Some topics about Nematic and Smectic-A Liquid Crystals

Chillán, enero de 2010

Table of contents

1. Introduction
2. The Models
3. Statement of the Problems
4. Nematic Case
5. Smectic-A Case

Table of contents

1. Introduction
2. The Models
3. Statement of the Problems
4. Nematic Case
5. Smectic-A Case

Table of contents

1. Introduction
2. The Models
3. Statement of the Problems
4. Nematic Case
5. Smectic-A Case

Table of contents

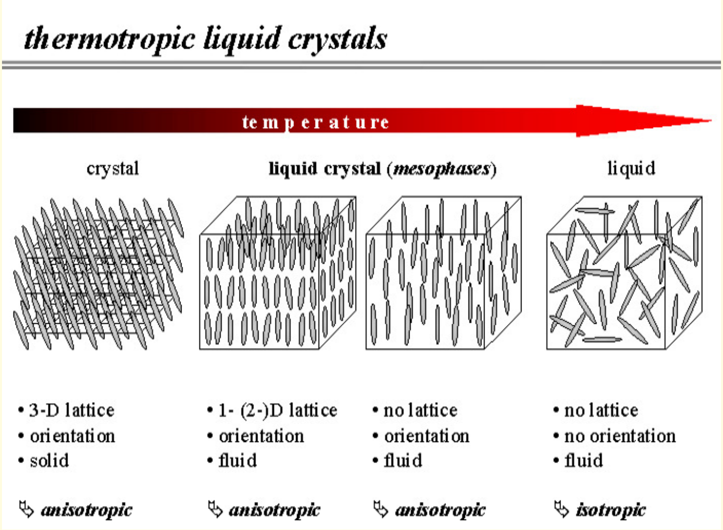
1. Introduction
2. The Models
3. Statement of the Problems
4. Nematic Case
5. Smectic-A Case

Table of contents

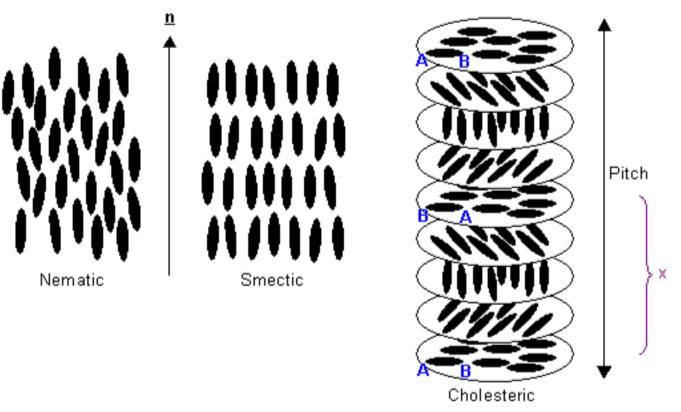
1. Introduction
2. The Models
3. Statement of the Problems
4. Nematic Case
5. Smectic-A Case

Table of contents

1. Introduction
2. The Models
3. Statement of the Problems
4. Nematic Case
5. Smectic-A Case



http://moebius.physik.tu-berlin.de/lc/lcs.html



http://www.doitpoms.ac.uk/

Introduction

Natural examples:

- Soap, soup
- Biological membranes
- The protein solution to generate silk of a spider
- DNA and polypeptides can form LC phases

Applications:

- Liquid Crystal Displays: wrist watches, pocket calculators, flat screens ...
- Liquid Crystal Thermometers: to show a map of temperatures to find tumors, bad connections on a circuit board ...
- Windows that can be changed from clear and opaque with the flip of a switch.
- To make a stable hydrocarbon foam.
- Optical Imaging and recording.

