Why (partial) differential equations? Why control problems? What for?

Enrique FERNÁNDEZ-CARA
Dpto. E.D.A.N. - Univ. of Sevilla, Spain

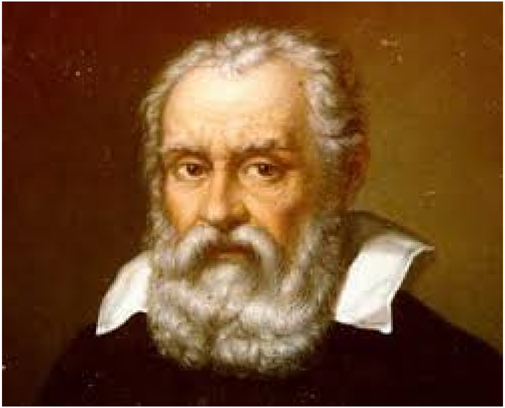
Using mathematics to describe real phenomena
A very relevant tool: differential equations
Introduce, solve and control

Outline

1. Preliminaries
   - Sentences
   - Fundamental concepts
2. Ordinary and partial differential equations
   - The early times: the motion of planets
   - More recent achievements in biology
   - Time-dependent phenomena
   - Basic ideas for existence, uniqueness, ...
3. Control issues
   - The meaning of control
   - Some examples and applications
   - General ideas to get controllability

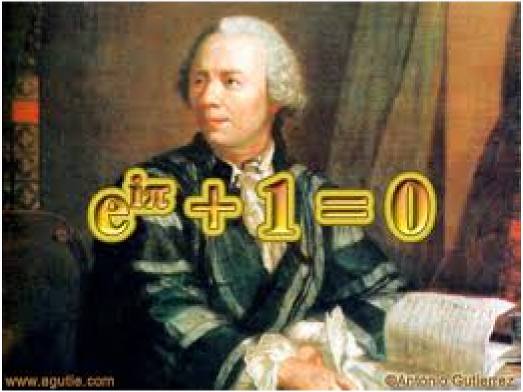
Galileo Galilei, 1564–1642:

God wrote the universe with mathematics



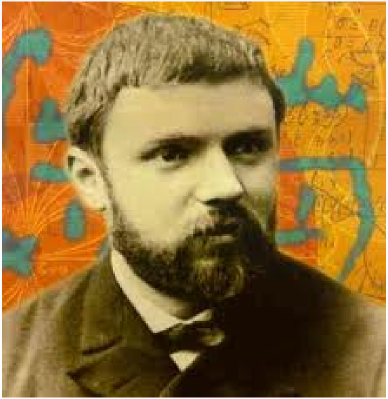
Leonhard Euler, 1707–1783:

In view of God's perfection, nothing happens in the universe without submission to a maximum or minimum rule



Henri Poincaré, 1854-1912:

- Mathematics is the art of giving the same name to different things.
- A science is a system of laws deduced from observation. The laws are, in sum, differential equations.



Functions:

Rules that assign quantities to other quantities

Examples:

- The position of a particle: $t \mapsto \mathbf{x}(t)$
- The temperature of a body: $(x,t) \mapsto \theta(x,t)$
- The pressure of a fluid: $(x_1, x_2, x_3, t) \mapsto p(x_1, x_2, x_3, t)$

Derivatives:

Tools that indicate how quickly a function changes

Examples:

- The velocity and the acceleration of a particle: $t \mapsto \dot{\mathbf{x}}(t)$, $t \mapsto \ddot{\mathbf{x}}(t)$
- The time rate change of the temperature: $(x, t) \mapsto \theta_t(x, t)$
- The changes in space of the pressure: $(x, t) \mapsto p_{x_i}(x, t)$, $i = 1, 2, 3$

Differential equations:

Identities where functions and their derivatives appear

Usually: motivted by physical, chemical, biological etc. laws

The first fundamental and starting point:

The description of planetary motion

Three relevant steps:

- Azarquiel, Cordoba (Spain), XI Cent
- Johannes Kepler (1577–1630), Central Europe, XVI Cent
- Sir Isaac Newton (1643–1727), United Kingdom, XVII Cent

Newton's contribution:

- Kepler's laws + reflexion $\Rightarrow$ UGL
- Then, UGL + conservation laws + mathematics $\Rightarrow$ KLs through ORDINARY DIFFERENTIAL EQUATIONS

With the notation of our days:

$$\mathbf{x} = \mathbf{x}(t), \quad t \in \mathbb{R}; \quad \text{the Sun } (M) \text{ and a planet } (m)$$

- Linear momentum: $m\ddot{\mathbf{x}} = F \quad (F = -\nabla U(\mathbf{x}))$
- Angular momentum: $m\mathbf{x} \times \dot{\mathbf{x}} = \mathbf{N} \quad (constant)$
- Energy: $\frac{1}{2}m|\dot{\mathbf{x}}|^2 + U(\mathbf{x}) = E \quad (constant)$
- UGL: $U(\mathbf{x}) = -GmM\frac{1}{|\mathbf{x}|} \quad (\mathbf{F} = -GmM\frac{\mathbf{x}}{|\mathbf{x}|^3})$

Computations give (among other things):

1. $\mathbf{x}(t) = (r(t)\,\cos\varphi(t))\,\mathbf{e}_1 + (r(t)\,\sin\varphi(t))\,\mathbf{e}_2$
   Orbits are on a plane
2. $r(t) = \frac{C_1}{1+C_2\cos\varphi(t)}$, for some C_1, C_2
   Orbits are elliptic and satisfy KLs

Major consequences:

Invention of differential equations, Birth of calculus (Leibniz?)
Explanations of phenomena (Hooke?)

