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Problemas de Homogeneización con Alto Contraste

High-Contrast Homogenization Problems

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Contents

Introduction

In the production of certain composite materials, the mixture of the components is carried out at a microscopic level or, more precisely, at a mesoscopic level (small from the macroscopic point of view but sufficiently large to neglect the quantum effects). The first difficulty involved is the numerical resolution of the partial differential equations that describe the behaviour of the related physical quantities. It would be necessary to use meshes whose elements are small compared to the measure of the structures formed by the components of the mixture. This would lead to systems of equations whose large sizes make their direct resolution virtually unattainable. Both physicians and engineers have usually tackled this kind of problems by inserting some small parameters with the purpose of making an asymptotic expansion with respect to them. As a consequence, they obtain much simpler problems whose solutions provide a good approximation of the solution to the original problem. In many occasions, a later mathematical justification for the resulting models has been obtained, proving some convergence results in certain functional spaces. In mathematics, the homogenization theory is the field that deals with this type of questions.

As an example, we recall the perhaps most classical result in the theory of homogenization. We consider the electric material obtained upon periodic repetition of a cell with small period $\varepsilon > 0$. The electrostatic theory states that the electrostatic potential u_ε is a solution to

$$-\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = \rho \quad \text{in } \Omega, \tag{1}$$

where Ω is an open subset of $\mathbb{R}^N$ ($N = 2, 3$ in practice) and ρ is the charge density. The matrix of coefficients A depends on the dielectric constant of the medium and is Y_N-periodic (where Y_N is the unit cube of $\mathbb{R}^N$). In order to have the uniqueness of solution to (1), an additional boundary condition is clearly needed. The generation of materials under this procedure is very common in engineering.

The method of asymptotic expansions (see e.g. [9], [65], [71], [84], [85]) applied to this problem consists in assuming that the function u_ε admits an expansion of the type

$$u_\varepsilon(x) \sim u_0(x) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \cdots,$$

with $u_1, u_2, \ldots$ periodic with respect to their second variable. By replacing it in (1) and identifying the coefficients with the same power of ε, one formally obtains that u_0 is a solution to

$$-\operatorname{div}(A_h \nabla u_0) = \rho \quad \text{in } \Omega, \tag{2}$$

where A_h (the homogenized matrix) is defined by

$$A_h\xi = \int_{Y_N} A(\xi + \nabla_y w_\xi)\, dy, \quad \forall\, \xi \in \mathbb{R}^N, \tag{3}$$

with w_ξ solution to

$$\begin{cases} -\mathrm{div}\,(A\nabla w_\xi) = 0 \;\; \text{in } \mathbb{R}^N, \\ w_\xi \; Y_N\text{-periodic.} \end{cases}$$

In addition, it is possible to prove

$$u_1(x,y) = w_{\nabla u_0(x)}(y).$$

The previous result explains the term homogenization. Whereas in (1) we had a strongly heterogeneous material, the constant matrix A_h in (2) corresponds to a homogeneous material. Note also that the numerical resolution of u_0 and u_1 is much simpler than that of u_ε. From a more theoretical point of view and on the macroscopic side, the electric properties of the material corresponding to $A(x/\varepsilon)$ are similar to the properties of the material modelled by A_h. If, for instance, the matrix A is the outcome of the mixture of two materials, i.e. there exist a measurable set $Z \subset Y_N$ and two matrices A_1, A_2 such that

$$A(y) = A_1\chi_Z(y) + A_2(1 - \chi_Z(y)), \quad \text{a.e. } y \in Y_N,$$

then, we build a new material, corresponding to A_h, whose properties depend not only on the proportion of the two mixed material (i.e. the measure of Z) but also on their geometric arrangement. Therefore, even if A_1 and A_2 are scalar matrices corresponding to isotropic materials (i.e. their properties do not depend on the direction), the homogenized matrix A_h does not need to be scalar.

Even though the method described above for obtaining A_h is formal, some convergence results can be found in [9] and [65]. In fact, due to its importance in architecture and engineering, many methods have been developed in order to mathematically solve problems with some periodicity assumption like the one above. We would like to highlight the two-scale convergence and the unfolding methods ([2], [4], [34], [36], [41], [81]).

The previous example shows how the process of obtaining new materials through the mixture of existing ones can be analysed using highly oscillating distributions. This is done by studying the asymptotic behaviour of PDE with varying coefficients. Although we talked about a periodic problem before, it is also of great importance to know the behaviour of similar problems under no periodicity condition in order to be able to obtain more general materials. In this context, the first question that arises is whether or not the kind of equations that we are dealing with is stable in the limit. Otherwise we would need more general models.

To our knowledge, the first results regarding the stability in the limit of a sequence of PDE with varying coefficients deal with the case of a sequence of second-order elliptic linear equations in the divergence form. S. Spagnolo showed in [87] (see also

[52]) that if the sequence of symmetric matrix-valued functions A_n is bounded in $L^\infty(\Omega)^{N\times N}$ and is uniformly elliptic in the sense that there exists $\alpha > 0$ satisfying

$$A_n\xi \cdot \xi \geq \alpha|\xi|^2, \quad \forall n \in \mathbb{N},\ \forall \xi \in \mathbb{R}^N, \text{ a.e. } \Omega, \tag{4}$$

then there exist a subsequence of A_n, still denoted by A_n, and a symmetric matrix function $A \in L^\infty(\Omega)^{N\times N}$ also fulfilling (4) such that for every $f \in H^{-1}(\Omega)$, the solutions to

$$\begin{cases} -\mathrm{div}\,(A_n\nabla u_n) = f & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

weakly converge in $H_0^1(\Omega)$ to the solution u of the problem obtained upon substitution of A_n by A. The extension of this result to the corresponding parabolic operator is also shown in the cited reference (the extension to the hyperbolic case appears in [43]). F. Murat and L. Tartar later generalised this result to the case of general matrices without any assumption of symmetry ([76]), also proving the convergence of $A_n\nabla u_n$ to $A\nabla u$ in $L^2(\Omega)^N$. This result can be easily extended to systems of elliptic equations and, especially to the linear elasticity system that describes the elastic deformation of a solid (supposing that the derivatives of the deformations are negligible). We refer to the works of G. Francfort [59], E. Sánchez-Palencia [85] and G. Duvaut (unavailable reference). The proof of this result relies on the oscillating functions method and the key idea is to use specific sequences of test functions (the previously mentioned two-scale convergence is also based on this idea). An essential tool in this proof is the div-curl theorem, which is the best known result of the compensated compactness theory, also introduced by F. Murat and L. Tartar ([77], [89]). The div-curl theorem states that for $p \in (1, \infty)$, if

$$\begin{aligned} \sigma_n \rightharpoonup \sigma \ \text{ in } L^p(\Omega)^N, \qquad & \tau_n \rightharpoonup \tau \ \text{ in } L^{p'}(\Omega)^N, \\ \mathrm{div}\,\sigma_n \to \mathrm{div}\,\sigma \ \text{ in } W^{-1,p}(\Omega), \qquad & \mathrm{curl}\,\tau_n \rightharpoonup \mathrm{curl}\,\tau \ \text{ in } W^{-1,p'}(\Omega)^{N\times N}, \end{aligned} \tag{6}$$

then

$$\sigma_n \cdot \tau_n \rightharpoonup \sigma \cdot \tau \ \text{ in } \mathcal{D}'(\Omega).$$

Although the convergence result for (5) is usually stated with homogeneous Dirichlet boundary conditions as we did, it also holds for other kinds of boundary conditions. In addition, the result is local in the sense that the value of the homogenized matrix A in an arbitrary open subset of Ω only depends on the values of A_n in that subset. Some extensions to nonlinear equations can be found e.g. in [82] and [53].

It is also worth mentioning that this sort of results is applied to the resolution of optimal material design problems by providing relaxed formulations (see e.g. [2], [35], [80]).

A common question that emerges from the cited results is what happens if the sequence A_n is not uniformly bounded or uniformly elliptic. This is known as high-contrast homogenization.

A very useful tool that allows to tackle this kind of problems is the Γ-convergence that was introduced by E. De Giorgi (see e.g. [12], [14], [48], [51]). Let X be a metric space (the definition can be extended to non metric spaces) and $F_n : X \to \mathbb{R}\cup\{+\infty\}$

a sequence of functionals, F_n is said to Γ-converge to F in X if the two following conditions hold:

$$\begin{cases} x_n \to x \text{ in } X \Longrightarrow \liminf_{n\to\infty} F_n(x_n) \geq F(x), \\ \forall\, x \in X,\ \exists x_n \to x \ \text{ such that } \limsup_{n\to\infty} F_n(x_n) \leq F(x). \end{cases}$$

The most important result in the Γ-convergence theory states that if F_n reaches a minimum at x_n and the sequence x_n is compact in X, then every limit point of x_n is a point of minimum of F. Therefore, if we go back to problem (5) and assume that A_n is symmetric, then u_n is a solution if and only if it is a solution to

$$\min_{u \in H_0^1(\Omega)} \left\{ \int_\Omega A_n \nabla u \cdot \nabla u \, dx - 2\langle f, u\rangle \right\}.$$

Furthermore, taking into account that, thanks to (4), the solutions to (5) are bounded in $L^2(\Omega)$, we can conclude that the result by S. Spagnolo can be deduced by showing (assuming that the right-hand side belongs to $L^2(\Omega)$)

$$\left[u \mapsto \int_\Omega \big(A_n \nabla u \cdot \nabla u - 2fu\big)\, dx\right] \xrightarrow{\Gamma} \left[u \mapsto \int_\Omega \big(A \nabla u \cdot \nabla u - 2fu\big) dx\right] \quad \text{in } L^2(\Omega),$$

or, equivalently (as a consequence of considering f as an element of the dual of $L^2(\Omega)$)

$$\left[u \mapsto \int_\Omega A_n \nabla u \cdot \nabla u \, dx\right] \xrightarrow{\Gamma} \left[u \mapsto \int_\Omega A \nabla u \cdot \nabla u \, dx\right] \quad \text{in } L^2(\Omega).$$

This formulation has the advantage that the functional

$$u \mapsto \int_\Omega A_n \nabla u \cdot \nabla u \, dx, \tag{7}$$

is well defined even though the integral might be infinite. This allows to work with the case of A_n not being in $L^\infty(\Omega)^{N\times N}$ more easily. However, the disadvantage is that it must be possible to write the problem as a minimization problem.

As a classic example of applicability of the theory of Γ-convergence to the resolution of homogenization problems, we point out the work [33] by L. Carbone and C. Sbordone, where they analyse the Γ-convergence in $L^\infty(\Omega)$ of the sequence of functionals

$$u \mapsto \int_\Omega F_n(x, u, \nabla u)\, dx, \tag{8}$$

with $F_n : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ a sequence of Carathéodory functions (measurable in the first variable and continuous in the other two), convex with respect to the last variable and such that

$$0 \leq F_n(x, s, \xi) \leq a_n(x)(1 + |s|^p + |\xi|^p), \quad \forall\,(s, \xi) \in \mathbb{R} \times \mathbb{R}^N, \ \text{ a.e. } x \in \Omega, \tag{9}$$

with $p > 1$ and a_n bounded in $L^1(\Omega)$. The authors show that, for a subsequence of n, there exists the Γ-limit of these functionals in $L^\infty(\Omega)$ and that it admits an integral representation of the same type, at least for the regular functions. Moreover, if a_n is equi-integrable then the Γ-limit in $L^\infty(\Omega)$ coincides with the Γ-limit in $L^1(\Omega)$. In addition the homogenization process is local as in the previous cases.

If we wanted to apply this result to the convergence of minima, then these functionals would need to attain a minimum and, also, these minima would have to be contained in a compact set of the considered topology. Thus, if a_n is equi-integrable, it is enough to have the boundedness of the sequence of minima in $W^{1,1}(\Omega)$. This can be achieved imposing some suitable coercivity condition, for instance,

$$0 \le b_n(x)|\xi|^p \le F_n(x,s,\xi),\ \forall\,(s,\xi)\in\mathbb{R}\times\mathbb{R}^N,\ \text{a.e. } x\in\Omega,\ b_n^{-\frac{1}{p}}\ \text{ bounded in } L^{p'}(\Omega).$$

If a_n were only bounded in $L^1(\Omega)$, we would need the sequence of minima to be compact in $L^\infty(\Omega)$, which, essentially, would mean to take $p > N$ and a coercivity condition such as

$$\alpha|\xi|^p \le F_n(x,s,\xi),\ \forall\,(s,\xi)\in\mathbb{R}\times\mathbb{R}^N,\ \text{a.e. } x\in\Omega,\ \alpha>0.$$

As an example, the results in [33] can be applied to problem (5), deducing that, for $N \ge 2$ and A_n symmetric satisfying

$$\begin{gathered} b_n(x)|\xi|^2 \le A_n(x)\xi\cdot\xi \le a_n(x)|\xi|^2, \quad \forall\,\xi\in\mathbb{R}^N, \quad \text{a.e. } x\in\Omega, \\ a_n, b_n \ge 0, \quad a_n \text{ bounded in } L^1(\Omega),\ \text{equi-integrable},\ b_n^{-1} \text{ bounded in } L^1(\Omega), \end{gathered}$$

and f regular enough, then the solutions to (5) converge weakly-$*$ in $BV(\Omega)$ to the solution to a problem of the same type.

In [56] (see also [8], [28]) V. N. Fenchenko and E. Ya. Khruslov provide an example where a_n is a function bounded in $L^1(\Omega)$ (but not equi-integrable) with $\Omega = \omega\times(0,1)$ and ω is an open bounded subset of $\mathbb{R}^2$, satisfying that the solutions to

$$\begin{cases} -\operatorname{div}(a_n\nabla u_n) = f & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

converge in $H_0^1(\Omega)$-weak to the solution to

$$\begin{cases} -\Delta u + 2\pi\left(u + \displaystyle\int_0^1 h(x_3,t)u(x_1,x_2,t)dt\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where h is a nonzero function. This is a case where the limit equation changes. In the limit we find a term of zero order and a nonlocal term. A general result in the same vein has been obtained by U. Mosco in [74] where, making use of the Beurling-Deny representation formula of Dirichlet forms ([10]), it is proved that the Γ-limit in $L^2(\Omega)$ of the sequence of functionals given by (7), with A_n nonnegative, bounded in $L^1(\Omega)^{N\times N}$ and symmetric, converge to a functional of the type

$$u \mapsto \int_\Omega A\nabla u\cdot\nabla u\,d\mu(x) + \int_\Omega u^2 d\nu(x) + \int_{\Omega\times\Omega}\big(u(x)-u(y)\big)^2 d\eta(x,y), \tag{10}$$

with μ, ν and η nonnegative bounded Borel measures. In general, the homogenization process thus leads us to nonlocal terms even if one starts with strongly local terms.

Thanks to a generalisation of the div-curl theorem, it has been proved later in [17], [19] that, in dimension $N = 2$, assuming that A_n is uniformly elliptic, the two last terms are actually zero, i.e. the functional does not change of form upon Γ-convergence and thus, the homogeneization process remains local. This result has been subsequently generalised in [20], where the authors show that it is not even necessary to impose the condition of boundedness in $L^1(\Omega)^{N\times N}$. Some related results concerning equations in the periodic case and the appearance of zero-order terms can be found in [13] and [21] respectively. All these works make use of certain recent results of uniform convergence for the solutions to elliptic PDE ([22], [72]). In fact, with these ideas it has been obtained in [23] an extension of the results by L. Carbone and C. Sbordone in [33] where the condition $p > N - 1$ (instead of $p > N$) implies the equivalence between the Γ-limit in $L^1(\Omega)$ and $L^\infty(\Omega)$ of the functionals defined by (8).

The results of uniform convergence in the references [13], [20], [21], [23] and [33] rely on the maximum principle, and so does the Beurling-Deny formula that leads to expression (10). For this reason, the generalisation of these results to the case of systems of equations does not hold. As a consequence, contrary to (10), the absence of a uniform bound of the coefficients in the linear elasticity may cause the appearance of second-order derivatives in the Γ-limit as proved by C. Pideri and P. Seppecher in [83]. Furthermore, M. Camar-Eddine and P. Seppecher showed in [32] that it is possible to reach any lower-semicontinuous quadratic functional that vanishes for the rigid movements.

Due to the lack of the maximum principle, there are not general results, to our knowledge, about what assumptions of boundedness or ellipticity on the coefficients are needed in order for a system of PDE to keep its structure in the limit and for the homogenization process to be local. It is worth mentioning the existence of some particular results for the linear case via Γ-convergence. For $N = 2$, it has been proved in [18] the stability of the linear elasticity system assuming that the coefficients are uniformly elliptic and bounded in L^1. This result is based on the generalisation of the div-curl theorem in [26]. Another result relative to a general elliptic system corresponding to M equations in an open set $\Omega \subset \mathbb{R}^N$ has been obtained in [24], where the authors consider a sequence of coefficients tensors A_n such that there exists another sequence of uniformly elliptic and bounded tensors B_n in such a way that $A_n - B_n$ strongly converges to zero in $L^1(\Omega; \mathcal{L}(\mathbb{R}^{M\times N}))$. Note that the uniform ellipticity is imposed in an integral way, i.e.

$$\alpha \int_\Omega |Du|^2 dx \leq \int_\Omega A_n Du : Du\, dx, \quad \forall\, u \in H_0^1(\Omega)^M, \tag{11}$$

with $\alpha > 0$. It is known (see e.g. [48]) that this implies

$$A_n\xi : \xi \geq \alpha|\xi|^2, \quad \forall\, \xi \in \mathbb{R}^{M\times N},\ \text{Rank}(\xi) = 1, \ \text{ a.e. } \Omega, \tag{12}$$

and thus, in the case of equations ($M = 1$), it is equivalent to (4). However, this is not the case for systems. In order to distinguish these cases, in the literature, it is common to say that a tensor which satisfies (12) is strongly elliptic whereas, in the case when this condition holds for all $\xi \in \mathbb{R}^{M\times N}$, then it is said to be very strongly elliptic. The theory of compensated compactness shows that if A_n is a regular function in Ω then conditions (12) and (11) are equivalent.

The main problem that we are going to tackle in the two first chapters of this thesis is to obtain some ellipticity and/or boundedness conditions in an arbitrary dimension, for linear and nonlinear systems, that lead to a local limit system. For that, we will make use of certain extensions of the div-curl theorem ([25], [26]). In the third chapter we will go on with this question but when there is also a reduction of dimension in the domain. Namely, we consider the elasticity system for the thin beam $\Omega_\varepsilon = (0,1)\times(\varepsilon\omega)$ where ω is an open bounded regular subset of $\mathbb{R}^{N-1}$. Contrary to the previous chapters where the problem is posed in a fixed domain, now we intend to deduce a uni-dimensional limit system. This is a classical problem in engineering. When trying to directly solve a problem of PDE posed in a domain where at least one of the dimensions is much smaller than the rest, we usually come across the previously mentioned difficulty of having to use very fine meshes. The idea in homogenization is to approximate the solutions of the problem by those of a problem posed in a domain of smaller dimension. Therefore, in the case of a beam, the problem that is usually solved, consists in two uncoupled elliptic equations of fourth order. From the mathematical point of view (see e.g. [68], [92]) these equations are obtained by passing to the limit in the elasticity system corresponding to a homogeneous isotropic material in dimension 3 when the thickness of the beam tens to zero. The solution to the limit problem provides an approximation of the transverse deformations of the beam. More generally, in [79] (see also [37]) the authors consider an elasticity tensor of the form $A(x_1, x_2/\varepsilon, x_3/\varepsilon)$, where A is an element of $L^\infty((0,1)\times\omega; \mathcal{L}(\mathbb{R}^{3\times 3}_s))$ and satisfies the usual ellipticity condition. This allows, for instance, to deal with materials in which there is a kernel of a certain material surrounded by another one. In this case the obtained approximation of the deformation is more complex.

Continuing the discussion from the beginning of this introduction, an important problem is to know what happens when the thin domain (beam or plate) is formed by an arbitrary mixture of materials. This leads to the study of the asymptotic behaviour of a problem of PDE posed in a thin domain Ω_ε, where $\varepsilon > 0$ is a small value that measures the thickness and where the coefficients also depend on ε. Up to our knowledge, this problem has not been studied so deeply as the case where there is a fixed domain. Nonetheless, we can refer to some related works such as [5], [30] and [86], where the authors analyse this problem under certain periodicity conditions. As it has been previously explained, this allows to deal with materials that are usually present in engineering. However, if we were interested in deducing what materials can be constructed upon the mixture of given ones, we would need to remove the conditions of periodicity. In the case of diffusion problems in a beam $(0,1)\times(\varepsilon\omega)$ and assuming uniform ellipticity and boundedness, the problem has been studied in [45] under certain conditions on the structure that allow to apply a result of the div-curl type as well as in [39] for a general setting. In this last reference, the authors deal with very general right-hand-side terms and deduce a limit system posed in the domain $(0,1)\times\omega$ which is nonlocal in general. When we restrict to right-hand-side terms that do not strongly oscillate in the variables corresponding to the degenerating dimensions, the limit problem is reduced to a one-dimensional local problem. For the study of the asymptotic behaviour of the elasticity system with variable coefficients in a degenerating domain, we cite [50] where the case of a beam $\omega\times(0,\varepsilon)$ with $\omega\subset\mathbb{R}^2$ open and bounded, is considered. Under suitable conditions of isotropy and assuming that the coefficients are uniformly elliptic and bounded, it is obtained a fourth-order limit equation corresponding to the vertical displacement, which is similar to the usual case studied in engineering for plates formed by isotropic materials. The case when there is no isotropy but the coefficients only depend on the height variable is analysed in [62]. In the limit system for this case it is not possible to uncouple, in general, the deformations in the horizontal and vertical variables.

Along this introduction, we have mentioned many cases for which the structure of a problem of PDE, where the coefficients are variable, is preserved in the limit. Nevertheless, there are notable examples where some important properties are lost in the limiting process. This can be used to construct materials with very particular properties. In this sense, we analyse the difference between local and global coercivity that we mentioned above when we talked about the homogenization of systems. It is a known result that the formula of periodic homogenization (3) remains true for systems by imposing integral (instead of pointwise) coercivity. Moreover, for the case $M=N$ it has been proved in [61] that it suffices to have the existence of $\alpha>0$ such that (for A Y_N-periodic)

$$\begin{cases} \displaystyle\int_{Y_N} ADu:Du\,dy\geq\alpha\int_{Y_N}|Du|^2dy, \quad \forall\, u\in H^1_{loc}(\mathbb{R}^N)\ Y_N\text{-periodic},\\ \displaystyle\int_{\mathbb{R}^N} ADu:Du\,dy\geq 0, \quad \forall\, u\in\mathcal{D}(\mathbb{R}^N)^N. \end{cases} \tag{13}$$

An interesting question is what properties of ellipticity are fulfilled by the homogenised tensor. S. Gutiérrez proves in [64] that, a certain homogenization scheme (called 1*-convergence in [27]) applied to the lamination of a strongly elliptic isotropic material, in the sense that (12) holds, and a very strongly elliptic isotropic material (i.e. that (12) holds for all $\xi\in\mathbb{R}^{N\times N}$), can lead to a limit material that does not even satisfy the strong ellipticity condition. S. Gutiérrez carries out this study for the two- and three-dimensional cases. In some cases in dimension 3, it is in fact necessary to perform a second lamination with a third material (that can be chosen very strongly elliptic). However, the process followed by S. Gutiérrez requires *a priori* bounds in L^2 for the sequence of deformations, which is incompatible with the assumption of weak coercivity. Therefore, S. Gutiérrez' result does not address the asymptotic behaviour of the corresponding sequence of systems of PDE. In [27], the authors provide, for the two-dimensional case, a justification of this result in terms of Γ-convergence and show the canonical character of the lamination performed by S. Gutiérrez. Recall that if the tensor functions $x\mapsto A(x/\varepsilon)$ fulfilled the uniform integral ellipticity condition

$$\int_\Omega A\left(\frac{x}{\varepsilon}\right)Du:Du\,dx\geq\alpha\int_\Omega|Du|^2\,dx, \quad \forall\, u\in C_c^\infty(\Omega)^N, \tag{14}$$

with $\alpha > 0$ (independent of ε), then the Γ-limit would also satisfy this property. This means that the tensor A constructed by S. Gutiérrez does not satisfy condition (14), although each one of the homogeneous phases of A does. As it has been pointed out by M. Briane and G. Francfort in [27], there exist tensor functions $A : \mathbb{R}^N \to \mathcal{L}(\mathbb{R}^{N\times N})$ with jump discontinuities such that (12) holds for $\Omega = \mathbb{R}^N$ but where condition (11) fails. This can be easily seen with the change of variable $y = x/\varepsilon$. This means that the equivalence between the two definitions that we mentioned before for a regular tensor function A is not true in general.

In the fourth chapter of this thesis we provide justification for the results by S. Gutiérrez in the three-dimensional case through the Γ-convergence theory.

In the exposition that we have conducted so far, we have introduced the different problems that interest us in the present PhD project, their motivation and the existing related results carried out by other authors. In addition, we have outlined the precise questions that we intend to tackle. In what follows, we provide an explicit description of the problems that we study in each chapter of this PhD project, the results that we have obtained as well as the difficulties that arose and the methods and tools that we used to overcome them.

Chapter 1

We consider Ω an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, and an integer number $M \geq 1$. In this chapter we study the asymptotic behaviour of the following elliptic linear problems

$$\begin{cases} -\mathrm{Div}\,(A_n Du_n) = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{15}$$

Our purpose is to give conditions of integrability and ellipticity on the sequence of tensor functions $A_n \in L^p(\Omega; \mathcal{L}(\mathbb{R}^{M\times N}))$ in order for the homogenized problem to be of the same type, at least for sufficiently regular elements, and in order to have a local homogenization process. As mentioned above, in the case of equations ($M = 1$), it is enough to have A_n^{-1} bounded in $L^1(\Omega)^{N\times N}$ and A_n bounded in $L^1(\Omega)^{N\times N}$ and equi-integrable. In fact, the result is not true if the condition of equi-integrability of A_n is removed. The proof of these results uses the maximum principle and thus, it is not valid for systems.

In our case, we first show the existence of an abstract homogenization result when the coefficients A_n only fulfil

$$A_n \text{ bounded in } L^1(\Omega; \mathcal{L}(\mathbb{R}^{M\times N})), \tag{16}$$

$$A_n\xi : \xi \geq 0, \ \forall \xi \in \mathbb{R}^{M\times N}, \tag{17}$$

$$\exists K > 0, \quad \int_\Omega |Du| dx \leq K \left(\int_\Omega A_n Du : Du\, dx \right)^{\frac{1}{2}}, \ \forall u \in W_0^{1,1}(\Omega)^M. \tag{18}$$

In the proof we use some estimates which are based on the theory of Γ-convergence applied to the symmetric part of A_n. For that, we also assume that the skew-symmetric part of A_n can be uniformly controlled by the symmetric part, namely,

$$\exists R>0,\quad |A_n\xi:\eta|\leq R|A_n\xi:\xi|^{\frac12}|A_n\eta:\eta|^{\frac12},\quad \forall\xi,\eta\in\mathbb{R}^{M\times N},\ \forall n\in\mathbb{N},\ \text{a.e. }\Omega. \tag{19}$$

Moreover, note that thanks to condition (16) we can assume the existence of $\mathfrak{a}\in\mathcal{M}(\overline{\Omega})$ such that

$$|A_n|\overset{*}{\rightharpoonup}\mathfrak{a}\ \text{ en }\mathcal{M}(\overline{\Omega}). \tag{20}$$

The mentioned theorem (see Theorem 1.16 for further details) states

Theorem 0.1. *Assume $A_n\in L^\infty(\Omega;\mathcal{L}(\mathbb{R}^{M\times N}))$ satisfies* (16), (17), (18) *and* (19). *Then, there exist a subsequence of n, still denoted by n, a Hilbert space $H\subset W_0^{1,1}(\Omega)^M$ and a continuous linear operator $\tilde{\Sigma}:H\to L^1_{\mathfrak{a}}(\Omega)^{M\times N}$ such that for every sequence f_n weakly-∗ converging to f in $L^\infty(\Omega)^M$, the unique solution to* (15) *satisfies*

$$\begin{aligned} u_n &\overset{*}{\rightharpoonup} u\quad \text{in } BV(\overline{\Omega})^M,\\ A_nDu_n &\overset{*}{\rightharpoonup}\tilde{\Sigma}(u)\mathfrak{a}\ \text{in }\mathcal{M}(\overline{\Omega})^{M\times N}. \end{aligned}\tag{21}$$

Observe that (21), together with the convergence of f_n, establishes that u is a solution to the equation

$$-\mathrm{Div}\big(\tilde{\Sigma}(u)\mathfrak{a}\big)=f\ \text{ in }\Omega,$$

and thus, it gives the existence of a limit equation. However, it does not yield a representation of $\tilde{\Sigma}$. We recall that even for the case $M=1$, the limit $\tilde{\Sigma}$ is nonlocal in general, and therefore it does not have the form of $\tilde{\Sigma}(u)=ADu$ for some tensor function A.

The result that we show in this chapter (Theorem 1.16) is actually more general and, additionally, it gives the convergence of the energies in the sense that there exists a continuous bilinear operator $\tilde{\mathcal{B}}:H\times H\to\mathcal{M}(\overline{\Omega})$ such that if u_n is as in the theorem and v_n is a sequence in $W_0^{1,1}(\Omega)^M$ that fulfils

$$v_n\overset{*}{\rightharpoonup}v\ \text{ in }BV(\overline{\Omega})^M,\quad \limsup_{n\to\infty}\int_\Omega A_nDv_n:Dv_n\,dx<+\infty,$$

then

$$A_nDu_n:Dv_n\overset{*}{\rightharpoonup}\tilde{\mathcal{B}}(u,v)\ \text{ in }\mathcal{M}(\overline{\Omega}).$$

Furthermore, this operator $\tilde{\mathcal{B}}$ is related to $\tilde{\Sigma}$ by

$$\tilde{\mathcal{B}}(u,v)=\tilde{\Sigma}(u):Dv\,\mathfrak{a}\ \text{ in }\Omega,\quad \forall\, v\in C_0^1(\Omega)^M,$$

and u is the unique solution to

$$\begin{cases} u\in H,\\ \displaystyle\int_\Omega d\tilde{\mathcal{B}}(u,v)=\int_\Omega f\cdot v\,dx,\quad \forall\, v\in H. \end{cases}$$

Observe that the ellipticity condition (18) on A_n is integral instead of pointwise. As mentioned above, these two conditions are not equivalent in the case of systems. This allows us to apply our results to the linear elasticity, where pointwise ellipticity fails. A sufficient pointwise condition in order to have (18) would be to impose A_n^{-1} bounded in $L^1(\Omega;\mathcal{L}(\mathbb{R}^{M\times N}))$.

