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EQUI-ATTRACTION AND CONTINUITY OF ATTRACTORS FOR SKEW-PRODUCT SEMIFLOWS

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Dedicated to Björn Schmalfuß on occasion of his Sixtieth Birthday

ABSTRACT. In this paper we prove the equivalence between equi-attraction and continuity of attractors for skew-product semi-flows, and equi-attraction and continuity of uniform and cocycle attractors associated to non-autonomous dynamical systems. To this aim proper notions of equi-attraction have to be introduced in phase spaces where the driving systems depend on a parameter. Results on the upper and lower-semicontinuity of uniform and cocycle attractors are relatively new in the literature, as a deep understanding of the internal structure of these sets is needed, which is generically difficult to obtain. The notion of lifted invariance for uniform attractors allows us to compare the three types of attractors and introduce a common framework in which to study equi-attraction and continuity of attractors. We also include some results on the rate of attraction to the associated attractors.

1. **Introduction and preliminaries.** The skew-product semiflow is an important tool to understand the dynamics of some non-autonomous differential equations (see [24, 25, 26]). It consists of a very ingenious way of tracking the non-autonomous nature of the equation into an autonomous equation in a product space. Autonomous equations are nowadays quite well known and there are many results on local and global existence and regularity of solutions, also on existence of attractors and stability under perturbations, [1, 3, 6, 12, 14, 21, 28].

In order to analyse the asymptotic behavior of solutions, we concentrate our attention on the attractors. These are subsets of the phase space which contain most of the important

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information concerning the long time behavior of solutions. There are several notions of attractor for non-autonomous problems. On one hand, the pullback and the cocycle attractors refer to an invariant family of time-dependent sets (with the pullback attraction property) containing all global bounded solutions and, consequently, all interesting structures closely related to its dynamics. On the other hand, the uniform attractor is a compact bounded subset of the phase space which attracts bounded sets in the forwards sense, but not invariant. The pullback and the uniform attractors are different in general. All these notions can be related with the aid of the concept of skew-product semiflow, see [24, 25, 27].

The existence of attractors for autonomous dynamical systems has been studied in [1], [6], [11], [12], [14], [21] and [28], for example. The robustness of attractors under perturbation is a fundamental property and there are many works devoted to this problem, both in the autonomous ([1, 10, 15, 16, 17, 18]) and the non-autonomous cases ([2, 5, 6, 7, 9, 20, 22]). The continuity of attractors was soon related to the equi-attraction (see [1], also in a non-autonomous framework [13].

The relation between equi-attraction and continuity of attractors has not been clarified yet for uniform attractors. One of the aims of this paper is to study the relationship between equi-attraction and continuity of uniform, cocycle and pullback attractors. To this end we will treat all of these attractors in a skew-product semiflow framework. Though the skew-product semiflows are semigroups in a product space, note that, under perturbation, the phase space changes with the parameter, which was not considered in the previous literature.

Before going further, we will introduce some necessary concepts, definitions, terminology and results which will be crucial for our analysis.

Let A and B be two subsets of the metric space (X,d). We denote by $\mathrm{dist}(A,B)$ the Hausdorff semidistance between two subsets, that is, $\mathrm{dist}(A,B)=\sup_{a\in A}\mathrm{d}(a,B)$ and we write $\mathrm{dist}_{\mathrm{H}}(A,B)=\mathrm{dist}(A,B)+\mathrm{dist}(B,A)$ for the Hausdorff distance.

Skew-product semiflows appear in a natural way when studying non-autonomous differential equations. Indeed, let us consider a family of differential equations in a Banach space $(X,\|\cdot\|_X)$. For $\eta\in[0,1]$, let $f_\eta\in C_b(\mathbb{R}\times X,X)$, where $C_b(\mathbb{R}\times X,X)$ is the space of all continuous and bounded maps from $\mathbb{R}\times X$ to X with a suitable metric ρ.

Consider the semigroup $\{\theta_t : t\in\mathbb{R}\}$ in $C_b(\mathbb{R}\times X,X)$ where $(\theta_t\phi)(s,x):=\phi(t+s,x)$, for all $(s,x)\in\mathbb{R}\times X$ and $\phi\in C_b(\mathbb{R}\times X,X)$. Let us denote by Σ_η the closure of $\{\theta_t f_\eta : t\in\mathbb{R}\}$ in the metric space $(C_b(\mathbb{R}\times X,X),\rho)$, and we assume that $\{\theta_t : t\in\mathbb{R}\}$ is a group over Σ_η. Furthermore, suppose that $\bigcup_{\eta\in[0,1]}\Sigma_\eta$ is precompact so that, in particular, Σ_η is compact, for each $\eta\in[0,1]$.

For a given $\sigma\in\Sigma_\eta$ we consider the system

$$\begin{cases} \dot{x}=\sigma(t,x), t\in\mathbb{R} \\ x(0)=x_0\in X, \end{cases} \tag{1}$$

and for $x_0\in X$, let $x(t,x_0,\sigma)$ be the solution of (1) at time $t\in\mathbb{R}$ with non-autonomous function $\sigma\in\Sigma_\eta$ and initial condition $x(0,x_0,\sigma)=x_0$. We will use the notation $\varphi_\eta(t,\sigma)x_0 := x(t,x_0,\sigma)$, when the non-autonomous function belongs to Σ_η. Note that these solutions satisfy, for all $x_0\in X$ and $\sigma\in\Sigma_\eta$, the following properties:

1. $\varphi_\eta(0,\sigma)x_0=x_0$;
2. $[0,\infty)\times X\times\Sigma_\eta\ni(t,x_0,\sigma)\mapsto\varphi_\eta(t,\sigma)x_0\in X$ is continuous;
3. $\varphi_\eta(t+s,\sigma)x_0=\varphi_\eta(t,\theta_s\sigma)\varphi_\eta(s,\sigma)x_0$, the cocycle property.

A map $\varphi_\eta : [0,\infty)\times X\times\Sigma_\eta\to X$ together with a group $\{\theta_t : t\in\mathbb{R}\}$ with the properties above is called a non-autonomous dynamical system (or nds for short) and is denoted by $(\varphi_\eta,\theta)_{(X,\Sigma_\eta)}$. However, the are situations in which we can have a set of mathematical elements $\{X, \Sigma_\eta, \varphi_\eta, \theta\}$ satisfying the precedent properties and not being necessarily associated to a differential equation in the form described above. For this reason, we will develop now an abstract theory in which we do not need any relationship to any differential equation, although our primary interest will be related to one of them.

Thus, in an abstract way, suppose that $(\mathscr{C}, \rho)$ is a complete metric space with a group $\{\theta_t : t \in \mathbb{R}\}$. Take a family $\{\sigma_\eta : \eta \in [0,1]\}$ in $\mathscr{C}$ with the property that there is a global solution $\zeta_\eta(t)$ for $\{\theta_t : t \geq 0\}$ through each σ_η and $\sigma_\eta \to \sigma_0$, as $\eta \to 0^+$. From now on, we are going to write $\zeta_\eta(-t) = \theta_{-t}\sigma_\eta$, for $t > 0$, and assume that

$$\theta_t \sigma_\eta \to \theta_t \sigma_0, \text{ as } \eta \to 0^+, \text{ uniformly for } t \in \mathbb{R}. \tag{2}$$

For $\eta \in [0,1]$, we denote by $\Gamma_\eta = \{\theta_t \sigma_\eta : t \in \mathbb{R}\}$ the orbit of σ_η by $\{\theta_t : t \in \mathbb{R}\}$ on $\mathscr{C}$. Let Σ_η be the closure of Γ_η and we denote

$$\Sigma := \overline{\bigcup_{\eta \in [0,1]} \Sigma_\eta} \text{ is compact on } \mathscr{C}. \tag{3}$$

We can now define the concept of uniform attractor for a non-autonomous dynamical system. It is the minimal closed set that attracts all solutions uniformly with respect the non-autonomous functions on Σ_η and with respect to initial values x_0 in bounded subsets of X. Precisely:

Definition 1.1. Suppose $(\varphi_\eta, \theta)_{(X,\Sigma_\eta)}$ is a non-autonomous dynamical system. A uniform attractor $\mathscr{A}_\eta$ (if it exists) for this system is the minimal closed subset of X such that

$$\sup_{\sigma \in \Sigma_\eta} \operatorname{dist}(\varphi_\eta(t,\sigma)B, \mathscr{A}_\eta) \to 0, \text{ as } t \to +\infty,$$

for each bounded subset B of X.

A recent characterization of a uniform attractor ([2]) is given by the property of *lift invariance*. A global solution through $x \in X$ and $\sigma \in \Sigma_\eta$ for the nds $(\varphi_\eta, \theta)_{(X,\Sigma_\eta)}$ is a map $\xi : \mathbb{R} \to X$ that satisfies, for all $t \geq s$,

$$\varphi_\eta(t-s, \theta_s \sigma)\xi(s) = \xi(t) \text{ and } \xi(0) = x,$$

Definition 1.2. We say a subset M of X is *lifted invariant* by the nds $(\varphi_\eta, \theta)_{(X,\Sigma_\eta)}$ if for every $x \in M$ there is $\sigma \in \Sigma_\eta$ and a global solution $\xi : \mathbb{R} \to X$ through x and σ in M. Moreover, we say that M is an isolated lifted invariant if there is a neighborhood $\mathscr{U}$ of M such that M is the maximal lifted invariant set on $\mathscr{U}$.

Proposition 1.3. [2, Proposition 3.21] *The uniform attractor of the non-autonomous dynamical system $(\varphi_\eta, \theta)_{(X,\Sigma_\eta)}$, if it exists, is the maximal bounded isolated lifted invariant set of X.*

Now let us define the skew-product semiflow associated to a non-autonomous dynamical system $(\varphi_\eta, \theta)_{(X,\Sigma_\eta)}$. For $t \geq 0$ and $(x,\sigma) \in X \times \Sigma_\eta$ let

$$\pi_\eta(t)(x,\sigma) := (\varphi_\eta(t,\sigma)x, \theta_t\sigma) \in X \times \Sigma_\eta.$$

It is easy to see that $\{\pi_\eta(t) : t \geq 0\}$ is a semigroup on $X \times \Sigma_\eta$.