Growth models in population dynamics

T. Malthus (1766–1834), V. Volterra (1860–1940), P.F. Verhulst (1804–1849), B. Gompertz (1779–1865)

- Malthus (exponential) law: $\dot{N} = \rho N$
- Logistic law: $\dot{N} = \rho N - RN^2$
- Gompertzian law: $\dot{N} = \rho N \log \frac{\theta}{N}$, etc.

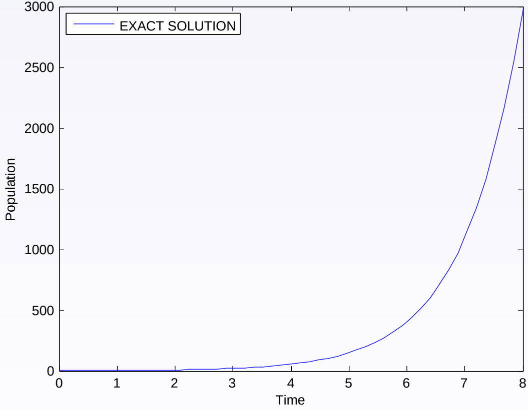


Figure: The evolution in time of a Mathusian population. $\dot{N} = \rho N$

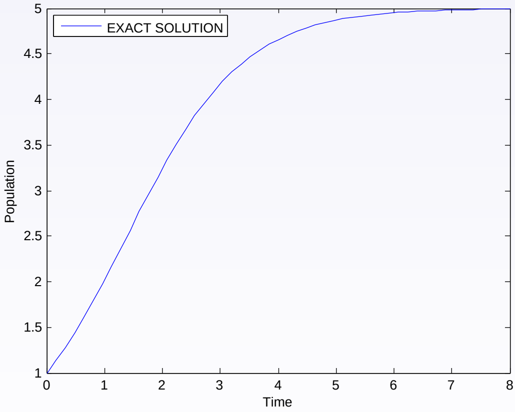


Figure: The evolution in time of a logistic population. $\dot{N} = \rho N - RN^2$

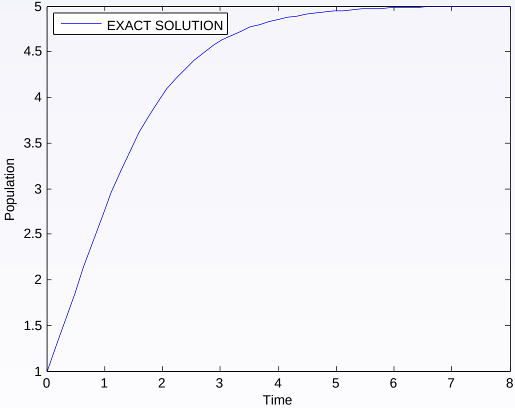


Figure: The evolution in time of a Gompertzian population. $\dot{N} = \rho N \log(\theta/N)$

Lotka-Volterra predator-prey models
Alfred J. Lotka (1880–1940) and Vito Volterra (1860–1940)

$$\dot{x} = ax - bxy, \quad \dot{y} = -cy + dxy$$

$x = x(t)$ and $y = y(t)$ are resp. the prey and predator populations

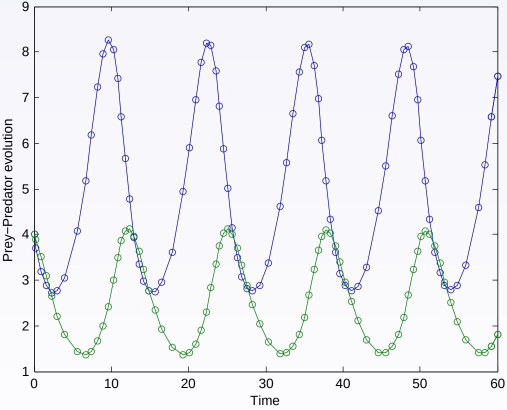


Figure: The evolution in time of a prey-predator system

The Laplace and Poisson (elliptic) PDEs

- Gravitational and electromagnetic fields in $\mathbf{R}^3$: $\mathbf{F}(\mathbf{x}) = -\nabla U(\mathbf{x})$, with

$$\begin{cases} -\Delta U = \rho(\mathbf{x})\, 1_D, & \mathbf{x} \in \mathbf{R}^3, \\ U \to 0 \quad \text{as } |\mathbf{x}| \to +\infty \end{cases}$$

Notation: $\Delta U := U_{x_1,x_1} + U_{x_2,x_2} + U_{x_3,x_3}$
Good strategy: compute U, then $\mathbf{F}$ (Laplace, Dirichlet, Poisson, …)

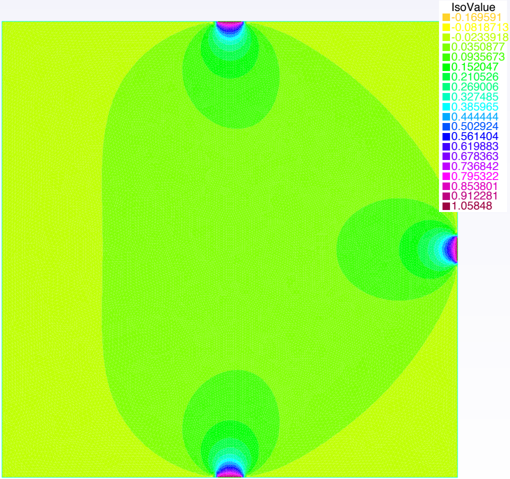
- Ideal fluid in $\mathbf{R}^2 \setminus B$: $\mathbf{v} = \nabla \times \psi$, with

$$\begin{cases} -\Delta\psi = 0, \quad (x_1, x_2) \in \mathbf{R}^2 \setminus B \\ \psi = 0, \quad (x_1, x_2) \in \partial B; \quad \psi \to \psi_\infty, \quad |(x_1, x_2)| \to +\infty \end{cases}$$

