Asymptotic behavior of a reaction-diffusion equation perturbed by multiplicative noise

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Introduction

Most real phenomena are better described if random, non-autonomous (or stochastic) terms are considered in the models

$$\frac{\mathrm{d}u}{\mathrm{d}t} = F(u)$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = F(t, u) + \text{noise}$$

Several questions:

- Are deterministic models good approximations of real ones?
- Which effects are caused by the noise in deterministic systems?
- What kind of noise is more appropriate?

Ito vs Stratonovich (different effects in long time behaviour)

The Chafee-Infante equation

We will compare the global asymptotic behaviour of

$$(Ch-I) \quad \begin{cases} u_t - \Delta u = \beta u - u^3 \ \text{ in } (0,+\infty)\times[0,L], \\ u(0,x) = u_0(x), \quad x\in[0,L], \\ u(t,0) = u(t,L) = 0, \quad t\geq 0. \end{cases}$$

with its Ito stochastic perturbation

$$(Ch{-}I{+}ito) \quad \begin{cases} u_t - \Delta u = \beta u - u^3 + \sigma u \dot{W}_t \ \text{ in } (0,+\infty)\times[0,L], \\ u(0,x) = u_0(x), \quad x\in[0,L], \\ u(t,0) = u(t,L) = 0, \quad t\geq 0. \end{cases}$$

and its Stratonovich one

$$(Ch{-}I{+}strat) \quad \begin{cases} u_t - \Delta u = \beta u - u^3 + \sigma u \circ \dot{W}_t \ \text{ in } (0,+\infty)\times[0,L], \\ u(0,x) = u_0(x), \quad x\in[0,L], \\ u(t,0) = u(t,L) = 0, \quad t\geq 0. \end{cases}$$

- Different effects.

Preliminaries on Dynamical Systems

(X, d) compl. met. space, $F : D(F) \subset X \longrightarrow X$

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = F(u(t)), \\ \\ u(0) = u_0. \end{cases}$$

Dynamical System in X:

$$S(t) : X \longrightarrow X, \qquad S(t)u_0 = u(t; u_0)$$

$$S(0) = Id_X,\ S(t+s) = S(t)S(s), \forall t, s \geq 0.$$

- $B \subset X$ *absorbing* if $\forall\, D \subset X$ bounded $\exists\, T(D)$:

$$S(t)D \subset B, \forall t \geq T(D).$$

- $B \subset X$ *attracting* if $\forall\, D \subset X$ bounded

$$\lim_{t \to +\infty} \mathrm{dist}_H(S(t)D, B) = 0.$$

Preliminaries on Dynamical Systems

$\mathcal{A} \subset X$ is the *global attractor* for $S(t)$ if

- is compact,
- $S(t)\mathcal{A} = \mathcal{A}, \forall t > 0$ (invariance),
- Attracts every bounded subset of X.

Theorem

The global attractor $\mathcal{A}$ exists if and only if there exists a compact attracting subset $K \subset X$.

Internal structure of the attractor determines the behavior.

For our $(Ch - I)$: If λ_n eigenvalues of $-\Delta$, we have:

- $\{0\}$ is a steady-state solution which is $\begin{cases} \text{stable if } \beta < \lambda_1 \\ \text{unstable if } \beta > \lambda_1 \end{cases}$
- There exists the global attractor $\mathcal{A}_0$ of (Ch-I) which is formed by the stationary points (which bifurcate from the origin when β passesses through λ_n–Pitchfork bifurcation) and the unstable manifolds joining them.

Random dynamical systems: Motivation

•**Problem :** solution paths are not globally bounded, in general.
Example: Consider (OU)

$$dz = -z\,dt + dW(t) \tag{1}$$

where $W(t)$ standard Wiener process[1] over the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, canonical probability space i.e.