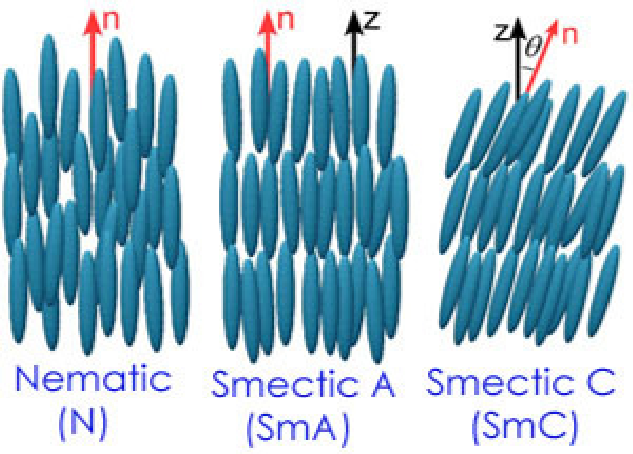
Natural examples:

- Soap, soup
- Biological membranes
- The protein solution to generate silk of a spider
- DNA and polypeptides can form LC phases

Applications:

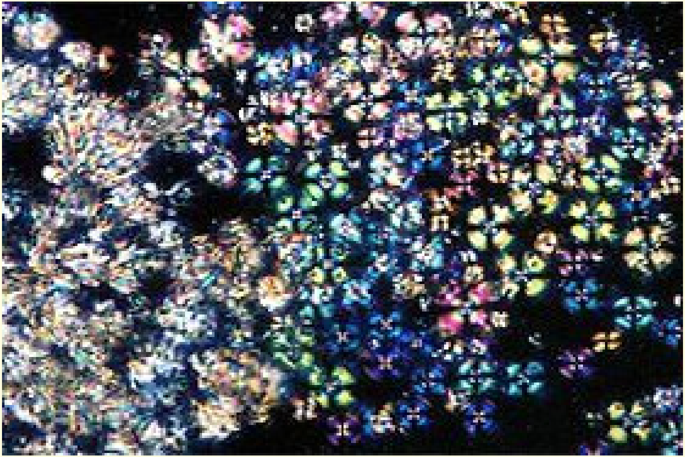
- Liquid Crystal Displays: wrist watches, pocket calculators, flat screens ...
- Liquid Crystal Thermometers: to show a map of temperatures to find tumors, bad connections on a circuit board ...
- Windows that can be changed from clear and opaque with the flip of a switch.
- To make a stable hydrocarbon foam.
- Optical Imaging and recording.

Introduction



http://atom.physics.calpoly.edu

Introduction



http://en.wikipedia.org/

Table of contents

1 Introduction

2 The Models

3 Statement of the Problems

4 Nematic Case

5 Smectic-A Case

- $\boldsymbol{d}$: Orientation of liquid crystal molecules (unit vector).
- $\boldsymbol{n}$: Single optical axis perpendicular to the layer.

- $|\boldsymbol{d}| = 1 \qquad \longrightarrow$ $\boldsymbol{f}$ Ginzburg-Landau penalization function

$$\boldsymbol{f}(\boldsymbol{d}) = \frac{1}{\varepsilon^2}(|\boldsymbol{d}|^2 - 1)\boldsymbol{d}$$

- $\nabla \times \boldsymbol{n} = 0 \qquad \longrightarrow \boldsymbol{n} = \nabla \varphi$

φ : Layer variable

- $\boldsymbol{d} = \boldsymbol{n} \qquad \longrightarrow |\nabla \varphi| = 1$

Penalized Oseen–Frank energy:

(NC) $E_e = \int_\Omega (\frac{1}{2}|\nabla \boldsymbol{d}|^2 + F(\varphi))$ **(SAC)** $E_e = \int_\Omega (\frac{1}{2}|\Delta \varphi|^2 + F(\nabla \varphi))$

where $\boldsymbol{f}(\boldsymbol{n}) = \nabla_{\boldsymbol{n}} F(\boldsymbol{n})$.

$F(\boldsymbol{n}) = \frac{1}{4\varepsilon^2}(|\boldsymbol{n}|^2 - 1)^2$ potential function of $\boldsymbol{f}(\boldsymbol{n}) = \frac{1}{\varepsilon^2}(|\boldsymbol{n}|^2 - 1)\boldsymbol{n}$.