The existence of uniform attractor for the non-autonomous dynamical system $(\varphi_\eta, \theta)_{(X,\Sigma_\eta)}$ is equivalent to the existence of a global attractor for the skew-product semiflow, see [2, Proposition 3.1]. Furthermore, we have the relation $\Pi_X \mathbb{A}_\eta = \mathscr{A}_\eta$, where $\mathbb{A}_\eta$ denotes the global attractor for the skew-product, Π_X is the continuous first coordinate projection of $X \times \Sigma_\eta$ and $\mathscr{A}_\eta$ the uniform attractor of the non-autonomous dynamical system.

There is also a relation between uniform attractors and pullback attractors which is shown by means of cocycle attractors. A non-autonomous set is a family of subsets of

X which is indexed in Σ_η, formally $\{B(\sigma) \subset X : \sigma \in \Sigma_\eta\}$. We say the non-autonomous set is bounded, closed or compact, if every fiber $B(\sigma)$ is bounded, closed or compact on X, respectively.

Definition 1.4. Suppose $\{(\varphi_\eta, \theta)\}_{(X,\Sigma_\eta)}$ is a non-autonomous dynamical system and $\{\theta_t : t \in \mathbb{R}\}$ is a driving group over Σ_η. We say that the compact non-autonomous set $\{A_\eta(\sigma) : \sigma \in \Sigma_\eta\}$ is a cocycle attractor for the non-autonomous dynamical system if

(i) $\{A_\eta(\sigma)\}_{\sigma\in\Sigma_\eta}$ is invariant, that is $\varphi_\eta(t,\sigma)A_\eta(\sigma) = A_\eta(\theta_t\sigma)$ for all $t \in \mathbb{R}^+$ and $\sigma \in \Sigma_\eta$;

(ii) $\{A_\eta(\sigma)\}_{\sigma\in\Sigma_\eta}$ Σ_η-pullback attracts bounded subsets of X, that is, for any bounded subset $D \subset X$, we have

$$\operatorname{dist}(\varphi_\eta(t,\theta_{-t}\sigma)D, A_\eta(\sigma)) \to 0, \text{ as } t \to \infty. \tag{4}$$

If $\bigcup_{\sigma\in\Sigma_\eta} A_\eta(\sigma)$ is precompact and the pullback attraction of (4) is uniform on $\sigma \in \Sigma_\eta$, then the existence of the cocycle attractor is equivalent to the existence of the global attractor for the skew-product semiflow on $X \times \Sigma_\eta$, therefore to the existence of uniform attractor, see [2, Theorem 3.11]).

Moreover, we can associate the cocycle attractor with a pullback attractor for an evolution process. Given $\sigma \in \Sigma_\eta$, define the evolution process $\{S^\eta_\sigma(t,s) : t \geq s \in \mathbb{R}\}$ by the expression

$$S^\eta_\sigma(t,s) := \varphi_\eta(t-s, \theta_s\sigma). \tag{5}$$

The family $\{A_\eta(\theta_t\sigma) : t \in \mathbb{R}\}$, where $\{A_\eta(\sigma) : \sigma \in \Sigma_\eta\}$ is the cocycle attractor described above, is a pullback attractor for the evolution process (5); indeed, we have that

(i) $\{A_\eta(\theta_t\sigma) : t \in \mathbb{R}\}$ is invariant over $S^\eta_\sigma(t,s)$, that is,

$$S^\eta_\sigma(t,s)A_\eta(\theta_s\sigma) = A_\eta(\theta_t\sigma), \text{ for all } t \geq s.$$

(ii) $A_\eta(\theta_t\sigma)$ pullback attracts bounded subsets B of X by $S^\eta_\sigma(t,s)$ at the time t,

$$\lim_{s\to-\infty} \operatorname{dist}(S^\eta_\sigma(t,s)B, A_\eta(\theta_t\sigma)) \to 0.$$

In this paper we study equi-attraction and continuity of attractors for all of these different notions of attractors. More particularly, in Section 2.1 we prove that equi-attraction is equivalent to continuity of global attractors for skew-product semiflows. It is remarkable that even though the skew-product is a semigroup on a product space, this space changes depending on a parameter on the driving system, so that our results cannot be deduced directly from previous ones in the literature. In Section 2.2 we show that uniform attraction of bounded subsets (uniformly with respect to parameters) is equivalent to continuity of uniform attractors, extending the results of [9] to the case of uniform attractors. Then, in Subsection 2.3 we discuss this equivalence for cocycle attractors, extending the results on [13]. In Section 3 we relate continuity of global attractors for skew-product semiflows with continuity of uniform and cocycle attractors for non-autonomous dynamical systems. An application to a non-autonomous perturbation of an autonomous differential equation is also shown to illustrate our abstract results.

2. **Equi-attraction for non-autonomous systems.**

This section is devoted to the definition of equi-attraction on the different settings for non-autonomous systems and relate equi-attraction with continuity of the attractor.

We start by stating the assumptions for the rest of the section. Let X be a Banach space and $(\mathscr{C}, \rho)$ a complete metric space with a semigroup $\{\theta_t : t \geq 0\}$. Then take a family of elements on $\mathscr{C}$, $\{\sigma_\eta \in \mathscr{C} : \eta \in [0,1]\}$, such that it holds (2) and (3) .

It is easy to see from (2) that the family $\{\Gamma_\eta\}_{\eta\in[0,1]}$ is continuous at $\eta = 0$ on $\mathscr{C}$.

Thus, suppose that $\{(\varphi_\eta,\sigma)_{(X,\Sigma_\eta)}\}_{\eta\in[0,1]}$ is a family of non-autonomous dynamical systems and define the skew-product semiflow $\pi_\eta(t)\colon X\times\Sigma_\eta\to X\times\Sigma_\eta$ by

$$\pi_\eta(t)(x,\sigma)=(\varphi_\eta(t,\sigma)x,\theta_t\sigma).$$

Suppose that for each η there is a global attractor $\mathbb{A}_\eta$ for the skew-product semiflow. So there exists a uniform attractor $\mathscr{A}_\eta=\Pi_X\mathbb{A}_\eta$ for the non-autonomous dynamical system $(\varphi_\eta,\theta)_{(X,\Sigma_\eta)}$ on X.

Then the global attractor of the skew-product semiflow $\mathbb{A}_\eta$ may be expressed in the product space $X\times\Sigma_\eta$ in terms of the cocycle attractor. We have that $\mathbb{A}_\eta\subset\mathscr{A}_\eta\times\Sigma_\eta$, for all $\eta\in[0,1]$, but the equality does not hold in general. However, if the non-autonomous set $\{A_\eta(\sigma):\sigma\in\Sigma_\eta\}$ is the cocycle attractor for the nds $(\varphi_\eta,\theta)_{(X,\Sigma_\eta)}$, then (see [4, Theorem 3.4] or [20, Propositions 3.30 and 3.31])

$$\mathbb{A}_\eta=\bigcup_{\sigma\in\Sigma_\eta}A_\eta(\sigma)\times\{\sigma\}.$$

We assume, furthermore, that for the family $\{\sigma_\eta\}_{\eta\in[0,1]}$ it holds that

$$\mathbb{A}_\eta=\overline{\bigcup_{\tau\in\mathbb{R}}A_\eta(\theta_\tau\sigma_\eta)\times\{\theta_\tau\sigma_\eta\}},\tag{6}$$

for each $\eta\in[0,1]$.

The next lemma provides us with a sufficient condition for (6) to hold true.

Lemma 2.1. *Assume that*

$$\lim_{t\to\infty}\sup_{\tau\in\mathbb{R}}\mathrm{dist}(\varphi_\eta(t,\theta_{\tau-t}\sigma_\eta)B,A_\eta(\theta_\tau\sigma_\eta))=0,\tag{7}$$

for any bounded subset B of X Then, the global attractor for the skew-product semiflow is given by

$$\mathbb{A}_\eta=\overline{\bigcup_{\tau\in\mathbb{R}}A_\eta(\theta_\tau\sigma_\eta)\times\{\theta_\tau\sigma_\eta\}}.$$

Proof. One of the inclusions is trivial, since $A_\eta(\theta_\tau\sigma_\eta)\times\{\theta_\tau\sigma_\eta\}\subset\mathbb{A}_\eta$, for every $\tau\in\mathbb{R}$ and $\mathbb{A}_\eta$ is closed in $X\times\Sigma$.

As for the other, let $(x,\sigma)\in\mathbb{A}_\eta$ be given. Note that, by our definition of Σ_η, there is a sequence $\tau_k\in\mathbb{R}$ such that $\theta_{\tau_k}\sigma_\eta\to\sigma$ on Σ_η. We must show that there is also a sequence $x_k\in A_\eta(\theta_{\tau_k}\sigma_\eta)$ such that $x_k\to x$ on X as $k\to\infty$.