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}, \qquad W(t, \omega) := \omega(t)$$

$\mathcal{F}$ the Borel σ-algebra, $\mathbb{P}$ the Wiener measure, and $\theta_t : \Omega \to \Omega$ given by

$$(\theta_t \omega)(\cdot) := \omega(t + \cdot) - \omega(t).$$

[1] family of ran. var. $W(t)(\cdot) : \omega \in \Omega \mapsto W(t)(\omega) \in \mathbb{R}, t \geq 0$ s.t. $\mathbb{P}-$a.s.

- $W(0) = 0$
- cont. paths (NOT bounded variation, a.s.): $t \in \mathbb{R}^+ \mapsto W(t)(\omega) \in \mathbb{R}$
- independent increments:
- stationarity: the joint distribution of $\{W(t_1 + t), \ldots, W(t_k + t)\}$ does not depend on t.
- $W(t) - W(s)$, $0 \leq s \leq t$, is a Gussian var.: mean 0, variance $t - s$.

Random dynamical systems: Motivation

Solution:

$$\begin{aligned} z(t) &= z(t_0)e^{-(t-t_0)} + \int_{t_0}^{t} e^{-(t-s)}dW(s) \\ &= z(t_0)e^{-(t-t_0)} + \omega(t) - e^{-(t-t_0)}\omega(t_0) \\ &\quad - \int_{t_0}^{t} e^{-(t-s)}\omega(s)ds \end{aligned} \tag{2}$$

If we take two solutions $z_1(\cdot), z_2(\cdot)$ then

$$z_1(t) - z_2(t) = (z_1(t_0) - z_2(t_0))e^{-(t-t_0)}$$

Random dynamical systems: Motivation

BUT, a special solution:
Denote

$$z^*(\omega) = -\int_{-\infty}^{0} e^s \omega(s)\, ds.$$

Define

$$\begin{aligned} z(t,\omega) &= z^*(\theta_t \omega) \\ &= -\int_{-\infty}^{0} e^s \left(\theta_t \omega\right)(s)\, ds \\ &= -\int_{-\infty}^{0} e^s \left(\omega(t+s) - \omega(t)\right)\, ds \\ &= \omega(t) - \int_{-\infty}^{t} e^{s-t} \omega(s)\, ds \end{aligned}$$

Same obtained in (2) when $t_0 \to -\infty$ (pullback limit)
Random attractor: $\{\mathcal{A}(\omega); \omega \in \Omega\}$ with $\mathcal{A}(\omega) = z^*(\omega)$.

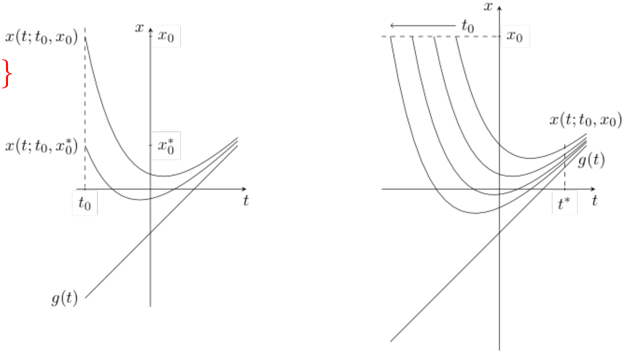
$$\frac{dx}{dt} = -x + t,\ x(t_0) = x_0$$

$$x(t; t_0, x_0) = e^{-(t-t_0)}(x_0 + 1 - t_0) + t - 1$$

$\mathcal{A}(t) = \{g(t) = t - 1\}$

$x(t; t_0, x_0) \to \infty$,
if $t \to +\infty$

$x(t; t_0, x_0) \to t - 1$,
if $t_0 \to -\infty$



Non-autonomous/Random dynamical systems

- New concept of attractor
- Differences between pullback and forward attractor
- Both convergences coincide in the autonomous case
- Pullback attractors are becoming popular in applications: e.g. Chesson proposes it for ecological models. Asymptotic environmentally-determined trajectories (aedts) are basically pullback attractors with singleton components.
- More suitable for stochastic/random cases (originally appeared in this context)

Random dynamical systems generated by random equations

Consider now a random differential equation:

$$\frac{dx}{dt} = f(\theta_t \omega, x) \qquad (3)$$

where $(\Omega, (\theta_t)_{t \in \mathbb{R}})$ are the ones defined previously.