- Probability of leaving a region $R \subset \mathbf{R}^2$ through $\Gamma \subset \partial R$:

$$\begin{cases} -\Delta u := -u_{x_1,x_1} - u_{x_2,x_2} = 0, \quad (x_1, x_2) \in R \\ u = 1_\Gamma, \quad (x_1, x_2) \in \partial R \end{cases}$$

The probability of leaving a room



Taylor's and D'Alembert's achievements:

A PDE for the elastic string and a formula for its solutions

$$u_{tt} - c^2 u_{xx} = 0, \quad (x,t) \in (0,1) \times (0,T)$$

$$u(x,t) = f(x+ct) + g(x-ct)$$

J. D'Alembert, "Refléxions sur la cause des vents", Prusian Academy Prize, 1747

Fourier's achievements:

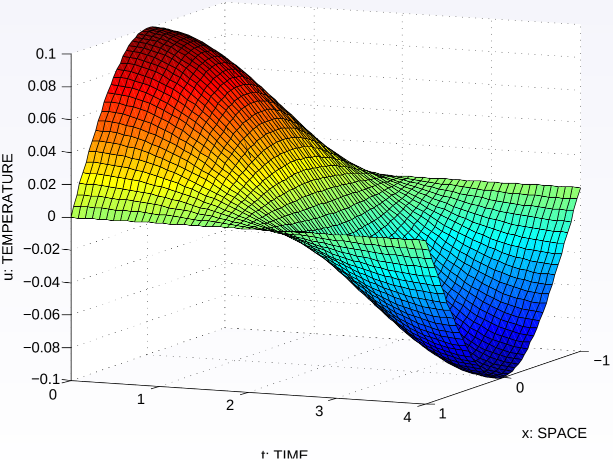
A PDE for heat propagation and its solutions

$$u_t - k u_{xx} = 0, \quad (x,t) \in (0,1) \times (0,T)$$

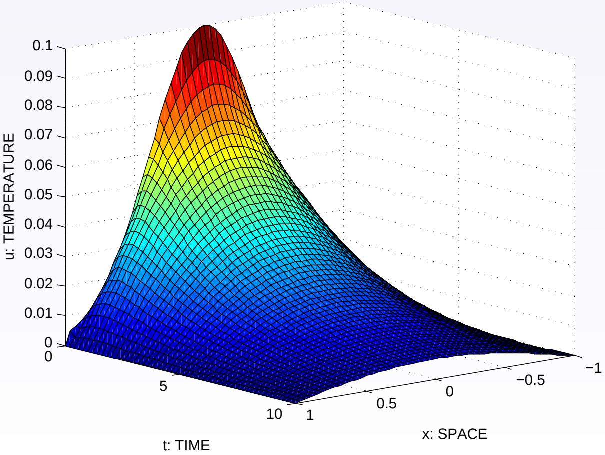
$$u(x,t) = \frac{u_0(t)}{2} + \sum_{n \geq 1} \{ u_n(t) \cos(n\pi x) + v_n(t) \sin(n\pi x) \}$$

J. Fourier, "Sur la propagation de la chaleur dans les solides", Paris Academy Prize, 1811

The vibrating string



The evolution of temperature



Other evolution PDEs: the Navier-Stokes system
C. Navier (1785–1836), G.G. Stokes (1819–1903)
The Navier-Stokes PDEs for a viscous incompressible fluid:

$$\begin{cases} \rho(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) - \mu\Delta\mathbf{u} + \nabla p = \rho\mathbf{f} \\ \nabla\cdot\mathbf{u} = 0 \end{cases}$$

$(\mathbf{x}, t) \in D \times (0, +\infty)$, with $D \subset \mathbf{R}^N$, $N = 2$ or $N = 3$
$\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ velocity, $p = p(\mathbf{x}, t)$ pressure, $\rho, \mu > 0$, $\mathbf{f}$ is given

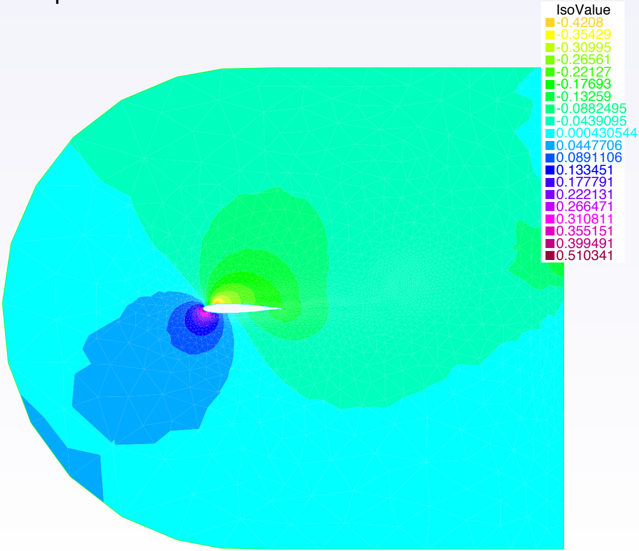
New (major) difficulty: nonlinearity – Much more difficult to solve!

Interesting questions: existence, uniqueness, regularity, additional properties

Clay Prize, 10^6 Dollars!

Fortunately: $\exists$ numerical methods!