Minimization problem $\longrightarrow$ Euler-Lagrange equation

(NC) $\omega \equiv \Delta \boldsymbol{d} - \boldsymbol{f}(\boldsymbol{d}) = 0,$ **(SAC)** $\omega \equiv \Delta^2 \varphi - \nabla \cdot \boldsymbol{f}(\nabla \varphi) = 0,$

The models

Penalized Oseen–Frank energy:

(NC) $E_e = \int_\Omega (\frac{1}{2}|\nabla \boldsymbol{d}|^2 + F(\varphi))$ **(SAC)** $E_e = \int_\Omega (\frac{1}{2}|\Delta\varphi|^2 + F(\nabla\varphi))$

where $\boldsymbol{f}(\boldsymbol{n}) = \nabla_{\boldsymbol{n}} F(\boldsymbol{n})$.

$F(\boldsymbol{n}) = \frac{1}{4\varepsilon^2}(|\boldsymbol{n}|^2 - 1)^2$ potential function of $\boldsymbol{f}(\boldsymbol{n}) = \frac{1}{\varepsilon^2}(|\boldsymbol{n}|^2 - 1)\boldsymbol{n}$.

Minimization problem $\longrightarrow$ Euler-Lagrange equation

(NC) $\omega \equiv \Delta \boldsymbol{d} - \boldsymbol{f}(\boldsymbol{d}) = 0,$ **(SAC)** $\omega \equiv \Delta^2\varphi - \nabla \cdot \boldsymbol{f}(\nabla\varphi) = 0,$

$\Omega \subset \mathbb{R}^N$ ($N = 2$ or 3), $\partial\Omega$ regular, $Q = \Omega \times (0, +\infty)$

(Ericksen-Leslie, Lin, E)

Angular momentum

(NC) $\partial_t \boldsymbol{d} + \boldsymbol{u} \cdot \nabla \boldsymbol{d} + \gamma\omega = 0$ **(SAC)** $\partial_t \varphi + \boldsymbol{u} \cdot \nabla \varphi + \gamma\omega = 0$

Linear momentum

$$\rho(\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) - \nabla \cdot (\sigma^d + \lambda\,\sigma^e) + \nabla p = 0, \quad \nabla \cdot \boldsymbol{u} = 0$$

where

$\Omega \subset \mathbb{R}^N$ ($N = 2$ or 3), $\partial\Omega$ regular, $Q = \Omega \times (0, +\infty)$

(Ericksen-Leslie, Lin, E)

Angular momentum

(NC) $\partial_t \boldsymbol{d} + \boldsymbol{u} \cdot \nabla \boldsymbol{d} + \gamma\omega = 0$ **(SAC)** $\partial_t \varphi + \boldsymbol{u} \cdot \nabla \varphi + \gamma\omega = 0$

Linear momentum

$$\rho(\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) - \nabla \cdot (\sigma^d + \lambda\,\sigma^e) + \nabla p = 0, \quad \nabla \cdot \boldsymbol{u} = 0$$

where

The models

(NC)

$$\sigma^d = \mu_4 D(\boldsymbol{u}),$$

$$\sigma^e = \lambda \nabla \cdot (\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d})$$

(SAC)

$$\sigma^d = \mu_1 (\boldsymbol{n}^t D(\boldsymbol{u}) \boldsymbol{n}) \boldsymbol{n} \otimes \boldsymbol{n} + \mu_4 D(\boldsymbol{u}) + \mu_5 (D(\boldsymbol{u}) \boldsymbol{n} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes D(\boldsymbol{u}) \boldsymbol{n}),$$

$$\sigma^e = -\boldsymbol{f}(\boldsymbol{n}) \otimes \boldsymbol{n} + \nabla(\nabla \cdot \boldsymbol{n}) \otimes \boldsymbol{n} - (\nabla \cdot \boldsymbol{n}) \nabla \boldsymbol{n}$$

Table of contents

1. Introduction
2. The Models
3. Statement of the Problems
4. Nematic Case
5. Smectic-A Case

Nematic Model

The equations

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = -\nabla\mathbf{d}^t\Delta\mathbf{d}, \\ \nabla\cdot\mathbf{u} = 0, \\ \partial_t\mathbf{d} + (\mathbf{u}\cdot\nabla)\mathbf{d} = (\Delta\mathbf{d} - f(\mathbf{d})), \quad |\mathbf{d}| \leq 1, \end{cases}$$

in Q

+ time-dependent (bc) on $\Sigma = (0,\infty)\times\partial\Omega$.
+ (iv) or (tp) in Ω.