Define $G(t, \omega)x_0 := x(t; 0, \omega, x_0)$ where $x(\cdot; s, \omega, x_0)$ is the solution of (3) s.t. $x(s) = x_0$.

NOTICE: $x(t - s; 0, \theta_s \omega, x_0) = G(t - s, \theta_s \omega)x_0 = x(t; s, \omega, x_0)$

Then, G is a cocycle

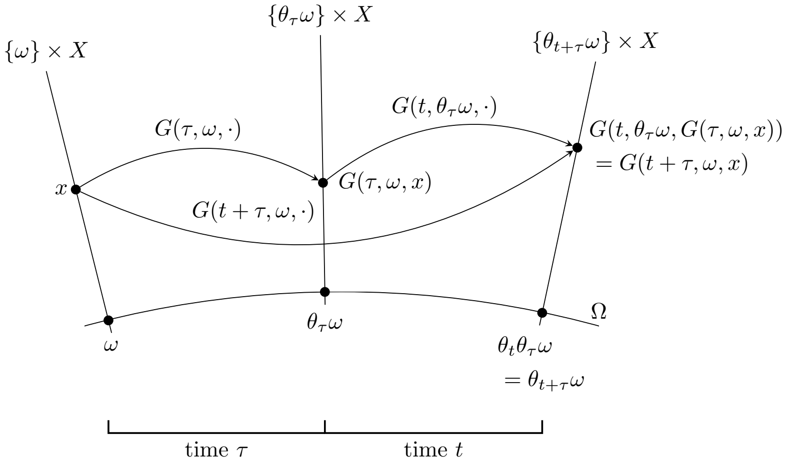
(i) the mapping $x \mapsto G(t, \omega)x$ is continuous for every $t \geq 0$;

(ii) $G(0, \omega)$ is the identity operator;

(iii) (cocycle property) $G(t + s, \omega) = G(t, \theta_s \omega)G(s, \omega)$ for all $s, t \geq 0$.

The pair (θ_t, G) is called a **random dynamical system.** Recall that θ_t is a **group**.

Graphical interpretation of the cocycle property using fibers



Random dynamical systems

Random sets

(i) A set-valued mapping $B : \omega \to 2^X \backslash \emptyset$ is said to be a random set if $\omega \mapsto \mathrm{dist}_X(x, B(\omega))$ is measurable for any $x \in X$.

(ii) A random set $B(\omega)$ is said to be bounded, compact or closed if $B(\omega)$ is bounded, compact or closed, for a.e. $\omega \in \Omega$.

(iii) A bounded random set $B(\omega) \subset X$ is said to be tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t} \sup_{x\in B(\theta_{-t}\omega)} \|x\|_X = 0, \quad \text{for all } \beta > 0;$$

a random variable $\omega \mapsto r(\omega) \in \mathbb{R}$ is said to be tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t} \sup_{t\in\mathbb{R}} |r(\theta_{-t}\omega)| = 0, \quad \text{for all } \beta > 0.$$

Random dynamical systems

In what follows $\mathcal{D}(X) :=$ set of all tempered random sets of X.