Indeed, if $\{S^\eta_\sigma(t,s):t\geq s\}$ denotes the evolution process defined in (5), we know that $x\in A_\eta(\sigma)$, which is the pullback attractor for $S^\eta_\sigma(\cdot,\cdot)$, at time $t=0$, thus there is a global solution γ for $\{S^\eta_\sigma(t,s):t\geq s\}$ such that $\gamma(t)\in A_\eta(\theta_t\sigma)$, for all $t\in\mathbb{R}$, $\gamma(0)=x$ and it holds that

$$S^\eta_\sigma(t,s)\gamma(s)=\gamma(t),\text{ for every }t\geq s.\tag{8}$$

Observe that $\Gamma:=\bigcup_{t\in\mathbb{R}}\gamma(t)$ is bounded. So, by (7),

$$\lim_{t\to\infty}\sup_{\tau\in\mathbb{R}}\mathrm{dist}(\varphi_\eta(t,\theta_{-t}\theta_\tau\sigma_\eta)\Gamma,A_\eta(\theta_\tau\sigma_\eta))=0.$$

And, given $\varepsilon>0$, there exists $t_0=t_0(\Gamma,\varepsilon)>0$ such that for $t\geq t_0$

$$\sup_{\tau\in\mathbb{R}}\sup_{r\in\mathbb{R}}\mathrm{dist}(\varphi_\eta(t,\theta_{-t}\theta_\tau\sigma_\eta)\gamma(r),A_\eta(\theta_\tau\sigma_\eta))<\varepsilon/2.$$

Thus, for every $r\in\mathbb{R}$, there exists $x^k_r\in A_\eta(\theta_{\tau_k}\sigma_\eta)$ with

$$\mathrm{d}(\varphi_\eta(t_0,\theta_{-t_0}\theta_{\tau_k}\sigma_\eta)\gamma(r),x^k_r)<\varepsilon/2.\tag{9}$$

Then, as $\theta_{\tau_k}\sigma_\eta \to \sigma$ in Σ_η and the nds $(\varphi_\eta, \theta)_{(X,\Sigma_\eta)}$ is uniformly continuous on compact subsets of X, we have that $\varphi_\eta(t_0, \theta_{-t_0}\theta_{\tau_k}\sigma_\eta)\gamma(r) \to \varphi_\eta(t_0, \theta_{-t_0}\sigma)\gamma(r)$, as $k \to \infty$ uniformly for $r \in \mathbb{R}$. Therefore, there is $k_0 \in \mathbb{N}$ such that if $k \geq k_0$ we have

$$\sup_{r\in\mathbb{R}} \mathrm{d}(\varphi_\eta(t_0, \theta_{-t_0}\theta_{\tau_k}\sigma_\eta)\gamma(r), \varphi_\eta(t_0, \theta_{-t_0}\sigma)\gamma(r)) < \varepsilon/2. \tag{10}$$

Hence, from (9) and (10), choosing $r = -t_0$ and taking $x_k := x^k_{-t_0} \in A_\eta(\theta_{\tau_k}\sigma_\eta)$, we have that

$$\mathrm{d}(\varphi_\eta(t_0, \theta_{-t_0}\sigma)\gamma(-t_0), x_k) < \varepsilon. \tag{11}$$

Therefore, by (8), noting that $x = S^\eta_\sigma(0, -t_0)\gamma(-t_0) = \varphi_\eta(t_0, \theta_{-t_0}\sigma)\gamma(-t_0)$, and thanks to (11) the proof is concluded. □

Hypothesis (7) is needed in order to prove that existence of cocycle attractors implies existence of global attractors for skew-product semiflows. It plays an important role on the relation of equi-attraction and continuity.

2.1. **On skew-product semiflows.** Skew-product semiflows are semigroups in a product space $X \times \Sigma_\eta$, so one could think that we can just apply the known results for continuity and equi-attraction in [13]. The difference here is that the family π_η acts on the phase space $X \times \Sigma_\eta$ and they change with the parameter $\eta \in [0, 1]$. As seen before Σ_η is the closure of the orbit of an element of a fixed metric space $\mathscr{C}$ (which in the case of the non-autonomous differential equation is $\mathscr{C}_b(\mathbb{R} \times X, X)$), thus possessing a common metric.

2.1.1. *Equi-attraction and continuity.* We prove here results of the equivalence between equi-attraction and continuity for skew-product semiflows.

Consider $(\Lambda, \mathrm{d}_\Lambda)$ a metric space (the parameter space). We recall that a family of subsets $\{C_\lambda\}_{\lambda\in\Lambda}$ of the metric space (X, d) is said to be

- upper semicontinuous at $\lambda = \lambda_0$ if
$$\mathrm{dist}(C_\lambda, C_{\lambda_0}) \to 0, \text{ as } \lambda \to \lambda_0;$$
- lower semicontinuous at $\lambda = \lambda_0$ if
$$\mathrm{dist}(C_{\lambda_0}, C_\lambda) \to 0, \text{ as } \lambda \to \lambda_0.$$

If $\{C_\lambda\}_{\lambda\in\Lambda}$ is both upper and lower semicontinuous at λ_0 it is said to be continuous at λ_0.

Suppose a continuity hypothesis on the skew-product semiflows, that is

$$\sup_{t\in[0,T]} \sup_{x\in K} \sup_{\tau\in\mathbb{R}} \mathrm{d}(\pi_\eta(t)(x, \theta_\tau\sigma_\eta), \pi_0(t)(x, \theta_\tau\sigma_0)) \to 0, \text{ as } \eta \to 0, \tag{12}$$

for any compact subset K of X and $T > 0$. Since, for $\tau \in \mathbb{R}$, $\pi_\eta(t)(x, \theta_\tau\sigma_\eta) \in X \times \Sigma_\eta$ and $\pi_0(t)(x, \theta_\tau\sigma_0) \in X \times \Sigma_0$, we compare both elements on $X \times \Sigma$.

Moreover, suppose that

$$\bigcup_{\eta\in[0,1]} \mathbb{A}_\eta \text{ is precompact in } X \times \Sigma. \tag{13}$$

We say that the family of global attractors $\{\mathbb{A}_\eta : \eta \in [0, 1]\}$ for a family of skew-product semiflows $\{\pi_\eta(t) : t \geq 0\}$ is *equi-attracting* if for every bounded B of X we have

$$\lim_{t\to\infty} \sup_{\eta\in[0,1]} \sup_{x\in B} \sup_{\sigma\in\Sigma_\eta} \mathrm{dist}(\pi_\eta(t)(x, \sigma), \mathbb{A}_\eta) \to 0. \tag{14}$$

Theorem 2.2. *Suppose the family of skew-product semiflows* $\{\pi_\eta(t) : t \geq 0\}_{\eta\in[0,1]}$ *with global attractors* $\{\mathbb{A}_\eta : \eta \in [0,1]\}$ *satisfy* (12) *and* (13). *If* $\{\mathbb{A}_\eta : \eta \in [0,1]\}$ *is equi-attracting, then this family of attractors is continuous at* $\eta = 0$*, that is,*

$$\lim_{\eta\to 0} \mathrm{dist}_{\mathrm{H}}(\mathbb{A}_\eta, \mathbb{A}_0) = 0.$$

Proof. Let $\mathscr{B} = \overline{\bigcup_{\eta\in[0,1]} \Pi_X \mathbb{A}_\eta} \subset X$, by (13) we know that $\mathscr{B}$ is compact in X. So, given $\varepsilon > 0$, there is $t_0 = t_0(\mathscr{B}, \varepsilon) > 0$ such that

$$\sup_{\eta\in[0,1]} \sup_{\sigma\in\Sigma_\eta} \sup_{x\in\mathscr{B}} \mathrm{dist}(\pi_\eta(t)(x,\sigma), \mathbb{A}_\eta) < \varepsilon/2, \text{ for } t \geq t_0, \tag{15}$$

from the equi-attraction of the global attractors.

On the other hand, our continuity assumption (12) on the skew-product semiflows implies that

$$\begin{aligned}
\mathrm{dist}(\mathbb{A}_\eta, \mathbb{A}_0) &\leq \mathrm{dist}(\pi_\eta(t_0)\mathbb{A}_\eta, \pi_0(t_0)[\mathscr{B}\times\Sigma_0]) + \mathrm{dist}(\pi_0(t_0)[\mathscr{B}\times\Sigma_0], \mathbb{A}_0) \\
&\leq \sup_{\tau\in\mathbb{R}} \mathrm{dist}(\pi_\eta(t_0)[\overline{A_\eta(\theta_\tau\sigma_\eta)\times\{\theta_\tau\sigma_\eta\}}], \pi_0(t_0)[\mathscr{B}\times\Sigma_0]) + \varepsilon/2 \\
&\leq \varepsilon.
\end{aligned}$$

For the other term of the Hausdorff distance, notice that

$$\begin{aligned}
&\mathrm{dist}(\mathbb{A}_0, \mathbb{A}_\eta) = \mathrm{dist}(\pi_0(t_0)\mathbb{A}_0, \mathbb{A}_\eta) \\
&\quad\leq \sup_{\tau\in\mathbb{R}} \mathrm{dist}(\pi_0(t_0)[\overline{A_0(\theta_\tau\sigma_0)\times\{\theta_\tau\sigma_0\}}], \pi_\eta(t_0)[\mathscr{B}\times\Sigma_\eta]) + \mathrm{dist}(\pi_\eta(t_0)[\mathscr{B}\times\Sigma_\eta], \mathbb{A}_\eta) \\
&\quad\leq \varepsilon.
\end{aligned}$$

Thanks to both inequalities the theorem is proved. □

We emphasize here that it is not an easy task to define and prove a result as above when the family of semigroups is defined on phase spaces that vary with the parameter. The problem arises as it is not possible to define a suitable notion of continuity of semigroups. An important feature of skew-product semiflows is that even though the phase space changes, there is a common one such that we are able to keep fixed.

We recall that continuity of global attractors does not imply that the family is equi-attracting, so we need the following additional hypothesis on the semigroups. Moreover we shall adapt the definitions to our skew-product semiflow, since the spaces change as the parameter η varies.

Definition 2.3. The family of skew-product semiflows $\{\pi_\eta(t) : t \geq 0\}_{\eta\in[0,1]}$ is said to be uniformly bounded if

$$\bigcup_{\eta\in[0,1]} \bigcup_{t\geq 0} \pi_\eta(t)[B\times\Upsilon_\eta] \text{ is bounded in } X\times\Sigma \tag{16}$$

whenever $B \subset X$ and $\Upsilon_\eta \subset \Sigma_\eta$ are bounded.

In this case, we ask $\bigcup_{\eta\in[0,1]} \Sigma_\eta$ to be precompact in $\mathscr{C}$, therefore it is bounded. As each Σ_η is invariant over $\{\theta_t : t \geq 0\}$, if

$$\bigcup_{\eta\in[0,1]} \bigcup_{t\geq 0} \bigcup_{\sigma\in\Sigma_\eta} \varphi_\eta(t,\sigma)B \text{ is bounded,}$$

whenever $B \subset X$ is bounded, then the family $\{\pi_\eta(t) : t \geq 0\}_{\eta\in[0,1]}$ is uniformly bounded.