The equations

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} - \nabla\cdot\sigma^d_{nl} \\ \qquad -(\Delta^2\varphi - \nabla\cdot\boldsymbol{f}(\nabla\varphi))\nabla\varphi + \nabla q = 0, \\ \nabla\cdot\boldsymbol{u} = 0, \\ \partial_t\varphi + \boldsymbol{u}\cdot\nabla\varphi + (\Delta^2\varphi - \nabla\cdot\boldsymbol{f}(\nabla\varphi)) = 0, \end{cases}$$

in Q
where $\sigma^d_{nl} := (\boldsymbol{n}^t D(\boldsymbol{u})\boldsymbol{n})\boldsymbol{n}\otimes\boldsymbol{n} + D(\boldsymbol{u})\boldsymbol{n}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes D(\boldsymbol{u})\boldsymbol{n}$

+ time-dependent (bc) on $\Sigma = (0,\infty)\times\partial\Omega$.
+ (iv) or (tp) in Ω.

Lifting functions

Boundary condition depending on the time $\rightsquigarrow$

$$\textbf{(NC)} \quad \widetilde{\boldsymbol{d}} = \widetilde{\boldsymbol{d}}(t)$$

stationary (for weak norms) or non-stationary (for regular norms) lifting function

$$\widehat{\boldsymbol{d}}(t) = \boldsymbol{d}(t) - \widetilde{\boldsymbol{d}}(t), \qquad \widehat{\boldsymbol{d}} = 0 \text{ on } \partial\Omega$$

$$\text{Unknows:} \quad \boldsymbol{u}, p, \widehat{\boldsymbol{d}}$$

$$\textbf{(SAC)} \quad \widetilde{\varphi} = \widetilde{\varphi}(t)$$

non stationary lifting function

$$\widehat{\varphi}(t) = \varphi(t) - \widetilde{\varphi}(t), \qquad \widehat{\varphi} = 0 \text{ on } \partial\Omega$$

$$\text{Unknows:} \quad \boldsymbol{u}, p, \widehat{\varphi}$$

Table of contents

1 Introduction

2 The Models

3 Statement of the Problems

4 Nematic Case

5 Smectic-A Case

Weak solution:

$$\boldsymbol{u} \in L^\infty(0,+\infty;\boldsymbol{L}^2) \cap L^2_w(0,+\infty;\boldsymbol{H}^1),$$
$$\boldsymbol{d} \in L^\infty(0,+\infty;\boldsymbol{H}^1) \cap L^2_w(0,+\infty;\boldsymbol{H}^2)$$

Regular solution:

$$\boldsymbol{u} \in L^\infty(0,+\infty;\boldsymbol{H}^1) \cap L^2_w(0,+\infty;\boldsymbol{H}^2),$$
$$\boldsymbol{d} \in L^\infty(0,+\infty;\boldsymbol{H}^2) \cap L^2_w(0,+\infty;\boldsymbol{H}^3)$$

Nematic Case

Asymptotic Stability:

$$E(t) = \frac{1}{2}|\boldsymbol{u}(t)|^2 + E_e(t) \to E_\infty,$$

$$\boldsymbol{u}(t) \to 0 \text{ in } \mathbf{H}_0^1(\Omega), \qquad \omega(t) = (\Delta\varphi - f(\varphi))(t) \to 0 \text{ in } L^2(\Omega)$$

when $t \uparrow +\infty$, where $E_\infty = E_{e,\overline{\boldsymbol{d}}} = \frac{1}{2}|\nabla\overline{\boldsymbol{d}}|^2 + \int_\Omega F(\overline{\boldsymbol{d}})$ and $\overline{\boldsymbol{d}}$ is a critical point of E_e. Moreover, $\boldsymbol{d} \to \overline{\boldsymbol{d}}$ "for subsequences" in $H^2(\Omega)$-weak.