Absorbing sets

A random set $K(\omega) \subset X$ is a random absorbing set in $\mathcal{D}(X)$ if for any $B \in \mathcal{D}(X)$ and a.e. $\omega \in \Omega$, there exists $T_B(\omega) > 0$ s.t.

$$G(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subset K(\omega), \quad \forall t \geq T_B(\omega).$$

Random Attractor

Let $\{G(t,\omega)\}_{t\geq 0, \omega\in\Omega}$ an RDS and $\mathcal{A}(\omega)(\subset X)$ a random set. Then $\mathcal{A}(\omega)$ is a global random $\mathcal{D}$ attractor (or pullback $\mathcal{D}$ attractor) for $\{G(t,\omega)\}_{t\geq 0, \omega\in\Omega}$ if $\omega \mapsto \mathcal{A}(\omega)$ satisfies

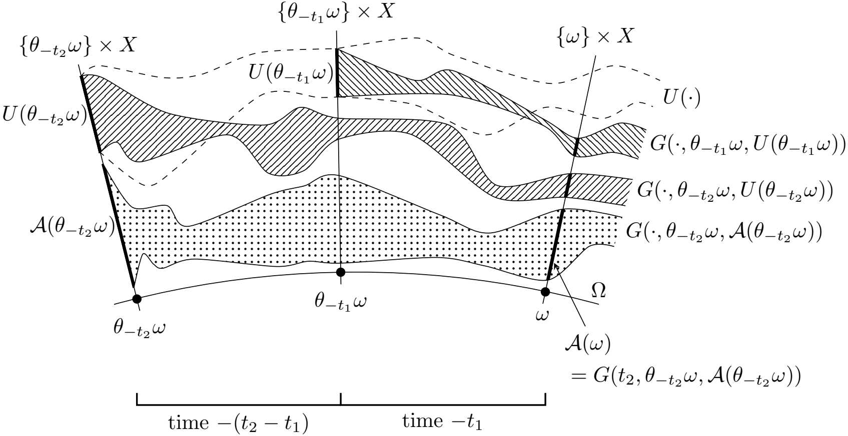
(i) $\mathcal{A}(\omega)$ is a compact set of X for a.e. $\omega \in \Omega$;

(ii) for a.e. $\omega \in \Omega$ and all $t \geq 0$, it holds $G(t,\omega)\mathcal{A}(\omega) = \mathcal{A}(\theta_t\omega)$;

(iii) for any $B \in \mathcal{D}(X)$ and a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} \mathrm{dist}_X(G(t, \theta_{-t}\omega)B(\theta_{-t}\omega), \mathcal{A}(\omega)) = 0,$$

Graphical interpretation of the random attractor



Conditions ensuring the existence of Random Attractors

Existence of Random Attractor[2] Let $B \in \mathcal{D}(X)$ be absorbing closed set for $\{G(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ and satisfies the asymptotic compactness condition for *a.e.* $\omega \in \Omega$, i.e., each sequence $x_n \in G(t_n, \theta_{-t_n}, B(\theta_{-t_n}\omega))$ has a convergent subsequence in X when $t_n \to \infty$. Then the cocycle G has a unique random attractor

$$\mathcal{A}(\omega) = \bigcap_{\tau \geq t_B(\omega)} \overline{\bigcup_{t\geq\tau} G(t, \theta_{-t}\omega) B(\theta_{-t}\omega)}.$$

If the pullback absorbing set is positively invariant, i.e., $G(t,\omega)B(\omega) \subset B(\theta_t\omega))$ for all $t \geq 0$, then

$$\mathcal{A}(\omega) = \bigcap_{t\geq 0} G(t, \theta_{-t}\omega) B(\theta_{-t}\omega).$$

[2][Bates, Lisei & Lu (2006), Caraballo, Lukaszewicz & Real (2006), Flandoli & Schmalfuß(1996)]

Conditions ensuring the existence of Random Attractors

For state space $X = \mathbb{R}^d$, the asymp. compactness follows trivially.