Definition 2.4. The family of skew-product semiflows $\{\pi_\eta(t) : t \geq 0\}_{\eta\in[0,1]}$ is said to be collectively asymptotically compact if, for $t_k \overset{k\to\infty}{\longrightarrow} \infty$ in $(0,\infty)$, $\eta_k \overset{k\to\infty}{\longrightarrow} 0^+$ in $(0,1]$, $\{x_k\}$ bounded in X and $\sigma_k \in \Sigma_{\eta_k}$ are such that $\{\pi_{\eta_k}(t_k)(x_k,\sigma_k)\}$ is bounded in $X\times\Sigma$, the sequence $\{\pi_{\eta_k}(t_k)(x_k,\sigma_k)\}$ has a convergent subsequence.

Again, as $\bigcup_{\eta\in[0,1]}\Sigma_\eta$ is precompact in $\mathscr{C}$ and Σ_η is invariant over $\{\theta_t : t \geq 0\}$, every sequence $\{\sigma_k\}_{k\in\mathbb{N}}$ (or $\{\theta_{\tau_k}\sigma_k\}_{k\in\mathbb{N}}$) has a convergent subsequence. Thus, in order to verify collective asymptotic compactness we may only care about the term of $\{\pi_{\eta_k}(t_k)(x_k,\sigma_k)\}$ that lies on X. More precisely, if for every sequences $t_k \to \infty$ and $\{x_k\}$ bounded in X, for which the corresponding sequence $\{\varphi_{\eta_k}(t_k,\sigma_k)x_k\}$ is also bounded in X, we have that the sequence $\{\varphi_{\eta_k}(t_k,\sigma_k)x_k\}$ admits a convergent subsequence, then the family of skew-product semiflows is collectively asymptotically compact.

The following result is the converse to Theorem 2.2.

Theorem 2.5. *Suppose $\{\pi_\eta(t) : t \geq 0\}_{\eta\in[0,1]}$ is uniformly bounded and collectively asymptotic compact with a family of global attractors that is continuous at $\eta = 0$. Then the family of global attractors $\{\mathbb{A}_\eta : \eta \in [0,1]\}$ is equi-attracting.*

Proof. Suppose by contradiction that there exist $\varepsilon > 0$, sequences $\eta_k \to 0$, $t_k \to \infty$, $\tau_k \in \mathbb{R}$ and $\{x_k\} \subset X$ bounded such that

$$\operatorname{dist}(\pi_{\eta_k}(t_k)(x_k,\theta_{\tau_k}\sigma_\eta),\mathbb{A}_0) \geq \varepsilon, \tag{17}$$

for all $k \in \mathbb{N}$. Note at first that there exists a sequence $\sigma_k \in \Sigma_{\eta_k}$, although, as $\{\theta_\tau\sigma_{\eta_k} : \tau\in\mathbb{R}\}$ is dense on Σ_{η_k} and the skew-product semiflow is continuous, we can assume that the sequence in (17) may be taken in $\{\theta_\tau\sigma_{\eta_k} : \tau\in\mathbb{R}\}$.

Observe that

$$\mathscr{B} = \overline{\bigcup_{k\in\mathbb{N}}\bigcup_{t\geq 0}\pi_{\eta_k}(t)(x_k,\sigma_k)}$$

is bounded on $X\times\Sigma$ since the skew-product semiflows are uniformly bounded.

Fix $t > 0$. Let $s_k := t_k - t > 0$, the family of skew-product semiflows is collectively asymptotically compact, thus $\{\pi_{\eta_k}(s_k)(x_k,\theta_{\tau_k}\sigma_{\eta_k}) : k \in \mathbb{N}\}$ is relatively compact. Assume then that $\pi_{\eta_k}(s_k)(x_k,\theta_{\tau_k}\sigma_{\eta_k}) \to (b,\xi) \in X\times\Sigma_0$, on the topology of $X\times\Sigma$, as $k\to\infty$.

Therefore, by (12) and the above,

$$\begin{aligned}\pi_0(t)(b,\xi) &= \lim_{k\to\infty}\pi_{\eta_k}(t)\pi_{\eta_k}(s_k)(x_k,\theta_{\tau_k}\sigma_{\eta_k})\\ &= \lim_{k\to\infty}\pi_{\eta_k}(t_k)(x_k,\theta_{\tau_k}\sigma_{\eta_k}).\end{aligned}$$

That is, for any given $t > 0$, there is $(b,\xi)\in X\times\Sigma_0$ such that $\operatorname{dist}(\pi_0(t)(b,\xi),\mathbb{A}_0) \geq \varepsilon$ and that contradicts the fact that $\mathbb{A}_0$ is the global attractor for $\{\pi_0(t) : t\geq 0\}$. ☐

2.1.2. *Rates of convergence.* We can use the equi-attraction for semigroups to obtain a better estimate and rate of convergence of the global attractors for the skew-product semiflows. This is a consequence of the theorem for semigroups.

Theorem 2.6. *Suppose $\{\pi_\eta(t) : t \geq 0\}$ is a skew-product semiflow on the space $X\times\Sigma_\eta$, with global attractor $\mathbb{A}_\eta$, $\eta\in[0,1]$, such that $\mathbb{B} = \bigcup_{\eta\in[0,1]}\mathbb{A}_\eta \subset X\times\Sigma$ is precompact and define $\mathscr{B} = \Pi_X(\mathbb{B}) \subset X$. Assume that there exists a decreasing function $\zeta : [0,\infty)\to(0,\infty)$ such that $\zeta(0) = \zeta_0$, $\lim_{t\to\infty}\zeta(t) = 0$ and*

$$\sup_{\eta\in[0,1]}\operatorname{dist}(\pi_\eta(t)[\mathscr{B}\times\Sigma_\eta],\mathbb{A}_\eta) \leq \zeta(t), \tag{18}$$

for all $t \geq 0$.

Also, suppose that

$$\sup_{\tau\in\mathbb{R}}\sup_{x\in\mathscr{B}} \mathrm{d}(\pi_\eta(t)(x,\theta_\tau\sigma_\eta),\pi_0(t)(x,\theta_\tau\sigma_0)) \leq E_\eta(t), \text{ for all } t\geq 0, \tag{19}$$

where $E_\eta(t)\to 0$*, as* $\eta\to 0$*, for each* $t\geq 0$.

Then

$$\mathrm{dist}(\mathbb{A}_\eta,\mathbb{A}_0) \leq \inf_{\varepsilon\in(0,\zeta_0)} 2\{E_\eta(\zeta^{-1}(\varepsilon))+\varepsilon\}. \tag{20}$$

Proof. Fix $t\geq 0$. Remember that $\mathbb{A}_\eta=\pi_\eta(t)\mathbb{A}_\eta$ and that $\mathscr{B}=\Pi_X\mathbb{B}$ is precompact in X. Thus, we have that

$$\mathrm{dist}(\mathbb{A}_\eta,\mathbb{A}_0)\leq \mathrm{dist}(\pi_\eta(t)\mathbb{A}_\eta,\pi_0(t)\mathbb{B})+\mathrm{dist}(\pi_0(t)\mathbb{B},\mathbb{A}_0).$$

From Lemma 2.1

$$\mathrm{dist}(\pi_\eta(t)\mathbb{A}_\eta,\pi_0(t)\mathbb{B})=\sup_{\tau\in\mathbb{R}}\mathrm{dist}(\pi_\eta(t)[A_\eta(\theta_\tau\sigma_\eta)\times\{\theta_\tau\sigma_\eta\}],\pi_0(t)\mathbb{B}).$$

Since $\mathbb{B}\subset\bigcup_{\eta\in[0,1]}\mathscr{B}\times\Sigma_\eta$, the above inequalities ensure that

$$\mathrm{dist}(\mathbb{A}_\eta,\mathbb{A}_0)\leq E_\eta(t)+\zeta(t). \tag{21}$$

Analogously, we have that

$$\mathrm{dist}(\mathbb{A}_0,\mathbb{A}_\eta)\leq \mathrm{dist}(\pi_0(t)\mathbb{A}_0,\pi_\eta(t)[\mathscr{B}\times\Sigma_\eta])+\mathrm{dist}(\pi_\eta(t)[\mathscr{B}\times\Sigma_\eta],\mathbb{A}_\eta). \tag{22}$$

In order to complete the proof, from (21) and (22) we choose $\varepsilon\leq\zeta_0$ and $t=\zeta^{-1}(\varepsilon)$. □

Corollary 2.7. [6, Corollary 3.20] *Suppose, in addition to the hypotheses in the last theorem, that there exist* $c>0$ *and* $v>0$ *such that* $\zeta(t)=ce^{-vt}$, $t\geq 0$*, and that* $E_\eta(t)=\rho(\eta)e^{Lt}$*, with* $L>0$ *and* $\rho\colon[0,1]\to[0,\infty)$ *continuous with* $\rho(0)=0$.

Then, there is a constant $\overline{c}>0$ *such that*

$$\mathrm{dist}_{\mathrm{H}}(\mathbb{A}_\eta,\mathbb{A}_0)\leq\overline{c}\rho(\eta)^{\frac{v}{v+L}}. \tag{23}$$

2.2. **On uniform attractors.** In this section we state a definition for equi-attraction in the sense of uniform attractors and show the equivalence between equi-attraction and continuity for uniform attractors.

We exploit the connection between uniform attractors and global attractors for skew-product semiflows, and also prove that continuity for attractors of skew-product semiflow is equivalent to continuity of uniform attractors, as long as the driving group also possesses continuous attractors.

The sense of equi-attraction we are going to use for uniform attractors of a family of non-autonomous dynamical systems is defined as follows:

Definition 2.8. Let $\{(\varphi_\eta,\theta)_{(X,\Sigma_\eta)}\}_{\eta\in[0,1]}$ be a family of non-autonomous dynamical systems with uniform attractors $\mathscr{A}_\eta$. $\{\mathscr{A}_\eta:\eta\in[0,1]\}$ is said to uniformly equi-attract bounded subsets of X, if for all bounded $B\subset X$ we have

$$\lim_{t\to\infty}\sup_{\eta\in[0,1]}\sup_{\sigma\in\Sigma_\eta}\mathrm{dist}(\varphi_\eta(t,\sigma)B,\mathscr{A}_\eta)=0. \tag{24}$$

We define this concept as uniform equi-attraction since the uniform attractor $\mathscr{A}_\eta$ of $(\varphi_\eta,\theta)_{(X,\Sigma_\eta)}$ is the minimal closed set of X that uniformly (for $\sigma\in\Sigma_\eta$) attracts all bounded subsets of X.