In order to have a local representation of the operator $\tilde{\Sigma}$ (and of $\tilde{\mathcal{B}}$) it is necessary to assume some integrability conditions on A_n. The obtained result is based on the div-curl theorem in [26], which, contrary to the classical result (see (6)), is applicable to the case of σ_n bounded in $L^p(\Omega)^N$ and τ_n bounded in $L^q(\Omega)^N$ with

$$\frac{1}{p}+\frac{1}{q}\le 1+\frac{1}{N}. \tag{22}$$

We have (see Theorem 1.11 for further details)

Theorem 0.2. *Under the assumptions of Theorem 0.1, let us also assume that*

$$A_n \text{ bounded in } L^p(\Omega;\mathcal{L}(\mathbb{R}^{M\times N})),\quad p\in\left[\frac{N}{2},\infty\right],$$

$$\int_\Omega |Du|^r dx \le \int_\Omega \gamma_n (A_n Du : Du)^{\frac{r}{2}} dx,\quad \forall u\in W_0^{1,r}(\Omega)^M,\ \forall n\in\mathbb{N},$$

with

$$r=\frac{2Np}{(N+2)p-N},\quad \gamma_n \text{ bounded in } L^{\frac{2}{2-r}}(\Omega),$$

then there exists $A\in L^p(\Omega;\mathcal{L}(\mathbb{R}^{M\times N}))$ such that

$$\tilde{\Sigma}(u)\mathfrak{a} = ADu,\quad \forall\, u\in H\cap W^{1,\frac{2p}{p-1}}(\Omega)^M.$$

It is worth pointing out that if a weaker integrability is imposed on A_n (i.e. smaller p), then a stronger ellipticity (larger r) is required for the integral representation and, conversely, a stronger integrability condition would allow a weaker ellipticity.

In addition, this theorem also includes, in particular, the results in [18] for the two-dimensional elasticity system with coefficients uniformly elliptic and bounded in L^1, which also uses the version of the div-curl theorem in [26].

Chapter 2

As in the previous chapter, we consider an open bounded set $\Omega\subset\mathbb{R}^N$ with $N\ge 2$ and an integer number $M\ge 1$. In this chapter we analyse the Γ-limit in $L^p(\Omega)^M$, $p>1$, of sequences of nonlinear functionals defined over vector functions of the type

$$\mathscr{F}_n(v) := \int_\Omega F_n(x,Dv)\,dx \quad \text{for } v\in W_0^{1,p}(\Omega)^M. \tag{23}$$

We assume that the energy densities $F_n : \Omega \times \mathbb{R}^{M\times N} \to [0,\infty)$ are Carathéodory functions such that there exist $\alpha, \beta, \gamma > 0$ and two sequences of non-negative measurable functions h_n, a_n, with h_n bounded in $L^1(\Omega)$ and a_n bounded in $L^r(\Omega)$, where

$$\begin{cases} r > \frac{N-1}{p}, & \text{if } 1 < p \le N-1, \\ r = 1, & \text{if } p > N-1, \end{cases}$$

satisfying the following assumptions of (integral) ellipticity, growth and Lipschitzianity

$$F_n(\cdot, 0) = 0, \quad \text{a.e. } \Omega, \tag{24}$$

$$\int_\Omega F_n(x, Du)\, dx \ge \alpha \int_\Omega |Du|^p\, dx - \beta, \quad \forall u \in W_0^{1,p}(\Omega)^M, \tag{25}$$

$$F_n(x, \lambda\xi) \le h_n(x) + \gamma F_n(x,\xi), \quad \forall \lambda \in [0,1],\ \forall \xi \in \mathbb{R}^{M\times N},\ \text{a.e. } x \in \Omega, \tag{26}$$

$$\begin{cases} \big|F_n(x,\xi) - F_n(x,\eta)\big| \\ \qquad \le \big(h_n(x) + F_n(x,\xi) + F_n(x,\eta) + |\xi|^p + |\eta|^p\big)^{\frac{p-1}{p}} a_n(x)^{\frac{1}{p}}\, |\xi - \eta|, \\ \forall\, \xi, \eta \in \mathbb{R}^{M\times N},\ \text{a.e. } x \in \Omega. \end{cases} \tag{27}$$

Condition (24) implies that the functionals (23) reach a minimum for $v = 0$ which is usual in nonlinear elasticity. This means that in the equilibrium (no displacements) the elastic energy is zero. Concerning the rest of the assumptions, they are also fulfilled in the usual models of nonlinear elasticity, for instance, some hyperelastic materials such as the Saint Venant-Kirchhoff materials and some Ogden's type hyper-elastic materials ([40], Vol. 1). As a prototypical example, consider

$$F_n(x,\xi) = |A_n(x)\xi_s : \xi_s|^{\frac{p}{2}}, \quad \forall\, \xi \in \mathbb{R}^{M\times N}, \quad \text{a.e. } x \in \Omega,$$

with ξ_s the symmetric part of ξ. In this case, one can take

$$a_n(x) = |A_n(x)|^{\frac{p}{2}},$$

which shows that a_n essentially measures how big the coefficients are.

We do not impose the convexity of F_n with respect to its second variable as it is usual in equations. In fact, it is known that the Γ-limit of a sequence of functionals in a given topology agrees with the Γ-limit of the lower-semicontinuous hull of these functionals. It is also known that if a functional of the type (23) is lower semicontinuous for the topology of $L^p(\Omega)$ then F_n, as a function of its second variable, is convex for the rank-one matrices (F_n is rank-one convex). For this reason, contrary to the case of equations, the assumption of convexity is more restrictive for systems.

As a consequence of the nonlinearity of the problem, the div-curl theorem cannot be applied directly as we did in the previous chapter. Nevertheless, we make use of a lemma in [25] which is essential for the proof of the version of the div-curl theorem that appears in the same reference. It is a compactness result for bounded sequences in $W^{1,q}$ based on the embedding $W^{1,q}(S^{N-1}) \subset L^{q^*}(S^{N-1})$, where S^{N-1} is the unit sphere of $\mathbb{R}^{N-1}$. Whereas in the div-curl theorem in [26] condition (22) is assumed, in [25] it is only necessary to have

$$\frac{1}{p} + \frac{1}{q} < 1 + \frac{1}{N-1}.$$

As a result, if we applied the results in this chapter to the linear case (i.e. F_n quadratic with respect to its second variable), we could improve the main theorem of the previous chapter when $r = 2$, A_n symmetric and $N \geq 3$, showing that the assumption $p \geq N/2$ can be relaxed by replacing it by $p > (N-1)/2$.

The main results of this chapter (see Theorems 2.3 and 2.4 for further details) show the existence of a function $F : \Omega \times \mathbb{R}^{M\times N} \to \mathbb{R}$ which satisfies similar properties to those of F_n such that, at least for regular functions, the Γ-limit $\mathscr{F}$ in $L^p(\Omega)^M$ of the sequence $\mathscr{F}_n$ satisfies

$$\mathscr{F}(v) = \int_\Omega F(x, Dv)\, dx.$$

Furthermore, the result is local in the sense that the value of F in an open subset of Ω only depends on the value of F_n in that subset.

Chapter 3

In this chapter we consider the linear elasticity system posed in a thin beam of thickness $\varepsilon > 0$, $\Omega_\varepsilon := (0,1) \times (\varepsilon\omega)$, when the tensor of coefficients also depends on ε. Specifically, we study the problem

$$\begin{cases} -\operatorname{div}(A_\varepsilon e(u_\varepsilon)) = h_\varepsilon & \text{in } \Omega_\varepsilon, \\ A_\varepsilon e(u_\varepsilon)\nu = 0 & \text{on } (0,1) \times (\varepsilon\partial\omega), \end{cases} \tag{28}$$

where $\omega \subset \mathbb{R}^{N-1}$ is a regular, connected, bounded domain (in practice $N = 2, 3$), ν is the unitary outward normal vector to ω on $\partial\omega$, u_ε is the deformation of the beam, $e(u_\varepsilon)$ is the strain tensor and $h_\varepsilon = (h_{\varepsilon,1}, h'_\varepsilon)$ is the exterior force that will be assumed of the type

$$h_{\varepsilon,1}(x) = f_1\left(x_1, \frac{x'}{\varepsilon}\right), \; h'_\varepsilon(x) = \varepsilon f'\left(x_1, \frac{x'}{\varepsilon}\right) + g'\left(x_1, \frac{x'}{\varepsilon}\right), \quad \text{a.e. } x \in \Omega_\varepsilon,$$

with $f \in L^2(\Omega)^N$ and $g' \in L^2(\Omega)^{N-1}$ (where $\Omega := \Omega_1$) such that

$$\int_\omega g' dy' = 0, \quad \text{a.e. } y_1 \in (0,1).$$

Observe that, in order to have the uniqueness of solution, it would be necessary to impose some boundary condition on $\{0,1\} \times (\varepsilon\omega)$. Our results remain true with different boundary conditions.

Our aim is to find a one-dimensional limit system whose solution provides an approximation of the solutions to (28) without any assumption of isotropy or homogeneity on the elasticity coefficients A_ε.

For the sake of simplicity, we assume uniform ellipticity, that is

$$\exists\alpha > 0, \quad A_\varepsilon\xi : \xi \geq \alpha|\xi|^2, \quad \forall\xi \in \mathbb{R}^{N\times N}_s, \quad \text{a.e. } (0,1)\times(\varepsilon\omega).$$

Nevertheless, as done in the previous chapters, we do not require the coefficients to be uniformly bounded. Namely, we just impose

$$\varepsilon\|A_\varepsilon\|_{L^\infty(\Omega_\varepsilon;\mathcal{L}(\mathbb{R}^{N\times N}_s))} \to 0, \qquad \|A_\varepsilon\|_{L^1(\Omega_\varepsilon;\mathcal{L}(\mathbb{R}^{N\times N}_s))} \text{ bounded.}$$

The main result that we obtain (see Theorem 3.1) gives an approximation for the solutions of the type

$$\begin{cases} u_{\varepsilon,1}(x) \sim u_1(x_1) - \displaystyle\sum_{j=2}^N \frac{du_j}{dx_1}(x_1)\frac{x_j}{\varepsilon}, \\ u_{\varepsilon,j}(x) \sim \dfrac{1}{\varepsilon}u_j(x_1) + \displaystyle\sum_{i=2}^N Z_{ji}(x_1)\frac{x_i}{\varepsilon}, \quad j \in \{2,\cdots,N\}. \end{cases} \tag{29}$$

This approximation consists in the sum of a deformation of Bernouilli-Navier's type given by the function $u = (u_1,\ldots,u_N)$ plus a torsion term given by the matrix function Z, which is skew-symmetric. The latter corresponds to an infinitesimal rotation around the axis of the beam. We show that the functions u and Z are solutions to a one-dimensional linear system that, in variational form, reads as

$$\begin{cases} \displaystyle\int_0^1 Ae_0(u,Z):e_0(\tilde u,\tilde Z)\,dy_1 = \frac{1}{|\omega|}\int_\Omega \left(f_1\left(\tilde u_1 - \frac{d\tilde u'}{dy_1}\cdot y'\right) + f'\cdot\tilde u' + g'\cdot(\tilde Z\, y')\right)dy, \\ \forall(\tilde u,\tilde Z) \in H^1_0(0,1)\times H^2_0(0,1)^{N-1}\times H^1_0(0,1;\mathbb{R}^{(N-1)\times(N-1)}_{sk}), \\ \qquad \text{with } \displaystyle\int_0^1 Ae_0(\tilde u,\tilde Z):e_0(\tilde u,\tilde Z)\,dx_1 < \infty, \end{cases} \tag{30}$$

where the subindex sk refers to skew-symmetric matrices and the operator e_0 is defined by

$$e_0(u,Z) := \begin{pmatrix} \dfrac{du_1}{dx_1} & \left(\dfrac{d^2u'}{dx_1^2}\right)^T \\ \dfrac{d^2u'}{dx_1^2} & \dfrac{dZ}{dx_1} \end{pmatrix}.$$

In addition, the tensor function A belongs to $L^1(0,1;\mathcal{L}(\mathbb{R}^{N\times N}_{s_1sk'}))$ and is such that there exist $\beta,\gamma > 0$ and $a \in L^1(0,1)$, $a \geq 0$, satisfying

$$|AE| \leq \beta\big(AE:E\big)^{\frac12}a^{\frac12}, \quad \forall\, E \in \mathbb{R}^{N\times N}_{s_1sk'}, \text{ a.e. } (0,1),$$

$$|E|^2 \leq \gamma AE:E, \quad \forall\, E \in \mathbb{R}^{N\times N}_{s_1sk'}, \text{ a.e. } (0,1),$$

where $\mathbb{R}^{N\times N}_{s_1 sk'}$ is the subspace of the matrices $M \in \mathbb{R}^{N\times N}$ that satisfy

$$M_{1i} = M_{i1},\ i = 1, \dots, N, \qquad M_{ij} = -M_{ji},\ i, j = 2, \dots, N.$$

Observe that, even though the sequence A_ε is only bounded in L^1, the limit tensor A also belongs to L^1. The proof of this result is an adaptation of the classical proof of the H-convergence theorem by F. Murat and L. Tartar (cf. [76, 91]) combined with a decomposition result for sequences of deformations in thin domains that can be found in [38].

The limit system (30) provides a general model for strongly heterogeneous beams that do not satisfy any isotropy condition. Recall that for a homogeneous isotropic material, the model used in architecture or engineering corresponds (in dimension 3) to a system of two fourth-order equations (given by the functions u_2 and u_3 in (30)).

Chapter 4

In this chapter we focus on the homogenization, via Γ-convergence, of weakly coercive integral energies with densities $\mathbb{L}(x/\varepsilon)Dv : Dv$, where $\mathbb{L} \in L^\infty_{\rm per}(Y_N; \mathscr{L}_s(\mathbb{R}^{N\times N}_s))$ is a periodic, symmetric, tensor function.

This chapter is divided into two main parts.

In the first part of Chapter 4, we analyse condition (13) (with A replaced by $\mathbb{L}$) which, as previously mentioned, is enough in order for the periodic homogenization formula (3) to hold for systems. In [27], the authors give a class of examples in dimension 2 that fulfil (13) but such that $\mathbb{L}$ is not very strongly elliptic (i.e. (12) does not hold for all $\xi \in \mathbb{R}^{N\times N}$). Following the same ideas, in Theorem 4.4 we show a set of mixtures in dimension 3 that satisfy (13) and are not very strongly elliptic. In addition, Theorem 4.5 improves condition (13) showing that it is enough to have

$$\int_{\mathbb{R}^N} \mathbb{L}Du : Du\, dy \geq 0, \quad \forall\, u \in \mathcal{D}(\mathbb{R}^N)^N. \tag{31}$$

for the Γ-convergence result to hold true.

The second part of this chapter focuses on the loss of strong ellipticity through the homogenization process in the case of linear elasticity in dimension 3. We make a deep study of the lamination process carried out by S. Gutiérrez in [64] and we try to justify it, in terms of Γ-convergence, by using Theorem 4.5. In order to apply this theorem we need the relaxed functional coercivity (31) and, for that, we make use of the translation method for the null-Lagrangians. This method consists in finding a matrix $D \in \mathbb{R}^{3\times 3}$ such that

$$\mathbb{L}M : M + D : \mathrm{Adj}(M) \geq 0 \quad \forall M \in \mathbb{R}^{3\times 3}, \ \text{ a.e. } Y_N, \tag{32}$$

as it was done in [27] for the two-dimensional case. Surprisingly, contrary to what happens in dimension 2, we prove in Theorem 4.8 that if a strongly elliptic, laminated (i.e. $\mathbb{L}(y) = \mathbb{L}(y_1)$) material fulfils (32), then it is impossible to obtain an effective material for which the strong ellipticity condition fails. Therefore, we need to perform a second lamination (in a new direction), as done by S. Gutiérrez in [64], in order to produce a limit material that losses strong ellipticity. Indeed, Theorem 4.14 shows that there exist certain strongly elliptic materials for which the strong ellipticity can be lost after a rank-two lamination with some specific very strongly elliptic materials.

Introducción

En la elaboración de ciertos materiales compuestos, la mezcla de los distintos componentes se realiza a nivel microscópico, o más exactamente mesoscópico (pequeño desde el punto de vista macroscópico pero suficientemente grande para que se puedan despreciar los efectos cuánticos). La primera dificultad que esto entraña es la resolución numérica de las ecuaciones en derivadas parciales que describen el comportamiento de las distintas magnitudes físicas relacionadas. Para ello, es necesario usar mallas cuyos elementos sean pequeños con respecto a la medida de las estructuras que forman los compuestos que aparecen en la mezcla. Esto da lugar a sistemas de ecuaciones tan grandes que su resolución directa puede ser imposible. Tanto físicos como ingenieros han atacado usualmente este tipo de problemas mediante la introducción de pequeños parámetros con la idea de más tarde llevar a cabo un desarrollo asintótico con respecto a ellos. Ello conduce a la resolución de problemas mucho más simples, los cuales proporcionan una buena aproximación de la solución del problema original. En muchos casos, se ha dado posteriormente justificación matemática a los distintos modelos aproximados obtenidos, probándose resultados de convergencia en ciertos espacios funcionales. La parte de la Matemática que se ocupa de este tipo de cuestiones se conoce como teoría de la homogeneización.

Como ejemplo recordamos el que probablemente es el problema más clásico en homogeneización. Por fijar ideas consideramos un material eléctrico que se obtiene repitiendo una célula de forma periódica con un pequeño periodo $\varepsilon > 0$. Las ecuaciones de la electrostática nos dicen que el potencial eléctrico u_ε es solución de

$$-\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = \rho \ \text{ en } \Omega, \tag{1}$$

donde Ω es un abierto de $\mathbb{R}^N$ (en la práctica $N = 2, 3$) y ρ es la densidad de carga. La matriz de coeficientes A depende de la constante dieléctrica del medio y es periódica de periodo el cubo unidad. Claramente, a fin de tener unicidad de solución para (1) es necesario añadir alguna condición de contorno. La construcción de materiales mediante este procedimiento es usual en Ingeniería.

El método de desarrollos asintóticos (ver e.g. [9], [65], [71], [84], [85]) aplicado a este problema consiste en suponer que la función u_ε admite un desarrollo del tipo

$$u_\varepsilon(x) \sim u_0(x) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \cdots,$$

con las funciones $u_1, u_2, \ldots$ periódicas en la segunda variable. Sustituyendo en (1) e igualando los coeficientes con el mismo exponente en ε se obtiene formalmente que

u_0 es solución del problema

$$-\operatorname{div}(A_h \nabla u_0) = \rho \ \text{ en } \Omega, \tag{2}$$

donde A_h (matriz homogeneizada) viene dada por

$$A_h \xi = \int_{Y_N} A(\xi + \nabla_y w_\xi)\, dy, \quad \forall\, \xi \in \mathbb{R}^N, \tag{3}$$

con w_ξ solución de

$$\begin{cases} -\operatorname{div}(A \nabla w_\xi) = 0 \ \text{ en } \mathbb{R}^N, \\ w_\xi \text{ periódica de periodo el cubo unidad } Y_N. \end{cases}$$

Además se puede probar

$$u_1(x, y) = w_{\nabla u_0(x)}(y).$$

El resultado anterior nos da una muestra de por qué usar el término homogeneización. Mientras que en (1) nos encontrábamos con un material fuertemente heterogéneo, en (2) nos encontramos con un material homogéneo dado por la matriz constante A_h. Observar que la resolución numérica de las funciones u_0 y u_1 es mucho más simple que la de u_ε. El resultado merece también ser analizado desde un punto de vista más teórico. Desde el punto de vista macroscópico, las propiedades eléctricas del material correspondiente a la matriz $A(x/\varepsilon)$ son similares a las del material correspondiente a A_h. Si pensamos por ejemplo que la matriz A se obtiene mezclando dos materiales, i.e. existen $Z \subset Y_N$ medible y A_1, A_2 matrices tales que

$$A(y) = A_1 \chi_Z(y) + A_2(1 - \chi_Z(y)), \quad \text{e.c.t. } y \in Y_N,$$

entonces, al mezclar estos materiales hemos construido uno nuevo, correspondiente a la matriz A_h, cuyas propiedades no dependen solamente de la proporción de ambos (i.e. de la medida de Z) sino también de su disposición geométrica. Así por ejemplo aunque A_1 y A_2 sean matrices escalares, correspondientes a materiales isótropos (i.e. sus propiedades no dependen de la dirección), la matriz A_h no tiene por qué ser escalar.

Aunque el método descrito anteriormente para la obtención de A_h es formal, resultados de convergencia se pueden encontrar por ejemplo en [9] y [65]. De hecho debido a su importancia especialmente en Ingeniería y Arquitectura, se han desarrollado diversos métodos para poder resolver matemáticamente problemas como el anterior donde hay algún tipo de periodicidad. Destacar los métodos de convergencia en dos escalas y "unfolding" ([2], [4], [34], [36], [41], [81]).

El ejemplo anterior nos muestra cómo podemos analizar desde el punto de vista matemático la obtención de nuevos materiales mediante la mezcla de otros ya existentes, usando distribuciones que suelen ser altamente oscilantes. La idea es estudiar la convergencia de ecuaciones en derivadas parciales con coeficientes variables. Si bien en el caso anterior nos encontrábamos con un problema periódico, a fin de obtener materiales generales, es importante conocer qué ocurre cuando no hay ningún tipo de periodicidad. La primera pregunta que surge es si el tipo de ecuaciones que estamos considerando es estable cuando pasamos al límite. En caso contrario deberemos usar modelos más generales.

Los primeros resultados, en nuestro conocimiento, referentes a la estabilidad en el paso al límite de una sucesión de EDP con coeficientes variables, se refieren al caso de una sucesión de ecuaciones lineales elípticas de segundo orden escritas en forma de divergencia. Así, en [87] (ver también [52]) S. Spagnolo mostró que si A_n es una sucesión acotada en $L^\infty(\Omega)^{N\times N}$ con valores en las matrices simétricas y tal que es uniformemente elíptica en el sentido de que existe $\alpha > 0$ con

$$A_n\xi \cdot \xi \geq \alpha|\xi|^2, \quad \forall n \in \mathbb{N},\ \forall\, \xi \in \mathbb{R}^N,\ \text{e.c.t. } \Omega, \tag{4}$$

entonces, existe una subsucesión de A_n, que seguimos denotando por A_n, y una función matricial simtrica $A \in L^\infty(\Omega)^{N\times N}$, verificando también (4), tal que para toda $f \in H^{-1}(\Omega)$, las soluciones de

$$\begin{cases} -\text{div}\,(A_n\nabla u_n) = f & \text{en } \Omega, \\ u_n = 0 & \text{sobre } \partial\Omega, \end{cases} \tag{5}$$

convergen en $H_0^1(\Omega)$ débil hacia la solución u del problema resultante de cambiar A_n por A. Se muestra además cómo el resultado se extiende al operador parabólico correspondiente (la extensión al caso hiperbólico aparece en [43]). F. Murat y L. Tartar extendieron más adelante este resultado al caso de matrices no necesariamente simétricas ([76]) mostrando además que se tiene la convergencia de $A_n\nabla u_n$ a $A\nabla u$ en $L^2(\Omega)^N$. El resultado se extiende fácilmente a sistemas de ecuaciones elípticas y en particular al sistema de la elasticidad lineal que nos describe la deformación elástica de un sólido (suponiendo que las derivadas de las deformaciones son pequeñas). En este sentido mencionamos los trabajos de G. Francfort [59], E. Sánchez-Palencia [85] y G. Duvaut (referencia no disponible). La demostración de este resultado se basa en lo que actualmente se denomina método de las funciones oscilantes y consiste en usar sucesiones especiales de funciones test (la convergencia en dos escalas mencionada anteriormente también se basa en esta idea). Una herramienta importante en la demostración es el teorema del div-rot que es el resultado más conocido de lo que se conoce como compacidad por compensación, también introducida por F. Murat y L. Tartar ([77], [89]) y que establece que dado $p \in (1, \infty)$, si

$$\begin{gathered} \sigma_n \rightharpoonup \sigma \ \text{ en } L^p(\Omega)^N, \qquad \tau_n \rightharpoonup \tau \ \text{ en } L^{p'}(\Omega)^N, \\ \text{div}\,\sigma_n \to \text{div}\,\sigma \ \text{ en } W^{-1,p}(\Omega), \qquad \text{rot}\,\tau_n \rightharpoonup \text{rot}\,\tau \ \text{ en } W^{-1,p'}(\Omega)^{N\times N}, \end{gathered} \tag{6}$$

entonces

$$\sigma_n \cdot \tau_n \rightharpoonup \sigma \cdot \tau \ \text{ en } \mathcal{D}'(\Omega).$$

Aunque el resultado de convergencia para (5) se suele enunciar, tal y como hemos hecho, con condiciones de contorno de tipo Dirichlet homogéneas, también es cierto con otras condiciones de contorno. Además es local en el sentido de que el valor de la matriz A en un subconjunto abierto arbitrario de Ω solo depende de los valores de A_n en ese conjunto. Extensiones a ecuaciones no lineales aparecen por ejemplo en [53] y [82].

Mencionar que este tipo de resultados se usa en la resolución de problemas de diseño óptimo de materiales proporcionando formulaciones relajadas (ver e.g. [2], [35], [80]).

Una pregunta que surge a partir de los resultados mencionados es qué ocurre si la sucesión A_n no está uniformemente acotada y/o no es uniformemente elíptica. Es lo que se conoce como homogeneización con alto contraste.

Una herramienta importante para tratar con este tipo de problemas es la Γ-convergencia introducida por E. De Giorgi (ver e.g. [12], [14], [48], [51]). Dado un espacio métrico X (la definición se extiende a espacios no métricos) y una sucesión de funcionales $F_n : X \to \mathbb{R} \cup \{+\infty\}$, se dice que F_n Γ-converge a F en X si se cumple

$$\begin{cases} x_n \to x \text{ en } X \Longrightarrow \liminf\limits_{n\to\infty} F_n(x_n) \geq F(x), \\ \forall\, x \in X,\ \exists x_n \to x \ \text{ tal que } \limsup\limits_{n\to\infty} F_n(x_n) \leq F(x). \end{cases}$$

El resultado más importante de la Γ-convergencia establece que si F_n alcanza mínimo en x_n y si la sucesión x_n es compacta en X, entonces todos los puntos de acumulación de x_n son puntos de mínimo para F. Así, si volvemos al problema (5) y suponemos A_n simétrica, sabemos que u_n es solución si y sólo si lo es del problema

$$\min_{u\in H_0^1(\Omega)} \left\{ \int_\Omega A_n \nabla u \cdot \nabla u \, dx - 2\langle f, u\rangle \right\}.$$

Teniendo además en cuenta que gracias a (4) las soluciones de (5) están acotadas en $H_0^1(\Omega)$ y por tanto son compactas en $L^2(\Omega)$, deducimos que el resultado de S. Spagnolo se puede obtener probando (suponemos el segundo miembro en $L^2(\Omega)$)

$$\left[u \mapsto \int_\Omega \big(A_n \nabla u \cdot \nabla u - 2fu\big)\, dx\right] \xrightarrow{\Gamma} \left[u \mapsto \int_\Omega \big(A \nabla u \cdot \nabla u - 2fu\big) dx\right] \quad \text{en } L^2(\Omega),$$

o equivalentemente (es consecuencia de considerar f en el dual de $L^2(\Omega)$) a que

$$\left[u \mapsto \int_\Omega A_n \nabla u \cdot \nabla u \, dx\right] \xrightarrow{\Gamma} \left[u \mapsto \int_\Omega A \nabla u \cdot \nabla u \, dx\right] \quad \text{en } L^2(\Omega).$$

Una ventaja de esta formulación es que el funcional

$$u \mapsto \int_\Omega A_n \nabla u \cdot \nabla u \, dx, \tag{7}$$

está bien definido aunque la integral pueda ser infinita, lo que permite tratar más fácilmente el caso en que A_n no está en $L^\infty(\Omega)^{N\times N}$. La desventaja es que el problema tiene que poder plantearse como un problema de mínimo.

Como ejemplo clásico de aplicación de la teoría de Γ-convergencia a la resolución de problemas de homogeneización, destacamos el artículo [33] de L. Carbone y C. Sbordone, donde se estudia la Γ-convergencia en $L^\infty(\Omega)$ de la sucesión de funcionales

$$u \mapsto \int_\Omega F_n(x, u, \nabla u)\, dx, \tag{8}$$

con $F_n : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ una sucesión de funciones de Carathéodory (medibles en la primera variable y continuas en las otras dos), convexas en la última variable y tales que se verifica

$$0 \leq F_n(x,s,\xi) \leq a_n(x)(1+|s|^p+|\xi|^p), \quad \forall\,(s,\xi) \in \mathbb{R} \times \mathbb{R}^N, \quad \text{e.c.t. } x \in \Omega, \tag{9}$$

con $p > 1$ y a_n acotada en $L^1(\Omega)$. Los autores muestran que, para una subsucesión de n, existe el Γ-límite de estos funcionales en $L^\infty(\Omega)$ y que al menos para las funciones regulares admite una representación integral del mismo tipo. Además, si a_n es equi-integrable entonces el Γ-límite en $L^\infty(\Omega)$ coincide con el Γ-límite en $L^1(\Omega)$. Comentar que como en los casos anteriores, el proceso de homogeneización es además local.

Si queremos aplicar este resultado a la convergencia de mínimos, necesitamos también que estos funcionales admitan mínimo y que los mínimos se encuentren en un compacto de la topología que estamos considerando. Así, si suponemos a_n equi-integrable, nos basta que la sucesión de mínimos esté acotada en $W^{1,1}(\Omega)$, lo que se puede obtener mediante alguna hipótesis de coercitividad adecuada como por ejemplo

$$0 \leq b_n(x)|\xi|^p \leq F_n(x,s,\xi),\ \forall\,(s,\xi) \in \mathbb{R} \times \mathbb{R}^N,\ \text{e.c.t. } x \in \Omega,\ b_n^{-\frac{1}{p}} \text{ acotado en } L^{p'}(\Omega).$$

Si a_n está solo acotada en $L^1(\Omega)$ necesitamos que la sucesión de mínimos sea compacta en $L^\infty(\Omega)$, lo que nos llevará esencialmente a tomar $p > N$ y una hipótesis de coercitividad tal como

$$\alpha|\xi|^p \leq F_n(x,s,\xi),\ \forall\,(s,\xi) \in \mathbb{R} \times \mathbb{R}^N,\ \text{e.c.t. } x \in \Omega,\ \alpha > 0.$$

Como ejemplo se pueden aplicar los resultados de [33] al problema (5), deduciéndose que para $N \geq 2$ y A_n simétrica, verificando

$$\begin{gathered} b_n(x)|\xi|^2 \leq A_n(x)\xi\cdot\xi \leq a_n(x)|\xi|^2, \quad \forall\,\xi \in \mathbb{R}^N, \quad \text{e.c.t. } x \in \Omega, \\ a_n, b_n \geq 0, \quad a_n \text{ acotada en } L^1(\Omega),\ \text{equi-integrable},\ b_n^{-1} \text{ acotada en } L^1(\Omega), \end{gathered}$$

y f suficientemente regular, las soluciones de (5) convergen $*$-débil en $BV(\Omega)$ hacia la solución de un problema del mismo tipo.