When the cocycle mapping is strictly uniformly contracting[3], i.e., there exists $K > 0$ such that

$$\|G(t,\omega)x_0 - G(t,\omega)y_0\|_X \ \leq \ e^{-Kt}\,\|x_0 - y_0\|_X$$

for all $t \geq 0$, $\omega \in \Omega$ and x_0, $y_0 \in X$, then the random attractor consists of singleton subsets $\mathcal{A}(\omega) = \{a(\omega)\}$ (as in our motivating example)

[3] [Caraballo, Kloeden & Schmalfuß(2004), Kloeden & Lorenz (2013)]

Random dynamical systems generated by stochastic equations

- Not every stochastic equation has been proved to generate a random dynamical system.
- Main idea is to perform a transformation (change of variable)
- For stochastic PDEs, only additive or multiplicative (linear) noise has been considered.

$$dx = F(x)\,dt + \sigma x dW(t), \quad \text{(linear multiplicative)} \tag{4}$$

$$dx = F(x)\,dt + dW(t), \quad \text{(additive)} \tag{5}$$

- Different interpretations of the stochastic integrals may yield to completely different results.

Random dynamical systems generated by stochastic equations

Consider the linear n-dimensional ODE

$$\dot{x} = F(x), \tag{6}$$

and the stochastic versions

$$\mathrm{d}x = F(x)\,\mathrm{d}t + \sigma x \circ \mathrm{d}W(t) \quad \text{(Stratonovich)} \tag{7}$$

$$\mathrm{d}y = F(y)\,\mathrm{d}t + \sigma y\,\mathrm{d}W(t), \quad \text{(Ito)} \tag{8}$$

These equations must be interpreted in integral formulation:

$$x(t) = x_0 + \int_0^t F(x(s))\,\mathrm{d}s + \int_0^t \sigma x(s) \circ \mathrm{d}W(s) \quad \text{(Stratonovich)} \tag{9}$$

$$y(t) = y_0 + \int_0^t F(y(s))\,\mathrm{d}s + \int_0^t \sigma y(s)\,\mathrm{d}W(s), \quad \text{(Ito)} \tag{10}$$

Random dynamical systems generated by stochastic equations

We must be careful with interpreting/defining the stochastic term: Main difficulty is that paths are NOT of bounded variation, we CANNOT use the Riemann-Stieltjes sums to define the stochastic integral. Instead, we have to define the stochastic integral

$$\int_0^T \phi(t)\, dW(t)$$

as a limit in $L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$.
We will not construct the stochastic integral but let us consider an illustrative example:
Consider the one-dimensional standard Wiener process $W(t) = W_t$, and let us try to define $\int_0^T W_s\, dW_s$ using the Riemann-Stieltjes sums.

Random dynamical systems generated by stochastic equations

Let us fix a sequence of partitions (Δ_n) of $[0, T]$,

$$\Delta_n = \{0 = t_0^n < t_1^n < \ldots < t_n^n = T\},$$

s.t. $\delta_n = \max_{0 \le k \le n-1}(t_{k+1}^n - t_k^n)$, satisfies $\lim_{n\to\infty} \delta_n = 0$.
Pick $a \in [0,1]$, and denote $\tau_k^n = a t_k^n + (1-a) t_{k-1}^n$, y

$$S_n = \sum_{k=1}^{n} W_{\tau_k^n}(W_{t_k^n} - W_{t_{k-1}^n}).$$

Using the decomposition

$$S_n = W_T^2/2 - 1/2 \sum_{k=1}^{n} (W_{t_k^n} - W_{t_{k-1}^n})^2 + \sum_{k=1}^{n} (W_{\tau_k^n} - W_{t_{k-1}^n})^2$$

$$+ \sum_{k=1}^{n} (W_{t_k^n} - W_{\tau_k^n})(W_{\tau_k^n} - W_{t_{k-1}^n}),$$

Random dynamical systems generated by stochastic equations

it is not difficult to check that

$$\lim_{n\to\infty} S_n = W_T^2/2 - (1-2a)T/2 \text{ in } L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}).$$

Consequently, trying to define $\int_0^T W_s\, dW_s$ as a limit in mean square, the integral depends on τ_k^n (i.e. on a).
Denoting $(a)\int_0^T W_s\, dW_s$ the obtained integral for the choice of each $a \in [0,1]$, we have

$$(a)\int_0^t W_s\, dW_s = W_t^2/2 - (1-2a)t/2.$$

- Notice:

1. The classical expected results for $a = 1/2$ (Stratonovich)
2. To get that its mean value becomes ZERO (in fact, a martingale) only when $a = 0$. (Ito)

Random dynamical systems generated by stochastic equations

- Each interpretation possesses advantages and disadvantages;
- A rule which permits to pass from one to the other;
- Additional term in Ito's formula;
- Main difference is for long-time behaviour: stabilization or destabilization.

