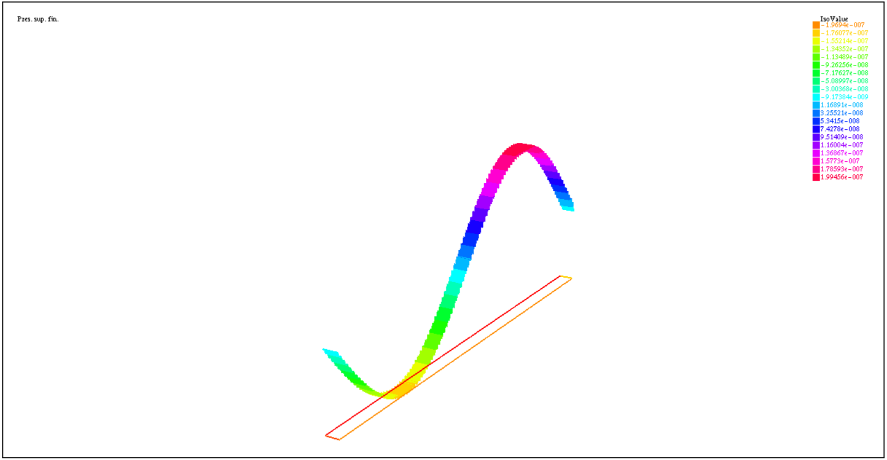


Final Fluid Flow 2D



Final Density 2D

Final Surface Pressure



Observations:

- The 2D model justifies the assumptions leading to the 1D model (lower computational cost).
- The introduction of an imposed pressure gradient in the 1D model permits to improve the accuracy in the computation of the velocity, if the initial conditions are not 1D.

References:

T. Chacón Rebollo, M. Gómez Mármol, S. Rubino, *On the existence and asymptotic stability of solutions for unsteady mixing-layer models*, submitted to DCDS, (2012).

T. Chacón Rebollo, M. Gómez Mármol, S. Rubino, *Numerical analysis of pressure gradient effects on oceanic turbulent mixing-layer models*, submitted to M3AS, (2012).

S. Rubino, *Numerical modelling of oceanic turbulent mixing layers considering pressure gradient effects*, Mascot10 Proceedings: IMACS Series in Comp. and Appl. Math., **16** (2011), 229-238.