The pressure of a NS fluid around an airfoil



A more complex system: cancer angiogenesis (N, h) and blood viscoelasticity $(\mathbf{u}, p, \tau)$:

$$\begin{cases} N_t + \mathbf{u} \cdot \nabla N - \nabla \cdot (D(N)\nabla N) = -\nabla \cdot (N\nabla h) + H(N) \\ h_t + \mathbf{u} \cdot \nabla h - \nabla \cdot (E(h)\nabla h) = K(N, h) \\ \rho(\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}) - \mu\Delta\mathbf{u} + \nabla p = \nabla \cdot \tau + \rho\mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \\ \tau_t + (\mathbf{u} \cdot \nabla)\tau + a\tau + g(\nabla\mathbf{u}, \tau) = 2b\, D\mathbf{u} \end{cases}$$

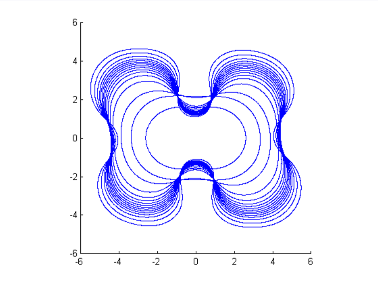
$g(\nabla\mathbf{u}, \tau)$: bilinear, taking into account frame invariance

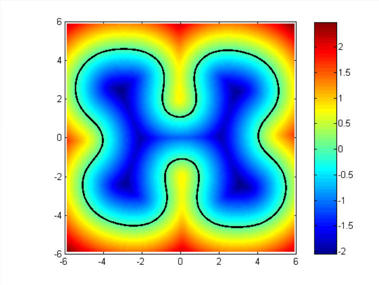
Much more difficult to analyze and solve! – Keller, Segal, ????

Again many questions: existence, uniqueness, regularity, etc.
For instance: do classical (regular) solutions exist? (unknown)

Again: numerical methods give results

Tumor growth evolution in low vascularization regime
(results by MC Calzada and others, 2011)





ANALYSIS OF NONLINEAR PDEs

$$E(U) = F \quad + \dots$$

Existence (steps):

- Approximation: $E(U_h) = F_h \; + \dots, \quad h \to 0$
- Estimates: $\|U_h\|_X \leq C$
- Compactness: $U_h \to U$ weakly in X, strongly in Y, with $X \hookrightarrow Y$
- Conclusion: $\exists$ solutions $U \in X$

OK for Navier-Stokes and many variants
Not so easy for Oldroyd-like systems:

$$\begin{cases} \rho(\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}) - \mu\Delta\mathbf{u} + \nabla p = \nabla \cdot \tau + \rho\mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \\ \tau_t + (\mathbf{u} \cdot \nabla)\tau + a\,\tau + g(\nabla\mathbf{u}, \tau) = 2b\,D\mathbf{u} \end{cases}$$

Estimates: only sometimes; Compactness is not immediate
(contributions by EFC, F Guillén and others)

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Again unclear for temperature-dependent flows:

$$\begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla \cdot (\nu(\theta) D\mathbf{u}) + \nabla p = \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \\ \theta_t + \mathbf{u} \cdot \nabla\theta - \nabla \cdot (\kappa(\theta) D\theta) = \nu(\theta) D\mathbf{u}\, D\mathbf{u} \end{cases}$$

Estimates: only in L^1 !
(contributions by B Climent, EFC and others)

ANALYSIS OF NONLINEAR PDEs

$$E(U) = F \quad + \ldots$$

Existence (steps):

- Approximation: $E(U_h) = F_h \; + \ldots, \quad h \to 0$
- Estimates: $\|U_h\|_X \le C$
- Compactness: $U_h \to U$ weakly in X, strongly in Y, with $X \hookrightarrow Y$
- Conclusion: $\exists$ solutions $U \in X$

Uniqueness and regularity:

- Rely on (very) good estimates: small X
- Usual argument for uniqueness:

$$0 = E(U_1) - E(U_2) = \tilde{E}(U_1, U_2) \cdot (U_1 - U_2) \Rightarrow U_1 = U_2$$

Unknown for 3D Navier-Stokes

Up to now: analysis and (numerical) resolution of

$$\begin{cases} E(U) = F \\ + \dots \end{cases}$$

From now on: control, i.e. acting to get good (or the best) results …

What is easier? Solving? Controlling?

James Watt, 1736–1819 — The steam engine
(later analyzed by G. Airy, J.C. Maxwell)

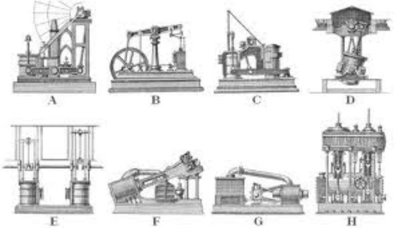


Figure: Steam engines

Main ideas: balls rotate around an axis, increasing velocity open valves, scaping vapor diminishes velocity.

This way: autoregulation, optimal performance, constant velocity, etc.

The (general) optimal control problem; recall Euler's sentence!

$$\begin{array}{l}\text{Minimize } \; J(v,y) \\ \text{Subject to } \; v \in \mathcal{V}_{ad}, \;\; y \in \mathcal{Y}_{ad}, \; (v,y) \; \text{satisfies} \; (S)\end{array}$$

with

$$E(y) = F(v) \quad + \; \dots \qquad (S)$$

The (general) controllability problem:

$$\text{Find } v \in \mathcal{V}_{ad} \text{ such that } \; Ry \in \mathcal{Z}_{ad}$$

Main questions: $\exists$, uniqueness/multiplicity, characterization, computation, ...

Lian-Martin model for tumor growth: Without and with radiotherapy

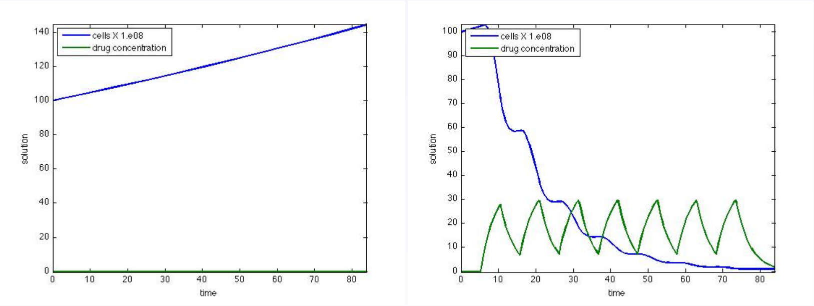


Figure: Controlling tumor growth (towards optimal therapy strategies, I). The state (cells+drug) solves $y_t = \lambda y \log(\theta/y) - k(v - v_*)_+ y, \;\; v_t = u - \gamma v$