En [56] (ver también [8], [28]) V. N. Fenchenko y E. Ya. Khruslov muestran un ejemplo de una función $a_n \geq 1$, acotada en $L^1(\Omega)$ (pero no equi-integrable) con $\Omega = \omega \times (0,1)$, $\omega \subset \mathbb{R}^2$ abierto acotado, tal que las soluciones del problema

$$\begin{cases} -\operatorname{div}(a_n \nabla u_n) = f & \text{en } \Omega, \\ u_n = 0 & \text{sobre } \partial\Omega, \end{cases}$$

convergen débilmente en $H_0^1(\Omega)$ hacia la solución de

$$\begin{cases} -\Delta u + 2\pi \left(u + \displaystyle\int_0^1 h(x_3,t)u(x_1,x_2,t)\,dt \right) = f & \text{en } \Omega, \\ u = 0 & \text{sobre } \partial\Omega, \end{cases}$$

con h una función no nula. Vemos por tanto cómo ahora la ecuación cambia de forma. En el límite encontramos un término de orden cero y un término no local. Un resultado general en este sentido ha sido obtenido por U. Mosco en [74], donde usando la fórmula de representación de Beurling-Deny para formas de Dirichlet ([10]) se prueba que el Γ-límite en $L^2(\Omega)$ de la sucesión de funcionales definidos por (7) con A_n no negativa, acotada en $L^1(\Omega)^{N\times N}$ y simétrica converge hacia un funcional del tipo

$$u \mapsto \int_\Omega A\nabla u \cdot \nabla u \, d\mu(x) + \int_\Omega u^2 d\nu(x) + \int_{\Omega\times\Omega} \big(u(x) - u(y)\big)^2 d\eta(x, y), \qquad (10)$$

con μ, ν y η medidas Borelianas no negativas y acotadas. En general, el proceso de homogeneización lleva a la aparición de términos no locales incluso partiendo de términos fuertemente locales.

Gracias a una generalización del teorema del div-rot se ha probado más tarde en [17], [19] que en realidad en dimensión $N = 2$, suponiendo A_n uniformemente elíptica, los dos últimos términos son siempre nulos, i.e. el funcional no cambia de forma por Γ-convergencia y el proceso de homogeneización sigue siendo local. Este resultado ha sido generalizado posteriormente en [20] mostrando que ni siquiera es necesario suponer la acotación en $L^1(\Omega)^{N\times N}$. Resultados relacionados referentes a ecuaciones en el caso periódico y a la aparición de términos de orden cero pueden encontrarse en [13] y [21] respectivamente. Todos estos trabajos usan ciertos resultados recientes de convergencia uniforme para las soluciones de EDP elípticas ([22], [72]). De hecho con estas ideas se ha obtenido en [23] una extensión de los resultados de L. Carbone y C. Sbordone en [33] donde se muestra que para la equivalencia entre el Γ-límite en $L^1(\Omega)$ y $L^\infty(\Omega)$ de los funcionales que aparecen en (8) basta en realidad tomar $p > N - 1$ en lugar de $p > N$.

Los resultados de convergencia uniforme que se usan en las referencias [13], [20], [21], [23] y [33] están basados en el principio del máximo. También la fórmula de Beurling-Deny que conduce a la expresión (10) está basada en él. Ello hace que en principio no se puedan generalizar los resultados que aparecen en estos trabajos al caso de sistemas de ecuaciones. Así, contrariamente a (10), en el caso de la elasticidad lineal la ausencia de acotación uniforme de los coeficientes puede provocar la aparición en el Γ-límite de derivadas de segundo orden como probaron C. Pideri y P. Seppecher en [83]. Es más, M. Camar-Eddine y P. Seppecher probaron en [32] que se puede alcanzar cualquier funcional cuadrático semicontinuo inferiormente que sea nulo para los movimientos rígidos.

Debido a la falta de principio del máximo, no hay resultados generales, en nuestro conocimiento, acerca de qué hipótesis de acotación o elipticidad son necesarias en los coeficientes de un sistema de EDP de forma que en el límite mantenga su estructura y el proceso de homogeneización sea local. Comentar la existencia de algunos resultados particulares en el caso lineal usando Γ-convergencia. Así, para $N = 2$, se ha probado en [18] la estabilidad del sistema de la elasticidad lineal suponiendo que los coeficientes son uniformemente elípticos y acotados en L^1. El resultado se basa en la generalización del teorema del div-rot que aparece en [26]. Otro resultado relativo a un sistema elíptico general correspondiente a M ecuaciones en un abierto

Ω de $\mathbb{R}^N$ ha sido obtenido en [24] donde se supone que el tensor de coeficientes A_n es tal que existe otra sucesión de tensores B_n uniformemente elípticos y acotados de forma que $A_n - B_n$ converge fuertemente a cero en $L^1(\Omega; \mathcal{L}(\mathbb{R}^{M\times N}))$. Comentar que la elipticidad uniforme solo se impone en forma integral, i.e.

$$\alpha \int_\Omega |Du|^2 dx \leq \int_\Omega A_n Du : Du\, dx, \quad \forall\, u \in H_0^1(\Omega)^M, \tag{11}$$

con $\alpha > 0$. Es conocido (ver e.g. [48]) que esto implica

$$A_n\xi : \xi \geq \alpha|\xi|^2, \quad \forall\, \xi \in \mathbb{R}^{M\times N},\ \text{Rang}(\xi) = 1, \quad \text{e.c.t. } \Omega, \tag{12}$$

y por tanto en el caso de ecuaciones, $M = 1$, es equivalente a (4). Sin embargo esto no es así para sistemas. Para distinguir estos casos, en la literatura, es usual decir que un tensor que verifica la condición (12) es fuertemente elíptico mientras que en el caso en que esta condición es satisfecha para todo $\xi \in \mathbb{R}^{M\times N}$, se dice que es muy fuertemente elíptico. Cuando A_n es una función regular en Ω, la teoría de compacidad por compensación (ver e.g. [76], [89]) muestra que (12) es equivalente a (11).

El problema principal en el que nos interesamos en los dos primeros capítulos de la tesis es obtener condiciones de elipticidad y/o acotación generales en dimensión arbitraria, primero para sistemas lineales y posteriormente para no lineales, que conduzcan a un sistema límite local para lo que usaremos extensiones del teorema del div-rot ([25], [26]). En el tercer capítulo continuaremos con esta cuestión pero en el caso en que además hay una reducción de dimensión. Concretamente consideraremos el sistema de la elasticidad para barras delgadas $\Omega_\varepsilon = (0,1) \times (\varepsilon\omega)$ con ω un abierto acotado regular de $\mathbb{R}^{N-1}$. A diferencia de los casos mencionados anteriormente donde el abierto en el que planteamos la ecuación está fijo, ahora lo que se pretende es obtener un problema límite uni-dimensional. Esta es una cuestión clásica en Ingeniería. Al tratar de resolver directamente un problema de EDP en un dominio donde al menos una de sus dimensiones es mucho menor que las demás, nos encontramos con la dificultad anteriormente mencionada de tener que utilizar mallas muy finas. La idea es aproximar las soluciones del problema por las de otro planteado en un dominio con menor dimensión. Así, en el caso de vigas, el problema que se resuelve usualmente consiste en un sistema formado por dos ecuaciones elípticas de cuarto orden desacopladas. Desde el punto de vista matemático (ver e.g. [68], [92]) estas ecuaciones se obtienen pasando al límite cuando el grosor de la viga tiende a cero en el sistema de la elasticidad correspondiente a un material homogéneo e isótropo en dimensión 3 y su solución proporciona una aproximación de las deformaciones transversales a la viga. Más generalmente, en [79] (ver también [37]) se ha considerado el caso de un tensor de la forma $A(x_1, x_2/\varepsilon, x_3/\varepsilon)$, donde A pertenece a $L^\infty((0,1)\times\omega; \mathcal{L}(\mathbb{R}_s^{3\times 3}))$ y verifica la hipótesis de elipticidad usual. Esto permite por ejemplo tratar con materiales en los que aparece un núcleo de un determinado material rodeado por otro. En este caso los autores obtienen una aproximación más compleja de las soluciones.

Siguiendo con la discusión planteada al principio de esta introducción, un problema importante es saber qué ocurre cuando el dominio delgado (viga o placa) está formado por una mezcla arbitraria de materiales. Esto lleva a estudiar el comportamiento asintótico de un problema de EDP planteado en un dominio delgado Ω_ε, con $\varepsilon > 0$ un valor pequeño, que nos mide el grosor, en el cual los coeficientes también dependen de ε. Aunque en nuestro conocimiento este problema no ha sido tan estudiado como el caso en que el dominio está fijo, podemos sin embargo referenciar ciertos trabajos en este sentido. Así, en [5], [30] y [86] se analiza este problema imponiendo ciertas hipótesis de periodicidad. Como ya explicamos anteriormente, esto permite tratar con varios materiales que aparecen usualmente en Ingeniería. Sin embargo, si queremos saber qué tipo de materiales generales se pueden obtener a partir de unos dados tendremos que eliminar la hipótesis de periodicidad. En el caso de problemas de difusión en una viga $(0,1) \times (\varepsilon\omega)$ e imponiendo hipótesis de elipticidad y acotación uniformes, el problema ha sido tratado en [45] bajo ciertas hipótesis de estructura que permiten aplicar un resultado de tipo div-rot y en [39] de forma general. En esta última referencia se trata con segundos miembros muy generales que conducen a un sistema límite planteado en el dominio $(0,1)\times\omega$, el cual es no local en general. Cuando nos restringimos a segundos miembros que no oscilan fuertemente en la variable correspondiente a las dimensiones que están degenerando, se puede comprobar cómo el problema se reduce a un problema local unidimensional. En el caso del comportamiento asintótico del sistema de la elasticidad con coeficientes variables en un dominio que degenera, debemos citar la referencia [50] donde se considera el caso de una placa $\omega \times (0, \varepsilon)$ con $\omega \subset \mathbb{R}^2$ abierto regular. Imponiendo ciertas hipótesis de isotropía y suponiendo que los coeficientes son uniformemente elípticos y acotados, se obtiene una ecuación límite de cuarto orden correspondiente al desplazamiento vertical, lo que es similar al caso que normalmente se trata en Ingeniería para placas formadas por materiales isótropos. En [62] se considera el caso en que no hay ninguna isotropía pero los coeficientes sólo dependen de la variable en altura de la placa. Ahora en el sistema límite no se pueden desacoplar en general las deformaciones en las variables horizontal y vertical y por tanto el problema límite tiene una estructura distinta.

A lo largo de esta introducción hemos visto cómo en muchos casos la estructura de un problema de EDP donde los coeficientes varían se conserva por paso al límite. Sin embargo algunos ejemplos notables conducen a casos en los cuales algunas propiedades importantes no se conservan. Ello puede ser usado para obtener materiales con características muy particulares. En este sentido, consideramos la diferencia entre coercitividad local y coercitividad global que expusimos anteriormente al hablar de la homogeneización de sistemas. Recordar que la fórmula de homogeneización periódica del comienzo de esta introducción, (3), sigue siendo cierta para sistemas imponiendo la coercitividad integral en lugar de la puntual. Más aún, en el caso $M = N$, ha sido mostrado en [61] que el resultado es cierto imponiendo simplemente la existencia de $\alpha > 0$ tal que (para A periódica de periodo el cubo unidad Y_N)

$$\begin{cases} \displaystyle\int_{Y_N} ADu\!:\!Du\,dy \geq \alpha \int_{Y_N} |Du|^2 dy, \quad \forall\, u \in H^1_{loc}(\mathbb{R}^N) \text{ periódica de periodo } Y_N, \\ \displaystyle\int_{\mathbb{R}^N} ADu\!:\!Du\,dy \geq 0, \quad \forall\, u \in \mathcal{D}(\mathbb{R}^N)^N. \end{cases} \tag{13}$$

Una importante pregunta es qué propiedades de elipticidad verifica el tensor homogeneizado. S. Gutiérrez en [64] prueba que, en un cierto marco de homogeneización (llamado 1*-convergencia en [27]), a partir de la laminación de un material isotrópo fuertemente elíptico, en el sentido de que se satisface (12), con uno muy fuertemente elíptico (i.e. que (12) se verifica para toda $\xi \in \mathbb{R}^{N\times N}$), se puede obtener un material para el cual ni siquiera la elipticidad fuerte es satisfecha. S. Gutiérrez realiza este estudio en los casos bidimensional y tridimensional. En algunos casos en dimensión 3, es necesario además realizar una segunda laminación con un tercer material (que puede ser elegido muy fuertemente elíptico). Sin embargo, el proceso seguido por S. Gutiérrez requiere cotas *a priori* en L^2 para la sucesión de deformaciones, lo cual es incompatible con la hipótesis de coercitividad débil. Por tanto, el resultado de S. Gutiérrez no se refiere al paso al límite en la sucesión de sistemas de EDP correspondientes. En [27] los autores proporcionan en el caso bidimensional una justificación de este resultado en términos de Γ-convergencia y muestran el carácter canónico de la laminación llevada a cabo por S. Gutiérrez. En este sentido recordar que si las funciones tensoriales $x \mapsto A(x/\varepsilon)$ verificaran la propiedad de elipticidad integral uniforme

$$\int_\Omega A\left(\frac{x}{\varepsilon}\right) Du : Du\, dx \geq \alpha \int_\Omega |Du|^2\, dx, \quad \forall\, u \in C_c^\infty(\Omega)^N, \tag{14}$$

con α positiva (independiente de ε), el Γ-límite también verificaría esta propiedad. Esto significa que el tensor A propuesto por S. Gutiérrez no cumple (14), aunque sí lo cumple cada una de las fases que constituyen el tensor A. Tal y como observan M. Briane y G. Francfort en [27], realizando el cambio de variables $y = x/\varepsilon$, esto significa que existen funciones tensoriales $A : \mathbb{R}^N \to \mathcal{L}(\mathbb{R}^{N\times N})$, con discontinuidades de salto, las cuales verifican (12) con $\Omega = \mathbb{R}^N$ pero no cumplen (11). Es decir, la equivalencia entre estas definiciones que expusimos anteriormente para A regular, no es cierta en general.

En el cuarto capítulo de la presente memoria formalizamos los resultados de S. Gutiérrez en el caso tridimensional en el marco de la Γ-convergencia.

En la exposición que hemos llevado a cabo anteriormente hemos realizado una introducción a los distintos problemas que nos interesan en la presente memoria, su motivación y lo resultados previos obtenidos por otros autores. También hemos esquematizado cuáles son las cuestiones precisas que pretendemos abordar. Realizamos a continuación una descripción explícita, desglosada por capítulos, de los distintos resultados que hemos obtenido a lo largo de la memoria, las dificultades que se presentan y los métodos que hemos usado para abordarlas:

Capítulo 1

Consideramos Ω un subconjunto abierto y acotado de $\mathbb{R}^N$, $N \geq 2$, y un número entero $M \geq 1$. En este capítulo nos proponemos obtener condiciones de integrabilidad y elipticidad sobre la sucesión de funciones tensoriales $A_n \in L^p(\Omega; \mathcal{L}(\mathbb{R}^{M\times N}))$

de forma que podamos asegurar que el problema homogeneizado correspondiente a los problemas elípticos lineales

$$\begin{cases} -\mathrm{Div}\,(A_n Du_n) = f_n & \text{en } \Omega, \\ u_n = 0 & \text{sobre } \partial\Omega, \end{cases} \tag{15}$$

sea del mismo tipo, al menos para las funciones suficientemente regulares y que además el proceso de homogeneización sea local. Como se ha mencionado anteriormente, en el caso de ecuaciones ($M = 1$), basta que A_n^{-1} esté acotado en $L^1(\Omega)^{N\times N}$ y A_n esté acotado en $L^1(\Omega)^{N\times N}$ y sea equi-integrable. El resultado además es falso si se elimina la hipótesis de equi-integrabilidad. La demostración de estos resultados usa el principio del máximo que no es válido para sistemas.

En nuestro caso, comenzamos probando la existencia de un resultado abstracto de homogeneización cuando los coeficientes A_n solamente verifican las propiedades

$$A_n \text{ acotado en } L^1(\Omega; \mathcal{L}(\mathbb{R}^{M\times N})), \tag{16}$$

$$A_n\xi : \xi \geq 0,\ \forall\, \xi \in \mathbb{R}^{M\times N}, \tag{17}$$

$$\exists K > 0, \quad \int_\Omega |Du| dx \leq K \left(\int_\Omega A_n Du : Du\, dx \right)^{\frac{1}{2}},\ \forall\, u \in W_0^{1,1}(\Omega)^M. \tag{18}$$

La demostración usa estimaciones que están basadas en la teoría de la Γ-convergencia aplicada a la parte simétrica de A_n. Para ello, suponemos también que la parte antisimétrica de A_n está uniformemente controlada por su parte simétrica, concretamente

$$\exists R > 0,\ \ |A_n\xi : \eta| \leq R|A_n\xi : \xi|^{\frac{1}{2}}|A_n\eta : \eta|^{\frac{1}{2}},\ \ \forall \xi, \eta \in \mathbb{R}^{M\times N},\ \forall n \in \mathbb{N},\ \text{e.c.t. } \Omega. \tag{19}$$

Observar también que gracias a (16) podemos suponer la existencia de $\mathfrak{a} \in \mathcal{M}(\overline{\Omega})$ tal que

$$|A_n| \stackrel{*}{\rightharpoonup} \mathfrak{a} \ \text{ en } \mathcal{M}(\overline{\Omega}). \tag{20}$$

El teorema en cuestión establece (ver Teorema 1.16 para más detalles)

Theorem 0.1. *Supongamos que $A_n \in L^\infty(\Omega; \mathcal{L}(\mathbb{R}^{M\times N}))$ verifica (16), (17), (18) y (19). Entonces, existe una subsucesión de n, que seguimos denotando por n, un espacio del Hilbert $H \subset W_0^{1,1}(\Omega)^M$ y un operador lineal continuo $\tilde{\Sigma} : H \to L^1_{\mathfrak{a}}(\Omega)^{M\times N}$ tal que para toda sucesión f_n que converge $*$-débil a f en $L^\infty(\Omega)^M$, se tiene que la única solución de* (15) *verifica*

$$u_n \stackrel{*}{\rightharpoonup} u \ \ \textit{en } BV(\overline{\Omega})^M,$$

$$A_n Du_n \stackrel{*}{\rightharpoonup} \tilde{\Sigma}(u)\mathfrak{a} \ \textit{en } \mathcal{M}(\overline{\Omega})^{M\times N}. \tag{21}$$

Observar que (21) junto con la convergencia de f_n, establece que u es solución de la ecuación

$$-\mathrm{Div}\big(\tilde{\Sigma}(u)\mathfrak{a}\big) = f \ \text{ en } \Omega,$$

y por tanto nos proporciona la existencia de una ecuación límite. Sin embargo no tenemos una representación de $\tilde{\Sigma}$. Recordamos que ya en el caso $M = 1$ se tiene que $\tilde{\Sigma}$ es en general no local y por tanto no es de la forma $\tilde{\Sigma}(u) = ADu$ para una cierta función tensorial A.

El resultado que probamos en el capítulo (Teorema 1.16) es en realidad más general y en particular proporciona también la convergencia de las energías en el sentido de que existe un operador bilineal, continuo $\tilde{\mathcal{B}} : H \times H \to \mathcal{M}(\overline{\Omega})$ tal que si u_n es como en el teorema y v_n es una sucesión en $W_0^{1,1}(\Omega)^M$ tal que

$$v_n \overset{*}{\rightharpoonup} v \text{ en } BV(\overline{\Omega})^M, \quad \limsup_{n\to\infty} \int_\Omega A_n Dv_n : Dv_n \, dx < +\infty,$$

entonces

$$A_n Du_n : Dv_n \overset{*}{\rightharpoonup} \tilde{\mathcal{B}}(u,v) \text{ en } \mathcal{M}(\overline{\Omega}).$$

Además, este operador $\tilde{\mathcal{B}}$ está relacionado con $\tilde{\Sigma}$ mediante

$$\tilde{\mathcal{B}}(u,v) = \tilde{\Sigma}(u) : Dv \, \mathfrak{a} \text{ en } \Omega, \quad \forall\, v \in C_0^1(\Omega)^M,$$

y se tiene que u es la única solución de

$$\begin{cases} u \in H, \\ \displaystyle\int_\Omega d\tilde{\mathcal{B}}(u,v) = \int_\Omega f \cdot v \, dx, \quad \forall\, v \in H. \end{cases}$$

Nótese también que la condición de elipticidad (18) sobre A_n está escrita en forma integral y no en forma puntual. Como ya hemos indicado anteriormente, estas dos condiciones no son equivalentes en el caso de sistemas. Esto permite, en particular, aplicar nuestros resultados al caso de la elasticidad lineal, donde la elipticidad puntual falla. Una condición puntual suficiente para asegurar (18) sería imponer que A_n^{-1} estuviese acotada en $L^1(\Omega; \mathcal{L}(\mathbb{R}^{M\times N}))$.

A fin de obtener una representación local para el operador $\tilde{\Sigma}$ (y para $\tilde{\mathcal{B}}$) es necesario suponer algunas hipótesis de integrabilidad sobre A_n. El resultado que obtenemos está basado en el teorema del div-rot que aparece en [26], el cual a diferencia del caso clásico (ver (6)) permite tratar el caso σ_n acotado en $L^p(\Omega)^N$ y τ_n acotado en $L^q(\Omega)^N$ con

$$\frac{1}{p} + \frac{1}{q} \leq 1 + \frac{1}{N}. \tag{22}$$

Se tiene (ver Teorema 1.11 para más detalles)

Theorem 0.2. *En las condiciones del Teorema 0.1, supongamos además*

$$A_n \text{ acotada en } L^p(\Omega; \mathcal{L}(\mathbb{R}^{M\times N})), \quad p \in \left[\frac{N}{2}, \infty\right],$$

$$\int_\Omega |Du|^r dx \leq \int_\Omega \gamma_n (A_n Du : Du)^{\frac{r}{2}} dx, \quad \forall u \in W_0^{1,r}(\Omega)^M, \ \forall n \in \mathbb{N},$$

con

$$r = \frac{2Np}{(N+2)p - N}, \quad \gamma_n \text{ acotada en } L^{\frac{2}{2-r}}(\Omega),$$

entonces existe $A \in L^p(\Omega; \mathcal{L}(\mathbb{R}^{M\times N}))$ *tal que*

$$\tilde{\Sigma}(u)\mathfrak{a} = ADu, \quad \forall\, u \in H \cap W^{1,\frac{2p}{p-1}}(\Omega)^M.$$

Nótese también que si imponemos una menor integrabilidad de A_n (p más pequeño), necesitamos una elipticidad más fuerte (r mayor) para la representación integral, y al contrario, tener mayor integrabilidad permite una elipticidad menor.

Comentar que este teorema incluye, en particular, los resultados obtenidos en [18] para el sistema de la elasticidad en dimensión 2 con coeficientes uniformemente elípticos y acotados en L^1, teorema que también usa la versión del div-rot que aparece en [26].

Capítulo 2

Como en el capítulo anterior, consideramos un subconjunto abierto y acotado $\Omega \subset \mathbb{R}^N$ con $N \geq 2$ y un número entero $M \geq 1$. En este capítulo analizamos el Γ-límite en $L^p(\Omega)^M$, $p > 1$, de sucesiones de funcionales no lineales definidos sobre funciones vectoriales del tipo

$$\mathscr{F}_n(v) := \int_\Omega F_n(x, Dv)\, dx \quad \text{para } v \in W_0^{1,p}(\Omega)^M. \tag{23}$$

Suponemos que las densidades de energía $F_n : \Omega \times \mathbb{R}^{M\times N} \to [0,\infty)$ son funciones de Carathéodory tales que existen $\alpha, \beta, \gamma > 0$ y dos sucesiones de funciones medibles no negativas h_n, a_n, con h_n acotada en $L^1(\Omega)$ y a_n acotada en $L^r(\Omega)$, donde

$$\begin{cases} r > \frac{N-1}{p}, & \text{si } 1 < p \leq N-1, \\ r = 1, & \text{si } p > N-1. \end{cases}$$

de forma que se satisfacen las siguientes hipótesis de elipticidad (integral), crecimiento y Lipschitzianidad

$$F_n(\cdot, 0) = 0, \quad \text{e.c.t. } \Omega, \tag{24}$$

$$\int_\Omega F_n(x, Du)\, dx \geq \alpha \int_\Omega |Du|^p\, dx - \beta, \quad \forall u \in W_0^{1,p}(\Omega)^M, \tag{25}$$

$$F_n(x, \lambda\xi) \leq h_n(x) + \gamma F_n(x,\xi), \quad \forall \lambda \in [0,1],\ \forall \xi \in \mathbb{R}^{M\times N},\ \text{e.c.t. } x \in \Omega, \tag{26}$$

$$\begin{cases} \big|F_n(x,\xi) - F_n(x,\eta)\big| \\ \qquad \leq \big(h_n(x) + F_n(x,\xi) + F_n(x,\eta) + |\xi|^p + |\eta|^p\big)^{\frac{p-1}{p}} a_n(x)^{\frac{1}{p}}\, |\xi - \eta|, \\ \forall\, \xi, \eta \in \mathbb{R}^{M\times N},\ \text{e.c.t. } x \in \Omega. \end{cases} \tag{27}$$

La hipótesis (24) implica que los funcionales definidos por (23) alcanzan un mínimo para $v = 0$, lo cual es usual en elasticidad no lineal. Esto significa que en la posición de reposo (sin desplazamientos) la energía elástica es nula. Respecto a las demás hipótesis, también se satisfacen en modelos usuales en elasticidad no lineal como por ejemplo ciertos materiales hiperelásticos como los materiales de Saint Venant-Kirchhoff y algunos materiales de tipo Ogden ([40], Vol. 1). Como ejemplo modelo considerar

$$F_n(x,\xi) = |A_n(x)\xi_s : \xi_s|^{\frac{p}{2}}, \quad \forall \xi \in \mathbb{R}^{M\times N}, \ \text{ e.c.t. } x \in \Omega,$$

con ξ_s la parte simétrica de ξ. En este caso se puede tomar

$$a_n(x) = |A_n(x)|^{\frac{p}{2}},$$

lo que nos muestra que a_n mide esencialmente cómo de grandes son los coeficientes.

Remarcar que no se impone la convexidad de F_n en la segunda variable como es normal en los trabajos dedicados a ecuaciones. En realidad, es conocido que el Γ-límite de una sucesión de funcionales en una determinada topología coincide con el Γ-límite de la envolvente semicontinua inferior de estos funcionales. Por otra parte se sabe que si un funcional del tipo (23) es semicontinuo inferiormente para la topología de $L^p(\Omega)$ entonces, F_n como función de la segunda variable es convexa sobre las matrices de rango 1 (rango-1 convexa). Por ello la hipótesis de convexidad no es restrictiva en el caso de ecuaciones pero sí para sistemas.

Debido a la no-linealidad del problema no se puede aplicar, como en el capítulo anterior, el teorema del div-rot. Sin embargo, usamos un lema que aparece en [25], el cual es fundamental para probar la versión del teorema del div-rot que aparece en esta referencia. Se trata de un resultado de compacidad para sucesiones acotadas en $W^{1,q}$ basado en la inyección $W^{1,q}(S^{N-1}) \subset L^{q^*}(S^{N-1})$, donde S^{N-1} es la esfera unidad en $\mathbb{R}^{N-1}$. Mientras que en el teorema del div-rot que aparece en [26] se imponía (22), en [25] sólo se necesita

$$\frac{1}{p} + \frac{1}{q} < 1 + \frac{1}{N-1}.$$

Gracias a esto, si aplicamos los resultados de este capítulo al caso lineal (F_n cuadrático en la segunda variable), podemos mejorar el teorema principal del capítulo anterior cuando $r = 2$, A_n simétricas y $N \geq 3$, mostrando que la hipótesis $p \geq N/2$ se puede relajar a $p > (N-1)/2$.

Los resultados principales de este capítulo (ver Teoremas 2.3 y 2.4 para más detalles) muestran la existencia de una función $F : \Omega \times \mathbb{R}^{M\times N} \to \mathbb{R}$ verificando propiedades similares a las de F_n de forma que, al menos sobre las funciones regulares, el funcional Γ-límite $\mathscr{F}$ en $L^p(\Omega)^M$ de la sucesión $\mathscr{F}_n$ verifica

$$\mathscr{F}(v) = \int_\Omega F(x, Dv)\, dx.$$

Además el resultado es local en el sentido que el valor de F en un subconjunto abierto de Ω sólo depende del valor de F_n en este subconjunto.

Capítulo 3

En este capítulo consideramos el sistema de la elasticidad lineal en una viga de grosor $\varepsilon > 0$, $\Omega_\varepsilon := (0,1) \times (\varepsilon\omega)$, cuando el tensor de coeficientes también varía con ε. En concreto, estudiamos el problema

$$\begin{cases} -\operatorname{div}(A_\varepsilon e(u_\varepsilon)) = h_\varepsilon & \text{en } \Omega_\varepsilon, \\ A_\varepsilon e(u_\varepsilon)\nu = 0 & \text{sobre } (0,1) \times (\varepsilon\partial\omega), \end{cases} \tag{28}$$

donde $\omega \subset \mathbb{R}^{N-1}$ es un dominio regular, conexo y acotado (en la práctica $N = 2, 3$), ν es el vector normal unitario exterior a ω sobre $\partial\omega$, u_ε es la deformación de la viga, $e(u_\varepsilon)$ es el tensor de esfuerzos y $h_\varepsilon = (h_{\varepsilon,1}, h'_\varepsilon)$ es la fuerza externa que se supone de la forma

$$h_{\varepsilon,1}(x) = f_1\left(x_1, \frac{x'}{\varepsilon}\right),\ h'_\varepsilon(x) = \varepsilon f'\left(x_1, \frac{x'}{\varepsilon}\right) + g'\left(x_1, \frac{x'}{\varepsilon}\right), \quad \text{e.c.t. } x \in \Omega_\varepsilon,$$

con $f \in L^2(\Omega)^N$ y $g' \in L^2(\Omega)^{N-1}$ (donde $\Omega := \Omega_1$) tal que

$$\int_\omega g' dy' = 0, \quad \text{e.c.t. } y_1 \in (0,1).$$

Obsérvese que para tener la unicidad de solución sería necesario imponer también condiciones de frontera sobre $\{0,1\} \times (\varepsilon\omega)$. Nuestros resultados permiten trabajar con distintas condiciones en este conjunto.
Nuestro objetivo es encontrar un sistema límite en dimensión 1 cuya solución aproxime las soluciones de (28) sin imponer ninguna hipótesis de isotropía ni homogeneidad sobre los coeficientes de elasticidad A_ε.