2.2.1. *Uniform equi-attraction and continuity of uniform attractors.* Let us consider a family of non-autonomous dynamical systems $\{(\varphi_\eta,\theta)_{(X,\Sigma_\eta)} : \eta \in [0,1]\}$ with uniform attractors $\mathscr{A}_\eta$, $\eta \in [0,1]$, and assume that

$$\bigcup_{\eta\in[0,1]} \mathscr{A}_\eta \text{ is precompact in } X. \tag{25}$$

Moreover, suppose that for all compact sets K of X and any $T > 0$ we have

$$\sup_{t\in[0,T]} \sup_{\tau\in\mathbb{R}} \sup_{x\in K} \mathrm{d}(\varphi_\eta(t,\theta_\tau\sigma_\eta)x, \varphi_0(t,\theta_\tau\sigma_0)x) \to 0, \text{ as } \eta \to 0^+. \tag{26}$$

The proof of the following result follows the same steps as in Theorem 2.2. However there are some differences which require our attention. For this reason, we prefer to fully prove it.

Theorem 2.9. *Assume that the family of uniform attractors $\{\mathscr{A}_\eta : \eta \in [0,1]\}$ associated to the non-autonomous dynamical systems $\{(\varphi_\eta,\theta)_{(X,\Sigma_\eta)} : \eta \in [0,1]\}$ uniformly equi-attracts bounded subsets of X, and that* (25) *and* (26) *hold. Then the family of uniform attractors is continuous at $\eta = 0$, that is*

$$\mathrm{dist_H}(\mathscr{A}_\eta,\mathscr{A}_0) \to 0 \textit{ as } \eta \to 0. \tag{27}$$

Proof. First, observe that the subset $\mathscr{B} = \overline{\bigcup_{\eta\in[0,1]} \mathscr{A}_\eta}$ is bounded in X. Therefore, given $\varepsilon > 0$, there exists $t_0 = t_0(\varepsilon,\mathscr{B}) \geq 0$ such that

$$\sup_{\eta\in[0,1]} \sup_{\sigma\in\Sigma_\eta} \mathrm{dist}(\varphi_\eta(t,\sigma)\mathscr{B},\mathscr{A}_\eta) \leq \varepsilon/2, \text{ for all } t \geq t_0. \tag{28}$$

From (26) there is $0 < \eta_0 < 1$ such that, for all $\eta \leq \eta_0$,

$$\sup_{\tau\in\mathbb{R}} \mathrm{dist}(\varphi_\eta(t_0,\theta_\tau\sigma_\eta)\mathscr{A}_\eta, \varphi_0(t_0,\theta_\tau\sigma_0)\mathscr{B}) \leq \varepsilon/2.$$

From the lifted invariance property of the uniform attractors and Lemma 2.1 it is clear that $\mathscr{A}_\eta \subseteq \bigcup_{\sigma\in\Sigma_\eta} \varphi_\eta(t,\sigma)\mathscr{A}_\eta$, from the continuity of the nds $\{(\varphi_\eta,\theta)\}_{(X,\Sigma_\eta)}$ we have that

$$\mathscr{A}_\eta \subseteq \overline{\bigcup_{\tau\in\mathbb{R}} \varphi_\eta(t_0,\theta_\tau\sigma_\eta)\mathscr{A}_\eta}.$$

Therefore,

$$\begin{aligned}
\mathrm{dist}(\mathscr{A}_\eta,\mathscr{A}_0) &\leq \mathrm{dist}(\mathscr{A}_\eta, \cup_{\sigma\in\Sigma_0}\varphi_0(t_0,\sigma)\mathscr{B}) + \mathrm{dist}(\cup_{\sigma\in\Sigma_0}\varphi_0(t_0,\sigma)\mathscr{B},\mathscr{A}_0) \\
&\leq \sup_{\tau\in\mathbb{R}} \mathrm{dist}(\varphi_\eta(t_0,\theta_\tau\sigma_\eta)\mathscr{A}_\eta, \cup_{\sigma\in\Sigma_0}\varphi_0(t_0,\sigma)\mathscr{B}) + \sup_{\sigma\in\Sigma_0} \mathrm{dist}(\varphi_0(t_0,\sigma)\mathscr{B},\mathscr{A}_0) \\
&\leq \varepsilon.
\end{aligned}$$

Now, to prove that $\mathrm{dist}(\mathscr{A}_0,\mathscr{A}_\eta) \to 0$ when $\eta \to 0$, we use similar arguments. As above, we have that

$$\mathrm{dist}(\mathscr{A}_0,\mathscr{A}_\eta) \leq \sup_{\tau\in\mathbb{R}} \mathrm{dist}(\varphi_0(t_0,\theta_\tau\sigma_0)\mathscr{A}_0, \cup_{\sigma\in\Sigma_\eta}\varphi_\eta(t_0,\sigma)\mathscr{A}_0) + \sup_{\sigma\in\Sigma_\eta} \mathrm{dist}(\varphi_\eta(t_0,\sigma)\mathscr{A}_0,\mathscr{A}_\eta).$$

And we recall that, as in (28), equi-attraction implies that $\mathrm{dist}(\varphi_\eta(t_0,\sigma)\mathscr{A}_0,\mathscr{A}_\eta) \leq \varepsilon/2$ for every $\sigma \in \Sigma_\eta$ and $\eta \in [0,1]$. Also, the first term on the inequality can be estimated in a similar way using our continuity hypothesis. □

In order to prove the converse, as done before in [9], we need more uniformity on the assumptions on the non-autonomous dynamical systems. Below we give an expected definition that is made necessary in the proof.

Definition 2.10. A family of non-autonomous dynamical systems $\{(\varphi_\eta,\theta)_{X,\Sigma_\eta}\}_{\eta\in[0,1]}$ is said to be uniformly bounded if

$$\bigcup_{\eta\in[0,1]}\bigcup_{\sigma\in\Sigma_\eta}\bigcup_{t\geq 0}\varphi_\eta(t,\sigma)B \text{ is bounded,} \tag{29}$$

for every bounded subset B of X.

Definition 2.11. A family of non-autonomous dynamical systems $\{(\varphi_\eta,\theta)_{X,\Sigma_\eta}\}_{\eta\in[0,1]}$ is said to be collectively asymptotically compact if for every sequences $\eta_k\to 0$, $t_k\to\infty$, $\sigma_k\in\Sigma_{\eta_k}$ and $\{x_k\}\subset X$ bounded such that $\{\varphi_{\eta_k}(t_k,\sigma_k)x_k\}$ is also bounded in X, then the set $\{\varphi_{\eta_k}(t_k,\sigma_k)x_k : k\in\mathbb{N}\}$ is relatively compact in X.

More precisely, compactness and invariance of Σ imply that these definitions are equivalent, as we discussed at Subsection 2.1.

Theorem 2.12. *Let the family of non-autonomous dynamical systems* $\{(\varphi_\eta,\theta)_{(X,\Sigma_\eta)}\}_{\eta\in[0,1]}$ *with uniform attractor* $\mathscr{A}_\eta$*,* $\eta\in[0,1]$*. Suppose that* (26) *holds and that* $\{(\varphi_\eta,\theta)_{(X,\Sigma_\eta)}\}_{\eta\in[0,1]}$ *is a uniformly bounded and collectively uniformly asymptotically compact family, and furthermore that*

$$\mathrm{dist_H}(\mathscr{A}_\eta,\mathscr{A}_0)\to 0, \text{ as } \eta\to 0. \tag{30}$$

Then, for every sequence $\{\eta_k\}_{k\in\mathbb{N}}$*, with* $\eta_k\to 0$ *when* $k\to\infty$*,* $\overline{\bigcup_{k\in\mathbb{N}}\mathscr{A}_{\eta_k}}$ *is compact and the sequence* $\{\mathscr{A}_{\eta_k} : k\in\mathbb{N}\}$ *uniformly equi-attracts bounded subsets of* X*. Consequently there exists* $\eta_0>0$ *such that* $\bigcup_{\eta\in[0,\eta_0]}\mathscr{A}_\eta$ *is precompact and the family* $\{\mathscr{A}_\eta\}_{\eta\in[0,\eta_0]}$ *uniformly equi-attracts bounded subsets of* X*.*

Proof. Suppose by contradiction that exist $\varepsilon>0$ and sequences $\eta_k\to 0$, $t_k\to\infty$, when $k\to\infty$, $\tau_k\in\mathbb{R}$ and bounded sequence $\{x_k\}_{k\in\mathbb{N}}$ such that $\mathrm{dist}(\varphi_{\eta_k}(t_k,\theta_{\tau_k}\sigma_{\eta_k})x_k,\mathscr{A}_0)\geq\varepsilon$, for all $k\in\mathbb{N}$.

The family of non-autonomous dynamical systems is uniformly bounded, so

$$B=\overline{\bigcup_{k\in\mathbb{N}}\bigcup_{t\geq 0}\varphi_{\eta_k}(t,\theta_{\tau_k}\sigma_{\eta_k})x_k} \text{ is bounded.}$$

Fix $t>0$ and define the sequence $s_k := t_k-t>0$. From the collective uniform asymptotic compactness property the sequence $\{\varphi_{\eta_k}(s_k,\theta_{\tau_k}\sigma_{\eta_k})x_k : k\in\mathbb{N}\}$ has a convergent subsequence, and assume without loss of generality that $\varphi_{\eta_k}(s_k,\theta_{\tau_k}\sigma_{\eta_k})x_k\to b$ in X, when $k\to\infty$.

It follows from the uniform asymptotic compactness and from (26), that for every $t>0$ there exists $b\in B$ and $\sigma\in\Sigma_0$ such that

$$\mathrm{dist}(\varphi_0(t,\sigma)b,\mathscr{A}_0)\geq\varepsilon.$$

The above contradicts the fact that $\mathscr{A}_0$ is the global attractor for φ_0 on X. □

2.2.2. *Rates of convergence.* One interesting remark about equi-attraction is that if we have an explicit bound for the attraction we are able to transfer it to get an upper bound on the closeness of the attractors. As expected, we have also this result in the case of uniform attractors.