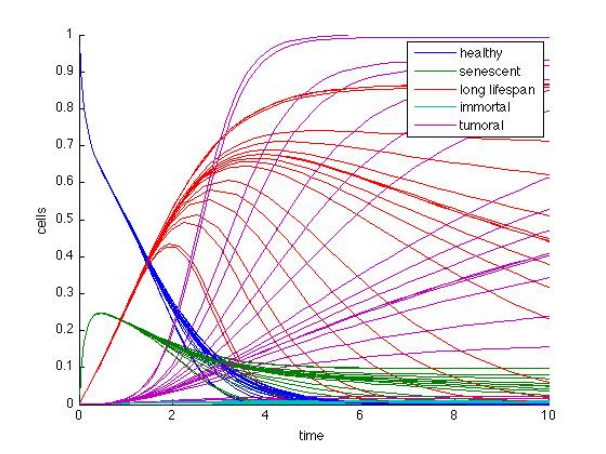
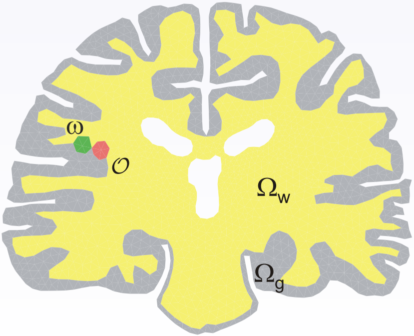


Figure: Controlling tumor growth, II). The state (healthy + senescent + long-life + inmortal + tumoral cells) solves a 5×5 ODE, controlled by u

MDELLING AND OPTIMIZING RADIOTHERAPY STRATEGIES

(glioblastoma, results by R Echevarría and others, 2007)

- Brain $\approx$ a two-dimensional crown section
- 2 subdomains
- Parameter values in agreement with [Alvord-Murray-Swanson 2000]



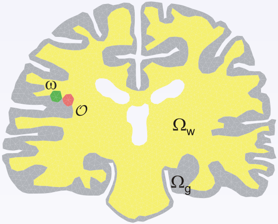
The state equation (description of the phenomenon):

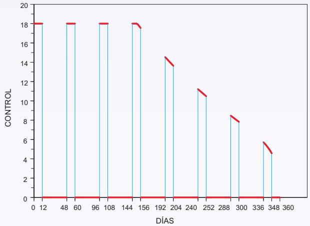
$$\left\{\begin{array}{l} c_t - \nabla\cdot(D(x)\nabla c) = (\rho - v 1_\omega)\, c, \quad (x,t)\in\Omega\times(0,T) \\ c|_{t=0} = c_0, \quad x\in\Omega \\ +\dots \end{array}\right. \qquad (E)$$

$c = c(x,t)$ is the state: a cancer cell population density
$v = v(x,t)$ is the control: a radiotherapy administration dose
Glioblastoma [Murray-Swanson, 90's], $D(x) = D_w$ or D_g (white and grey matters)

The optimal control problem:

$$\left\{\begin{array}{l} \text{Minimize } \; J(v,y) = \frac{1}{2}\int_\Omega |c(x,T)|^2 + \frac{1}{2}\iint_{\omega\times(0,T)} |v|^2 \\ \text{Subject to } \; 0\le v\le M,\; \iint_{\omega\times(0,T)} v \le R,\; \dots,\; (v,y) \text{ satisfies } (E) \end{array}\right.$$





Evolution after detection (no therapy)

Evolution after detection (optimal therapy)

See more in:

- http://mathematicalneurooncology.org/
  Kristin Swanson
- http://mathcancer.org/
  Paul Macklin

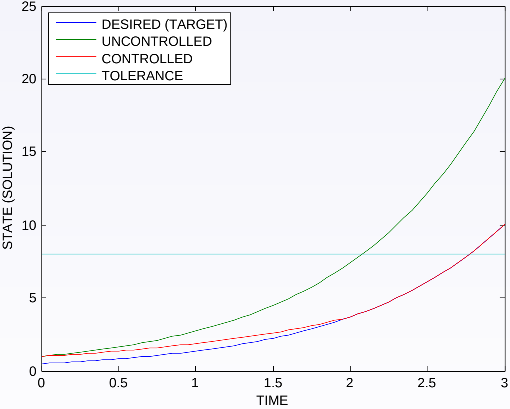


Figure: The desired, the uncontrolled and the controlled trajectories

Local exact controllability to a fixed flow

Again Navier-Stokes, local ECT:

$$(NS) \qquad \begin{cases} \mathbf{y}_t + (\mathbf{y}\cdot\nabla)\mathbf{y} - \Delta\mathbf{y} + \nabla p = v1_\omega, \quad \nabla\cdot\mathbf{y} = 0 \\ \mathbf{y}(x,t) = 0, \quad (x,t)\in\partial\Omega\times(0,T) \\ \mathbf{y}(x,0) = \mathbf{y}^0(x) \end{cases}$$

Fix a solution $(\overline{\mathbf{y}}, \overline{p})$, with $\overline{\mathbf{y}} \in L^\infty$
Goal: Find v such that $\mathbf{y}(T) = \overline{\mathbf{y}}(T)$

Fortunately: possible, at least if $\mathbf{y}^0$ is not too far from $\overline{\mathbf{y}}(0)$
Question: is this always possible? (unknown)