Para simplificar, suponemos la hipótesis de elipticidad uniforme

$$\exists \alpha > 0, \quad A_\varepsilon \xi : \xi \geq \alpha|\xi|^2, \quad \forall \xi \in \mathbb{R}^{N\times N}_s, \quad \text{e.c.t. } (0,1) \times (\varepsilon\omega),$$

pero, como en los capítulos anteriores, no imponemos que los coeficientes estén uniformemente acotados. Concretamente sólo imponemos

$$\varepsilon \|A_\varepsilon\|_{L^\infty(\Omega_\varepsilon;\mathcal{L}(\mathbb{R}^{N\times N}_s))} \to 0, \qquad \|A_\varepsilon\|_{L^1(\Omega_\varepsilon;\mathcal{L}(\mathbb{R}^{N\times N}_s))} \text{ acotada.}$$

El resultado principal que obtenemos (véase Teorema 3.1) proporciona una aproximación de las soluciones del tipo

$$\begin{cases} u_{\varepsilon,1}(x) \sim u_1(x_1) - \displaystyle\sum_{j=2}^N \frac{du_j}{dx_1}(x_1)\frac{x_j}{\varepsilon}, \\ u_{\varepsilon,j}(x) \sim \dfrac{1}{\varepsilon} u_j(x_1) + \displaystyle\sum_{i=2}^N Z_{ji}(x_1)\frac{x_i}{\varepsilon}, \quad j \in \{2, \cdots, N\}, \end{cases} \tag{29}$$

que consiste en la suma de una deformación de tipo Bernouilli-Navier dada por la función $u = (u_1, \ldots, u_N)$ más un término de torsión dado por la función matricial Z, la cual es antisimétrica. Este último se corresponde con una rotación infinitesimal alrededor del eje de la viga. Probamos que las funciones u y Z son soluciones de un sistema lineal unidimensional que en forma variacional se puede escribir como

$$\begin{cases} \displaystyle\int_0^1 Ae_0(u,Z):e_0(\tilde{u},\tilde{Z})\,dy_1 = \frac{1}{|\omega|}\int_\Omega \left(f_1\left(\tilde{u}_1 - \frac{d\tilde{u}'}{dy_1}\cdot y'\right) + f'\cdot\tilde{u}' + g'\cdot(\tilde{Z}\,y')\right) dy, \\ \forall(\tilde{u},\tilde{Z}) \in H_0^1(0,1)\times H_0^2(0,1)^{N-1}\times H_0^1(0,1;\mathbb{R}_{sk}^{(N-1)\times(N-1)}), \\ \qquad \text{con } \displaystyle\int_0^1 Ae_0(\tilde{u},\tilde{Z}) : e_0(\tilde{u},\tilde{Z})\,dx_1 < \infty, \end{cases} \tag{30}$$

donde el subíndice sk se refiere a matrices antisimétricas y donde el operador e_0 está dado por

$$e_0(u,Z) := \begin{pmatrix} \dfrac{du_1}{dx_1} & \left(\dfrac{d^2u'}{dx_1^2}\right)^T \\ \dfrac{d^2u'}{dx_1^2} & \dfrac{dZ}{dx_1} \end{pmatrix}.$$

Además la función tensorial A está en $L^1(0,1;\mathcal{L}(\mathbb{R}^{N\times N}_{s_1sk'}))$ y es tal que existen $\beta,\gamma>0$ y $a\in L^1(0,1)$, no negativa tales que

$$|AE| \le \beta\big(AE:E\big)^{\frac{1}{2}}a^{\frac{1}{2}}, \quad \forall\, E\in\mathbb{R}^{N\times N}_{s_1sk'},\ \text{e.c.t. } (0,1),$$

$$|E|^2 \le \gamma AE:E, \quad \forall\, E\in\mathbb{R}^{N\times N}_{s_1sk'},\ \text{e.c.t. } (0,1),$$

donde $\mathbb{R}^{N\times N}_{s_1sk'}$ son las matrices $M\in\mathbb{R}^{N\times N}$ tales que

$$M_{1i} = M_{i1},\ i=1,\dots,N, \qquad M_{ij} = -M_{ji},\ i,j=2,\dots,N.$$

Observar que aunque la sucesión A_ε está acotada solo en L^1, el tensor límite A tiene los coeficientes en L^1. La prueba de este resultado es una adaptación de la prueba clásica del teorema de H-convergencia de F. Murat y L. Tartar (cf. [76, 91]) combinado con un resultado de descomposición para sucesiones de deformaciones en dominios finos que puede encontrarse en [38].
El sistema límite (30) proporciona un modelo general para vigas fuertemente heterogéneas que no verifican ninguna hipótesis de isotropía. Recordar que en el caso de un material isótropo homogéneo, el sistema que se usa en Arquitectura o Ingeniería corresponde (en dimensión 3) a dos ecuaciones de cuarto orden (que proporcionarían las funciones u_2 y u_3 que aparecen en (30)).

Capítulo 4

En este capítulo nos centramos en la homogeneización por medio de Γ-convergencia de energías integrales débilmente coercitivas con densidades $\mathbb{L}(x/\varepsilon)Dv:Dv$, donde $\mathbb{L}\in L^\infty_{\rm per}(Y_N;\mathscr{L}_s(\mathbb{R}^{N\times N}_s))$ es una función tensorial simétrica periódica.

Este capítulo está dividido en dos partes bien diferenciadas.

En la primera parte del Capítulo 4, analizamos la condición (13) (con A reemplazado por $\mathbb{L}$) que, como expusimos anteriormente, es suficiente para que la fórmula de homogeneización periódica (3) se verifique para sistemas. En [27], los autores presentan una clase de ejemplos en dimensión 2 que verifican (13) pero tales que $\mathbb{L}$ no es muy fuertemente elíptica (es decir, no verifica (12) para todo $\xi \in \mathbb{R}^{N\times N}$). Siguiendo las mismas ideas, en el Teorema 4.4 proporcionamos un conjunto de mezclas en dimensión 3 que satisfacen (13) y no son muy fuertemente elípticas. Además, con el Teorema 4.5 proporcionamos una mejora de la condición (13) obteniendo el mismo resultado de Γ-convergencia suponiendo únicamente que se verifica

$$\int_{\mathbb{R}^N} \mathbb{L}Du : Du\, dy \geq 0, \quad \forall\, u \in \mathcal{D}(\mathbb{R}^N)^N. \tag{31}$$

En la segunda parte del capítulo, analizamos la pérdida de elipticidad fuerte a través de la homogeneización en el caso de la elasticidad lineal en dimensión 3. Hacemos un estudio exhaustivo del proceso de laminación llevado a cabo por S. Gutiérrez en [64] e intentamos darle justificación, en términos de Γ-convergencia, haciendo uso del Teorema 4.5. Para poder aplicar este teorema necesitamos tener la condición relajada de coercitividad funcional (31) y para obtenerla empleamos el método de traslación para los Lagrangianos nulos cuadráticos, es decir, probar la existencia de una matriz $D \in \mathbb{R}^{3\times 3}$ tal que

$$\mathbb{L}M : M + D : \mathrm{Adj}(M) \geq 0 \quad \forall M \in \mathbb{R}^{3\times 3}, \quad \text{e.c.t. } Y_N, \tag{32}$$

al igual que se hizo en [27] en el caso bidimensional. Sorprendentemente, al contrario de lo que ocurre para dimensión 2, en el Teorema 4.8 probamos que si un material fuertemente elíptico con estructura laminada (i.e. $\mathbb{L}(y) = \mathbb{L}(y_1)$) satisface la condición (32), entonces es imposible obtener un material efectivo para el que la condición de elipticidad fuerte falle. Este resultado justifica la necesidad de realizar una segunda laminación (en una nueva dirección) como hizo S. Gutiérrez en [64] para poder generar materiales límite que perdieran la elipticidad fuerte. Efectivamente, en el Teorema 4.14 probamos la existencia de ciertos materiales fuertemente elípticos para los que la elipticidad fuerte puede perderse tras un proceso de laminación de segundo rango (en dos pasos) si es mezclado con determinados materiales que pueden ser incluso muy fuertemente elípticos.

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Chapter 1

High-contrast homogenization of linear systems of partial differential equations

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Abstract.

We give some integrability conditions for the coefficients of a sequence of elliptic systems with varying coefficients in order to get the stability for homogenization. In the case of equations, it is well known that equi-integrability and bound in L^1 is enough for this purpose, however this is based on the maximum principle and then, it does not work for systems. Here, we use an extension of the Murat-Tartar div-curl Lemma due to M. Briane, J. Casado-Díaz and F. Murat in order to get the stability by homogenization for systems uniformly elliptic, with bounded coefficients in $L^{\frac{N}{2}}$, with N the dimension of the space. We also show that a weaker ellipticity condition can be assumed but then, we need a stronger integrability for the coefficients.

1.1 Introduction

Composite materials play an important role in many branches of Mechanics, Physics, Chemistry and Engineering. In such materials, some physical parameters, such as the conductivity or the elasticity coefficients, are usually discontinuous and may present oscillations between the characteristic values of each one of their components. When these components are very mixed, these parameters vary very rapidly, complicating then the microscopic structure of the material. It is reasonable to think that a good approximation of the macroscopic behaviour of such heterogeneous materials can be achieved by making the parameter ε, which describes the fineness of the microscopic structure, tend to zero in the equation describing phenomena, for instance, elasticity or thermal conductivity. The homogenization theory (see e.g. [1]) finds its purpose in performing this limit process. It provides a good mathematical framework for the analysis of composite media with complete generality without imposing any geometric or periodicity assumptions. Homogenization problems have been studied by mathematicians since the seventies and by physicists and engineers since earlier, although they only focused their interest on very specific cases such as periodic structures. For non-necessarily periodic problems, the most classical results refer to a sequence of elliptic problems with uniformly elliptic and uniformly bounded varying diffusion matrices. We refer to S. Spagnolo ([2]) in the case of symmetric matrices and to F. Murat and L. Tartar ([3]) in the general case. Assuming Ω a bounded open set in $\mathbb{R}^N$ and A_n bounded in $L^\infty(\Omega)^{N\times N}$, such that there exists $\alpha > 0$ with

$$A_n\xi\cdot\xi \geq \alpha|\xi|^2, \quad \forall n\in\mathbb{N},\ \forall\xi\in\mathbb{R}^N,\ \text{a.e. in } \Omega, \tag{1.1}$$

it is proved the existence of $A \in L^\infty(\Omega)^{N\times N}$ also satisfying (1.1) and a subsequence of n, still denoted by n, such that for every $f\in H^{-1}(\Omega)$, the solutions of

$$\begin{cases} -\operatorname{div}(A_n\nabla u_n) = f \quad \text{in } \Omega, \\ u_n = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

converge weakly in $H_0^1(\Omega)$ to the solution of the analogous problem with A_n replaced by A. Other boundary conditions can also be considered. For the case of matrices non-necesarily bounded in $L^\infty(\Omega)^{N\times N}$ we refer to [4], where it is studied the Γ-limit in $L^1(\Omega)$ of the sequence of functionals

$$v \mapsto \int_\Omega f_n(x,v,\nabla v)\,dx.$$

Assuming f_n convex in the second variable and such that there exist $p > 1$ and h_n bounded in $L^1(\Omega)$ and equi-integrable such that

$$0 \leq f_n(x,s,\xi) \leq h_n(x)\left(1+|s|^p+|\xi|^p\right), \quad \forall(s,\xi)\in\mathbb{R}\times\mathbb{R}^N, \ \text{ a.e. } x\in\Omega,$$

it is proved that the Γ-limit of these functionals has the same structure, at least for smooth functions v. Applied to $f_n = A_n(x)\xi\cdot\xi$, this result implies that the limit equation of (1.2) is still of the same form for A_n symmetric, bounded and equi-integrable in $L^1(\Omega)^{N\times N}$ and satisfying an elliptic condition in such way that the sequence of solutions of (1.2) becomes compact in $L^1(\Omega)$.

If $N \geq 3$ and A_n is bounded in $L^1(\Omega)^{N\times N}$ but not equi-integrable, the limit problem of (1.2) is not of the same type anymore. Some counterexamples can be found in [5], [6], [7]. A general result about the structure of the limit in this case can be found in [8]. If $N = 2$, it has been proved in [9] (see also [10], [11]) that the stability by homogenization of problem (1.2) holds without any bound on A_n.

The results in [4] and [9] are based on the maximum principle and therefore, they cannot be extended to systems. For this reason the stability of (1.2) when the function u_n is valued in $\mathbb{R}^M$, with $M > 1$ and A_n is a sequence of tensors bounded in $L^1(\Omega; \mathcal{L}(\mathbb{R}^{M\times N}))$ and equi-integrable is an open question to our knowledge. A partial result in this sense has been obtained in [12], where it is proved by assuming that there exists a sequence of tensor functions B_n uniformly bounded and uniformly elliptic (in an integral way) such that $\|A_n - B_n\|_{L^\infty(\Omega;\mathcal{L}(\mathbb{R}^{M\times N}))}$ tends to zero.

A useful tool in homogenization which does not use the maximum principle is the div-curl Lemma by F. Murat and L. Tartar ([13], [14]) which was already used in [3]. An extension of this result is presented in [15], where it is applied to the homogenization of monotone operators in $W^{1,N}(\Omega)^N$, showing that in this case a bound of the coefficients in $L^1(\Omega)$ (without the equi-integrability condition) is enough to get a local homogenization result. In the case of systems, this result has also been applied in [16] to get the homogenization of the linear elasticity system in dimension 2, with bounded coefficients in $L^1(\Omega)$. A related result has also been used in [17] to carry out the homogenization of the plate equation and the Stokes system in dimension 2.

Our purpose in the present paper is to use the div-curl Lemma in [15] to give some sufficient conditions on the integrability and ellipticity of the tensor functions A_n assuring that the homogenized system corresponding to the problems

$$\begin{cases} -\mathrm{Div}(A_n Du_n) = f \quad \text{in } \Omega, \\ u_n = 0 \quad \text{on } \Omega, \end{cases} \tag{1.3}$$

has the same structure at least for smooth functions. Contrary to the above mentioned papers which are also based on the div-curl Lemma, here the reasoning is different. Instead of applying the G-convergence theory, we show that, assuming that the non-symmetric part of A_n can be controlled by the symmetric one, the Γ-convergence theory allows us to get an abstract non-local homogenization result for problem (1.3) (see Theorem 1.16) which just assumes A_n bounded in $L^1(\Omega; \mathcal{L}(\mathbb{R}^{M\times N}))$ and uniformly elliptic in $W_0^{1,1}(\Omega)^M$. Then, using the div-curl Lemma we show that the homogenization result becomes local if A_n is bounded in $L^p(\Omega; \mathcal{L}(\mathbb{R}^{M\times N}))$ for some $p \geq N/2$, non-negative, and is such that

$$\int_\Omega |Du|^r dx \leq \int_\Omega \gamma_n (A_n Du : Du)^{\frac{r}{2}} dx, \tag{1.4}$$

for $r = 2Np/((N+2)p - N)$ and γ_n bounded in $L^{\frac{2}{2-r}}(\Omega)$. Assumption (1.4) holds if we suppose that A_n^{-1} is bounded in $L^{\frac{Np}{2p-N}}(\Omega; \mathcal{L}(\mathbb{R}^{M\times N}))$, which is a pointwise hypothesis while (1.4) is an integral one. In the case of equations, pointwise ellipticity and integral ellipticity are equivalent but this is not true for systems. We observe that if we impose a weaker integrability on A_n we need a stronger ellipticity and reciprocally. Namely, assuming uniform ellipticity in $H_0^1(\Omega)$, we just need A_n bounded in $L^{\frac{N}{2}}(\Omega;\mathcal{L}(\mathbb{R}^{M\times N}))$, while assuming ellipticity in $W^{1,\frac{2N}{N+2}}(\Omega)$, we need A_n bounded in $L^\infty(\Omega;\mathcal{L}(\mathbb{R}^{M\times N}))$.

More generally than (1.3), we can replace f by a sequence of right-hand sides f_n which can vary with n and converges in a certain sense we define in Section 1.2 (see Definition 1.8).

Our results apply in particular to the linear elasticity system (where pointwise ellipticity does not hold), extending the results obtained in [16] for $N = 2$.

Finally, we recall that, in the homogenization of the elasticity system, if we do not impose any bound in the coefficients, then it has been proved in [18] that any quadratic semicontiunous functional in L^2 can be obtained as Γ-limit.

Notation

- $|E|$ denotes the Lebesgue measure of any measurable set $E\subset\mathbb{R}^N$.
- $:$ denotes the euclidean inner product in $\mathbb{R}^{M\times N}$, i.e. $\xi:\eta=\mathrm{tr}(\xi^T\eta)$ for any $\xi,\eta\in\mathbb{R}^{M\times N}$.
- Du denotes the Jacobian matrix of a function u valued in $\mathbb{R}^M$. For $M=1$, we denote Du as ∇u, the gradient of u.
- $|\xi|$ denotes the euclidean norm of a matrix $\xi\in\mathbb{R}^{M\times N}$, i.e. $|\xi|=|\xi:\xi|^{\frac{1}{2}}$.
- $\mathbb{R}^{N\times N}_s$ denotes the space of symmetric matrices in $\mathbb{R}^N$.
- $\mathcal{L}(X)$ denotes the space of linear functions from the space X into itself.
- $|A|$ denotes the norm of $A\in\mathcal{L}(\mathbb{R}^{M\times N})$ induced by the euclidean norm of $\mathbb{R}^{M\times N}$, i.e.
$$|A|=\sup_{\xi\in\mathbb{R}^{M\times N}\setminus\{0\}}\frac{|A\xi|}{|\xi|}$$
- A^t denotes the transposed tensor of $A\in\mathcal{L}(\mathbb{R}^{M\times N})$
- A^s denotes the symmetric part of a tensor $A\in\mathcal{L}(\mathbb{R}^{M\times N})$. It also denotes the symmetric part of a matrix $A\in\mathbb{R}^{N\times N}$.
- $\mathcal{M}(\Omega)$ denotes the space of bounded Radon measures in the bounded open set $\Omega\subset\mathbb{R}^N$, which is the dual space of the continuous functions in Ω, vanishing on $\partial\Omega$, $C_0^0(\Omega)$.
- $\mathcal{M}(\overline{\Omega})$ denotes the space of Radon measures in $\overline{\Omega}$, with $\Omega\subset\mathbb{R}^N$ open and bounded. It is the dual space of the continuous functions in $\overline{\Omega}$, $C^0(\overline{\Omega})$.
- $\int_E f d\mu$ denotes the integral of f with respect to a measure μ in a set E. If μ is the Lebesgue measure, we just write $\int_E f dx$.

1.2 Main result

In the present section let us state the main result of the paper, Theorem 1.11. It refers to the asymptotic behavior of the solutions of the sequence of elliptic partial differential systems given by

$$\begin{cases} -\operatorname{Div}(A_n Du_n) = f_n & \text{in } \Omega, \\ u_n = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

where Ω is a bounded open set of $\mathbb{R}^N$ and u_n is valued in $\mathbb{R}^M$, $M \geq 1$. In the case $M = 1$, assuming that the coefficient tensors A_n are bounded in $L^1(\Omega; \mathcal{L}(\mathbb{R}^N))$ and equi-integrable and imposing some ellipticity conditions in such way that the solutions of (1.5) become compact in $L^1(\Omega)$, it is well known that the limit problem of (1.5) has the same structure (see e.g. [3], [2], [4]). Moreover, in dimension 2, some bound on A_n needs to be imposed ([9]). However the proof of these results is based on the maximum principle and thus it does not work for the case of elliptic systems considered here. Our purpose is to get some integrability and ellipticity conditions on A_n in order to have the stability by homogenization of the solutions of (1.5).

Let us assume the existence of $R > 0$, $p \in [\frac{N}{2}, \infty]$, and $\gamma_n \in L^{\frac{2}{2-r}}(\Omega)$, with

$$r = \frac{2Np}{(N+2)p - N} \in \left[\frac{2N}{N+2},\ 2\right], \tag{1.6}$$

$\gamma_n \geq 0$, such that

$$\{A_n\} \text{ is bounded in } L^p(\Omega; \mathcal{L}(\mathbb{R}^{M\times N})), \tag{1.7}$$

$$A_n\xi : \xi \geq 0, \quad \forall\, \xi \in \mathbb{R}^{M\times N},\ \forall\, n \in \mathbb{N},\ \text{a.e. in } \Omega, \tag{1.8}$$

$$|A_n\xi : \eta| \leq R|A_n\xi : \xi|^{\frac{1}{2}}|A_n\eta : \eta|^{\frac{1}{2}}, \quad \forall \xi, \eta \in \mathbb{R}^{M\times N},\ \forall n \in \mathbb{N},\ \text{a.e. in } \Omega, \tag{1.9}$$

$$\{\gamma_n\} \text{ is bounded in } L^{\frac{2}{2-r}}(\Omega), \tag{1.10}$$

$$\int_\Omega |Du|^r dx \leq \int_\Omega \gamma_n (A_n Du : Du)^{\frac{r}{2}} dx, \quad \forall u \in W_0^{1,r}(\Omega)^M,\ \forall n \in \mathbb{N}. \tag{1.11}$$

Remark 1.1. *The tensors A_n are not necessarily supposed to be symmetric but assumption* (1.9) *means that the antisymmetric part of A_n can be controlled by the symmetric one. We observe that this assumption always holds in the classical setting, i.e. when A_n is bounded in $L^\infty(\Omega; \mathcal{L}(\mathbb{R}^{M\times N}))$ and uniformly elliptic.*

Remark 1.2. *Assumption (1.11) is an ellipticity condition on A_n. If $M = 1$ (see e.g. [19]), it is equivalent to assuming that*

$$|\xi|^2 \leq |\gamma_n|^{\frac{2}{r}} A_n\xi : \xi, \quad \forall\, \xi \in \mathbb{R}^N,\ a.e.\ in\ \Omega. \tag{1.12}$$

However this is not true for $M \geq 2$. In fact, for the most classical example of elliptic system, the ellipticity system, assumption (1.12) does not hold because A_n vanishes on the antisymmetric matrices. In fact, as a model example of a sequence

A_n satisfying the above assumptions we can consider the following example in linear elasticity:

Let B_n be a bounded sequence in $L^p(\Omega;\mathcal{L}(\mathbb{R}^{N\times N}_s))$, $p\in[N/2,\infty]$ if $N>2$, $p\in[1,\infty)$ if $N=2$, such that B_n^{-1} is bounded in $L^{\frac{Np}{2p-N}}(\Omega;\mathcal{L}(\mathbb{R}^{N\times N}_s))$, and such that (1.9) is satisfied with A_n replaced by B_n. Then, defining $A_n\in L^p(\Omega;\mathcal{L}(\mathbb{R}^{N\times N}))$ by

$$A_n\xi = B_n\xi^s,\quad \forall\,\xi\in\mathbb{R}^{N\times N},\ \text{a.e. in }\Omega,$$

and taking into account that Korn's inequality implies

$$\begin{aligned}\int_\Omega |Du|^r dx &\le C\int_\Omega |e(u)|^r dx\\ &\le C\int_\Omega \left|B_n^{-1}\right|^{\frac{r}{2}}\big(B_n e(u):e(u)\big)^{\frac{r}{2}}dx,\quad \forall\,u\in W^{1,r}_0(\Omega)^N,\end{aligned}$$

it is simple to check that A_n satisfies Assumptions (1.8),..., (1.11).

Observe that if B_n is assumed to be just bounded in $L^{\frac{N}{2}}(\Omega;\mathcal{L}(\mathbb{R}^{N\times N}_s))$, then we need B_n^{-1} to be bounded in $L^\infty(\Omega;\mathcal{L}(\mathbb{R}^{N\times N}_s))$ which is equivalent to assuming that B_n is uniformly elliptic. By assuming a stronger integrability on B_n we can weaken the ellipticity condition to B_n^{-1} being just bounded in $L^{\frac{N}{2}}(\Omega;\mathcal{L}(\mathbb{R}^{N\times N}_s))$, which corresponds to B_n bounded in $L^\infty(\Omega;\mathcal{L}(\mathbb{R}^{N\times N}_s))$.

Since the sequence of tensor functions A_n is not assumed to be necessarily symmetric, problem (1.5) cannot be written in general as a minimum problem. Therefore, the asymptotic behavior of this problem is not reduced to the study of the Γ-convergence of a certain sequence of functionals. However, thanks to assumption (1.9) which permits to estimate the skew-symmetric part of A_n from its symmetric part, we will show that the Γ-convergence theory can be used to simplify the study of (1.5).

We recall the definition of Γ-convergence (see [20], [19], [21]).

Definition 1.3. *Let X be a metric space. A sequence of functionals $F_n:X\to\overline{\mathbb{R}}$ is said to Γ-converge to a functional $F:X\to\overline{\mathbb{R}}$ (denoted by $F_n\xrightarrow{\Gamma}F$) if for every $x\in X$, we have*

(i) for every sequence x_n converging to x in X

$$F(x)\le\liminf_{n\to\infty}F_n(x_n),$$

(ii) there exists a sequence $\hat{x}_n$ converging to x in X such that

$$\limsup_{n\to\infty}F_n(\hat{x}_n)\le F(x).$$

This sequence is said to be a recovery sequence for x.

In order to apply the Γ-convergence theory to problem (1.5), we introduce

Definition 1.4. *For every $n \in \mathbb{N}$, we define $F_n : W_0^{1,r}(\Omega)^M \to [0,\infty]$ by*

$$F_n(u) = \int_\Omega A_n Du : Du\, dx, \qquad \forall\, u \in W_0^{1,r}(\Omega)^M. \tag{1.13}$$

The domain of F_n is denoted by H_n

$$\begin{aligned} H_n &= \left\{ u \in W_0^{1,r}(\Omega)^M : \int_\Omega F_n(u) < +\infty \right\} \\ &= \left\{ u \in W_0^{1,r}(\Omega)^M : \int_\Omega A_n Du : Du\, dx < +\infty \right\}. \end{aligned} \tag{1.14}$$

It is a Hilbert space endowed with the norm

$$\|u\|_{H_n} = \left(\int_\Omega A_n Du : Du\, dx \right)^{\frac{1}{2}}, \quad \forall\, u \in H_n. \tag{1.15}$$

Since $W_0^{1,r}(\Omega)^M$ endowed with the norm of $L^r(\Omega)^M$ is a separable metric space, and F_n is non-negative and quadratic, Theorem 8.5 in [19] allows us to extract a subsequence of F_n, still denoted as F_n, such that there exists a non-negative quadratic functional $F : W_0^{1,r}(\Omega)^M \to [0,+\infty]$, which satisfies

$$F_n \xrightarrow{\Gamma} F. \tag{1.16}$$

We also recall that F is lower semicontinuous in $W_0^{1,r}(\Omega)^M$ endowed with the topology of $L^r(\Omega)^M$ and that, similarly to F_n, the space

$$H = D(F) = \left\{ u \in W_0^{1,r}(\Omega)^M : \ F(u) < +\infty \right\},$$

is a Hilbert space endowed with the norm

$$\|u\|_H = F(u)^{\frac{1}{2}}, \quad \forall\, u \in H. \tag{1.17}$$

We aso introduce

$$DH = \left\{ Du : u \in H \right\}. \tag{1.18}$$

Remark 1.5. *Thanks to assumption (1.11), if $u_n \in W_0^{1,r}(\Omega)^M$ is such that $F_n(u_n)$ is bounded, then u_n is bounded in $W_0^{1,r}(\Omega)^M$. Thus, by the Rellich-Kondrachov compactness theorem, we get the existence of a subsequence of u_n which converges strongly in $L^r(\Omega)^M$. This is the main reason for taking the Γ-convergence in the topology of $L^r(\Omega)^M$. Indeed, we observe that*

$$\left. \begin{array}{c} u_n \to u \ \text{ in } L^r(\Omega)^M \\ F_n(u_n) \le C \end{array} \right\} \Longrightarrow u_n \rightharpoonup u \ \begin{cases} \text{weakly in } W_0^{1,r}(\Omega)^M & \text{if } r > 1, \\ \text{weakly-}* \text{ in } BV(\Omega)^M & \text{if } r = 1, \end{cases} \tag{1.19}$$

and thus the Γ-convergence of F_n in the topology of $L^r(\Omega)^M$ is equivalent to the Γ-convergence in the weak topology of $W_0^{1,r}(\Omega)^M$ if $r > 1$ or $BV(\Omega)^M$ weak-$*$ if $r = 1$ (and then $N = 2$, $p = 1$), but this is not a convergence in a metric space. Thus, it is simpler to work with the convergence in $L^r(\Omega)^M$. We refer to ([19]) for the definition of Γ-convergence in an arbitrary topology not necessarily metric.

Using the spaces H_n we can also give the definition of solution for problem (1.5), which we will use in what follows.

Definition 1.6. *Given $f_n \in H_n'$, we say that $u_n \in H_n$ is the solution of problem* (1.5) *if it satisfies*

$$\int_\Omega A_n Du_n : Dv\,dx = \langle f_n, v\rangle_{H_n',H_n}, \quad \forall v \in H_n. \tag{1.20}$$

Remark 1.7. *The existence and uniqueness of solution for problem (1.5) is a simple consequence of Lax-Milgram's theorem.*

Let us introduce the following convergences for elements in the varying spaces H_n and H_n'.

Definition 1.8. *Given a sequence $v_n \in H_n$, and $v \in H$ we say that v_n H_n-converges weakly to v if*

$$\|v_n\|_{H_n} \ \text{bounded}, \quad v_n \to v \ \text{ in } L^r(\Omega)^M. \tag{1.21}$$

Given $f_n \in H_n'$, we say that f_n H_n'-converges to $f \in H'$ if

$$\langle f_n, v_n\rangle_{H_n',H_n} \to \langle f, v\rangle_{H',H}, \quad \forall\, v_n \in H_n \text{ which } H_n\text{-converges weakly to } v. \tag{1.22}$$

Remark 1.9. *As we observed in Remark 1.5, the conditions in (1.21) imply that v_n converges weakly to v in $W_0^{1,r}(\Omega)^M$ if $r > 1$ or in $BV(\Omega)^M$ weak-$*$ if $r = 1$. Thus, the simpler example of a weakly H_n-converging sequence f_n is given by a sequence which converges in $W^{-1,r'}(\Omega)^M$.*

Remark 1.10. *We will see in Proposition 1.18 below that if f_n H_n'-converges to f, then $\|f_n\|_{H_n}$ is bounded. In particular this implies that the solution u_n of problem (1.5) is such that $\|u_n\|_{H_n}$ is bounded.*

We are now in position to give the main result of the paper.