Theorem 2.13. *Consider a family of non-autonomous dynamical systems* $\{(\varphi_\eta,\theta)_{(X,\Sigma_\eta)}\}_{\eta\in[0,1]}$ *with uniform attractors* $\mathscr{A}_\eta$ *such that* $\mathscr{B}=\bigcup_{\eta\in[0,1]}\mathscr{A}_\eta$ *is precompact in* X*.*

Assume that there exists a strictly decreasing function $\zeta:[0,\infty)\to(0,\infty)$ *with* $\zeta(0)=\zeta_0$ *and* $\lim_{s\to\infty}\zeta(s)=0$ *which*

$$\sup_{\eta\in[0,1]}\sup_{\sigma\in\Sigma_\eta}\mathrm{dist}(\varphi_\eta(t,\sigma)\mathscr{B},\mathscr{A}_\eta)\leq\zeta(t) \tag{31}$$

for all $t \geq 0$.

Suppose also that

$$\sup_{x\in\mathscr{B}}\sup_{\tau\in\mathbb{R}} \mathrm{d}(\varphi_\eta(t,\theta_\tau\sigma_\eta)x, \varphi_0(t,\theta_\tau\sigma_0)x) \leq E_\eta(t), \text{ for all } t \geq 0, \tag{32}$$

where $E_\eta(t) \to 0$, as $\eta \to 0$, for each t.

Then

$$\mathrm{dist_H}(\mathscr{A}_\eta, \mathscr{A}_0) \leq \inf_{\varepsilon\in(0,\zeta_0]} 2\{E_\eta(\zeta^{-1}(\varepsilon)) + \varepsilon\}.$$

Proof. First, let us fix $t \geq 0$. Note that $\mathscr{A}_\eta \subset \bigcup_{\sigma\in\Sigma_\eta} \varphi_\eta(t,\sigma)\mathscr{A}_\eta$, so we must have

$$\mathrm{dist}(\mathscr{A}_\eta,\mathscr{A}_0) \leq \sup_{\tau\in\mathbb{R}} \mathrm{dist}(\varphi_\eta(t,\theta_\tau\sigma_\eta)\mathscr{A}_\eta, \cup_{\sigma\in\Sigma_0}\varphi_0(t,\sigma)\mathscr{B}) + \mathrm{dist}(\cup_{\sigma\in\Sigma_0}\varphi_0(t,\sigma)\mathscr{B},\mathscr{A}_0).$$

From (32) we derive

$$\begin{aligned}\sup_{\tau\in\mathbb{R}} \mathrm{dist}(\varphi_\eta(t,\theta_\tau\sigma_\eta)\mathscr{A}_\eta, \cup_{\sigma\in\Sigma_0}\varphi_0(t,\sigma)\mathscr{B}) &\leq \sup_{x\in\mathscr{B}}\sup_{\tau\in\mathbb{R}} \mathrm{d}(\varphi_\eta(t,\theta_\tau\sigma_\eta)x, \varphi_0(t,\theta_\tau\sigma_0)x)\\ &\leq E_\eta(t).\end{aligned}$$

Consequently, $\mathrm{dist}(\mathscr{A}_\eta,\mathscr{A}_0) \leq E_\eta(t) + \zeta(t)$.

Also notice that

$$\mathrm{dist}(\mathscr{A}_0,\mathscr{A}_\eta) \leq \sup_{\tau\in\mathbb{R}}\sup_{\sigma\in\Sigma_\eta} \mathrm{dist}(\varphi_0(t,\theta_\tau\sigma_0)\mathscr{A}_0, \varphi_\eta(t,\sigma)\mathscr{A}_0) + \sup_{\sigma\in\Sigma_\eta} \mathrm{dist}(\varphi_\eta(t,\sigma)\mathscr{A}_0,\mathscr{A}_\eta).$$

Taking $\varepsilon \leq \zeta_0$ and letting $t = \zeta^{-1}(\varepsilon)$ we combine the inequalities above to deduce

$$\mathrm{dist}_H(\mathscr{A}_\eta,\mathscr{A}_0) \leq \inf_{\varepsilon\in(0,\zeta_0]} 2\{E_\eta(\zeta^{-1}(\varepsilon) + \varepsilon\},$$

as it was claimed. □

Corollary 2.14. *In addition to the hypotheses in the theorem above suppose that there exist $c > 0$ and $\nu > 0$ such that $\zeta(t) = ce^{-\nu t}$, $t \geq 0$, and $E_\eta(t) = \rho(\eta)e^{Lt}$, with $L > 0$ and $\rho\colon [0,1] \to [0,\infty)$ continuous with $\rho(0) = 0$. Then, there is a constant $\bar{c} > 0$ such that*

$$\mathrm{dist_H}(\mathscr{A}_\eta,\mathscr{A}_0) \leq \bar{c}\rho(\eta)^{\frac{\nu}{\nu+L}}. \tag{33}$$

2.3. **On cocycle and pullback attractors for non-autonomous dynamical systems.** In this section we discuss equi-attraction for cocycle attractors. Some results on this topic have already been obtained as can be checked in the existing literature (see for example [4, 5, 6, 11, 13, 19, 20, 23]).

Let us assume that $\{\theta_t : t \in \mathbb{R}\}$ is a group over Σ_η, for each $\eta \in [0,1]$, which is compact and invariant, and also $\bigcup_{\eta\in[0,1]}\Sigma_\eta$ precompact. Suppose that the non-autonomous compact set $\{A_\eta(\sigma) : \sigma \in \Sigma_\eta\}$ is the cocycle attractor for each $\eta \in [0,1]$. Recall that $\mathscr{A}_\eta = \bigcup_{\sigma\in\Sigma_\eta} A_\eta(\sigma)$.

The uniform equi-attraction of cocycle attractors is in a way stronger thean the uniform attractors. That is made clear below, since continuity of cocycle attractors implies continuity of uniform attractors but the converse is not true in general.

2.3.1. *Equi-attraction and continuity of cocycle attractors.* We recall that the family of cocycle attractors have a close relation to pullback attractors. To be more clear, if we consider, for $\sigma \in \Sigma_\eta$, the evolution process

$$S_\sigma^\eta(t,s) = \varphi_\eta(t-s,\theta_s\sigma), \text{ for } t \geq s, \tag{34}$$

then if S_σ^η has pullback attractor $\{A_\sigma(t) : t \in \mathbb{R}\}$, for all $t \in \mathbb{R}$, if uniqueness is assumed it is not hard to see that $A_\sigma(t) = A_\eta(\theta_t\sigma)$, the right one being the cocycle attractor described above.

Here we define the equi-attraction for a family of cocycle attractors $\{A_\eta(\sigma) : \sigma \in \Sigma_\eta\}$. Indeed it is not the same equi-attraction for evolution processes, since we are asking for a uniformity on the family of processes. This will allow us to gain some uniformity on the continuity of attractors.

Definition 2.15. Let $\{(\varphi_\eta,\theta)_{(X,\Sigma_\eta)}\}_{\eta\in[0,1]}$ be a family of non-autonomous dynamical systems and for each $\eta \in [0,1]$, let the non-autonomous set $\{A_\eta(\sigma) : \sigma \in \Sigma_\eta\}_{\eta\in[0,1]}$ denote their corresponding cocycle attractors. We say that a family of cocycle attractors $\{A_\eta(\sigma) : \sigma \in \Sigma_\eta\}_{\eta\in[0,1]}$, uniformly equi-pullback attracts bounded sets of X if

$$\lim_{t\to\infty} \sup_{\eta\in[0,1]} \sup_{\sigma\in\Sigma_\eta} \operatorname{dist}(\varphi_\eta(t,\theta_{-t}\sigma)D, A_\eta(\sigma)) = 0, \tag{35}$$

for all bounded subsets $D \subset X$.

Even more, one can relate Definition 2.15 with Definition 2.8, since $\mathscr{A}_\eta = \bigcup_{\sigma\in\Sigma_\eta} A_\eta(\sigma)$. Recall that $\{\theta_t : t \in \mathbb{R}\}$ is a group over Σ_η, which is invariant. Therefore, for all fixed t, as we vary $\theta_{-t}\sigma$ on Σ_η we are covering all Σ_η, then we may exchange the supreme of $\theta_{-t}\sigma$ at (35) by $\sigma \in \Sigma_\eta$ to obtain that the family of uniform attractors $\{\mathscr{A}_\eta : \eta \in [0,1]\}$ equi-attracts bounded subsets of X. The converse, however, does not need to be true, because attraction of the union $\bigcup_{\sigma\in\Sigma_\eta} A_\eta(\sigma)$ does not guarantee that the fiber of the non-autonomous set $A_\eta(\sigma)$ attracts bounded subsets by the "action of" σ.

In the same way as before, equi-pullback attraction does imply continuity of cocycle attractors.

Theorem 2.16. *Suppose that the family $\{A_\eta(\theta_\tau\sigma_\eta) : \tau \in \mathbb{R}\}_{\eta\in[0,1]}$ uniform equi-pullback attracts bounded subsets of X, that $\bigcup_{\eta\in[0,1]}\bigcup_{\sigma\in\Sigma_\eta} A_\eta(\sigma)$ is precompact in X and that for all compact subset $K \subset X$ we have*

$$\sup_{t\in[0,T]} \sup_{\tau\in\mathbb{R}} \sup_{x\in K} \mathrm{d}(\varphi_\eta(t,\theta_\tau\sigma_\eta)x, \varphi_0(t,\theta_\tau\sigma_0)x) \to 0 \text{ as } \eta \to 0^+. \tag{36}$$

Then

$$\sup_{\tau\in\mathbb{R}} \operatorname{dist}_{\mathrm{H}}(A_\eta(\theta_\tau\sigma_\eta), \mathscr{A}_0(\theta_\tau\sigma_0)) \to 0, \text{ as } \eta \to 0.$$

For a proof of the previous theorem we refer the reader to [9] and [13]. Observe that using evolution processes it is easy to show that the family $\{A_\eta(\sigma_\eta)\}_{\eta\in[0,1]}$ is continuous at $\eta = 0$. Then, by [7] and our uniformity assumption, we have that the family $\{A_\eta(\theta_t\sigma_\eta) : t \in \mathbb{R}\}_{\eta\in[0,1]}$ is continuous at $\eta = 0$.