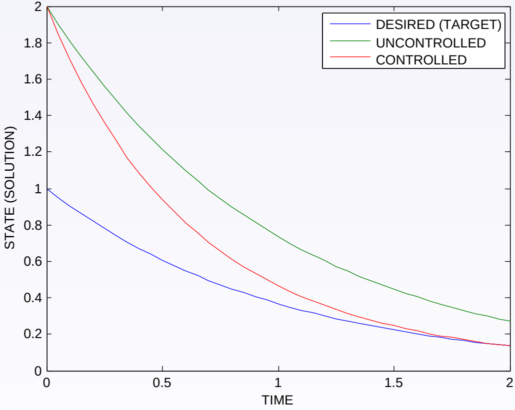


Figure: The desired, the uncontrolled and the controlled trajectories

Results by EFC and DA Souza, 2014

Test 1: Poiseuille flow

$$\overline{y} = (4x_2(1-x_2), 0), \quad \overline{p} = 4x_1$$

(stationary)

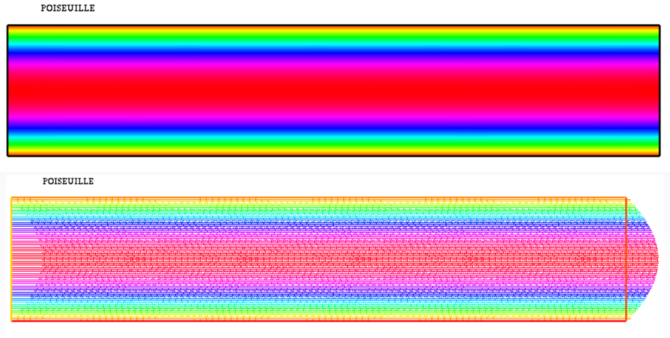


Figure: Poiseuille flow

Test 1: Poiseuille flow $\Omega = (0,5) \times (0,1)$, $\omega = (1,2) \times (0,1)$, $T = 2$
$y_0 = \overline{y} + m\,z$, $z = \nabla \times \psi$, $\psi = (1-y)^2 y^2 (5-x)^2 x^2$ ($m << 1$)
Approximation: P_2 in (x_1, x_2, t) + multipliers … – freefem++

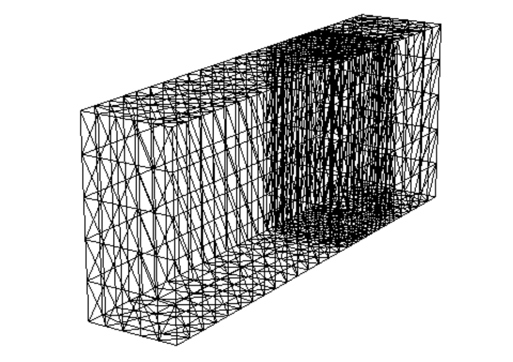
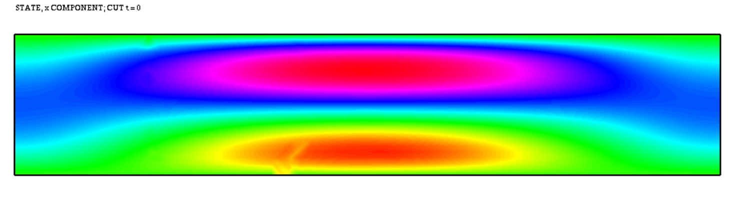