Theorem 1.11. *Assume that A_n satisfies* (1.7)–(1.11), *with $p > 1$. Then, there exist a subsequence of n, still denoted by n, a continuous bilinear operator $\mathcal{B} : DH \times DH \to \mathcal{M}(\overline{\Omega})$, a linear operator $\Sigma : DH \to L^{\frac{2p}{1+p}}(\Omega)^{M\times N}$ and a tensor function $A \in L^p(\Omega; \mathcal{L}(\mathbb{R}^{M\times N}))$ with the following properties:*

$$\mathcal{B}(Du, Du) \geq 0 \ \text{ in } \overline{\Omega}, \tag{1.23}$$

$$\int_{\overline{\Omega}} \varphi\, d|\mathcal{B}(Du, Dv)| \leq R \left(\int_{\overline{\Omega}} \varphi\, d\mathcal{B}(Du, Du)\right)^{\frac{1}{2}} \left(\int_{\overline{\Omega}} \varphi\, d\mathcal{B}(Dv, Dv)\right)^{\frac{1}{2}}, \tag{1.24}$$

for every $u, v \in H$ and every $\varphi \in C_0(\overline{\Omega})$, $\varphi \geq 0$.

$$\|u\|_H^2 \leq \int_{\overline{\Omega}} d\mathcal{B}(Du, Du), \quad \forall\, u \in H, \tag{1.25}$$

$$\int_\Omega |Du|^r dx \leq \left(\liminf_{n\to\infty} \|\gamma_n\|_{L^{\frac{2}{2-r}}(\Omega)}\right) \left(\int_{\overline{\Omega}} d\mathcal{B}(Du, Du)\right)^{\frac{r}{2}}, \quad \forall\, u \in H, \tag{1.26}$$

$$\|\mathcal{B}(Du,Du)\|_{\mathcal{M}(\overline{\Omega})} \leq R^4\|u\|_H^2, \quad \forall\, u \in H, \tag{1.27}$$

$$\|\Sigma(Du)\|_{L^{\frac{2p}{1+p}}(\Omega)^{M\times N}} \leq R^{\frac{1+5p}{2p}} \liminf_{n\to\infty} \|A_n\|^{\frac{1}{2}}_{L^p(\Omega,\mathcal{L}(\mathbb{R}^{M\times N}))}\|u\|_H, \quad \forall\, u \in H, \tag{1.28}$$

$$\mathcal{B}(Du,Dv) = \Sigma(Du)\!:\!Dv \;\; a.e. \;in\; \omega, \quad \forall\, \omega \subset \Omega \;open, \forall\, u \in H, \forall\, v \in H \cap W^{1,\frac{2p}{p-1}}(\omega)^M. \tag{1.29}$$

$$\Sigma(Du) = ADu \;\; a.e. \;in\; \omega, \quad \forall\, \omega \subset \Omega \;open, \forall\, u \in H \cap W^{1,\frac{2p}{p-1}}(\omega)^M. \tag{1.30}$$

Moreover, the operators $\mathcal{B}$ and Σ provide the following homogenization result for (1.5):

Let $f_n \in H_n'$ be a sequence which H_n'-converges to a functional $f \in H'$ and let u_n be the weak solution of (1.5). *Then, defining $u \in H$ as the unique solution of*

$$\int_{\overline{\Omega}} d\mathcal{B}(Du,Dv) = \langle f, v\rangle_{H',H}, \quad \forall v \in H, \tag{1.31}$$

we have

$$u_n \;\; H_n\text{-converges weakly to } u, \tag{1.32}$$

$$A_nDu_n \rightharpoonup \Sigma(Du) \;\; in\; L^{\frac{2p}{1+p}}(\Omega)^{M\times N}, \tag{1.33}$$

$$A_nDu_n\!:\!Dv_n \overset{*}{\rightharpoonup} \mathcal{B}(Du,Dv) \;in\; \mathcal{M}(\overline{\Omega}), \quad \forall\, v_n \in H_n \text{ which } H_n\text{-converges weakly to } v. \tag{1.34}$$

If $p = 1$ the result is analogous but now, taking a subsequence of n such that there exists $\mathfrak{a} \in \mathcal{M}(\overline{\Omega})$, such that

$$\|A_n\| \overset{*}{\rightharpoonup} \mathfrak{a} \quad weakly\text{-}* \;in\; \mathcal{M}(\overline{\Omega}), \tag{1.35}$$

we have that Σ is a linear operator from DH into $\mathcal{M}(\overline{\Omega})^{M\times N}$, $A \in L^\infty_{\mathfrak{a}}(\Omega, \mathcal{L}(\mathbb{R}^{M\times N}))$. Moreover, the following changes must be taken into account:

In (1.26), *$\int_\Omega |Du|\,dx$ must be replaced by $\|Du\|_{\overline{\otimes}}$.*

In (1.28), *the norm of $\Sigma(Du)$ must be taken in $\mathcal{M}(\overline{\Omega})$.*

In (1.29), *v must be taken in $H \cap C^1(\omega)$ and the equality $\mathcal{B}(Du,Dv) = \Sigma(Du) : Dv$ holds in the sense of the measures in ω.*

In (1.30), *u must be taken in $H \cap C^1(\omega)$ and the equality $\Sigma(Du) = ADu$ holds in the sense of the measures in ω.*

In (1.33) *the convergence holds in the weak-$*$ sense of the measures in $\overline{\Omega}$.*

Remark 1.12. *The equality $p = 1$ can only hold for $N = 2$.*

Remark 1.13. *From (1.29) and (1.31) we get that u is a solution of*

$$-\operatorname{Div}\Sigma(Du) = f \tag{1.36}$$

in the sense of the distributions in Ω, which thanks to (1.30) also implies

$$-\operatorname{Div} ADu = f \;\; in\; \Omega, \tag{1.37}$$

if u is smooth enough.

Remark 1.14. *Assertion (1.33) gives the convergence of the flux while (1.34) gives the convergence of the energy. Equalities (1.29) and (1.30) imply that if* u *and* v *are smooth enough, then* $\mathcal{B}$ *is given by*

$$\mathcal{B}(Du, Dv) = ADu : Dv \quad a.e. \ in\ \Omega.$$

Moreover, the operators Σ *and* $\mathcal{B}$ *are strongly local in the following sense: Assume* $u_1, u_2, v_1, v_2 \in H$, $\omega \subset \Omega$ *open such that* $u_1 = u_2$, $v_1 = v_2$ *in* ω, *then*

$$\Sigma(Du_1) = \Sigma(Du_2),\ \mathcal{B}(Du_1, Dv_1) = \mathcal{B}(Du_2, Dv_2) \quad in\ \omega.$$

Indeed, thanks to (1.30), we have

$$\Sigma(Du_1) - \Sigma(Du_2) = \Sigma(D(u_1 - u_2)) = \Sigma(0) = 0 \quad in\ \omega,$$

while (1.29) and (1.30) give

$$\begin{aligned}\mathcal{B}(Du_1, Dv_1) - \mathcal{B}(Du_2, Dv_2) &= \mathcal{B}(D(u_1 - u_2), Dv_1) - \mathcal{B}(Du_2, D(v_2 - v_1)) \\ &= \Sigma(D(u_1 - u_2)) : Dv_1 - \Sigma(Du_2) : D(v_2 - v_1) = 0 \quad in\ \omega.\end{aligned}$$

1.3 A first homogenization result

In this section, let us give a first homogenization result for problem (1.5) just by assuming boundness for the coefficients in $L^1(\Omega; \mathcal{L}(\mathbb{R}^{M\times N}))$ and ellipticity on $W^{1,1}(\Omega)^M$. Even for the case of equations it is well known that these assumptions are not enough to get a local limit (see e.g. ([7])). Thus, we just have a global homogenization theorem.

The assumptions on the coefficients we make in the present section are given by (1.8), (1.9) and

$$\{A_n\} \text{ is bounded in } L^1(\Omega; \mathcal{L}(\mathbb{R}^{M\times N})), \tag{1.38}$$

$$\exists K > 0 : \int_\Omega |Du| dx \le K \left(\int_\Omega A_n Du : Du\, dx \right)^{\frac{1}{2}}, \quad \forall u \in W_0^{1,1}(\Omega)^M,\ \forall n \in \mathbb{N}. \tag{1.39}$$

Remark 1.15. *Thanks to (1.38) and Theorem 8.5 in [19], extracting a subsequence if necessary, we can assume the existence of* $\mathfrak{a} \in \mathcal{M}(\overline{\Omega})$ *and a quadratic functional* $F : BV(\Omega)^M \to (0, \infty]$ *such that (1.16) and (1.35) hold. We will assume in the following that we have taken such a subsequence.*

The main result of the present section is given by the following theorem

Theorem 1.16. *Assume that* A_n *satisfies (1.8), (1.9), (1.38) and (1.39). Then, there exist a subsequence of* n, *still denoted by* n, *a continuous bilinear operator* $\tilde{\mathcal{B}} : H \times H \to \mathcal{M}(\overline{\Omega})$ *and a linear operator* $\tilde{\Sigma} : H \to L^1_{\mathfrak{a}}(\overline{\Omega})^{M\times N}$ *with the following properties:*

$$\tilde{\mathcal{B}}(u, u) \ge 0 \quad in\ \overline{\Omega}, \quad \forall u \in H, \tag{1.40}$$

$$\|u\|_H^2 \leq \int_{\Omega} d\tilde{\mathcal{B}}(u,u), \quad \forall\, u \in H, \tag{1.41}$$

$$\|\tilde{\mathcal{B}}(u,u)\|_{\mathcal{M}(\overline{\Omega})} \leq R^4 \|u\|_H^2, \quad \forall\, u \in H, \tag{1.42}$$

$$\|\tilde{\Sigma}(u)\|_{L^1_{\mathfrak{a}}(\overline{\Omega})^N} \leq R^3 \|\mathfrak{a}\|_{\mathcal{M}(\overline{\Omega})} \|u\|_H, \quad \forall\, u \in H, \tag{1.43}$$

$$\int_{\overline{\Omega}} \varphi\, d|\tilde{\mathcal{B}}(u,v)| \leq R \Big(\int_{\overline{\Omega}} \varphi\, d\tilde{\mathcal{B}}(u,u)\Big)^{\frac{1}{2}} \Big(\int_{\overline{\Omega}} \varphi\, d\tilde{\mathcal{B}}(v,v)\Big)^{\frac{1}{2}}, \quad \forall\, u,v \in H,\ \forall\, \varphi \in C^0(\overline{\Omega}),\ \varphi \geq 0, \tag{1.44}$$

$$\int_{\Omega} |Du|\, dx \leq K \left(\int_{\overline{\Omega}} d\tilde{\mathcal{B}}(u,u)\right)^{\frac{1}{2}}, \quad \forall\, u \in H, \tag{1.45}$$

$$\tilde{\mathcal{B}}(u,v) = \tilde{\Sigma}(u) : Dv \quad \text{in } \overline{\Omega}, \quad \forall u \in H, \quad \forall\, v \in C^1(\overline{\Omega})^M. \tag{1.46}$$

Moreover, the operators $\tilde{\mathcal{B}}$ and $\tilde{\Sigma}$ provide the following homogenization result for (1.5):

Let $f_n \in H'_n$ be a sequence which H'_n-converges to a functional $f \in H'$ and let u_n be the weak solution of (1.5)*. Then, defining $u \in H$ as the unique solution of*

$$\int_{\overline{\Omega}} d\tilde{\mathcal{B}}(u,v) = \langle f, v\rangle_{H',H}, \quad \forall v \in H, \tag{1.47}$$

we have

$$u_n \ H_n\text{-converges weakly to } u, \tag{1.48}$$

$$A_n Du_n \overset{*}{\rightharpoonup} \tilde{\Sigma}(u)\mathfrak{a} \quad \text{in } BV(\Omega), \tag{1.49}$$

$$A_n Du_n : Dv_n \overset{*}{\rightharpoonup} \tilde{\mathcal{B}}(u,v) \quad \text{in } \mathcal{M}(\overline{\Omega}), \quad \forall\, v_n \in H_n \text{ which } H_n\text{-converges weakly to } v. \tag{1.50}$$

The rest of this section is devoted to the proof of Theorem 1.16.

We start with the following inequality.

Lemma 1.17. *If A_n satisfies* (1.9) *and* (1.38)*, then, for every $n \in \mathbb{N}$, every $u \in W^{1,1}(\Omega)^M$, and every $\varphi \in C^0(\overline{\Omega})$, $\varphi \geq 0$ in $\overline{\Omega}$, we have*

$$\int_{\Omega} |A_n Du|\varphi\, dx \leq R \left(\int_{\Omega} |A_n|\varphi\, dx\right)^{\frac{1}{2}} \left(\int_{\Omega} A_n Du : Du\varphi dx\right)^{\frac{1}{2}}. \tag{1.51}$$

Proof. We can assume $A_n Du : Du$ in $L^1(\omega)$, otherwise (1.51) is obvious. Applying (1.9) and Cauchy-Schwarz inequality, we have

$$\begin{aligned} \int_{\Omega} |A_n Du|\varphi\, dx &= \int_{\Omega} \sup_{|\eta|=1} |A_n Du : \eta|\varphi\, dx \leq R \int_{\Omega} |A_n^s Du : Du|^{\frac{1}{2}} \sup_{|\eta|=1} |A_n^s \eta : \eta|^{\frac{1}{2}} \varphi\, dx \\ &\leq R \int_{\Omega} |A_n Du : Du|^{\frac{1}{2}} |A_n|^{\frac{1}{2}} \varphi\, dx \leq R \left(\int_{\Omega} |A_n|\varphi\, dx\right)^{\frac{1}{2}} \left(\int_{\Omega} A_n Du : Du\, \varphi\, dx\right)^{\frac{1}{2}}. \end{aligned} \tag{1.52}$$

□

Let us now prove the following result which in particular shows that a H'_n-converging sequence has bounded norm, as we mentioned in Remark 1.10.

Proposition 1.18. *Assume that the sequence A_n satisfies (1.8), (1.9), (1.38) and (1.39). Then, every sequence f_n which H'_n-converges to some $f \in H'$ satisfies*

$$\exists \lim_{n\to\infty} \|f_n\|_{H'_n} = \|f\|_{H'}. \tag{1.53}$$

Proof. By the Riesz Theorem, we know that the sequence u_n solution of

$$\begin{cases} -\operatorname{Div}(A^s_n Du_n) = f_n \quad \text{in } \Omega, \\ u_n = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.54}$$

is such that for all $n \in \mathbb{N}$

$$\|u_n\|_{H_n} = \|f_n\|_{H'_n}, \tag{1.55}$$

$$\langle f_n, \frac{u_n}{\|u_n\|_{H_n}} \rangle_{H'_n, H_n} = \frac{1}{\|u_n\|_{H_n}} \int_\Omega A^s_n Du_n : Du_n \, dx = \|f_n\|_{H'_n}. \tag{1.56}$$

Since

$$\left\| \frac{u_n}{\|u_n\|_{H_n}} \right\|_{H_n} = 1, \quad \forall n \in \mathbb{N},$$

thanks to (1.39), there exist a subsequence of n, still denoted by n, and $w \in BV(\Omega)^M$ such that $u_n/\|u_n\|_{H_n}$ converges weakly-$*$ to w in $BV(\Omega)^M$. Combined with (1.55), (1.56) and the definition of H'_n-convergence, this shows

$$\lim_{n\to\infty} \|u_n\|_{H_n} = \lim_{n\to\infty} \|f_n\|_{H_n} = \langle f, w \rangle_{H',H}. \tag{1.57}$$

In particular

$$u_n = \|u_n\|_{H_n} \frac{u_n}{\|u_n\|_{H_n}} \overset{*}{\rightharpoonup} u := \langle f, w \rangle_{H',H} w \ \text{ in } BV(\Omega)^M, \qquad \|u_n\|_{H_n} \text{ bounded.}$$

Using that u_n is defined by (1.54), we get that u_n is a recovery sequence for u and therefore,

$$\lim_{n\to\infty} \|u_n\|_{H_n} = \lim_{n\to\infty} F_n(u_n)^{\frac{1}{2}} = F(u)^{\frac{1}{2}} = \|u\|_H, \qquad (u, v)_H = \langle f, v \rangle_{H',H} \quad \forall\, v \in H,$$

By the Riesz Theorem $\|u\|_H = \|f\|_{H'}$ and thus

$$\lim_{n\to\infty} \|f_n\|_{H'_n} = \lim_{n\to\infty} \|u_n\|_{H_n} = \|u\|_H = \|f\|_{H'}.$$

□

Proof of Theorem 1.16. We divide the proof into four steps.

Step 1. We fix a sequence $f_n \in H'_n$ and an element $f \in H'$ in the conditions of (1.22), and we denote by u_n the solution of (1.5). Let us prove that

$$\limsup_{n\to\infty} \int_\Omega A_n Du_n : Du_n \, dx \le \|f\|^2_{H'}, \tag{1.58}$$

and that there exist a subsequence of n, still denoted by n, a function $u \in H$ and a function $\Xi \in L^1_{\mathfrak{a}}(\overline{\Omega})^{M\times N}$ such that

$$u_n \ H_n\text{-converges to } u, \tag{1.59}$$

$$A_n Du_n \overset{*}{\rightharpoonup} \Xi\mathfrak{a} \quad \text{weakly-}* \text{ in } \mathcal{M}(\overline{\Omega})^{M\times N}, \tag{1.60}$$

with

$$\|u\|_H^2 \leq \langle f, u\rangle_{H',H}, \tag{1.61}$$

$$\frac{1}{R^2}\|f\|_{H'} \leq \|u\|_H \leq \|f\|_{H'}, \tag{1.62}$$

$$\|\Xi\|_{L^1_{\mathfrak{a}}(\overline{\Omega})^{M\times N}} \leq R\|\mathfrak{a}\|_{\mathcal{M}(\overline{\Omega})}\|f\|_{H'}. \tag{1.63}$$

To prove these results, we use u_n as test function in (1.5). We get

$$\limsup_{n\to\infty} \|u_n\|_{H_n} = \limsup_{n\to\infty} \left(\int_\Omega A_n Du_n : Du_n \, dx\right)^{\frac{1}{2}} \leq \|f\|_{H'}. \tag{1.64}$$

By (1.39) and (1.51) with $\varphi = 1$, this proves the existence of a subsequence of n, $u \in H$, and $\sigma \in \mathcal{M}(\overline{\Omega})^{M\times N}$ which satisfy (1.59) and

$$A_n Du_n \overset{*}{\rightharpoonup} \sigma \quad \text{in } \mathcal{M}(\overline{\Omega})^{M\times N}. \tag{1.65}$$

Moreover, we observe that (1.51) shows

$$\int_{\overline{\Omega}} \varphi \, d|\sigma| \leq R\left(\int_{\overline{\Omega}} \varphi \, d\mathfrak{a}\right)\|f\|_{H'}, \quad \forall\, \varphi \in C^0(\overline{\Omega}). \tag{1.66}$$

Thus σ is absolutely continuous with respect to $\mathfrak{a}$ and then, by the Radon-Nikodym theorem, there exists $\Xi \in L^1_{\mathfrak{a}}(\overline{\Omega})^{M\times N}$ such that $\sigma = \Xi\mathfrak{a}$. Combined with (1.65), this proves (1.60). Moreover, taking $\varphi = 1$, we get (1.63).

On the other hand, by definition of Γ-convergence and (1.64), we have

$$\|u\|_H^2 \leq \liminf_{n\to\infty} \int_\Omega A_n Du_n : Du_n \, dx = \langle f, u\rangle_{H',H}.$$

This proves (1.61) and then, the second inequality in (1.62). For the first one, using Riesz Theorem, we define $\tilde{u} \in H$ by

$$(\tilde{u}, v)_H = \langle f, v\rangle_{H',H}, \quad \forall\, v \in H,$$

where $(\cdot,\cdot)_H$ denotes the inner product in H. Taking a recovery sequence $\tilde{u}_n$ for $\tilde{u}$, as test function in (1.5) and using (1.9) and (1.64), we have,

$$\begin{aligned}
\|f\|_{H'}^2 &= \langle f, \tilde{u}\rangle_{H',H} \\
&= \lim_{n\to\infty} \int_\Omega A_n Du_n : D\tilde{u}_n \, dx \\
&\leq R \lim_{n\to\infty} \left(\int_\Omega A_n Du_n : Du_n \, dx\right)^{\frac{1}{2}} \left(\int_\Omega A_n D\tilde{u}_n : D\tilde{u}_n \, dx\right)^{\frac{1}{2}} \\
&= R\langle f, u\rangle_{H',H}^{\frac{1}{2}}\|f\|_{H'}.
\end{aligned}$$

This shows
$$\|f\|_{H'}^2 \le R^2 \langle f, u\rangle_{H',H} \le R^2 \|f\|_{H'} \|u\|_H,$$
and then the first inequality in (1.62).

Step 2. Let Z be a countable dense subset of $C^0(\overline{\Omega})^M$. Observe that Z is dense in H' because if $v \in H$ is such that $\langle z, v\rangle_{H',H} = 0$, for every $z \in Z$, then
$$\int_\Omega zv\,dx = 0, \quad \forall z \in Z,$$
and therefore $v = 0$ a.e. in Ω.

We define SZ as the vector space generated by Z. Let us denote by w_n^f the solution of (1.5) with right-hand side $f \in SZ$. Using Step 1 and a diagonal argument, we deduce the existence of a subsequence of n, $w^f \in H$ and $\Upsilon^f \in L^1_{\mathfrak{a}}(\overline{\Omega})^{M\times N}$ such that (1.59)–(1.63) hold, for every $f \in SZ$, with u_n, u and Ξ replaced by w_n^f, w^f and Υ^f respectively. Taking into account that $A_n Dw_n^f : Dw_n^g$ is bounded in $L^1(\Omega)$, for every f and g in Z, we can also assume the existence of $Q^{f,g} \in \mathcal{M}(\overline{\Omega})$ such that
$$A_n Dw_n^f : Dw_n^g \overset{*}{\rightharpoonup} Q^{f,g} \ \text{ in } \mathcal{M}(\overline{\Omega}), \quad \forall\, f, g \in SZ. \tag{1.67}$$

It is clear that the operators $f \in SZ \mapsto w^f \in H$, $f \in SZ \mapsto \Upsilon^f \in L^1_{\mathfrak{a}}(\overline{\Omega})^{M\times N}$ are linear and the operator $(f,g) \in SZ \times SZ \mapsto Q^{f,g} \in \mathcal{M}(\overline{\Omega})$ is bilinear. Moreover, by (1.9) and (1.58), we have
$$\|Q^{f,f}\|_{\mathcal{M}(\overline{\Omega})} \le \|f\|_{H'}^2, \tag{1.68}$$
$$\int_{\overline{\Omega}} \varphi\, d|Q^{f,g}| \le R \left(\int_{\overline{\Omega}} \varphi\, dQ^{f,f}\right)^{\frac12} \left(\int_{\overline{\Omega}} \varphi\, dQ^{g,g}\right)^{\frac12}, \quad \forall \varphi \in C^0(\overline{\Omega}),\ \varphi \ge 0, \tag{1.69}$$
while (1.61), (1.62) and (1.63) give
$$\|w^f\|_H^2 \le \langle f, w^f\rangle_{H',H}, \tag{1.70}$$
$$\frac{1}{R^2}\|f\|_{H'} \le \|w^f\|_H \le \|f\|_{H'}, \tag{1.71}$$
$$\|\Upsilon^f\|_{L^1_{\mathfrak{a}}(\overline{\Omega})^{M\times N}} \le R\|\mathfrak{a}\|_{\mathcal{M}(\overline{\Omega})}\|f\|_{H'}. \tag{1.72}$$
Reasoning by density, we deduce that these operators can be extended to continuous operators on H', still denoted the same way.

Since the linear function $f \in H' \mapsto w^f \in H$ satisfies (1.71), we can apply Lax-Milgram's theorem to deduce that this function is one-to-one with a continuous inverse denoted by $\mathcal{L}$. We define $\tilde{\mathcal{B}} : H \times H \to \mathcal{M}(\overline{\Omega})$ and $\tilde{\Sigma} : H \to L^1_{\mathfrak{a}}(\overline{\Omega})^{M\times N}$ by
$$\tilde{\mathcal{B}}(u,v) = Q^{\mathcal{L}u,\mathcal{L}v}, \quad \tilde{\Sigma}(u) = \Upsilon^{\mathcal{L}u}. \tag{1.73}$$

By (1.68)–(1.72) and $Q^{f,f}$ being non-negative for every $f \in H'$, we easily deduce (1.40), (1.41), (1.42), (1.43) and (1.44).

For $u \in H$, with $u = w^f$ for some $f \in SZ$, properties (1.44) and (1.45) easily follow from (1.9), (1.39) and $A_n Dw_n^f : Dw_n^f$ converging weakly-$*$ to $\tilde{\mathcal{B}}(u,u)$ in $\mathcal{M}(\overline{\Omega})$. By continuity, these properties are in fact true for every $u \in H$.

Step 3. We consider $f \in SZ$ and a sequence v_n which H_n-converges weakly to a function v. Using $v_n - w_n^g$, with $g \in SZ$ as test function in the equation satisfied by w_n^f, taking into account the definition (1.67) of $Q^{f,g}$ and passing to the limit, we get

$$\lim_{n\to\infty}\int_\Omega A_n Dw_n^f : Dv_n\,dx - \int_{\overline{\Omega}} dQ^{f,g} = \lim_{n\to\infty}\int_\Omega A_n Dw_n^f : D(v_n - w_n^g)\,dx$$
$$= \langle f, v - w^g\rangle_{H',H}.$$

Replacing in this equality g by a sequence g_n which H_n'-converges to $\mathcal{L}v$, and taking into account the continuity of the function $(f,g)\mapsto Q^{f,g}$ and definition (1.73) of $\tilde{\mathcal{B}}$, we have then proved

$$\int_{\overline{\Omega}} d\tilde{\mathcal{B}}(w^f, v) = \lim_{n\to\infty}\int_\Omega A_n Dw_n^f : Dv_n\,dx, \tag{1.74}$$

for every $f \in SZ$ and every sequence v_n which H_n-converges weakly in H' to v. In particular, for every $v \in C^1(\overline{\Omega})$, we have

$$\int_{\overline{\Omega}} \tilde{\Sigma}(w^f) : Dv\,d\mathfrak{a} = \int_{\overline{\Omega}} \Upsilon^f : Dv\,d\mathfrak{a} = \lim_{n\to\infty}\int_\Omega A_n Dw_n^f : Dv\,dx = \int_{\overline{\Omega}} d\tilde{\mathcal{B}}(w^f, v).$$

Reasoning by density, this proves (1.46).

Step 4. Let $f_n \in H_n'$ be a sequence which H_n'-converges to a functional $f \in H'$ and let u_n be the weak solution of (1.5). We also consider a sequence v_n which H_n-converges weakly to some function $v \in H$. By using Step 1, we know that there exist a subsequence of n, $u \in H$ and $\Xi \in L^1_{\mathfrak{a}}(\overline{\Omega})^{M\times N}$ such that (1.59) and (1.60) hold. Since $A_n Du_n : Dv_n$ is bounded in $L^1(\Omega)$ we can also assume the existence of $\Lambda \in \mathcal{M}(\overline{\Omega})$ such that $A_n Du_n : Dv_n$ converges weakly-$*$ in $\mathcal{M}(\overline{\Omega})$ to Λ. Taking into account (1.9) and (1.58), (1.62), (1.63) with f_n replaced by $f_n - g$, we deduce

$$\|u - w^g\|_H \le \|f - g\|_{H'}, \quad \|\Xi - \tilde{\Sigma}(w^g)\|_{L^1_{\mathfrak{a}}(\overline{\Omega})^{M\times N}} \le R\|\mathfrak{a}\|_{\mathcal{M}(\overline{\Omega})}\|f - g\|_{H'},$$

$$\|\Lambda - Q^{g,\mathcal{L}v}\|_{\mathcal{M}(\overline{\Omega})} \le \limsup_{n\to\infty}\int_\Omega A_n D(u_n - w_n^g) : Dv_n\,dx$$
$$\le R\|f - g\|_{H'}\left(\int_\Omega A_n Dv_n : Dv_n\,dx\right)^{\frac{1}{2}}.$$

Then, by continuity and density, and definition (1.73) of $\tilde{\mathcal{B}}$ and $\tilde{\Sigma}$ we get

$$u = w^f, \quad \Xi = \tilde{\Sigma}(u), \quad \Lambda = \tilde{\mathcal{B}}(u, v).$$

In particular, we have (1.49) and (1.50), which taking into account that

$$\int_{\overline{\Omega}} d\tilde{\mathcal{B}}(u, v) = \lim_{n\to\infty}\int_\Omega A_n Du_n : Dv_n\,dx = \lim_{n\to\infty}\langle f_n, v_n\rangle_{H_n',H_n} = \langle f, v\rangle_{H',H},$$

and the arbitrariness of v, allow us to conclude that u is a solution of (1.47). Since this solution is unique by Lax-Milgram's Theorem, we conclude that it is not necessary to extract any further subsequence from the one considered in Step 2. □

1.4 Integral representation of the limit

This section is devoted to proving the main result of the present work, Theorem 1.11, showing that if the sequence of tensor functions A_n satisfies assumptions (1.7)–(1.11) then the homogenization result established in the previous section is a local process. The main tool we use to show this result is an extension of the classical Murat-Tartar div-curl Lemma ([13], [14]) obtained in [15] or more exactly its following corollary.

Theorem 1.19. *For $q, r \in [1, \infty)$ such that*

$$\frac{1}{q} + \frac{1}{r} \le 1 + \frac{1}{N}, \tag{1.75}$$

we consider two sequences $\sigma_n \in L^q(\Omega)^{M\times N}$, $u_n \in W_0^{1,r}(\Omega)^M$, and two functions $\sigma \in L^q(\Omega)^{M\times N}$, $u \in W_0^{1,r}(\Omega)^M$, such that

$$\begin{cases} \sigma_n \rightharpoonup \sigma & \text{in } L^q(\Omega)^{M\times N} \quad \text{if } q>1, \\ \sigma_n \overset{*}{\rightharpoonup} \sigma & \text{in } \mathcal{M}(\Omega)^{M\times N} \quad \text{if } q=1, \end{cases} \qquad \begin{cases} u_n \rightharpoonup u & \text{in } W^{1,r}(\Omega)^M \quad \text{if } r>1, \\ u_n \overset{*}{\rightharpoonup} u & \text{in } BV(\Omega)^M \quad \text{if } r=1, \end{cases} \tag{1.76}$$

$$\operatorname{Div}\sigma_n \to \operatorname{Div}\sigma \ \text{ in } \begin{cases} W^{-1,r'}(\Omega)^{M\times N} & \text{if } r>1, \\ L^N(\Omega)^{M\times N} & \text{if } r=1, \end{cases} \tag{1.77}$$

$$\sigma_n : Du_n \ \text{ is bounded in } \mathcal{M}(\Omega). \tag{1.78}$$

Then,

$$\sigma_n : Du_n \overset{*}{\rightharpoonup} \sigma : Du \ \text{ in } \mathcal{M}(\Omega). \tag{1.79}$$

Remark 1.20. *In Theorem 1.19, the sequence $\sigma_n : Du_n$ is defined as an element of $\mathcal{D}'(\Omega)$ by*

$$\begin{aligned} \langle \sigma_n : Du_n, \varphi\rangle_{\mathcal{D}'(\overline{\Omega}),\mathcal{D}(\overline{\Omega})} = &- \langle \operatorname{Div}\sigma_n, u_n\varphi\rangle_{\mathcal{D}'(\Omega)^M,\mathcal{D}(\Omega)^M} \\ &- \int_\Omega \sigma_n : (u_n \otimes \nabla\varphi)\, dx, \qquad \forall\, \varphi \in \mathcal{D}(\Omega). \end{aligned} \tag{1.80}$$

We observe that this definition makes sense thanks to $u_n \in W_0^{1,r}(\Omega)^M$ and Sobolev's inequality. In the case $q = 1$ it is also necessary to use a result by J. Bourgain and H. Brezis ([22]) showing that $\sigma_n \in L^1(\Omega)^{M\times N}$ and $\operatorname{Div}\sigma_n$ smooth imply $\sigma_n \in W^{-1,N'}(\Omega)^{M\times N}$. The definition of $\sigma : Du$ is similar.

Assumption (1.78) means that for every $n \in \mathbb{N}$, we can extend $\sigma_n : Du_n$ to an element of $C_0^0(\Omega)' = \mathcal{M}(\Omega)$ and that the corresponding sequence of measures is bounded.