Stability:

If initial data are small, $|\boldsymbol{u}|$, $|\omega|$, and $E(t)$ are small for each $t \geq 0$

Nematic Case

Previous result: (iv)-problem, boundary condition independent of time. Existence of a weak global solution. Existence and uniqueness of a regular solution for larger viscosity. [Lin,Liu'95]

Goal Time-dependent (bc) case

- [Climent,Guillén,Rojas'06]
  (tp)-problem. Existence of a weak periodic solution.
  Regularity $N = 2$
- [Climent,Guillén,Moreno'08]
  (iv)-problem. Existence of a weak global solution.
  Existence of a global strong solution, ν big enough.
  Uniqueness of strong/weak solutions.
  (tp)-problem. Existence of a regular solution, ν big enough.
- [Climent,Guillén,Rodríguez] Stability and asymptotic stability (tp)-problem, time-independent (bc) case

Previous result: (iv)-problem, boundary condition independent of time. Existence of a weak global solution. Existence and uniqueness of a regular solution for larger viscosity. [Lin,Liu'95]

Goal Time-dependent (bc) case

- [Climent,Guillén,Rojas'06]
  (tp)-problem. Existence of a weak periodic solution.
  Regularity $N = 2$
- [Climent,Guillén,Moreno'08]
  (iv)-problem. Existence of a weak global solution.
  Existence of a global strong solution, ν big enough.
  Uniqueness of strong/weak solutions.
  (tp)-problem. Existence of a regular solution, ν big enough.
- [Climent,Guillén,Rodríguez] Stability and asymptotic stability (tp)-problem, time-independent (bc) case

Table of contents

1 Introduction

2 The Models

3 Statement of the Problems

4 Nematic Case

5 Smectic-A Case

Weak solution:

$$\boldsymbol{u} \in L^\infty(0,+\infty;\boldsymbol{H}^1) \cap L^2_w(0,+\infty;\boldsymbol{H}^2),$$

$$\varphi \in L^\infty(0,+\infty;\boldsymbol{H}^4) \cap L^2_w(0,+\infty;\boldsymbol{H}^6)$$

Regular solution:

$$\boldsymbol{u} \in L^\infty(0,+\infty;\boldsymbol{H}^1) \cap L^2_w(0,+\infty;\boldsymbol{H}^2),$$

$$\varphi \in L^\infty(0,+\infty;\boldsymbol{H}^2) \cap L^2_w(0,+\infty;\boldsymbol{H}^3)$$

Asymptotic Stability:

$$\mathbf{u}(t) \to 0 \text{ in } \mathbf{H}_0^1(\Omega) \quad (\Delta^2\varphi - \nabla \cdot f(\nabla\varphi))(t) \to 0 \text{ in } L^2(\Omega)$$

when $t \uparrow +\infty$. Moreover, $\varphi \to \overline{\varphi}$ for "sequences" in $H^4(\Omega)$-weak, where $\overline{\varphi}$ is a solution of a Euler-Lagrange problem.

Stability:

If initial data are small, $\boldsymbol{u}$, φ, and ω are small for each $t \geq 0$

Previous result: (iv)-Problem, time-independent boundary conditions. Existence of weak solutions in $[0, T]$, global regularity of weak solutions (for big enough viscosity). [Liu'00]

Goal Time-dependent (bc) case [Climent,Guillén]

1. Uniqueness weak/strong solutions (iv)-Problem,
2. Existence of global weak solutions (iv)-Problem, "bounded" up to infinity time,
3. Existence of weak time-periodic solutions,
4. Existence of regular solutions for both previous cases (dominant viscosity coefficient).
5. Stability and asymptotic stability Time-independent (bc) case

Smectic-A Case

Previous result: (iv)-Problem, time-independent boundary conditions. Existence of weak solutions in $[0, T]$, global regularity of weak solutions (for big enough viscosity). [Liu'00]

Goal Time-dependent (bc) case [Climent,Guillén]

1. Uniqueness weak/strong solutions (iv)-Problem,
2. Existence of global weak solutions (iv)-Problem, "bounded" up to infinity time,
3. Existence of weak time-periodic solutions,
4. Existence of regular solutions for both previous cases (dominant viscosity coefficient).
5. Stability and asymptotic stability Time-independent (bc) case