Figure: The Mesh – Nodes: 1830, Elements: 7830, Variables: 7×1830

Test 1: Poiseuille flow



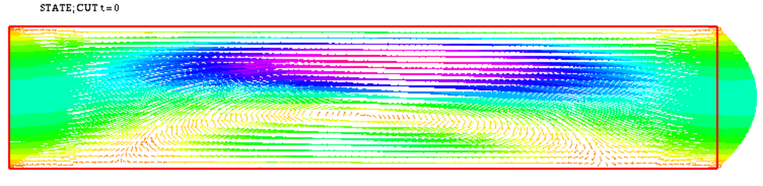


Figure: The initial state

Test 1: Poiseuille flow

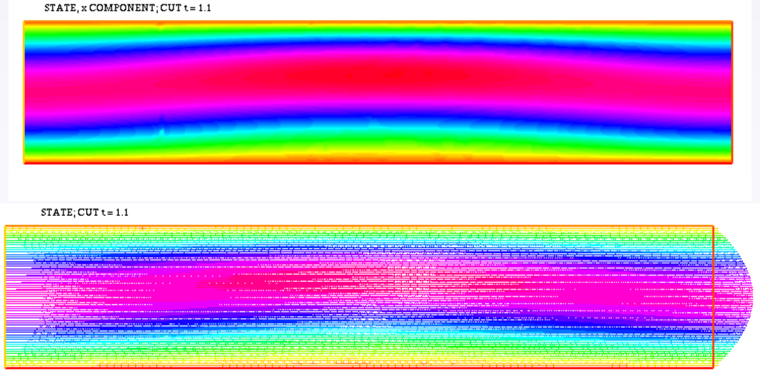


Figure: The state at $t = 1.1$

Test 1: Poiseuille flow

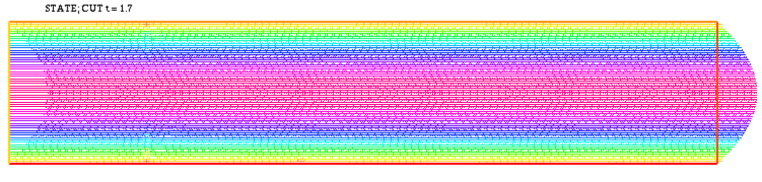


Figure: The state at $t = 1.7$

ZPoisseuille.edp

Test 2: Taylor-Green (vortex) flow

$$\bar{y} = (\sin(2x_1)\cos(2x_2)e^{-8t}, -\cos(2x_1)\sin(2x_2)e^{-8t})$$

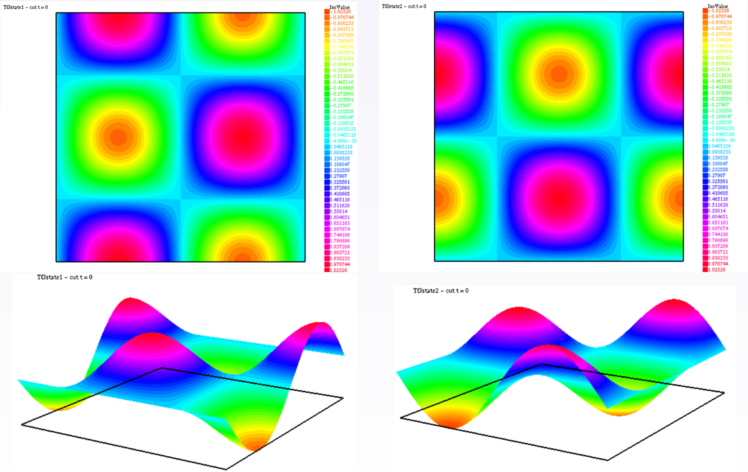


Figure: Taylor-Green flow

Test 2: Taylor-Green (vortex) flow

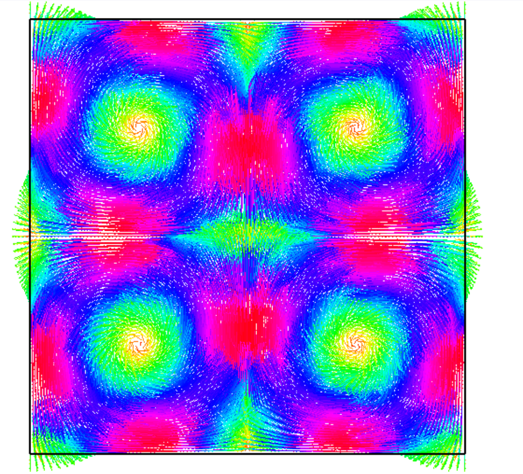


Figure: The Taylor-Green velocity field

Test 2: Taylor-Green (vortex) flow

$\Omega = (0,\pi)\times(0,\pi)$, $\omega = (\pi/3, 2\pi/3)\times(0,1)$, $T = 1$

$y_0 = \overline{y} + m\,z,\ \ z = \nabla\times\psi,\ \ \psi = (\pi - y)^2 y^2 (\pi - x)^2 x^2$ ($m << 1$)