Proof of Theorem 1.19. Thanks to the div-curl Lemma given in [15], there exist two sequences $x_k \in \Omega$ and $r_k \in \mathbb{R}$, such that

$$\sigma_n : Du_n \rightharpoonup \sigma : Du + \sum_{k=1}^{\infty} \operatorname{div}(r_k \delta_{x_k}) \ \text{ in } \mathcal{D}'(\Omega), \tag{1.81}$$

but by assumption $\sigma_n : Du_n$ bounded in $\mathcal{M}(\Omega)$, and then for a subsequence, it converges weakly-$*$ to a certain measure $\tilde{\mu}$ in $\mathcal{M}(\Omega)$. By the definition of $\sigma : Du$, we then get

$$\int_\Omega \varphi\, d\tilde{\mu} = -\langle \operatorname{Div}\sigma, u\varphi\rangle_{\mathcal{D}'(\Omega)^M,\mathcal{D}(\Omega)^M} - \int_\Omega \sigma : (u\otimes\nabla\varphi)\,dx - \sum_{k\in\mathbb{N}} r_k\nabla\varphi(x_k) \quad \forall\varphi\in\mathcal{D}(\Omega).$$

This proves the existence of a function $\Psi\in L^1(\Omega)^N$ and a measure $\mu\in\mathcal{M}(\Omega)$ such that

$$\sum_{k\in\mathbb{N}} r_k\nabla\varphi(x_k) = \int_\Omega \Psi\cdot\nabla\varphi\,dx + \int_\Omega \varphi\,d\mu \quad \forall\,\varphi\in\mathcal{D}(\Omega),$$

which is only possible if $r_k = 0$ for every $k\in\mathbb{N}$. This proves (1.79). □

Proof of Theorem 1.11. By Theorem 1.16, there exist a subsequence of n, still denoted by n, a continuous bilinear operator $\tilde{\mathcal{B}} : H\times H \to \mathcal{M}(\overline{\Omega})$ and a linear operator $\tilde{\Sigma} : H \to L^1_{\mathfrak{a}}(\overline{\Omega})^N$ satisfying (1.40)–(1.46) and such that if f_n is a sequence which H'_n-converges to a certain f, then the weak solution u_n of (1.5) is such that (1.48), (1.49) and (1.50) hold, with $u\in H$ the unique solution of (1.31).

Now, we observe that similarly to (1.52), and using that A_n is bounded in $L^p(\Omega;\mathcal{L}(\mathbb{R}^{M\times N}))$, we have

$$\begin{aligned}\int_\Omega |A_nDu|^{\frac{2p}{1+p}}\varphi\,dx &\le R\int_\Omega (A_nDu : Du)^{\frac{p}{1+p}}|A_n|^{\frac{p}{1+p}}\varphi\,dx\\ &\le R\left(\int_\Omega A_nDu : Du\,\varphi\,dx\right)^{\frac{p}{1+p}}\left(\int_\Omega |A_n|^p\varphi\,dx\right)^{\frac{1}{1+p}}, \quad \forall u\in H_n,\ \forall\varphi\in C^0(\overline{\Omega}).\end{aligned} \tag{1.82}$$

From this inequality, we deduce that if u_n is the solution of (1.5) for a right-hand side f_n which H'_n-converges to some f (and then, with bounded norm thanks to Proposition 1.18), then A_nDu_n is bounded in $L^{\frac{2p}{1+p}}(\Omega)^{M\times N}$. This proves that in Theorem 1.16, we have

$$\tilde{\Sigma}(u)\mathfrak{a}\in L^{\frac{2p}{1+p}}(\Omega)^{M\times N} \quad \text{if } p>1, \quad \forall u\in H. \tag{1.83}$$

Moreover, the solution u_n to problem (1.5) is such that A_nDu_n converges weakly in $L^{\frac{2p}{1+p}}(\Omega)^{M\times N}$ to $\tilde{\Sigma}(u)\mathfrak{a}$ if $p>1$.

We define

$$E_p = \begin{cases} L^{\frac{2p}{1+p}}(\Omega) & \text{if } p>1,\\ L^1_{\mathfrak{a}}(\overline{\Omega}) & \text{if } p=1.\end{cases} \tag{1.84}$$

Then, we define $\mathcal{B} : DH\times DH \to \mathcal{M}(\overline{\Omega})$ and $\Sigma : DH \to E_p^{M\times N}$ by

$$\mathcal{B}(Du,Dv) = \tilde{\mathcal{B}}(u,v), \quad \forall\,u,v\in H, \tag{1.85}$$

$$\Sigma(Du) = \begin{cases}\tilde{\Sigma}(u)\mathfrak{a} & \text{if } p>1,\\ \tilde{\Sigma}(u) & \text{if } p=1,\end{cases} \quad \forall\,u\in H. \tag{1.86}$$

Thanks to (1.40)–(1.46) and (1.82), it is clear that (1.23)–(1.28) hold. Therefore, in order to show Theorem 1.11, it just remains to prove (1.29) and the existence of a tensor function $A \in L^p(\Omega; \mathcal{L}(\mathbb{R}^{M\times N}))$, if $p>1$, $A \in L^\infty_{\mathfrak{a}}(\Omega; \mathcal{L}(\mathbb{R}^{M\times N}))$ if $p=1$, such that (1.30) holds. This will be given in the following three steps.

Step 1. Let us prove that for every $u \in H$, every $\omega \subset \Omega$ open and every $v \in H$ with

$$v \in \begin{cases} W^{1,\frac{2p}{p-1}}(\omega)^M & \text{if } p>1, \\ C^1(\omega)^M & \text{if } p=1, \end{cases} \tag{1.87}$$

we have

$$\mathcal{B}(Du, Dv) = \begin{cases} \Sigma(Du) : Dv & \text{if } p>1, \\ \Sigma(Du) : Dv\,\mathfrak{a} & \text{if } p=1, \end{cases} \quad \text{in } \omega. \tag{1.88}$$

To prove this result, we first assume that there exists $f \in C^0(\overline{\Omega})^M$ such that

$$\int_\Omega d\mathcal{B}(Du, Dw) = \int_\Omega f w\,dx \quad \forall\, w \in H, \tag{1.89}$$

and we consider the solution u_n of (1.5) with right-hand side f. We know that u_n H_n-converges weakly to u. Consider also a sequence v_n which H_n-converges weakly to v. Since $A_n Du_n : Dv_n$ is bounded in $\mathcal{M}(\Omega)$, we can apply Theorem 1.19 in ω to $\sigma_n = A_n Du_n$. Taking into account (1.33) and (1.34), we then deduce that for every $\varphi \in \mathcal{D}(\omega)$, we have

$$\int_\Omega \varphi\, d\mathcal{B}(Du, Dv) = \lim_{n\to\infty} \int_\Omega A_n Du_n : Dv_n \varphi\,dx = \begin{cases} \int_\Omega \Sigma(Du) : Dv\,\varphi\,dx & \text{if } p>1, \\ \int_\Omega \Sigma(Du) : Dv\,\varphi\,d\mathfrak{a} & \text{if } p=1. \end{cases}$$

This proves (1.88) for u satisfying (1.89), with $f \in C^0(\overline{\Omega})^M$. The general case then follows by using that the space of such u is dense in H and that $\mathcal{B}(\cdot, Dv)$ and Σ are continuous in DH.

Step 2. Assume $p>1$. We introduce the measure $\mathfrak{a}_p$ as (it is well defined up to a subsequence)

$$|A_n|^p \overset{*}{\rightharpoonup} \mathfrak{a}_p \ \text{ in } \mathcal{M}(\overline{\Omega}),$$

and observe that (1.82) implies

$$\begin{aligned} &\int_\Omega |\Sigma(Du)|^{\frac{2p}{1+p}} \varphi\,dx \\ &\quad \le R \left(\int_\Omega \varphi\, d\mathcal{B}(Du, Du) \right)^{\frac{p}{1+p}} \left(\int_\Omega \varphi\, d\mathfrak{a}_p \right)^{\frac{1}{1+p}}, \quad \forall\, \varphi \in C^0(\overline{\Omega}),\ \varphi \ge 0, \end{aligned} \tag{1.90}$$

and then, using (1.88) and Hölder's inequality, we deduce that for $\omega \subset \Omega$ open and $u \in H \cap W^{1,\frac{2p}{p-1}}(\omega)$, we have

$$\begin{aligned} &\int_\omega \varphi\, d\mathcal{B}(Du, Du) \\ &\le R^{\frac{1+p}{2p}} \left(\int_\omega \varphi\, d\mathcal{B}(Du, Du) \right)^{\frac{1}{2}} \left(\int_\omega \varphi\, d\mathfrak{a}_p \right)^{\frac{1}{2p}} \left(\int_\omega |Du|^{\frac{2p}{p-1}} \varphi\,dx \right)^{\frac{p-1}{2p}},\ \forall\, \varphi \in C^0_0(\omega),\ \varphi \ge 0, \end{aligned}$$

and then

$$\int_\omega \varphi\, d\mathcal{B}(Du,Du) \leq R^{\frac{1+p}{p}} \left(\int_\omega \varphi\, d\mathfrak{a}_p\right)^{\frac{1}{p}} \left(\int_\omega |Du|^{\frac{2p}{p-1}}\varphi\, dx\right)^{\frac{p-1}{p}}, \quad \forall \varphi \in C^0_0(\omega),\ \varphi \geq 0,$$

which by the derivation measure theorem, proves

$$\mathcal{B}(Du,Du) \leq R^{\frac{1+p}{p}} L(\mathfrak{a}_p)^{\frac{1}{p}} |Du|^2 \ \text{ a.e. in } \omega, \tag{1.91}$$

where $L(\mathfrak{a}_p)$ denotes the Lebesgue part of $\mathfrak{a}_p$. Taking into account this result in (1.90), we also deduce

$$|\Sigma(Du)| \leq R^{\frac{1+p}{p}} L(\mathfrak{a}_p)^{\frac{1}{p}} |Du|. \ \text{ a.e. in } \omega. \tag{1.92}$$

In the case $p = 1$, a similar reasoning taking into account (1.52), shows

$$\mathcal{B}(Du,Du) \leq R^2 |Du|^2 \mathfrak{a} \ \text{ in } \mathcal{M}(\omega), \quad \forall\, u \in H \cap C^1(\omega)^M, \tag{1.93}$$

$$|\Sigma(Du)| \leq R^2 |Du| \quad \mathfrak{a}\text{-a.e. in } \omega, \quad \forall\, u \in H \cap C^1(\omega)^M. \tag{1.94}$$

Step 3. We consider a sequence Ω_n of open sets contained in Ω such that

$$\Omega_0 = \emptyset, \qquad \overline{\Omega}_n \subset \Omega_{n+1}, \quad \forall\, n \in \mathbb{N}, \qquad \Omega = \bigcup_{n\in\mathbb{N}} \Omega_n,$$

and a sequence of functions $\varphi_n \in C^\infty_c(\Omega)$, such that $\varphi_n = 1$ in Ω_n. Then, we define a tensor function $A : \Omega \to \mathcal{L}(\mathbb{R}^{M\times N})$ by

$$A\xi = \sum_{n\in\mathbb{N}} \Sigma(D(\xi \cdot x\, \varphi_n))\chi_{\Omega_n}, \quad \forall\, \xi \in \mathbb{R}^{M\times N}, \text{ a.e. in } \Omega \quad (\mathfrak{a}\text{-a.e. in } \Omega \text{ if } p = 1).$$

Assume $u \in H \cap W^{1,\frac{2p}{p-1}}(\omega)^M$, if $p > 1$, $u \in C^1(\omega)^M$ if $p = 1$, with $\omega \subset \Omega$ open. Then, by the linearity of Σ, (1.92) and (1.94), we have

$$|\Sigma(Du) - A\xi| \leq R^{\frac{1+p}{p}} L(\mathfrak{a}_p)^{\frac{1}{p}} |Du - \xi| \ \text{ a.e. in } \omega \ (\mathfrak{a}\text{-a.e. in } \omega \text{ if } p = 1).$$

This proves

$$\Sigma(Du) = ADu \ \text{ in } \omega,$$

which finishes the proof of Theorem 1.11. □

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Chapter 2

Homogenization of equi-coercive nonlinear energies defined on vector-valued functions, with non-uniformly bounded coefficients

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Abstract.
The present paper deals with the asymptotic behavior of equi-coercive sequences $\{\mathscr{F}_n\}$ of nonlinear functionals defined over vector-valued functions in $W_0^{1,p}(\Omega)^M$, where $p > 1$, $M \geq 1$, and Ω is a bounded open set of $\mathbb{R}^N$, $N \geq 2$. The strongly local energy density $F_n(\cdot, Du)$ of the functional $\mathscr{F}_n$ satisfies a Lipschitz condition with respect to the second variable, which is controlled by a positive sequence $\{a_n\}$ which is only bounded in some suitable space $L^r(\Omega)$. We prove that the sequence $\{\mathscr{F}_n\}$ Γ-converges for the strong topology of $L^p(\Omega)^M$ to a functional $\mathscr{F}$ which has a strongly

local density $F(\cdot, Du)$ for sufficiently regular functions u. This compactness result extends former results on the topic, which are based either on maximum principle arguments in the nonlinear scalar case, or adapted div-curl lemmas in the linear case. Here, the vectorial character and the nonlinearity of the problem need a new approach based on a careful analysis of the asymptotic minimizers associated with the functional $\mathscr{F}_n$. The relevance of the conditions which are imposed to the energy density $F_n(\cdot, Du)$, is illustrated by several examples including some classical hyper-elastic energies.

2.1 Introduction

In this paper we study the asymptotic behavior of the sequence of nonlinear functionals, including some hyper-elastic energies (see the examples of Section 2.2.3), defined on vector-valued functions by

$$\mathscr{F}_n(v) := \int_\Omega F_n(x, Dv)\, dx \quad \text{for } v \in W_0^{1,p}(\Omega)^M, \quad \text{with } p \in (1, \infty),\ M \geq 1, \tag{2.1}$$

in a bounded open set Ω of $\mathbb{R}^N$, $N \geq 2$. The sequence $\mathscr{F}_n$ is assumed to be equi-coercive. Moreover, the associated density $F_n(\cdot, \xi)$ satisfies some Lipschitz condition with respect to $\xi \in \mathbb{R}^{M\times N}$, and its coefficients are not uniformly bounded in Ω.

The linear scalar case, *i.e.* when $F_n(\cdot, \xi)$ is quadratic with respect to $\xi \in \mathbb{R}^N$ ($M = 1$), with uniformly bounded coefficients was widely investigated in the seventies through G-convergence by Spagnolo [33], extended by Murat and Tartar with H-convergence [28, 35], and alternatively through Γ-convergence by De Giorgi [22, 23] (see also [21, 4]). The linear elasticity case was probably first derived by Duvaut (unavailable reference), and can be found in [32, 25]. In the nonlinear scalar case the first compactness results are due to Carbone, Sbordone [17] and Buttazzo, Dal Maso [14] by a Γ-convergence approach assuming the L^1-equi-integrability of the coefficients. More recently, these results were extended in [5, 9, 10] relaxing the L^1-boundedness of the coefficients but assuming that $p > N-1$ if $N \geq 3$, showing then the uniform convergence of the minimizers thanks to the maximum principle. In all these works the scalar framework combined with the condition $p > N-1$ if $N \geq 3$ and the equi-coercivity of the functionals, induce in terms of the Γ-convergence for the strong topology of $L^p(\Omega)$, a limit energy $\mathscr{F}$ of the same nature satisfying

$$\mathscr{F}(v) := \int_\Omega F(x, Dv)\, d\nu \quad \text{for } v \in W, \tag{2.2}$$

where $C_c^1(\Omega)^M \subset W$ is some suitable subspace of $W_0^{1,p}(\Omega)^M$, and ν is some Radon measure on Ω. Removing the L^1-equi-integrability of the coefficients in the three-dimensional linear scalar case (note that $p = N-1 = 2$ in this case), Fenchenko and Khruslov [24] (see also [26]) were, up to our knowledge, the first to obtain a violation of the compactness result due to the appearance of local and nonlocal terms in the limit energy $\mathscr{F}$. This seminal work was also revisited by Bellieud and Bouchitté [2]. Actually, the local and nonlocal terms in addition to the classical strongly local term come from the Beurling-Deny [3] representation formula of a Dirichlet form, and arise naturally in the homogenization process as shown by Mosco [27]. The complete picture of the attainable energies was obtained by Camar-Eddine and Seppecher [15] in the linear scalar case. The elasticity case is much more intricate even in the linear framework, since the loss of uniform boundedness of the elastic coefficients may induce the appearance of second gradient terms as Seppecher and Pideri proved in [30]. The situation is dramatically different from the scalar case, since the Beurling-Deny formula does not hold in the vector-valued case. In fact, Camar-Eddine and Seppecher [16] proved that any lower semi-continuous quadratic functional vanishing on the rigid displacements, can be attained. Compactness results were obtained in the linear elasticity case using some (strong) equi-integrability of the coefficients in [11], and using various extensions of the classical Murat-Tartar [28] div-curl result in [7, 13, 12, 29] (which were themselves initiated in the former works [6, 9] of the two first authors).

In our context the vectorial character of the problem and its nonlinearity prevent us from using the uniform convergence of [10] and the div-curl lemma of [12], which are (up to our knowledge) the more recent general compactness results on the topic. We assume that the nonnegative energy density $F_n(\cdot,\xi)$ of the functional (2.1) attains its minimum at $\xi = 0$, and satisfies the following Lipschitz condition with respect to $\xi \in \mathbb{R}^{M\times N}$:

$$\begin{cases} \big|F_n(x,\xi)-F_n(x,\eta)\big| \le \big(h_n(x)+F_n(x,\xi)+F_n(x,\eta)+|\xi|^p+|\eta|^p\big)^{\frac{p-1}{p}} a_n(x)^{\frac{1}{p}}\,|\xi-\eta| \\ \forall\,\xi,\eta\in\mathbb{R}^{M\times N},\ \text{a.e. } x\in\Omega, \end{cases}$$

which is controlled by a positive function $a_n(\cdot)$ (see the whole set of conditions (2.3) to (2.8) below). The sequence $\{a_n\}$ is assumed to be bounded in $L^r(\Omega)$ for some $r > (N-1)/p$ if $1 < p \le N-1$, and bounded in $L^1(\Omega)$ if $p > N-1$. Note that for $p > N-1$ our condition is better than the L^1-equi-integrability used in the scalar case of [17, 14], but not for $1 < p \le N-1$. Under these assumptions we prove (see Theorem 2.4) that the sequence $\{\mathscr{F}_n\}$ of (2.1) Γ-converges for the strong topology of $L^p(\Omega)^M$ (see Definition 2.1) to a functional of type (2.2) with

$$W \subset \begin{cases} W^{1,\frac{pr}{r-1}}(\Omega)^M & \text{if } 1 < p \le N-1, \\ C^1(\overline{\Omega})^M & \text{if } p > N-1, \end{cases}$$

and

$$\nu = \begin{cases} \text{Lebesgue measure} & \text{if } 1 < p \le N-1, \\ \mathscr{M}(\Omega)* - \lim\limits_{n\to\infty} a_n & \text{if } p > N-1. \end{cases}$$

Various types of boundary conditions can be taken into account in this Γ-convergence approach.

A preliminary result (see Theorem 2.3) allows us to prove that the sequence of energy density $\{F_n(\cdot,Du_n)\}$ converges in the sense of Radon measures to some strongly local energy density $F(\cdot,Du)$, when u_n is an asymptotic minimizer for $\mathscr{F}_n$ of limit u (see definition (2.17)). The proof of this new compactness result is based on an extension (see Lemma 2.6) of the fundamental estimate for recovery sequences in Γ-convergence (see, *e.g.*, [21], Chapters 18, 19), which provides a bound (see (2.26)) satisfied by the weak-$*$ limit of $\{F_n(\cdot, Du_n)\}$ with respect to the weak-$*$ limit of any sequence $\{F_n(\cdot, Dv_n)\}$ such that the sequence $\{v_n - u_n\}$ converges weakly to 0 in $W_0^{1,p}(\Omega)^M$. Rather than using fixed smooth cut-off functions as in the classical fundamental estimate, here we need to consider sequences of radial cut-off functions φ_n whose gradient has support in n-dependent sets on which $u_n - u$ satisfies some uniform estimate with respect to the radial coordinate (see Lemma 2.11 and its proof). This allows us to control the zero-order term $\nabla\varphi_n(u_n - u)$, when we put the trial function $\varphi_n(u_n - u)$ in the functional $\mathscr{F}_n$ of (2.1). The uniform estimate is a consequence of the Sobolev compact embedding for the $(N-1)$-dimensional sphere, and explains the role of the exponent $r > (N-1)/p$ if $1 < p \leq N-1$. A similar argument was used in the linear case [12] to obtain a new div-curl lemma which is the key-ingredient for the compactness of quadratic elasticity functionals of type (2.1).

Notations

- $\mathbb{R}_s^{N\times N}$ denotes the set of the symmetric matrices in $\mathbb{R}^{N\times N}$.
- For any $\xi \in \mathbb{R}^{N\times N}, \xi^T$ is the transposed matrix of ξ, and $\xi^s := \frac{1}{2}(\xi + \xi^T)$ is the symmetrized matrix of ξ.
- I_N denotes the unit matrix of $\mathbb{R}^{N\times N}$.
- $\cdot$ denotes the scalar product in $\mathbb{R}^N$, and $:$ denotes the scalar product in $\mathbb{R}^{M\times N}$ defined by
$$\xi : \eta := \mathrm{tr}\,(\xi^T\eta) \quad \text{for } \xi, \eta \in \mathbb{R}^{M\times N},$$
where tr is the trace.
- $|\cdot|$ denotes both the euclidian norm in $\mathbb{R}^N$, and the Frobenius norm in $\mathbb{R}^{M\times N}$, *i.e.*
$$|\xi| := \big(\mathrm{tr}\,(\xi^T\xi)\big)^{\frac{1}{2}} \quad \text{for } \xi \in \mathbb{R}^{M\times N}.$$
- For a bounded open set $\omega \subset \mathbb{R}^N$, $\mathscr{M}(\omega)$ denotes the space of the Radon measures on ω with bounded total variation. It agrees with the dual space of $C_0^0(\omega)$, namely the space of the continuous functions in $\bar{\omega}$ which vanish on $\partial\omega$. Moreover, $\mathscr{M}(\bar{\omega})$ denotes the space of the Radon measures on $\bar{\omega}$. It agrees with the dual space of $C^0(\bar{\omega})$.
- For any measures $\zeta, \mu \in \mathscr{M}(\omega)$, with $\omega \subset \mathbb{R}^N$, open, bounded, we define $\zeta^\mu \in L^1_\mu(\Omega)$ as the derivative of ζ with respect to μ. When μ is the Lebesgue measure, we write ζ^L.
- C is a positive constant which may vary from line to line.
- O_n is a real sequence which tends to zero as n tends to infinity. It can vary from line to line.

Recall the definition of the De Giorgi Γ-convergence (see, *e.g.*, [21, 4] for further details).

Definition 2.1. *Let V be a metric space, and let $\mathscr{F}_n, \mathscr{F} : V \to [0,\infty]$, $n \in \mathbb{N}$, be functionals defined on V. The sequence $\{\mathscr{F}_n\}$ is said to Γ-converge to $\mathscr{F}$ for the topology of V in a set $W \subset V$ and we write*

$$\mathscr{F}_n \xrightharpoonup{\Gamma} \mathscr{F} \quad \textit{in } W,$$

if

- *the Γ-liminf inequality holds*

$$\forall\, v \in W,\ \ \forall\, v_n \to v \quad \textit{in } V, \quad \mathscr{F}(v) \le \liminf_{n\to\infty} \mathscr{F}_n(v_n),$$

- *the Γ-limsup inequality holds*

$$\forall v \in W,\ \ \exists\, \overline{v}_n \to v \quad \textit{in } V, \quad \mathscr{F}(v) = \lim_{n\to\infty} \mathscr{F}_n(\overline{v}_n).$$

Any sequence $\overline{v}_n$ satisfying (2.1) *is called a* recovery sequence *for $\mathscr{F}_n$ of limit v.*

2.2 Statement of the results and examples

2.2.1 The main results

Consider a bounded open set $\Omega \subset \mathbb{R}^N$ with $N \ge 2$, M a positive integer, a sequence of nonnegative Carathéodory functions $F_n : \Omega \times \mathbb{R}^{M\times N} \to [0,\infty)$, and $p > 1$ with the following properties:

- There exist two constants $\alpha > 0$ and $\beta \ge 0$ such that

$$\int_\Omega F_n(x, Du)\, dx \ge \alpha \int_\Omega |Du|^p\, dx - \beta, \quad \forall\, u \in W_0^{1,p}(\Omega)^M, \tag{2.3}$$

  and

$$F_n(\cdot, 0) = 0 \ \text{ a.e. in } \Omega. \tag{2.4}$$

- There exist two sequences of measurable functions $h_n, a_n \ge 0$, and a constant $\gamma > 0$ such that

$$h_n \text{ is bounded in } L^1(\Omega), \tag{2.5}$$

$$a_n \text{ is bounded in } L^r(\Omega) \text{ with } \begin{cases} r > \dfrac{N-1}{p}, & \text{if } 1 < p \le N-1 \\ r = 1, & \text{if } p > N-1, \end{cases} \tag{2.6}$$

$$\begin{cases} \big|F_n(x,\xi) - F_n(x,\eta)\big| \\ \qquad \le \big(h_n(x) + F_n(x,\xi) + F_n(x,\eta) + |\xi|^p + |\eta|^p\big)^{\frac{p-1}{p}} a_n(x)^{\frac{1}{p}}\, |\xi - \eta| \\ \forall\, \xi, \eta \in \mathbb{R}^{M\times N}, \ \text{a.e. } x \in \Omega, \end{cases} \tag{2.7}$$

and

$$F_n(x,\lambda\xi) \le h_n(x) + \gamma\, F_n(x,\xi), \quad \forall\, \lambda \in [0,1],\ \forall\, \xi \in \mathbb{R}^{M\times N},\ \text{a.e. } x \in \Omega. \tag{2.8}$$

Remark 2.2. *From* (2.7) *and Young's inequality, we get that*

$$\begin{aligned} F_n(x,\xi) &\le F_n(x,\eta) + \big(h_n(x) + F_n(x,\xi) + F_n(x,\eta) + |\xi|^p + |\eta|^p\big)^{\frac{p-1}{p}} a_n(x)^{\frac{1}{p}}\, |\xi - \eta| \\ &\le F_n(x,\eta) + \frac{p-1}{p}\big(h_n(x) + F_n(x,\xi) + F_n(x,\eta) + |\xi|^p + |\eta|^p\big) + \frac{1}{p}\, a_n(x)\, |\xi - \eta|^p, \end{aligned}$$

and then

$$\begin{cases} F_n(x,\xi) \le (p-1)\, h_n(x) + (2p-1)\, F_n(x,\eta) + (p-1)\big(|\xi|^p + |\eta|^p\big) + a_n(x)\, |\xi - \eta|^p, \\ \forall\, \xi, \eta \in \mathbb{R}^{M\times N},\ \text{a.e. } x \in \Omega. \end{cases} \tag{2.9}$$

In particular, taking $\eta = 0$*, we have*

$$F_n(x,\xi) \le (p-1)\, h_n(x) + \big(p - 1 + a_n(x)\big)\, |\xi|^p, \quad \forall\, \xi \in \mathbb{R}^{M\times N},\ \text{a.e. } x \in \Omega, \tag{2.10}$$

where the right-hand side is a bounded sequence in $L^1(\Omega)$*.*

From now on, we assume that

$$a_n^r \overset{*}{\rightharpoonup} \mathrm{A} \ \text{ in } \mathscr{M}(\Omega) \quad \text{and} \quad h_n \overset{*}{\rightharpoonup} h \ \text{ in } \mathscr{M}(\Omega). \tag{2.11}$$

The paper deals with the asymptotic behavior of the sequence of functionals

$$\mathscr{F}_n(v) := \int_\Omega F_n(x, Dv)\, dx \quad \text{for } v \in W^{1,p}(\Omega)^M. \tag{2.12}$$

First of all, we have the following result on the convergence of the energy density $F_n(\cdot, Du_n)$, where u_n is an asymptotic minimizer associated with functional (2.12).

Theorem 2.3. *Let* $F_n : \Omega\times\mathbb{R}^{M\times N} \to [0,\infty)$ *be a sequence of Carathéodory functions satisfying* (2.3) *to* (2.8)*. Then, there exist a function* $F : \Omega \times \mathbb{R}^{M\times N} \to \mathbb{R}$ *and a subsequence of* n*, still denoted by* n*, such that for any* $\xi, \eta \in \mathbb{R}^N$*,*

$$\begin{cases} F(\cdot,\xi) \ \ \textit{is Lebesgue measurable}, & \textit{if } 1 < p \le N-1 \\ F(\cdot,\xi) \ \ \textit{is } \mathrm{A}\textit{-measurable}, & \textit{if } p > N-1, \end{cases} \tag{2.13}$$

$$\big|F(x,\xi)-F(x,\eta)\big| \leq$$
$$\begin{cases} C\big(h^L+F(x,\xi)+F(x,\eta)+(1+(\mathrm{A}^L)^{\frac{1}{r}})(|\xi|^p+|\eta|^p)\big)^{\frac{p-1}{p}}\cdot \\ \cdot\,(\mathrm{A}^L)^{\frac{1}{pr}}|\xi-\eta| \;\; a.e. \;\; in\;\Omega & if\;\; 1<p\leq N-1, \\ C\big(1+h^{\mathrm{A}}+F(x,\xi)+F(x,\eta)+|\xi|^p+|\eta|^p\big)^{\frac{p-1}{p}}\cdot \\ \cdot\,|\xi-\eta| \;\;\; \mathrm{A}\text{-}a.e. \;\; in\;\Omega & if\;\; p>N-1, \end{cases} \tag{2.14}$$

and

$$F(\cdot,0)=0 \;\; a.e. \;\; in\;\Omega. \tag{2.15}$$

For any open set $\omega\subset\Omega$, and any sequence $\{u_n\}$ in $W^{1,p}(\omega)^M$ which converges weakly in $W^{1,p}(\omega)^M$ to a function u satisfying

$$u\in\begin{cases} W^{1,\frac{pr}{r-1}}(\omega)^M, & if\; 1<p\leq N-1 \\ C^1(\omega)^M, & if\; p>N-1, \end{cases} \tag{2.16}$$

and such that

$$\begin{aligned}&\exists \lim_{n\to\infty}\int_\omega F_n(x,Du_n)\,dx \\ &\quad=\min\left\{\liminf_{n\to\infty}\int_\omega F_n(x,Dw_n)\,dx:\;\; w_n-u_n\rightharpoonup 0 \;\; in\; W_0^{1,p}(\omega)^M\right\}<\infty,\end{aligned} \tag{2.17}$$

we have

$$F_n(\cdot,Du_n)\overset{*}{\rightharpoonup}\begin{cases} F(\cdot,Du), & if\; 1<p\leq N-1 \\ F(\cdot,Du)\,\mathrm{A}, & if\; p>N-1 \end{cases} \quad in\;\mathscr{M}(\omega). \tag{2.18}$$

From Theorem 2.3 we may deduce the Γ-limit (see Definition 2.1) of the sequence of functionals (2.12) with various boundary conditions.