Definition 2.17. The family of non-autonomous dynamical systems $\{(\varphi_\eta,\theta)_{(X,\Sigma_\eta)}\}_{\eta\in[0,1]}$ is said to be collectively pullback asymptotically compact if for every sequences $\eta_k \to 0$, $t_k \to \infty$, $\sigma_k \in \Sigma_{\eta_k}$ and $\{x_k\} \subset X$ bounded such that $\{\varphi_{\eta_k}(t_k,\theta_{-t_k}\sigma_k)x_k\}$ is also bounded in X, then the set $\{\varphi_{\eta_k}(t_k,\theta_{-t_k}\sigma_k)x_k : k \in \mathbb{N}\}$ is relatively compact in X.

Theorem 2.18. *Let the family of non-autonomous dynamical systems $\{(\varphi_\eta,\theta)_{(X,\Sigma_\eta)}\}_{\eta\in[0,1]}$ with cocycle attractors $\{A_\eta(\sigma) : \sigma \in \Sigma_\eta\}$, $\eta \in [0,1]$. Suppose* (36) *holds for each compact subset K of X and $T > 0$.*

If $\{(\varphi_\eta,\theta)_{(X,\Sigma_\eta)}\}_{\eta\in[0,1]}$ is a uniformly bounded (Definition 2.10) and collectively pullback asymptotically compact family, and moreover

$$\sup_{\tau\in\mathbb{R}} \operatorname{dist}_{\mathrm{H}}(A_\eta(\theta_\tau\sigma_\eta), \mathscr{A}_0(\theta_\tau\sigma_0)) \to 0, \text{ as } \eta \to 0, \tag{37}$$

then there exists $\eta_0 > 0$ *such that*

$$\bigcup_{\eta\in[0,\eta_0]}\bigcup_{\tau\in\mathbb{R}} A_\eta(\theta_\tau\sigma_\eta) \text{ is precompact},$$

and the family of cocycle attractors $\{A_\eta(\theta_\tau\sigma_\eta) : \tau \in \mathbb{R}\}_{\eta\in[0,\eta_0]}$ *uniformly equi-pullback attracts bounded subsets of* X.

As the proof follows the same steps as those in theorems 2.5 and 2.12, we will omit them. The reader may check the paper [9] for a similar proof.

2.3.2. *Rates of convergence.* As one can imagine, a rate of equi-attraction is transferred to the proximity of the cocycle attractors too. This is what is shown in the next theorem.

Theorem 2.19. *Let* $\{(\varphi_\eta,\theta)_{(X,\Sigma_\eta)} : \eta \in [0,1]\}$ *be a family of non-autonomous dynamical systems such that* $\{\theta_t : t \in \mathbb{R}\}$ *is a group on* Σ_η *for each* $\eta \in [0,1]$. *Suppose* $\{A_\eta(\sigma) : \sigma \in \Sigma_\eta\}$ *is the cocycle attractor for* $(\varphi_\eta,\theta)_{(X,\Sigma_\eta)}$ *and that* $\mathscr{B} = \bigcup_{\eta\in[0,1]}\bigcup_{\sigma\in\Sigma_\eta} A_\eta(\sigma)$ *is precompact in* X.

Assume that there exists a strictly decreasing function $\zeta : [0,\infty) \to (0,\infty)$ *with* $\zeta(0) = \zeta_0$ *and* $\lim_{s\to\infty}\zeta(s) = 0$ *such that*

$$\sup_{\eta\in[0,1]}\sup_{\tau\in\mathbb{R}} \mathrm{dist}(\varphi_\eta(t,\theta_{\tau-t}\sigma\eta)\mathscr{B}, A_\eta(\theta_\tau\sigma_\eta)) \le \zeta(t), \tag{38}$$

for all $t \ge 0$. *Moreover, suppose that*

$$\sup_{x\in\mathscr{B}}\sup_{\tau\in\mathbb{R}} \mathrm{d}(\varphi_\eta(t,\theta_{\tau-t}\sigma_\eta)x, \varphi_0(t,\theta_{\tau-t}\sigma_0)x) \le E_\eta(t), \text{ for all } t \ge 0, \tag{39}$$

with $E_\eta(t) \to 0$*, as* $\eta \to 0$*, for each* $t \ge 0$.

Then

$$\sup_{\tau\in\mathbb{R}} \mathrm{dist_H}(A_\eta(\theta_\tau\sigma_\eta), \mathscr{A}_0(\theta_\tau\sigma_0)) \le \inf_{\varepsilon\in(0,\zeta_0]} 2\{E_\eta(\zeta^{-1}(\varepsilon)) + \varepsilon\}. \tag{40}$$

The proof is analogous to the proof of theorems 2.6 and 2.13.

Corollary 2.20. *In addition to the hypotheses of the preceding theorem, suppose that there exist* $c > 0$ *and* $\nu > 0$ *such that* $\zeta(t) = ce^{-\nu t}$*,* $t \ge 0$*, and that* $E_\eta(t) = \rho(\eta)e^{Lt}$*, with* $L > 0$ *and* $\rho : [0,1] \to [0,\infty)$ *is a continuous function with* $\rho(0) = 0$. *Then there exists a constant* $\bar{c} > 0$ *such that*

$$\sup_{\tau\in\mathbb{R}} \mathrm{dist_H}(A_\eta(\theta_\tau\sigma_\eta), A_0(\theta_\tau\sigma_0)) \le \bar{c}\rho(\eta)^{\frac{\nu}{\nu+L}}. \tag{41}$$

3. **Relationships on the continuity of attractors.** In this final section we want to relate continuity among the different kind of attractors described above. We know how existence of global attractors for skew-product semiflows are almost equivalent to existence of uniform and cocycle attractors for non-autonomous dynamical systems. Our question now is, if they all exist, wether continuity of global attractors for skew-product semiflows implies that the family of uniform attractors are continuous and vice-versa.

Given a family of non-autonomous dynamical systems $\{(\varphi_\eta,\theta)_{(X,\Sigma_\eta)}\}_{\eta\in[0,1]}$ let $\{\pi_\eta(t) : t \ge 0\}_{\eta\in[0,1]}$ denote the skew-product semiflow associated. Assume that for each $\eta \in [0,1]$ there exists a global attractor $\mathbb{A}_\eta \subset X \times \Sigma_\eta$ for the $\{\pi_\eta(t) : t \ge 0\}$. This implies that $\mathscr{A}_\eta = \Pi_X\mathbb{A}_\eta$ is the uniform attractor for the nds $(\varphi_\eta,\theta)_{(X,\Sigma_\eta)}$ and the non-autonomous set $\{A_\eta(\sigma)\}_{\sigma\in\Sigma}$ defined as $A_\eta(\sigma) := \{x \in X | (x,\sigma) \in \mathbb{A}_\eta\}$ is the cocycle attractor (see [4, Theorem 3.4]).

The phase space Σ_η is assumed to be the closure of a given global orbit, that is, there exists $\sigma_\eta \in \Sigma_\eta$ for which $\Sigma_\eta = \overline{\{\theta_t \sigma_\eta : t \in \mathbb{R}\}}$. Also, that

$$\theta_t \sigma_\eta \to \theta_t \sigma_0, \text{ as } \eta \to 0, \tag{42}$$

uniformly with respect to $t \in \mathbb{R}$. Hence, it is easy to see that the family $\{\Sigma_\eta\}_{\eta\in[0,1]}$ is continuous at $\eta = 0$.

Finally, let us suppose that for each $\eta \in [0,1]$ it holds

$$\mathbb{A}_\eta = \overline{\bigcup_{t\in\mathbb{R}} A_\eta(\theta_t \sigma_\eta) \times \{\theta_t \sigma_\eta\}}. \tag{43}$$

We assume that the skew-product semiflows enjoy a kind of continuity when η goes to 0, more precisely, suppose that for each compact subset K of X and each $T > 0$ we have that

$$\sup_{t\in[0,T]} \sup_{x\in K} \sup_{\tau\in\mathbb{R}} \mathrm{d}(\pi_\eta(t)(x, \theta_\tau \sigma_\eta), \pi_0(t)(x, \theta_\tau \sigma_0)) \to 0, \tag{44}$$

when $\eta \to 0$.

Also, we assume that

$$\bigcup_{\eta\in[0,1]} \mathbb{A}_\eta \subset X \times \Sigma \text{ is precompact.} \tag{45}$$

Lemma 3.1. *If $\{\pi_\eta(t) : t \geq 0\}_{\eta\in[0,1]}$ is a family of skew-product semiflows associated to the non-autonomous dynamical systems $\{(\varphi_\eta, \theta)_{(X,\Sigma_\eta)}\}_{\eta\in[0,1]}$ with global attractors $\mathbb{A}_\eta$, $\eta \in [0,1]$, that satisfies* (44) *and* (45)*, then the family $\{\mathbb{A}_\eta\}_{\eta\in[0,1]}$ is upper semicontinuous at $\eta = 0$.*

For a proof of this lemma the reader is referred to [6, Theorem 1.20] or [7, Theorem 2.11].

Proposition 3.2. *Suppose that the family of global attractors $\{\mathbb{A}_\eta\}_{\eta\in[0,1]}$ of the skew-product semiflow is continuous at $\eta = 0$. Therefore the family of uniform attractors $\{\mathscr{A}_\eta = \Pi_X \mathbb{A}_\eta\}_{\eta\in[0,1]}$ is continuous at $\eta = 0$. The same holds for the phase space of the driving semigroup, namely* $\mathrm{dist_H}(\Sigma_\eta, \Sigma_0) = 0$.