Approximation: P_2 in (x_1, x_2) and t + multipliers . . . – freefem++

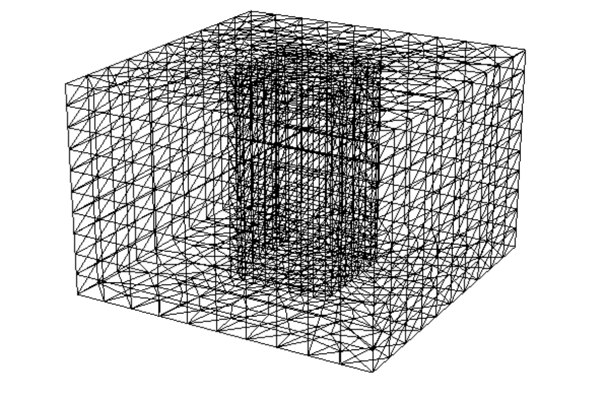


Figure: The mesh – Nodes: 3146, Elements: 15900, Variables: 7×3146

Test 2: Taylor-Green (vortex) flow

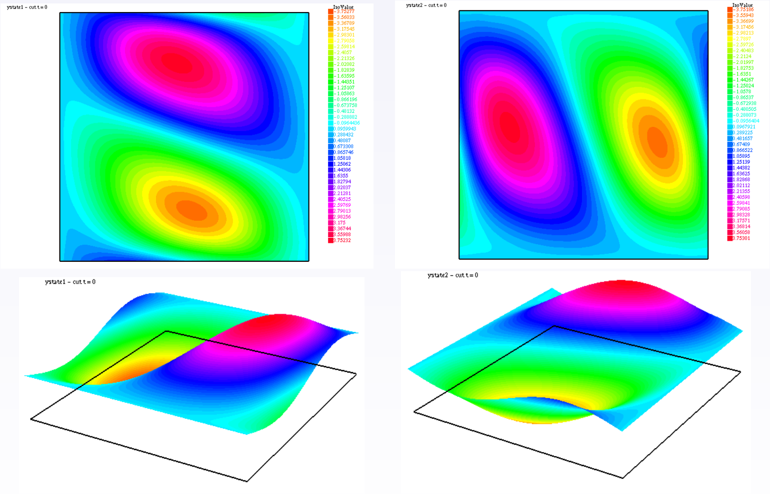


Figure: The initial state

Test 2: Taylor-Green (vortex) flow

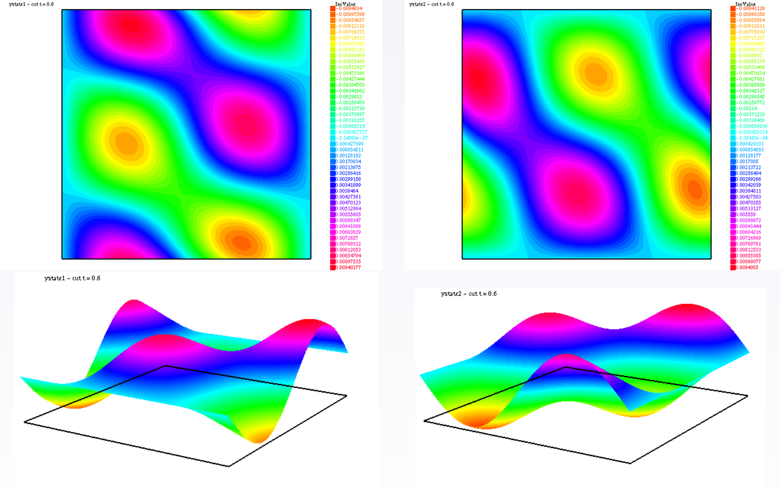


Figure: The state at $t = 0.6$

Test 2: Taylor-Green (vortex) flow

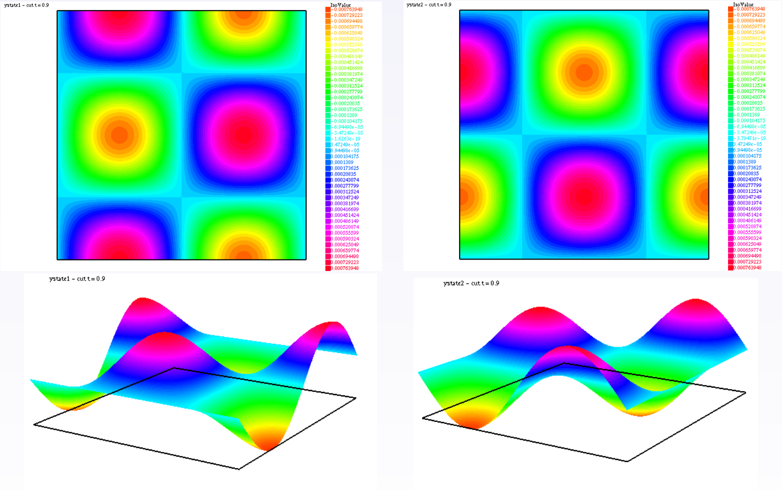


Figure: The state at $t = 0.9$

Taylor-Green Vortex.edp

NULL CONTROLLABILITY OF NONLINEAR EVOLUTION PDEs

$$y_t - A(y) = F(v), \quad y(0) = y_0, \quad y(T) = 0 \quad + \ \dots$$

Existence (steps):

- Linearization:
$$y_t - A'(0)y = F'(0)v, \quad y(0) = y_0, \quad y(T) = 0, \quad + \ \dots$$
- $\exists$ for the linearized problem:
$$NC \Leftrightarrow R(M) \hookrightarrow R(L) \Leftrightarrow \|\phi(0)\|_H^2 \leq C\|F'(0)^*\phi\|_U^2 \quad \forall \phi_0 \in H$$
$$-\phi_t - A'(0)^*\phi = 0, \quad \phi(T) = \phi_0 \quad + \ \dots$$ Carleman estimates
- Passage from linear to nonlinear: fixed-point, implicit function, ...

Unfortunately: in general, only local results, i.e. small y_0
(global Inverse Function Theorems?)

Contributions by Russell, J-L Lions, Fursikov, Imanuvilov, Lebeau, Zuazua, Coron, . . . ; also EFC, A Doubova, M. González-Burgos, DA Souza, . . . New ideas?

$\exists$ many applications of control theory to real-world problems

- Engineering
- Economics
- Biology and Medicine, etc.

Figure: Controlling an aerodynamic profile (I): a car

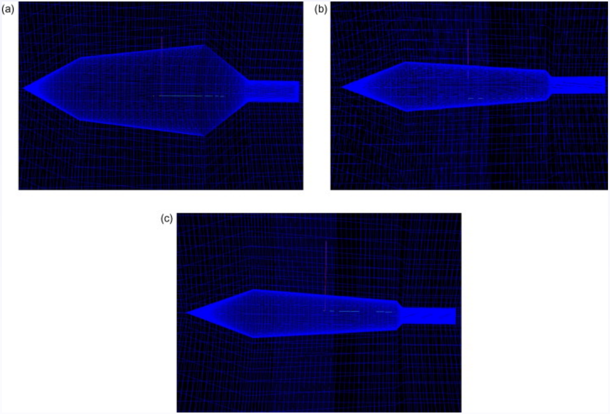


Figure: Controlling an aerodynamic shape (II): a rocket. (a) Initial design; (b) and (c) computed optimal designs

Figure: Controlling the trajectory of a space shuttle.

Figure: The POP project, INRIA, France. Automatic vision and expression.

Figure: The ICARE Project, INRIA, France. Autonomous car driving.

The autonomous car driving problem:

$$\dot{x} = f(x, u), \quad x(0) = x_0$$

with

$$\text{dist.}\,(x(t), Z(t)) \geq \varepsilon \quad \forall t$$
$$u \in \mathcal{U}_{ad} \quad (|u(t)| \leq C)$$

Goals (prescribed x_T and $\hat{x}$):

- $x(T) = x_T$
- Minimize $\sup_t |x(t) - \hat{x}(t)|$

Figure: The ICARE Project, INRIA, France. Autonomous car driving. Malis-Morin-Rives-Samson, 2004

The car in the street

Final comments

SOME CONCLUSIONS:

- Along the time: better descriptions (more precise, more complete) and more complex tools
- For many interesting problems we can get
  - Models
  - Theoretical results
  - Numerical (approximated) solutions
  - Qualitative and quantitative information on control properties

  Still many things to do . . .
- Why (partial) differential equations? Why control problems? What for?
  - To enlarge and improve scientific knowledge
  - To understand, describe and govern the behavior of real-life phenomena

Final comments

OUR GROUP (A SHORT DESCRIPTION, PEOPLE INVOLVED):

- M Delgado, I Gayte, M Molina, C Morales, A Suárez, . . .
  Theoretical results for PDE models concerning tumor growth: angiogenesis and metastasis modelling, stem cell models, etc.
- B Climent, F Guillén, JV Gutiérrez Santacreu, MA Rodríguez Bellido, G Tierra, . . .
  Theoretical and numerical control for PDE models from fluid mechanics: cristal liquids, solidification processes, etc.
- A Doubova, EFC, M González Burgos, DA Souza, . . .
  Theoretical and numerical analysis and control of linear and nonlinear PDEs and systems: Navier-Stokes-like controllability, non-scalar control problems, etc.

MUCHAS GRACIAS . . .