Theorem 2.4. *Let $F_n:\Omega\times\mathbb{R}^{M\times N}\to[0,\infty)$ be a sequence of Carathéodory functions satisfying (2.3) to (2.8). Let ω be an open set such that $\omega\subset\subset\Omega$, and let V be a subset of $W^{1,p}(\omega)^M$ such that*

$$\forall\, u\in V,\;\forall\, v\in W_0^{1,p}(\omega)^M,\quad u+v\in V. \tag{2.19}$$

Define the functional $\mathscr{F}_n^V:V\to[0,\infty)$ by

$$\mathscr{F}_n^V(v):=\int_\omega F_n(x,Dv)\,dx \quad for\; v\in V. \tag{2.20}$$

Assume that the open set ω satisfies

$$\begin{cases} |\partial\omega|=0, & if\; 1<p\leq N-1 \\ \mathrm{A}(\partial\omega)=0, & if\; p>N-1. \end{cases} \tag{2.21}$$

Then, for the subsequence of n *(still denoted by* n*) obtained in Theorem 2.3 we get*

$$\begin{cases} \mathscr{F}_n^V \overset{\Gamma}{\rightharpoonup} \mathscr{F}^V := \displaystyle\int_\omega F(x, Dv)\, dx \quad \text{in } V \cap W^{1,\frac{pr}{r-1}}(\omega)^M, & \text{if } 1 < p \leq N-1 \\ \mathscr{F}_n^V \overset{\Gamma}{\rightharpoonup} \mathscr{F}^V := \displaystyle\int_\omega F(x, Dv)\, dx \quad \text{in } V \cap C^1(\overline{\omega})^M, & \text{if } p > N-1, \end{cases} \tag{2.22}$$

for the strong topology of $L^p(\omega)^M$*, where* F *is given by convergence* (2.18)*.*

Remark 2.5. *The condition* (2.21) *on the open set* ω *is not so restrictive. Indeed, for any family* $(\omega)_{i\in I}$ *of open sets of* Ω *with two by two disjoint boundaries, at most a countable subfamily of* $(\partial\omega)_{i\in I}$ *does not satisfy* (2.21)*.*

2.2.2 Auxiliary lemmas

The proof of Theorem 2.3 is based on the following lemma which provides an estimate of the energy density for asymptotic minimizers. In our context it is equivalent to the fundamental estimate for recovery sequences (see Definition 2.1) in Γ-convergence theory (see, *e.g.*, [21], Chapters 18, 19).

Lemma 2.6. *Let* $F_n : \Omega \times \mathbb{R}^{M\times N} \to [0, \infty)$ *be a sequence of Carathéodory functions satisfying* (2.3) *to* (2.8)*. Consider an open set* $\omega \subset \Omega$*, and a sequence* $\{u_n\} \subset W^{1,p}(\omega)^M$ *converging weakly in* $W^{1,p}(\omega)^M$ *to a function* u *satisfying* (2.16)*, and such that*

$$F_n(\cdot, Du_n) \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathscr{M}(\omega),$$
$$|Du_n|^p \overset{*}{\rightharpoonup} \varrho \quad \text{in } \mathscr{M}(\omega).$$

Then, the measure ϱ *satisfies*

$$\varrho \leq \begin{cases} C\big(|Du|^p + |Du|^p(\mathrm{A}^L)^{\frac{1}{r}} + h + \mu + \mathrm{A}^L\big) & \text{a.e. in } \omega, \quad \text{if } 1 < p \leq N-1 \\ C\big(|Du|^p\mathrm{A} + h + \mu + \mathrm{A}\big) & \mathrm{A}\text{-a.e. in } \omega, \quad \text{if } p > N-1. \end{cases} \tag{2.23}$$

Moreover if u_n *satisfies*

$$\begin{aligned} &\exists \lim_{n\to\infty} \int_\omega F_n(x, Du_n)\, dx \\ &\quad = \min\left\{ \liminf_{n\to\infty} \int_\omega F_n(x, Dw_n)\, dx : \ w_n - u_n \rightharpoonup 0 \text{ in } W_0^{1,p}(\omega)^M \right\}, \end{aligned} \tag{2.24}$$

then for any sequence $\{v_n\} \subset W^{1,p}(\omega)^M$ *which converges weakly in* $W^{1,p}(\omega)^M$ *to a function*

$$v \in \begin{cases} W^{1,\frac{pr}{r-1}}(\omega)^M, & \text{if } 1 < p \leq N-1 \\ C^1(\omega)^M, & \text{if } p > N-1, \end{cases}$$

and such that

$$F_n(\cdot, Dv_n) \overset{*}{\rightharpoonup} \nu \quad \text{in } \mathscr{M}(\omega), \tag{2.25}$$

$$|Dv_n|^p \overset{*}{\rightharpoonup} \varpi \quad in\ \mathscr{M}(\omega),$$

we have

$$\mu \leq \begin{cases} \nu + C\big(h^L + \nu^L + \varpi^L + (1+(\mathrm{A}^L)^{\frac{1}{r}})|D(u-v)|^p\big)^{\frac{p-1}{p}} \cdot \\ \cdot\, (\mathrm{A}^L)^{\frac{1}{pr}}|D(u-v)| \quad a.e.\ in\ \omega & if\ \ 1<p\leq N-1, \\ \nu + C\big(1 + h^{\mathrm{A}} + \nu^{\mathrm{A}} + \varpi^{\mathrm{A}} + |D(u-v)|^p\big)^{\frac{p-1}{p}} \cdot \\ \cdot\, \mathrm{A}\,|D(u-v)| \quad \mathrm{A}\text{-}a.e.\ in\ \omega & if\ \ p>N-1. \end{cases} \tag{2.26}$$

We can improve the statement of Lemma 2.6 if we add a non-homogeneous Dirichlet boundary condition on $\partial\omega$.

Lemma 2.7. *Let ω be an open set such that $\omega \subset\subset \Omega$, and let u be a function satisfying*

$$u \in \begin{cases} W^{1,\frac{pr}{r-1}}(\Omega)^M, & if\ 1<p\leq N-1 \\ C^1(\overline{\Omega})^M, & if\ p>N-1. \end{cases} \tag{2.27}$$

Let $\{u_n\}$ and $\{v_n\}$ be two sequences in $W^{1,p}(\omega)^M$, such that u_n satisfies condition (2.24) *and*

$$u_n - u,\ v_n - u \in W^{1,p}_0(\omega)^M,$$

$$F_n(\cdot, Du_n) \overset{*}{\rightharpoonup} \mu \quad and \quad F_n(\cdot, Dv_n) \overset{*}{\rightharpoonup} \nu \quad in\ \mathscr{M}(\overline{\omega}), \tag{2.28}$$

$$|Du_n|^p \overset{*}{\rightharpoonup} \varrho \quad and \quad |Dv_n|^p \overset{*}{\rightharpoonup} \varpi \quad in\ \mathscr{M}(\overline{\omega}). \tag{2.29}$$

Then, estimates (2.23) *and* (2.26) *hold in $\overline{\omega}$.*

Remark 2.8. *Condition* (2.24) *means that u_n is a recovery sequence in ω for the functional*

$$w \in W^{1,p}(\omega)^M \mapsto \int_\omega F_n(x, Dw)\,dx, \tag{2.30}$$

with the Dirichlet condition $w - u_n \in W^{1,p}_0(\omega)^M$. Since $w = u_n$ clearly satisfies $w - u_n \in W^{1,p}_0(\omega)^M$, this makes u_n a recovery sequence without imposing any boundary condition. In particular, condition (2.24) *is fulfilled if for a fixed $f \in W^{-1,p}(\omega)^M$, u_n satisfies*

$$\int_\omega F_n(x, Du_n)\,dx = \min\left\{\int_\omega F_n\big(x, D(u_n+v)\big)\,dx - \langle f, v\rangle :\ v \in W^{1,p}_0(\omega)^M\right\}.$$

Assuming the differentiability of F_n with respect to the second variable, it follows that u_n satisfies the variational equation

$$\int_\omega D_\xi F_n(x, Du_n) : Dv\,dx - \langle f, v\rangle = 0, \quad \forall\, v \in W^{1,p}_0(\omega)^M,$$

i.e. u_n is a solution of

$$-\operatorname{Div}\big(D_\xi F_n(x, Du)\big) = f \quad in\ \omega,$$

where no boundary condition is imposed.

Assumption (2.24) *allows us to take into account very general boundary conditions. For example, if u_n is a recovery sequence for* (2.30) *with (non necessarily homogeneous) Dirichlet or Neumann boundary condition, then it also satisfies* (2.24).

Remark 2.9. *Condition* (2.24) *is equivalent to the asymptotic minimizer property satisfied by u_n:*

$$\int_\omega F_n(x, Du_n)\,dx \le \int_\omega F_n(x, Dw_n)\,dx + O_n, \quad \forall\, w_n \ \text{with}\ w_n - u_n \rightharpoonup 0 \ \text{in}\ W_0^{1,p}(\omega)^M.$$

We can check that if u_n satisfies this condition in ω, then u_n satisfies it in any open subset $\hat\omega \subset \omega$. To this end, it is enough to consider for a sequence $\hat w_n$ with $\hat w_n - u_n \in W_0^{1,p}(\hat\omega)^M$, the extension

$$w_n := \begin{cases} \hat w_n & \text{in}\ \hat\omega \\ u_n & \text{in}\ \omega \setminus \hat\omega. \end{cases}$$

Corollary 2.10. *Let $F_n : \Omega \times \mathbb{R}^{M\times N} \to [0,\infty)$ be a sequence of Carathéodory functions satisfying* (2.3) *to* (2.8). *Consider two open sets $\omega_1, \omega_2 \subset \Omega$ such that $\omega_1 \cap \omega_2 \neq \emptyset$, a sequence u_n converging weakly in $W^{1,p}(\omega_1)^M$ to a function u and a sequence v_n converging weakly in $W^{1,p}(\omega_2)^M$ to a function v, such that*

$$u, v \in \begin{cases} W^{1,\frac{pr}{r-1}}(\omega_1 \cap \omega_2)^M, & \text{if}\ 1 < p \le N-1 \\ C^1(\omega_1 \cap \omega_2)^M, & \text{if}\ p > N-1, \end{cases}$$

$$\begin{aligned} |Du_n|^p \overset{*}{\rightharpoonup} \varrho, \quad F_n(\cdot, Du_n) \overset{*}{\rightharpoonup} \mu \quad \text{in}\ \mathscr{M}(\omega_1), \\ |Dv_n|^p \overset{*}{\rightharpoonup} \varpi, \quad F_n(\cdot, Dv_n) \overset{*}{\rightharpoonup} \nu \quad \text{in}\ \mathscr{M}(\omega_2), \end{aligned}$$

$$\begin{aligned} &\exists \lim_{n\to\infty} \int_{\omega_1} F_n(x, Du_n)\,dx \\ &\quad = \min\left\{ \liminf_{n\to\infty} \int_{\omega_1} F_n(x, Dw_n)\,dx : \ w_n - u_n \rightharpoonup 0 \ \text{in}\ W_0^{1,p}(\omega_1)^M \right\}, \end{aligned}$$

$$\begin{aligned} &\exists \lim_{n\to\infty} \int_{\omega_2} F_n(x, Dv_n)\,dx \\ &\quad = \min\left\{ \liminf_{n\to\infty} \int_{\omega_2} F_n(x, Dw_n)\,dx : \ w_n - v_n \rightharpoonup 0 \ \text{in}\ W_0^{1,p}(\omega_2)^M \right\}. \end{aligned}$$

Then, we have

$$|\mu - \nu| \le \begin{cases} C\big(h^L + \mu^L + \nu^L + \varrho^L + \varpi^L + (1 + (\mathrm{A}^L)^{\frac{1}{r}})|D(u-v)|^p\big)^{\frac{p-1}{p}} \cdot \\ \cdot\, (\mathrm{A}^L)^{\frac{1}{pr}} |D(u-v)| \quad \text{a.e. in}\ \omega_1 \cap \omega_2 & \text{if}\ 1 < p \le N-1, \\ C\big(1 + h^{\mathrm{A}} + \mu^{\mathrm{A}} + \nu^{\mathrm{A}} + \varrho^{\mathrm{A}} + \varpi^{\mathrm{A}} + |D(u-v)|^p\big)^{\frac{p-1}{p}} \cdot \\ \cdot\, \mathrm{A}\, |D(u-v)| \quad \mathrm{A}\text{-a.e. in}\ \omega_1 \cap \omega_2 & \text{if}\ p > N-1. \end{cases} \tag{2.31}$$

Lemma 2.6 is itself based on the following compactness result.

Lemma 2.11. *Let $F_n : \Omega\times\mathbb{R}^{M\times N} \to [0,\infty)$ be a sequence of Carathéodory functions satisfying (2.3) to (2.8), and let ω be an open subset of Ω. Consider a sequence $\{\xi_n\} \subset L^p(\omega)^{M\times N}$ such that*

$$F_n(\cdot,\xi_n) \overset{*}{\rightharpoonup} \Lambda \quad \text{and} \quad |\xi_n|^p \overset{*}{\rightharpoonup} \Xi \quad \text{in } \mathscr{M}(\omega). \tag{2.32}$$

- *If $1 < p \leq N-1$ and the sequence $\{\rho_n\}$ converges strongly to ρ in $L^{\frac{pr}{r-1}}(\omega)^{M\times N}$, then there exist a subsequence of n and a function $\vartheta \in L^1(\omega)$ such that*

$$F_n(\cdot,\xi_n+\rho_n) - F_n(\cdot,\xi_n) \rightharpoonup \vartheta \quad \text{weakly in } L^1(\omega), \tag{2.33}$$

where ϑ satisfies

$$|\vartheta| \leq C\big(h^L + \Lambda^L + \Xi^L + (1+(\mathrm{A}^L)^{\frac{1}{r}})|\rho|^p\big)^{\frac{p-1}{p}} (\mathrm{A}^L)^{\frac{1}{pr}}|\rho| \quad \text{a.e. in } \omega. \tag{2.34}$$

- *If $p > N-1$ and the sequence $\{\rho_n\}$ converges strongly to ρ in $C^0(\overline{\omega})^{M\times N}$, then there exist a subsequence of n and a function $\vartheta \in L^1_{\mathrm{A}}(\omega)$ such that*

$$F_n(\cdot,\xi_n+\rho_n) \overset{*}{\rightharpoonup} \Lambda + \vartheta\,\mathrm{A} \quad \text{in } \mathscr{M}(\omega),$$

where ϑ satisfies

$$|\vartheta| \leq C\big(1 + h^{\mathrm{A}} + \Lambda^{\mathrm{A}} + \Xi^{\mathrm{A}} + |\rho|^p\big)^{\frac{p-1}{p}}|\rho| \quad \mathrm{A}\text{-a.e. in } \omega. \tag{2.35}$$

2.2.3 Examples

In this section we give three examples of functionals $\mathscr{F}_n$ satisfying the assumptions (2.3) to (2.8) of Theorem 2.3.

1. The first example illuminates the Lipschitz estimate (2.7). It is also based on a functional coercivity of type (2.3) rather than a pointwise coercivity.

2. The second example deals with the Saint Venant-Kirchhoff hyper-elastic energy (see, *e.g.*, [18] Chapter 4).

3. The third example deals with an Ogden's type hyper-elastic energy (see, *e.g.*, [18] Chapter 4).

Let Ω be a bounded set of $\mathbb{R}^N$, $N \geq 2$. We denote for any function $u : \Omega \to \mathbb{R}^N$,

$$\begin{gathered} e(u) := \tfrac{1}{2}\left(Du + Du^T\right), \quad E(u) := \tfrac{1}{2}\left(Du + Du^T + Du^T Du\right), \\ C(u) := (I_N + Du)^T(I_N + Du). \end{gathered} \tag{2.36}$$

Example 1

Let $p \in (1,\infty)$, and let A_n be a sequence of symmetric tensor-valued functions in $L^\infty\big(\Omega;\mathscr{L}(\mathbb{R}^{N\times N}_s)\big)$. We consider the energy density function defined by

$$F_n(x,\xi) := \big|A_n(x)\xi^s : \xi^s\big|^{\frac{p}{2}} \quad \text{a.e. } x\in\Omega,\ \forall\,\xi\in\mathbb{R}^{N\times N}.$$

We assume that there exists $\alpha>0$ such that

$$A_n(x)\xi:\xi \geq \alpha\,|\xi|^2, \quad \text{a.e. } x\in\Omega,\ \forall\,\xi\in\mathbb{R}^{N\times N}_s, \tag{2.37}$$

and that

$$|A_n|^{\frac{p}{2}} \text{ is bounded in } L^r(\Omega) \text{ with } r \text{ defined by (2.6).} \tag{2.38}$$

Then, the density F_n and the associated functional

$$\mathscr{F}_n(u) := \int_\Omega \big|A_n e(u):e(u)\big|^{\frac{p}{2}} dx \quad \text{for } u\in W^{1,p}_0(\Omega)^N,$$

satisfy the conditions (2.3) to (2.8) of Theorem 2.3.

Proof. Using successively (2.37) and the Korn inequality in $W^{1,p}_0(\Omega)^N$ for $p>1$ (see, *e.g.*, [34]), we have for any $u\in W^{1,p}_0(\Omega)^N$,

$$\mathscr{F}_n(u) = \int_\Omega \big|A_n e(u):e(u)\big|^{\frac{p}{2}}\,dx \geq \alpha\int_\Omega |e(u)|^p\,dx \geq \alpha\,C\int_\Omega |Du|^p\,dx,$$

which implies (2.3). Conditions (3.14) and (2.8) are immediate. It remains to prove condition (2.7) with estimate (2.6). Taking into account that

$$\begin{aligned}|D_\xi F_n(x,\xi)| &= p\,\big|(A_n(x)\xi^s:\xi^s)^{\frac{p-2}{2}}A_n(x)\xi^s\big| \\ &\leq p\,\big|A_n(x)\xi^s:\xi^s\big|^{\frac{p-1}{2}}|A_n(x)|^{\frac12}, \qquad \forall\,\xi\in\mathbb{R}^{N\times N},\ \text{a.e. } x\in\Omega,\end{aligned}$$

then using the mean value theorem and Hölder's inequality, we get

$$\begin{aligned}\big|F_n(x,\xi)-F_n(x,\eta)\big| &\leq p\left((A_n\xi^s:\xi^s)^{\frac12}+(A_n\eta^s:\eta^s)^{\frac12}\right)^{p-1}|A_n|^{\frac12}|\xi^s-\eta^s| \\ &\leq p\,2^{\frac{(p-1)^2}{p}}\big(F_n(x,\xi)+F_n(x,\eta)\big)^{\frac{p-1}{p}}|A_n|^{\frac12}|\xi-\eta|,\end{aligned}$$

for every $\xi,\eta\in\mathbb{R}^{N\times N}$ and a.e. $x\in\Omega$. This implies estimate (2.7) with $h_n=0$ and $a_n=|A_n|^{\frac{p}{2}}$ bounded in $L^r(\Omega)$. □

The two next examples belong to the class of hyper-elastic materials (see, *e.g.*, [18], Chapter 4).

Example 2

For $N = 3$, we consider the Saint Venant-Kirchhoff energy density defined by

$$F_n(x, \xi) := \frac{\lambda_n(x)}{2} \left[\operatorname{tr}\big(\tilde{E}(\xi)\big)\right]^2 + \mu_n(x) \left|\tilde{E}(\xi)\right|^2, \quad \text{a.e. } x \in \Omega, \ \forall \xi \in \mathbb{R}^{3\times 3}, \tag{2.39}$$

where $\tilde{E}(\xi) := \frac{1}{2}\left(\xi + \xi^T + \xi^T \xi\right)$, and λ_n, μ_n are the Lamé coefficients.

We assume that there exists a constant $C > 1$ such that

$$\lambda_n, \mu_n \geq 0 \ \text{ a.e. in } \Omega, \quad \operatorname*{ess\text{-}inf}_{\Omega} (\lambda_n + \mu_n) > C^{-1}, \quad \int_\Omega (\lambda_n + \mu_n)\, dx \leq C. \tag{2.40}$$

Then, the density F_n and the associated functional (see definition (2.36))

$$\mathscr{F}_n(u) := \int_\Omega \left(\frac{\lambda_n}{2} \left[\operatorname{tr}\big(E(u)\big)\right]^2 + \mu_n \left|E(u)\right|^2 \right) dx \quad \text{for } u \in W_0^{1,4}(\Omega)^3, \tag{2.41}$$

satisfy the conditions (2.3) to (2.8) of Theorem 2.3.

Proof. There exists a constant $C > 1$ such that we have for a.e. $x \in \Omega$ and any $\xi \in \mathbb{R}^{3\times 3}$,

$$C^{-1}(\lambda_n + \mu_n)\,|\xi|^4 - C\,(\lambda_n + \mu_n) \leq F_n(x, \xi) \leq C\,(\lambda_n + \mu_n)\,|\xi|^4 + C\,(\lambda_n + \mu_n). \tag{2.42}$$

Hence, we deduce that for a.e. $x \in \Omega$ and any $\xi, \eta \in \mathbb{R}^{3\times 3}$,

$$\begin{aligned}
&\left|F_n(x, \xi) - F_n(x, \eta)\right| \\
&\leq C\,(\lambda_n + \mu_n)\left(1 + |\xi|^2 + |\eta|^2\right)^{\frac{3}{2}} |\xi - \eta| \\
&= C\left((\lambda_n + \mu_n)^{\frac{1}{2}} + (\lambda_n + \mu_n)^{\frac{1}{2}}|\xi|^2 + (\lambda_n + \mu_n)^{\frac{1}{2}}|\eta|^2\right)^{\frac{3}{2}} (\lambda_n + \mu_n)^{\frac{1}{4}}|\xi - \eta| \\
&\leq C\left((\lambda_n + \mu_n)^{\frac{1}{2}} + F_n(x, \xi)^{\frac{1}{2}} + F_n(x, \eta)^{\frac{1}{2}}\right)^{\frac{3}{2}} (\lambda_n + \mu_n)^{\frac{1}{4}}|\xi - \eta| \\
&\leq C\left(\lambda_n + \mu_n + F_n(x, \xi) + F_n(x, \eta)\right)^{\frac{3}{4}} (\lambda_n + \mu_n)^{\frac{1}{4}}\,|\xi - \eta|,
\end{aligned}$$

which implies estimate (2.7) with $p = 4$ and $h_n = a_n = \lambda_n + \mu_n$, while (2.5) and (2.6) are a straightforward consequence of (2.40). Moreover, by the first inequality of (2.42) combined with (2.40) we get that the functional (2.41) satisfies the coercivity condition (2.3). Condition (3.14) is immediate. Finally, since we have

$$\left[\operatorname{tr}\big(\tilde{E}(\lambda\xi)\big)\right]^2 + \left|\tilde{E}(\lambda\xi)\right|^2 \leq C\big(1 + |\xi|^4\big), \quad \forall \lambda \in [0, 1], \ \forall \xi \in \mathbb{R}^{3\times 3},$$

condition (2.8) follows from the first inequality of (2.42), which concludes the proof of the second example. □

Remark 2.12. *The default of the Saint Venant-Kirchhoff model is that the function $F_n(x, \cdot)$ of* (2.39) *is not* polyconvex *(see [31]). Hence, we do not know if it is* quasiconvex, *or equivalently, if the functional $\mathscr{F}_n$ of* (2.41) *is lower semi-continuous for the weak topology of $W^{1,4}(\Omega)^3$ (see,* e.g. *[20], Chapter 4, for the notions of polyconvexity and quasiconvexity).*

Example 3

For $N = 3$ and $p \in [2, \infty)$, we consider the Ogden's type energy density defined by

$$F_n(x,\xi) := a_n(x)\Big[\mathrm{tr}\big(\tilde{C}(\xi)^{\frac{p}{2}} - I_3\big)\Big]^+ \quad \text{a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^{3\times 3}, \tag{2.43}$$

where $\tilde{C}(\xi) := (I_3 + \xi)^T(I_3 + \xi)$, and $t^+ := \max(t, 0)$ for $t \in \mathbb{R}$. We assume that there exists a constant $C > 1$ such that

$$\operatorname*{ess\text{-}inf}_{\Omega} a_n > C^{-1} \quad \text{and} \quad \int_\Omega a_n^r\, dx \le C \quad \text{with} \begin{cases} r > 1, & \text{if } p = 2 \\ r = 1, & \text{if } p > 2. \end{cases} \tag{2.44}$$

Then, the density F_n and the associated functional (see definition (2.36))

$$\mathscr{F}_n(u) := \int_\Omega a_n(x)\Big[\mathrm{tr}\big(C(u)^{\frac{p}{2}} - I_3\big)\Big]^+ dx \quad \text{for } u \in W_0^{1,p}(\Omega)^3, \tag{2.45}$$

satisfy the conditions (2.3) to (2.8) of Theorem 2.3.

Proof. There exists a constant $C > 1$ such that we have for a.e. $x \in \Omega$ and any $\xi \in \mathbb{R}^{3\times 3}$,

$$C^{-1} a_n\,|\xi|^p - C\,a_n \le F_n(x,\xi) \le C\,a_n\,|\xi|^p + C\,a_n. \tag{2.46}$$

This combined with the fact that the (well-ordered) eigenvalues of a symmetric matrix are Lipschitz functions (see, *e.g.*, [19], Theorem 2.3-2), implies that for a.e. $x \in \Omega$ and any $\xi, \eta \in \mathbb{R}^N$, we have

$$\begin{aligned}\big|F_n(x,\xi) - F_n(x,\eta)\big| &\le C\,a_n(1 + |\xi| + |\eta|)^{p-1}|\xi - \eta| \\ &\le C\left(a_n + a_n|\xi|^p + a_n|\eta|^p\right)^{\frac{p-1}{p}} a_n^{\frac{1}{p}}\,|\xi - \eta| \\ &\le C\left(a_n + F_n(x,\xi) + F_n(x,\eta)\right)^{\frac{p-1}{p}} a_n^{\frac{1}{p}}\,|\xi - \eta|,\end{aligned}$$

which implies estimate (2.7) with $h_n = a_n$, while (2.5) and (2.6) are a straightforward consequence of (2.44). Moreover, by the first inequality of (2.46) combined with (2.44) we get that the functional (2.45) satisfies the coercivity condition (2.3). Condition (3.14) is immediate. Finally, since we have

$$\mathrm{tr}\big(\tilde{C}(\lambda\xi)^{\frac{p}{2}}\big) \le C\big(1 + |\xi|^p\big), \quad \forall \lambda \in [0,1],\ \forall \xi \in \mathbb{R}^{3\times 3},$$

condition (2.8) follows from the first inequality of (2.46), which concludes the proof of the third example. □

Remark 2.13. *Contrary to Example 2, the function $F_n(x, \cdot)$ of* (2.43) *is polyconvex since it is the composition of the Ogden density energy defined for a.e. $x \in \Omega$, by*

$$W_n(x,\xi) := a_n(x)\Big[\mathrm{tr}\big(\tilde{C}(\xi)^{\frac{p}{2}} - I_3\big)\Big]^+ \quad \textit{for } \xi \in \mathbb{R}^{3\times 3}, \tag{2.47}$$

which is known to be polyconvex (see [1]), by the non-decreasing convex function $t \mapsto t^+$. However, in contrast with (2.47) *the function* (2.43) *does attain its minimum at $\xi = 0$, namely in the absence of strain.*

2.3 Proof of the results

2.3.1 Proof of the main results

Proof of Theorem 2.3. The proof is divided into two steps. In the first step we construct the limit functional F and we prove the properties (2.13), (2.14), (2.15) satisfied by the function F. The second step is devoted to convergence (2.18).

First step: Construction of F.
Let $\mathscr{F}_n : W^{1,p}(\Omega)^M \to [0,\infty]$ be the functional defined by

$$\mathscr{F}_n(v) = \int_\Omega F_n(x, Dv)\,dx \quad \text{for } v \in W^{1,p}(\Omega)^M.$$

By the compactness Γ-convergence theorem (see *e.g.* [21], Theorem 8.5), there exists a subsequence of n, still denoted by n, such that $\mathscr{F}_n$ Γ-converges for the strong topology of $L^p(\Omega)^M$ to a functional $\mathscr{F} : W^{1,p}(\Omega)^M \to [0,\infty]$ with domain $\mathscr{D}(\mathscr{F})$.

Let ξ be a matrix of a countable dense subset D of $\mathbb{R}^{M\times N}$ with $0 \in D$. Since the linear function $x \mapsto \xi x$ belongs to $\mathscr{D}(\mathscr{F})$ by (2.10), up to the extraction of a new subsequence, for any $\xi \in D$ there exists a recovery sequence w_n^ξ in $W^{1,p}(\Omega)^M$ which converges strongly to ξx in $L^p(\Omega)^M$ and such that

$$F_n(\cdot, Dw_n^\xi) \overset{*}{\rightharpoonup} \mu^\xi \quad \text{and} \quad |Dw_n^\xi|^p \overset{*}{\rightharpoonup} \varrho^\xi \quad \text{in } \mathscr{M}(\Omega).$$

In particular, since $F_n(\cdot, 0) = 0$ we have $\mu^0 = 0$. Moreover, by estimates (2.23) and (2.31) we have for any $\xi, \eta \in D$,

$$\varrho^\xi \le \begin{cases} C\big(|\xi|^p + |\xi|^p(\mathrm{A}^L)^{\frac{1}{r}} + h + \mu^\xi + \mathrm{A}^L\big) & \text{a.e. in } \omega, \quad \text{if } 1 < p \le N-1 \\ C\big(|\xi|^p\mathrm{A} + h + \mu^\xi + \mathrm{A}\big) \quad \mathrm{A}\text{-a.e. in } \omega, & \text{if } p > N-1, \end{cases} \tag{2.48}$$

$$|\mu^\xi - \mu^\eta| \le$$
$$\begin{cases} C\big(h^L + (\mu^\xi)^L + (\mu^\eta)^L + (\varrho^\xi)^L + (\rho^\eta)^L + (1 + (\mathrm{A}^L)^{\frac{1}{r}})|\xi - \eta|^p\big)^{\frac{p-1}{p}} \cdot \\ \cdot\,(\mathrm{A}^L)^{\frac{1}{pr}}|\xi - \eta| \quad \text{a.e. in } \Omega & \text{if } 1 < p \le N-1, \\ C\left(1 + h^{\mathrm{A}} + (\mu^\xi)^{\mathrm{A}} + (\mu^\eta)^{\mathrm{A}} + (\varrho^\xi)^{\mathrm{A}} + (\varrho^\eta)^{\mathrm{A}} + |\xi - \eta)|^p\right)^{\frac{p-1}{p}} \cdot \\ \cdot\,\mathrm{A}\,|\xi - \eta| \quad \mathrm{A}\text{-a.e. in } \Omega & \text{if } p > N-1. \end{cases} \tag{2.49}$$

Hence, by a continuity argument we can define a function $F : \Omega \times \mathbb{R}^{M\times N} \to [0,\infty)$ satisfying (2.13), (2.15) and such that

$$\mu^\xi = \begin{cases} F(\cdot, \xi), & \text{if } 1 < p \le N-1 \\ F(\cdot, \xi)\,\mathrm{A}, & \text{if } p > N-1, \end{cases} \qquad \forall\, \xi \in D, \tag{2.50}$$

where the property (2.14) is deduced from (2.48), (2.49).