Proof. Notice that for each $\eta \in [0,1]$ we have that $\mathbb{A}_\eta \subset \mathscr{A}_\eta \times \Sigma_\eta$. Thus,

$$\begin{aligned}
\mathrm{dist}(\mathbb{A}_\eta, \mathbb{A}_0) &\geq \mathrm{dist}(\mathbb{A}_\eta, \mathscr{A}_0 \times \Sigma_0) \\
&= \sup_{\sigma\in\Sigma_\eta} \sup_{x\in A_\eta(\sigma)} \mathrm{d}((x,\sigma), \mathscr{A}_0 \times \Sigma_0) \\
&= \sup_{\sigma\in\Sigma_\eta} \sup_{x\in A_\eta(\sigma)} [\mathrm{d}(x, \mathscr{A}_0) + \mathrm{d}(\sigma, \Sigma_0)]
\end{aligned}$$

Therefore, as $\{\mathbb{A}_\eta\}_{\eta\in[0,1]}$ is upper semicontinuous we must have that $\mathrm{dist}(\Sigma_\eta, \Sigma_0) \to 0$, then also $\mathrm{dist}(\cup_{\sigma\in\Sigma_\eta} A_\eta(\sigma), \mathscr{A}_0) \to 0$. That is, the family $\{\mathscr{A}_\eta\}_{\eta\in[0,1]}$ and $\{\Sigma_\eta\}_{\eta\in[0,1]}$ are upper semicontinuous.

In the equation above, note that we may exchange $\mathbb{A}_\eta$ with $\mathbb{A}_0$ and obtain the same estimates. Thus, lower semicontinuity of $\{\mathbb{A}_\eta\}_{\eta\in[0,1]}$ implies lower semicontinuity of the families we want. In this way we conclude the proof. □

Corollary 3.3. *Suppose that $\{\pi_\eta(t) : t \geq 0\}_{\eta\in[0,1]}$ is a family of skew-product semiflows, with global attractors $\mathbb{A}_\eta$, that verifies* (43)*,* (44) *and* (45)*. Then the family of uniform attractors, $\mathscr{A}_\eta = \Pi_X \mathbb{A}_\eta$, for the non-autonomous dynamical system $(\varphi_\eta, \theta)_{(X,\Sigma_\eta)}$ is upper semicontinuous at $\eta = 0$.*

Proof. This is a straightforward consequence of Lemma 3.1 and Proposition 3.2. □

However, to ensure that the family $\{\mathbb{A}_\eta\}_{\eta\in[0,1]}$ of global attractors for the skew-product semiflows is lower semicontinuous in terms of the uniform attractors we need more information on the structure of cocycle attractors.

Proposition 3.4. *Let $\{\pi_\eta(t) : t \geq 0\}_{\eta\in[0,1]}$ be a family of skew-product semiflows, with global attractors $\mathbb{A}_\eta$, that fulfills* (43), (44) *and* (45). *Moreover, suppose that the family $\{A_\eta(\theta_t\sigma_\eta)\}_{\eta\in[0,1]}$ of cocycle attractors is lower semicontinuous at $\eta = 0$ uniformly with respect to $t \in \mathbb{R}$.*

Then the family of global attractors for the skew-product semiflow $\{\mathbb{A}_\eta\}_{\eta\in[0,1]}$ is lower semicontinuous at $\eta = 0$.

Proof. Note that, from (43), we have

$$\mathrm{dist}(\mathbb{A}_0,\mathbb{A}_\eta) = \sup_{t\in\mathbb{R}} \mathrm{dist}(A_0(\theta_t\sigma_0)\times\{\theta_t\sigma_0\},\mathbb{A}_\eta) \tag{46}$$

As the families $\{\Sigma_\eta\}_{\eta\in[0,1]}$ and $\{\mathbb{A}_\eta(\theta_t\sigma_\eta) : t\in\mathbb{R}\}_{\eta\in[0,1]}$ are lower semicontinuous at $\eta = 0$, uniformly on $t \in \mathbb{R}$, we are able to conclude that $\mathrm{dist}(\mathbb{A}_0,\mathbb{A}_\eta) \to 0$, as $\eta \to 0$, since the right hand side of (46) is controlled by the lower semicontinuity from the cocycle attractors. □

Now we will show an example of a non-autonomous perturbation of an autonomous dynamical system.

Consider the semilinear problem on the Banach space X

$$\begin{cases} \dot{x} = f_\eta(t,x) \\ x(0) = x_0 \in X. \end{cases} \tag{47}$$

We assume that, for $\eta \in [0,1]$, $f_\eta : \mathbb{R}\times X \to X$ is continuous and uniformly Lipschitz for $t \in \mathbb{R}$ in bounded subsets of X, so the problem (47) is locally well posed. Hence $f_\eta \in C_b(\mathbb{R}\times X, X)$ and for $\tau \in \mathbb{R}$ define

$$\theta_\tau f(t,x) := f(t+\tau,x),$$

for $(t,x) \in \mathbb{R}\times X$ and let $\Sigma_\eta := \overline{\{\theta_\tau f_\eta : t\in\mathbb{R}\}}$.

Let us assume that $f_0(t,x) = f_0(x)$, for all $t\in\mathbb{R}$ and $x \in X$, Σ_η is compact for all $\eta\in[0,1]$ and $\bigcup_{\eta\in[0,1]}\Sigma_\eta$ is precompact, furthermore, we assume, for the perturbation, that

$$\lim_{\eta\to 0} \sup_{(t,x)\in\mathbb{R}\times B(0,r)} \|f_\eta(t,x) - f_0\|_X + \|D_x f_\eta(t,x) - D_x f_0(x)\|_{\mathscr{L}(X)} = 0, \tag{48}$$

for all $r > 0$.

It follows from (48) that

$$\lim_{\eta\to 0} \sup_{(t,x)\in\mathbb{R}\times B(0,r)} \sup_{\sigma_\eta\in\Sigma_\eta} \|\sigma_\eta(t,x) - f_0\|_X + \|D_x\sigma_\eta(t,x) - D_x f_0(x)\|_{\mathscr{L}(X)} = 0. \tag{49}$$

We consider the problem

$$\begin{cases} \dot{x} = \sigma(t,x) \\ x(0) = x_0 \in X, \end{cases} \tag{50}$$

for $\sigma\in\Sigma_\eta$, and we denote by $\varphi_\eta(t,\sigma)x_0 = x(t,\sigma,x_0)$, the solution for (50) at time t with non-autonomous function equals to $\sigma\in\Sigma_\eta$.

Hence, by (49), it is easy to see that, for each $T > 0$ and B bounded in X,

$$\sup_{t\in[0,T]} \sup_{\sigma\in\Sigma_\eta} \sup_{x\in B} \|\varphi_\eta(t,\sigma)x - \varphi_0(t)x\|_X \to 0, \text{ as } \eta\to 0. \tag{51}$$

We also know, from (49), that $\theta_\tau\sigma_\eta \to f_0$, uniformly (for $\tau\in\mathbb{R}$) as $\eta\to 0$. Let $\Sigma_0 = \{f_0\}$.

We assume that the skew-product semiflow $\pi_\eta(t) \colon X\times\Sigma_\eta \to X\times\Sigma_\eta$, for $t\geq 0$ and $\eta\in[0,1]$, defined as $\pi_\eta(t)(x,\sigma) = (\varphi_\eta(t,\sigma)x,\theta_t\sigma)$ has a global attractor $\mathbb{A}_\eta$, for each $\eta\in$

$[0,1]$; this implies that there is uniform attractor, $\mathscr{A}_\eta = \Pi_X \mathbb{A}_\eta$, for each non-autonomous dynamical system $\{(\varphi_\eta, \theta)_{(X,\Sigma_\eta)}\}$.

Therefore we consider the evolution process on X

$$S^\eta_\sigma(t,s)x := \varphi_\eta(t-s, \theta_s\sigma)x, \; t \geq s. \tag{52}$$

For each $\sigma \in \Sigma_\eta$ there is a unique pullback attractor (that coincides with a subfamily of the cocycle attractors) $\{A_\eta(\theta_t\sigma) : t \in \mathbb{R}\}$ for (52).

With this assumptions, let $\{\varphi_0(t) : t \geq 0\}$ be the limit semigroup on X. Suppose that $\{\varphi_0(t) : t \geq 0\}$ is a gradient dynamical system, in the sense of [14], for which all the stationary points $\{e^*_j : 1 \leq j \leq n\}$ are hyperbolic. If for the unstable manifolds of e^*_j it holds that there exists a $\delta > 0$ such that for any $\varepsilon > 0$ there exists an η_0 such that for all $0 < \eta < \eta_0$ there exists a global hyperbolic solution $\xi^*_{j,\eta}(\cdot)$ of $S^\eta_{f_\eta}$ with

$$\sup_j \sup_{t\in\mathbb{R}} |\xi^*_{j,\eta}(t) - e^*_j| < \varepsilon,$$

and within a δ neighbourhood of e^*_j

$$\sup_j \operatorname{dist_H}(W^u(\xi^*_{j,\eta}(\cdot))(t), W^u(e^*_j)) < \varepsilon, \text{ for all } t \in \mathbb{R},$$

that is, the unstable manifolds of e^*_j behave continuously. Then

$$\sup_{t\in\mathbb{R}} \operatorname{dist_H}(A_\eta(\theta_t f_\eta), A_0) \to 0, \text{ as } \eta \to 0. \tag{53}$$

See [8, Theorem 7.1].

Therefore, since (53) holds, Theorem 2.18 implies that the family $\{A_\eta(\theta_t f_\eta) : t \in \mathbb{R}\}_{\eta\in[0,1]}$ uniformly equi-pullback attracts bounded subsets of X (in terms of Definition 2.15). Thus, the hypothesis of Lemma 2.1 are fulfilled and, for every $\eta \in [0,1]$, it holds

$$\mathbb{A}_\eta = \overline{\bigcup_{t\in\mathbb{R}} A_\eta(\theta_t f_\eta) \times \{\theta_t f_\eta\}}.$$

Hence, we conclude, thanks to Proposition 3.4, that the family of global attractors for the skew-product semiflows $\{\mathbb{A}_\eta\}_{\eta\in[0,1]}$ is continuous at $\eta = 0$ and, consequently, the family of uniform attractors $\{\mathscr{A}_\eta\}_{\eta\in[0,1]}$ is continuous at $\eta = 0$.

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