Random dynamical systems generated by stochastic equations

Transformations into random differential equations:

$$\mathrm{d}u = F(u)\,\mathrm{d}t + \sigma u \circ \mathrm{d}W(t) \quad \text{(multiplicative)} \tag{11}$$

$$\mathrm{d}u = F(u)\,\mathrm{d}t + \mathrm{d}W(t), \quad \text{(additive)} \tag{12}$$

For a fixed *one-dimensional Wiener process* W, consider the one-dimensional SDE

$$dz = -z\,dt + dW(t) \tag{13}$$

for some $\lambda > 0$.

• There exists a random fixed point generating a stationary solution: Ornstein-Uhlenbeck

$$z^*(\omega) = -\int_{-\infty}^{0} e^s \omega(s)\,ds.$$

Random dynamical systems generated by stochastic equations

- Multiplicative case: Perform the change

$$v(t) = u(t)e^{-\sigma z^*(\theta_t\omega)}$$

Then we obtain the random equation

$$dv(t) = (e^{-\sigma z^*(\theta_t\omega)}F(e^{\sigma z^*(\theta_t\omega)}v(t)) + \sigma z^*(\theta_t\omega)v(t))dt$$

or

$$\frac{dv(t)}{dt} = e^{-\sigma z^*(\theta_t\omega)}F(e^{\sigma z^*(\theta_t\omega)}v(t)) + \sigma z^*(\theta_t\omega)v(t)$$

- Additive case: Perform the change

$$v(t) = u(t) - W(t)$$

The Chafee-Infante equation

Consider

$$(Ch-I) \quad \begin{cases} u_t - \Delta u = \beta u - u^3 \ \text{ in } (0,+\infty)\times[0,L], \\ u(0,x) = u_0(x), \quad x\in[0,L], \\ u(t,0) = u(t,L) = 0, \quad t\geq 0. \end{cases}$$

Denoting by λ_n the eigenvalues of $-\Delta$, we have:

- $\{0\}$ is a steady-state solution which is $\begin{cases} \text{stable if } \beta < \lambda_1 \\ \text{unstable if } \beta > \lambda_1 \end{cases}$
- There exists the global attractor $\mathcal{A}_0$ of (Ch-I) which is formed by the stationary points (which bifurcate from the origin when β passesses through λ_n–Pitchfork bifurcation) and the unstable manifolds joining them.

The Chafee-Infante equation

Now, we consider the perturbed versions:

1. $u_t - \Delta u = \beta u - u^3 + \sigma u \dot{W}_t$ (Ito) (DCDS (2000))
   - it generates a random dynamical system.
   - for any σ there exists $\mathcal{A}_\sigma(\omega)$ and $\dim_H(\mathcal{A}_\sigma(\omega)) < +\infty$.
   - for σ large enough, $\mathcal{A}_\sigma(\omega) = \{0\}$ (and $\{0\}$ expon. stable)
2. $u_t - \Delta u = \beta u - u^3 + \sigma u \circ \dot{W}_t$ (Stratonovich) (PRSL (2001))
   - for any σ there exists $\mathcal{A}_\sigma(\omega)$ and $\dim_H(\mathcal{A}_\sigma(\omega)) \sim \dim_H \mathcal{A}_0$.
3. What happens if we add a more general noise?
   - $+\sum_{i=1}^{d} B_i u \circ \dot{W}_t^i$ (Ito or Stratonovich)
   - + additive noise (collapse to a random fixed point: Crauel & Flandoli (1998), Caraballo et al. PAMS (2007))