Second step: Proof of convergence (2.18).
Let ω be an open set of Ω, let $\{u_n\}$ be a sequence fulfilling (2.17), which converges weakly in $W^{1,p}(\omega)^M$ to a function u satisfying (2.16), and let $\xi \in D$. Since $F_n(\cdot, Du_n)$ is bounded in $L^1(\Omega)$, there exists a subsequence of n, still denoted by n, such that

$$F_n(\cdot, Du_n) \overset{*}{\rightharpoonup} \mu \quad \text{and} \quad |Du_n|^p \overset{*}{\rightharpoonup} \varrho \quad \text{in } \mathscr{M}(\Omega). \tag{2.51}$$

Applying Corollary 2.10 to the sequences u_n and $v_n = w_n^\xi$, we have

$$|\mu - \mu^\xi| \leq$$
$$\begin{cases} C\big(h^L + \mu^L + (\mu^\xi)^L + \varrho^L + (\varrho^\xi)^L + (1 + (\mathrm{A}^L)^{\frac{1}{r}})|Du - \xi|^p\big)^{\frac{p-1}{p}} \cdot & \\ \cdot\, (\mathrm{A}^L)^{\frac{1}{pr}}|Du - \xi| \quad \text{a.e. in } \omega & \text{if } 1 < p \leq N-1, \\ C\big(1 + h^{\mathrm{A}} + \mu^{\mathrm{A}} + (\mu^\xi)^{\mathrm{A}} + \varrho^{\mathrm{A}} + (\varrho^\xi)^{\mathrm{A}} + |Du - \xi|^p\big)^{\frac{p-1}{p}} \cdot & \\ \cdot\, \mathrm{A}\,|Du - \xi| \quad \mathrm{A}\text{-a.e. in } \omega & \text{if } p > N-1. \end{cases}$$

Using (2.48), (2.50) and the continuity of $F(x, \xi)$ with respect to ξ, we get that

$$\mu = \begin{cases} F(\cdot, Du), & \text{if } 1 < p \leq N-1 \\ F(\cdot, Du)\,\mathrm{A}, & \text{if } p > N-1. \end{cases} \tag{2.52}$$

Note that since the limit μ is completely determined by F, the first convergence of (2.51) holds for the whole sequence, which concludes the proof. □

Proof of Theorem 2.4. The proof is divided into two steps.

First step: The case where $V = \{\hat{u}\} + W_0^{1,p}(\omega)^M$.
Fix a function $\hat{u}$ satisfying (2.27), and define the set $V := \{\hat{u}\} + W_0^{1,p}(\omega)^M$. Let $u \in V$ such that

$$u \in \begin{cases} W^{1,\frac{pr}{r-1}}(\omega)^M, & \text{if } 1 < p \leq N-1 \\ C^1(\overline{\omega})^M, & \text{if } p > N-1. \end{cases}$$

which is extended by $\hat{u}$ in $\Omega \setminus \omega$, and consider a recovery sequence $\{u_n\}$ for $\mathscr{F}_n^V$ of limit u. There exists a subsequence of n, still denoted by n, such that the first convergences of (2.28) and (2.29) hold. By Theorem 2.3 convergences (2.18) are satisfied in ω, which implies (2.52). Now, applying the estimate (2.26) of Lemma 2.7 with u_n and $v_n = u$, it follows that

$$\mu \leq \nu \ \text{ in } \overline{\omega} \quad \text{with} \quad F_n(\cdot, Dv_n) \overset{*}{\rightharpoonup} \nu \ \text{ in } \mathscr{M}(\Omega),$$

where the convergence holds up to a subsequence. Then, using estimate (2.7) with $\eta = 0$ and Hölder's inequality, we have for any $\varphi \in L^\infty\big(\Omega; [0,1]\big)$ with compact support in Ω,

$$\int_\Omega \varphi\, F_n(x,Du)\,dx \le \begin{cases} \left(\int_\Omega \varphi\left(h_n + F_n(x,Du) + |Du|^p\right)dx\right)^{\frac{p-1}{p}} \cdot & \\ \cdot \left(\int_\Omega \varphi\, a_n^r\,dx\right)^{\frac{1}{pr}} \left(\varphi\,|Du|^{\frac{pr}{r-1}}dx\right)^{\frac{r-1}{pr}} & \text{if } 1 < p \le N-1, \\ \left(\int_\Omega \varphi\left(h_n + F_n(x,Du) + |Du|^p\right)dx\right)^{\frac{p-1}{p}} \cdot & \\ \cdot \left(\int_\Omega \varphi\, a_n\,dx\right)^{\frac{1}{p}} \|Du\|_{L^\infty(\Omega)^M} & \text{if } p > N-1, \end{cases}$$

which implies that ν is absolutely continuous with respect to the Lebesgue measure if $1 < p \le N–1$, and absolutely continuous with respect to measure A if $p > N–1$. Due to condition (2.21) in both cases the equality $\nu(\partial\omega) = 0$ holds, so does with μ. This combined with (2.18) and (2.52) yields

$$\lim_{n\to\infty} \int_\omega F_n(x,Du_n)\,dx = \begin{cases} \displaystyle\int_\omega F(x,Du)\,dx, & \text{if } 1 < p \le N-1 \\ \displaystyle\int_\omega F(x,Du)\,d\mathrm{A}, & \text{if } p > N-1, \end{cases}$$

which concludes the first step.

Second step: The general case.
Let V be a subset of $W^{1,p}(\omega)^M$ satisfying (2.19). Let u be a function such that

$$u \in \begin{cases} V \cap W^{1,\frac{pr}{r-1}}(\Omega)^M, & \text{if } 1 < p \le N-1 \\ V \cap C^1(\overline{\Omega})^M, & \text{if } p > N-1, \end{cases}$$

and define the set $\tilde{V} := \{u\} + W_0^{1,p}(\omega)^M$. Consider a recovery sequence $\{u_n\}$ for $\mathscr{F}_n^V$ given by (2.20) of limit u, and a recovery sequence $\{\tilde{u}_n\}$ for $\mathscr{F}_n^{\tilde{V}}$ of limit u. By virtue of Theorem 2.3 the convergences (2.18) hold for both sequences $\{u_n\}$ and $\{\tilde{u}_n\}$. Hence, since ω is an open set, and $F_n(x,Du_n)$ is non-negative, we have

$$\left.\begin{array}{ll} \text{if } 1 < p \le N-1, & \displaystyle\int_\omega F(x,Du)\,dx \\ \text{if } p > N-1, & \displaystyle\int_\omega F(x,Du)\,d\mathrm{A} \end{array}\right\} \le \liminf_{n\to\infty} \int_\omega F_n(x,Du_n)\,dx. \tag{2.53}$$

Moreover, since $\tilde{u}_n - u_n \rightharpoonup 0$ in $W_0^{1,p}(\omega)^M$, $\tilde{u}_n \in V$ by property (2.19) and because $\{u_n\}$ is a recovery sequence for $\mathscr{F}_n^V$, $\{\tilde{u}_n\}$ is an admissible sequence for the minimization problem (2.17), which implies that

$$\exists \lim_{n\to\infty} \int_\omega F_n(x,Du_n)\,dx \le \liminf_{n\to\infty} \int_\omega F_n(x,D\tilde{u}_n)\,dx. \tag{2.54}$$

On the other hand, by the first step applied with $\tilde{u} = u$ and the set $\tilde{V}$, we have

$$\lim_{n\to\infty} \int_\omega F_n(x, D\tilde{u}_n)\, dx = \begin{cases} \displaystyle\int_\omega F(x, Du)\, dx, & \text{if } 1 < p \le N-1 \\ \displaystyle\int_\omega F(x, Du)\, d\mathrm{A}, & \text{if } p > N-1. \end{cases} \tag{2.55}$$

Therefore, combining (2.53), (2.54), (2.55), for the sequence n obtained in Theorem 2.3, the sequence $\{\mathscr{F}_n^V\}$ Γ-converges to some functional $\mathscr{F}^V$ satisfying (2.22) with $v = u$, which concludes the proof of Theorem 2.4. □

2.3.2 Proof of the lemmas

Proof of Lemma 2.11. Assume that $1 < p \le N-1$. Using (2.9), we have

$$\begin{aligned} &F_n(x, \xi_n + \rho_n) \\ &\quad \le (p-1)h_n + (2p-1)F_n(x, \xi_n) + (p-1)\big(|\xi_n + \rho_n|^p + |\xi_n|^p\big) + a_n|\rho_n|^p \quad \text{a.e. in } \omega. \end{aligned}$$

From this we deduce that $\{F_n(\cdot, \xi_n + \rho_n)\}$ is bounded in $L^1(\omega)$. Moreover, by (2.7), we have

$$\begin{aligned} &\big|F_n(x, \xi_n + \rho_n) - F_n(x, \xi_n)\big| \\ &\qquad \le \big(h_n + F_n(x, \xi_n + \rho_n) + F_n(x, \xi_n) + |\xi_n + \rho_n|^p + |\xi_n|^p\big)^{\frac{p-1}{p}} a_n^{\frac{1}{p}} |\rho_n| \quad \text{a.e. in } \omega, \end{aligned}$$

where, thanks to the strong convergence of $\{\rho_n\}$ in $L^{\frac{pr}{r-1}}(\omega)^{M\times N}$, we can show that the right-hand side is bounded in $L^1(\omega)$ and equi-integrable. Indeed, taking into account

$$\frac{p-1}{p} + \frac{1}{pr} + \frac{r-1}{pr} = 1,$$

we have the boundedness in $L^1(\omega)$ of the right-hand side, while the strong convergence of $\{\rho_n\}$ in $L^{\frac{pr}{r-1}}(\omega)^{M\times N}$ implies that $\{|\rho_n|^{\frac{pr}{r-1}}\}$ is equi-integrable and therefore, the equi-integrability of the right-hand side. By the Dunford-Pettis theorem, extracting a subsequence if necessary, we conclude (2.33), which, together with (2.32), in particular implies

$$F_n(\cdot, \xi_n + \rho_n) \overset{*}{\rightharpoonup} \Lambda + \vartheta \quad \text{in } \mathscr{M}(\omega).$$

Moreover, for any ball $B \subset \omega$, we have

$$\begin{aligned} &\int_B \big|F_n(x, \xi_n + \rho_n) - F_n(x, \xi_n)\big|\, dx \\ &\le \int_B \big(h_n + F_n(x, \xi_n + \rho_n) + F_n(x, \xi_n) + |\xi_n + \rho_n|^p + |\xi_n|^p\big)^{\frac{p-1}{p}} a_n^{\frac{1}{p}} |\rho_n|\, dx \\ &\le \left(\int_B \big(h_n + F_n(x, \xi_n + \rho_n) + F_n(x, \xi_n) + C|\xi_n|^p + C|\rho_n|^p\big)\, dx\right)^{\frac{p-1}{p}} \cdot \\ &\quad \cdot \left(\int_B a_n^r\, dx\right)^{\frac{1}{pr}} \left(\int_B |\rho_n|^{\frac{pr}{r-1}}\, dx\right)^{\frac{r-1}{pr}}, \end{aligned}$$

which, passing to the limit, implies

$$\int_B |\vartheta| dx \le \left((h + 2\Lambda + \vartheta + C\Xi)(\overline{B}) + C \int_B |\rho|^p dx \right)^{\frac{p-1}{p}} \mathrm{A}(\overline{B})^{\frac{1}{pr}} \left(\int_B |\rho|^{\frac{pr}{r-1}} dx \right)^{\frac{r-1}{pr}},$$

and then, dividing by $|B|$, the measures differentiation theorem shows that

$$|\vartheta| \le \left(h^L + 2\Lambda^L + \vartheta + C\Xi + C|\rho|^p\right)^{\frac{p-1}{p}} (\mathrm{A}^L)^{\frac{1}{pr}} |\rho| \quad \text{a.e. in } \omega. \tag{2.56}$$

Using Young's inequality in (2.56)

$$|\vartheta| \le \frac{p-1}{p}\left(h^L + 2\Lambda^L + \vartheta + C\Xi^L + C|\rho|^p\right) + \frac{1}{p}(\mathrm{A}^L)^{\frac{1}{r}} |\rho|^p \quad \text{a.e. in } \omega,$$

and then

$$|\vartheta| \le C\left(h^L + \Lambda^L + \Xi^L + (1 + (\mathrm{A}^L)^{\frac{1}{r}})|\rho|^p\right) \quad \text{a.e. in } \omega,$$

which substituted in (2.56) shows (2.34).

Assume now that $p > N-1$. Again, using (2.9) we deduce that $\{F_n(\cdot, \xi_n + \rho_n)\}$ is bounded in $L^1(\omega)$, and thanks to (2.7) we get

$$\begin{aligned} &\left|F_n(x, \xi_n + \rho_n) - F_n(x, \xi_n)\right| \\ &\quad \le \left(h_n + F_n(x, \xi_n + \rho_n) + F_n(x, \xi_n) + |\xi_n + \rho_n|^p + |\xi_n|^p\right)^{\frac{p-1}{p}} a_n^{\frac{1}{p}} |\rho_n| \quad \text{a.e. in } \omega. \end{aligned}$$

Consequently, the sequence $\{F_n(\cdot, \xi_n + \rho_n) - F_n(\cdot, \xi_n)\}$ is bounded in $L^1(\omega)$. Extracting a subsequence if necessary, the sequence $\{F_n(\cdot, \xi_n + \rho_n) - F_n(\cdot, \xi_n)\}$ weakly-$*$ converges in $\mathscr{M}(\omega)$ to a measure Θ, which, together with (2.32), implies

$$F_n(\cdot, \xi_n + \rho_n) \stackrel{*}{\rightharpoonup} \Lambda + \Theta \quad \text{in } \mathscr{M}(\omega).$$

Furthermore, if E is a measurable subset of ω, then, using Hölder's inequality, we have

$$\begin{aligned} &\int_E \left|F_n(x, \xi_n + \rho_n) - F_n(x, \xi_n)\right| dx \\ &\le \int_E \left(h_n + F_n(x, \xi_n + \rho_n) + F_n(x, \xi_n) + |\xi_n + \rho_n|^p + |\xi_n|^p\right)^{\frac{p-1}{p}} a_n^{\frac{1}{p}} |\rho_n| \, dx \\ &\le \left(C\|\rho_n\|^p_{L^\infty(\omega)^{M\times N}} + \int_E \left(h_n + F_n(x, \xi_n + \rho_n) + F_n(x, \xi_n) + C|\xi_n|^p\right) dx \right)^{\frac{p-1}{p}} \cdot \\ &\quad \cdot \left(\int_E a_n dx \right)^{\frac{1}{p}} \|\rho_n\|_{L^\infty(\omega)^{M\times N}}, \end{aligned}$$

which, passing to the limit, shows that Θ is absolutely continuous with respect to A. By the Radon-Nikodym theorem, there exists $\vartheta \in L^1_{\mathrm{A}}(\omega)$ such that

$$\Theta = \vartheta \mathrm{A} \quad \text{in } \mathscr{M}(\omega).$$

From the previous expression and using the measures differentiation theorem, we get (2.35). □

Proof of Lemma 2.6. Let $x_0 \in \omega$ and two numbers $0 < R_1 < R_2$ with $B(x_0, R_2) \subset \omega$. Lemma 2.6 in [12] gives the existence of a sequence of closed sets

$$U_n \subset [R_1, R_2], \text{ with } |U_n| \geq \frac{1}{2}(R_2 - R_1),$$

such that defining

$$\bar{u}_n(r, z) = u_n(x_0 + rz), \quad \bar{u}(r, z) = u(x_0 + rz), \quad r \in (0, R_2),\ z \in S_{N-1},$$

we have

$$\|\bar{u}_n - \bar{u}\|_{C^0(U_n;X)} \to 0, \tag{2.57}$$

where X is the space defined by

$$X := \begin{cases} L^s(S_{N-1})^M, \text{ with } 1 \leq s < \frac{(N-1)p}{N-1-p}, & \text{if } 1 < p < N-1, \\ L^s(S_{N-1})^M, \text{ with } 1 \leq s < \infty, & \text{if } p = N-1, \\ C^0(S_{N-1})^M, & \text{if } p > N-1. \end{cases}$$

For the rest of the prove we assume $1 < p \leq N-1$ because the case $p > N-1$ is quite similar.

We define $\bar{\varphi}_n \in W^{1,\infty}(0, \infty)$ by

$$\bar{\varphi}_n(r) = \begin{cases} 1, & \text{if } 0 < r < R_1, \\ \dfrac{1}{|U_n|} \displaystyle\int_r^{R_2} \chi_{U_n} ds, & \text{if } R_1 < r < R_2, \\ 0, & \text{if } R_2 < r, \end{cases} \tag{2.58}$$

and

$$\varphi_n(x) = \bar{\varphi}_n(|x - x_0|).$$

Applying the coercivity inequality (2.3) to the sequence $\varphi_n(u_n - u)$ and using $F_n(\cdot, 0) = 0$, $\varphi_n = 1$ in $B(x_0, R_1)$, we get

$$\begin{aligned} \alpha \int_{B(x_0,R_1)} |Du_n - Du|^p \, dx &\leq \alpha \int_{B(x_0,R_2)} \big|D\big(\varphi_n(u_n - u)\big)\big|^p \, dx \\ &\leq \int_{B(x_0,R_2)} F_n\big(x, D(\varphi(u_n - u))\big) \, dx \\ &= \int_{B(x_0,R_2)} F_n\big(x, \varphi_n Du_n - \varphi_n Du + (u_n - u) \otimes \nabla\varphi_n\big) \, dx. \end{aligned}$$

By the convergence (2.33) with $\xi_n := \varphi_n Du_n$, $\rho_n := -\varphi_n Du + (u_n - u) \otimes \nabla\varphi_n$, and by estimate (2.8) we obtain up to a subsequence

$$\begin{aligned}
&\lim_{n\to\infty} \int_{B(x_0,R_2)} F_n\big(x, \varphi_n Du_n - \varphi_n Du + (u_n - u) \otimes \nabla\varphi_n\big)\, dx \\
&\leq \lim_{n\to\infty} \int_{B(x_0,R_2)} F_n\big(x, \varphi_n Du_n\big)\, dx + \int_{B(x_0,R_2)} \vartheta\, dx \\
&\leq C(h+\mu)\big(\overline{B}(x_0, R_2)\big) + \int_{B(x_0,R_2)} \vartheta\, dx,
\end{aligned}$$

with

$$|\vartheta| \leq C\big(h^L + \mu^L + \varrho^L + (1 + (\mathrm{A}^L)^{\frac{1}{r}})|Du|^p\big)^{\frac{p-1}{p}} (\mathrm{A}^L)^{\frac{1}{pr}} |Du| \text{ a.e. in } \omega.$$

Indeed, thanks to (2.57) the sequence $(u_n - u) \otimes \nabla\varphi_n$ converges strongly to 0 in $L^{\frac{pr}{r-1}}(\omega)^{M\times N}$ taking into account the inequality

$$\frac{(N-1)p}{N-1-p} \geq \frac{pr}{r-1}.$$

Hence, we deduce from the previous estimates that

$$\begin{aligned}
\varrho\big(B(x_0, R_1)\big) \leq C(h+\mu)\big(\overline{B}(x_0, R_2)\big) + C\int_{B(x_0,R_1)} |Du|^p\, dx \\
+ C\int_{B(x_0,R_2)} \Big(\big(h^L + \mu^L + \varrho^L + (1 + (\mathrm{A}^L)^{\frac{1}{r}})|Du|^p\big)^{\frac{p-1}{p}} (\mathrm{A}^L)^{\frac{1}{pr}} |Du|\Big)\, dx.
\end{aligned}$$

Taking R_2 such that

$$(h+\mu)\big(\{|x - x_0| = R_2\}\big) = 0,$$

which holds true except for a countable set $E_{x_0} \subset \big(0, \mathrm{dist}(x_0, \partial\omega)\big)$, and making R_1 tend to R_2, we get that

$$\begin{aligned}
\varrho\big(B(x_0, R_2)\big) \leq C(h+\mu)\big(B(x_0, R_2)\big) + C\int_{B(x_0,R_2)} |Du|^p\, dx \\
+ C\int_{B(x_0,R_2)} \Big(\big(h^L + \mu^L + \varrho^L + (1 + (\mathrm{A}^L)^{\frac{1}{r}})|Du|^p\big)^{\frac{p-1}{p}} (\mathrm{A}^L)^{\frac{1}{pr}} |Du|\Big)\, dx,
\end{aligned}$$

for any $R_2 \in \big(0, \mathrm{dist}(x_0, \partial\omega)\big) \setminus E_{x_0}$. Then, by the measures differentiation theorem it follows that

$$\varrho \leq C\left(|Du|^p + h + \mu\right) + C\left(\big(h^L + \mu^L + \varrho^L + (1 + (\mathrm{A}^L)^{\frac{1}{r}})|Du|^p\big)^{\frac{p-1}{p}}\right) (\mathrm{A}^L)^{\frac{1}{pr}} |Du|.$$

Finally, the Young inequality yields the desired estimate (2.23).

Now consider $\{u_n\}$ and $\{v_n\}$ as in the statement of the lemma. Let $x_0 \in \omega$ and $0 < R_0 < R_1 < R_2$ with $B(x_0, R_2) \subset \omega$. Again using Lemma 2.6 in [12] there exist two sequences of closed sets

$$V_n \subset [R_0, R_1], \quad U_n \subset [R_1, R_2],$$

with

$$|V_n| \geq \frac{1}{2}(R_1 - R_0), \quad |U_n| \geq \frac{1}{2}(R_2 - R_1),$$

such that defining

$$\begin{array}{lll} \bar{u}_n(r,z) = u_n(x_0 + rz), & \bar{v}_n(r,z) = v_n(x_0 + rz), & r \in (0, R_2),\ z \in S_{N-1}, \\ \bar{u}(r,z) = u(x_0 + rz), & \bar{v}(r,z) = v(x_0 + rz), & r \in (0, R_2),\ z \in S_{N-1}, \end{array}$$

we have

$$\|\bar{u}_n - \bar{u}\|_{C^0(U_n;X)} \to 0, \quad \|\bar{v}_n - \bar{v}\|_{C^0(V_n;X)} \to 0.$$

Then, consider the function $\bar{\varphi}_n$ defined by (2.58) and the function $\bar{\psi}_n \in W^{1,\infty}(0,\infty)$ defined by

$$\bar{\psi}_n(r) = \begin{cases} 1, & \text{if } 0 < r < R_0, \\ \dfrac{1}{|V_n|} \displaystyle\int_r^{R_1} \chi_{V_n} ds, & \text{if } R_0 < r < R_1, \\ 0, & \text{if } R_1 < r. \end{cases}$$

From these sequences we define $w_n \in W^{1,p}(\omega)^M$ by

$$w_n = \psi_n(v_n - v + u) + \varphi_n(1 - \psi_n)u + (1 - \varphi_n)u_n,$$

with

$$\varphi_n(x) = \bar{\varphi}_n(|x - x_0|), \quad \psi_n(x) = \bar{\psi}_n(|x - x_0|),$$

i.e.

$$w_n = \begin{cases} v_n - v + u, & \text{if } |x - x_0| < R_0, \\ \psi_n(v_n - v) + u, & \text{if } R_0 < |x - x_0| < R_1, \\ \varphi_n u + (1 - \varphi_n)u_n, & \text{if } R_1 < |x - x_0| < R_2, \\ u_n, & \text{if } R_2 < |x - x_0|, x \in \omega. \end{cases} \tag{2.59}$$

It is clear that, for a subsequence, w_n converges a.e. to u. Using then that $w_n - u_n$ is in $W_0^{1,p}(\omega)^M$ and that, thanks to φ_n, ψ_n bounded in $W^{1,\infty}(\Omega)$, w_n is bounded in $W^{1,p}(\omega)^M$, we get

$$w_n - u_n \rightharpoonup 0 \quad \text{weakly in } W_0^{1,p}(\omega).$$

Thus, from (2.24) we deduce

$$\begin{aligned} &\int_\omega F_n(x, Du_n)\, dx \\ &\leq \int_\omega F_n(x, Dw_n)\, dx + O_n \\ &= \int_{B(x_0,R_0)} F_n\big(x, D(v_n - v + u)\big)\, dx + \int_{\{R_2 < |x-x_0|\} \cap \omega} F_n(x, Du_n)\, dx \\ &\quad + \int_{\{R_0 < |x-x_0| < R_1\}} F_n\big(x, \psi_n D(v_n - v) + Du + (v_n - v) \otimes \nabla\psi_n\big)\, dx \\ &\quad + \int_{\{R_1 < |x-x_0| < R_2\}} F_n\big(x, \varphi_n Du + (1 - \varphi_n)Du_n + (u - u_n) \otimes \nabla\varphi_n\big)\, dx + O_n, \end{aligned}$$

what implies, in particular

$$\begin{aligned}&\int_{B(x_0,R_2)} F_n(x, Du_n)\,dx\\ &\le \int_{B(x_0,R_0)} F_n\big(x, D(v_n - v + u)\big)\,dx\\ &\quad + \int_{\{R_0<|x-x_0|<R_1\}} F_n\big(x, \psi_n D(v_n - v) + Du + (v_n - v)\otimes\nabla\psi_n\big)\,dx\\ &\quad + \int_{\{R_1<|x-x_0|<R_2\}} F_n\big(x, \varphi_n Du + (1-\varphi_n)Du_n + (u-u_n)\otimes\nabla\varphi_n\big)\,dx + O_n.\end{aligned} \tag{2.60}$$

To estimate the first term on the right-hand side of this inequality, we use Lemma 2.11 with $\xi_n = Dv_n$, $\rho_n = D(-v+u)$, which take into account (2.25), gives

$$\begin{aligned}&\int_{B(x_0,R_0)} F_n\big(x, D(v_n - v + u)\big)\,dx\\ &\le \nu\big(\overline{B}(x_0,R_0)\big)\\ &\quad + C\int_{B(x_0,R_0)} \big(h^L + \nu^L + \varpi^L + (1+(\mathrm{A}^L)^{\frac{1}{r}})|D(u-v)|^p\big)^{\frac{p-1}{p}} (\mathrm{A}^L)^{\frac{1}{pr}}|D(u-v)|\,dx + O_n.\end{aligned} \tag{2.61}$$

For the second term, we use again Lemma 2.11 with $\xi_n = \psi_n Dv_n$ and $\rho_n = -\psi_n Dv + Du + (v_n - v)\otimes\nabla\psi_n$. Therefore, up to subsequence it holds

$$\begin{aligned}&\int_{\{R_0<|x-x_0|<R_1\}} F_n\big(x, \psi_n D(v_n - v) + Du + (v_n - v)\otimes\nabla\psi_n\big)\,dx\\ &\le C(h+\nu+\varpi)\big(\{R_0\le|x-x_0|\le R_1\}\big)\\ &\quad + C\int_{\{R_0<|x-x_0|<R_1\}} \big(h^L + \nu^L + \varpi^L + (1+(\mathrm{A}^L)^{\frac{1}{r}})(|Dv|^p + |Du|^p)\big)^{\frac{p-1}{p}}\cdot\\ &\quad \cdot(\mathrm{A}^L)^{\frac{1}{pr}}(|Du| + |Dv|)\,dx + O_n.\end{aligned} \tag{2.62}$$

The third term is analogously estimated by Lemma 2.11 with $\xi_n = (1-\varphi_n)Du_n$ and $\rho_n = \varphi_n Du + (u-u_n)\otimes\nabla\varphi_n$. Extracting a subsequence if necessary, it yields

$$\begin{aligned}&\int_{\{R_1<|x-x_0|<R_2\}} F_n\big(x, \varphi_n Du + (1-\varphi_n)Du_n + (u-u_n)\otimes\nabla\varphi_n\big)\,dx\\ &\le C(h+\mu+\varrho)\big(\{R_1\le|x-x_0|\le R_2\}\big)\\ &\quad + C\int_{\{R_1<|x-x_0|<R_2\}} \big(h^L + \mu^L + \varrho^L + (1+(\mathrm{A}^L)^{\frac{1}{r}})|Du|^p\big)^{\frac{p-1}{p}}(\mathrm{A}^L)^{\frac{1}{pr}}|Du|dx + O_n.\end{aligned} \tag{2.63}$$

From (3.52), (2.61), (2.62) and (2.63) we deduce that

$$\begin{aligned}
&\mu\big(B(x_0,R_2)\big)\\
&\leq \nu\big(\overline{B}(x_0,R_0)\big)\\
&\quad + C\int_{B(x_0,R_0)} \big(h^L+\nu^L+\varpi+(1+(\mathrm{A}^L)^{\frac{1}{r}})|D(u-v)|^p\big)^{\frac{p-1}{p}}(\mathrm{A}^L)^{\frac{1}{pr}}|D(u-v)|dx\\
&\quad + C(h+\nu+\varpi)\big(\{R_0\leq |x-x_0|\leq R_1\}\big)\\
&\quad + C\int_{\{R_0<|x-x_0|<R_1\}} \big(h^L+\nu^L+\varpi^L+(1+(\mathrm{A}^L)^{\frac{1}{r}})(|Dv|^p+|Du|^p)\big)^{\frac{p-1}{p}}\cdot\\
&\quad \cdot(\mathrm{A}^L)^{\frac{1}{pr}}(|Du|+|Dv|)\,dx\\
&\quad + C(h+\mu+\varrho)\big(\{R_1\leq |x-x_0|\leq R_2\}\big)\\
&\quad + C\int_{\{R_1<|x-x_0|<R_2\}} \big(h^L+\mu^L+\varrho^L+(1+(\mathrm{A}^L)^{\frac{1}{r}})|Du|^p\big)^{\frac{p-1}{p}}(\mathrm{A}^L)^{\frac{1}{pr}}|Du|dx.
\end{aligned} \tag{2.64}$$

Taking R_0 such that

$$(h+\nu+\varpi+\mu+\varrho)\big(\{|x-x_0|=R_0\}\big)=0,$$

which holds true except for a countable set $E_{x_0}\subset\big(0,\operatorname{dist}(x_0,\partial\omega)\big)$, and making R_1,R_2 tend to R_0, from (2.64) we deduce that

$$\begin{aligned}
&\mu\big(B(x_0,R_0)\big)\\
&\leq \nu\big(B(x_0,R_0)\big)\\
&\quad + C\int_{B(x_0,R_0)} \big(h^L+\nu^L+\varpi^L+(1+(\mathrm{A}^L)^{\frac{1}{r}})|D(u-v)|^p\big)^{\frac{p-1}{p}}(\mathrm{A}^L)^{\frac{1}{pr}}|D(u-v)|dx,
\end{aligned}$$

for any $R_0\in\big(0,\operatorname{dist}(x_0,\partial\omega)\big)\setminus E_{x_0}$ (observe that the right term in the integral is well defined as an element of $L^1(\omega)$). Therefore, the measures differentiation theorem shows (2.26).

□

Proof of Lemma 2.7. The proof is the same as the proof of Lemma 2.6 choosing any point x_0 in Ω rather than ω, extending the functions u_n,v_n by u in $\Omega\setminus\omega$, and then noting that the function w_n defined by (2.59) in Ω is also equal to u in $\Omega\setminus\omega$. □

Proof of Corollary 2.10. Assume that $1<p\leq N-1$. Applying Lemma 2.6 with $\omega=\omega_1$ (see also Remark 2.8 about the subsets of ω) we obtain

$$\mu\leq\nu+C\big(h^L+\nu^L+\varpi^L+(1+(\mathrm{A}^L)^{\frac{1}{r}})|D(u-v)^p|\big)^{\frac{p-1}{p}}(\mathrm{A}^L)^{\frac{1}{pr}}|D(u-v)|\quad\text{in }\omega_1\cap\omega_2.$$

Analogously with $\omega = \omega_2$, we get

$$\nu \leq \mu + C\big(h^L + \mu^L + \varrho^L + (1 + (\mathrm{A}^L)^{\frac{1}{r}})|D(u-v)^p|\big)^{\frac{p-1}{p}} (\mathrm{A}^L)^{\frac{1}{pr}} |D(u-v)| \quad \text{in } \omega_1 \cap \omega_2.$$

These two expressions prove the first estimate of (2.31). The proof of the second estimate is similar. □

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Chapter 3

Asymptotic behavior of the linear elasticity system with varying and unbounded coefficients in a thin beam

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Abstract.

We study the asymptotic behavior of the solutions of the linear elasticity system in a thin beam of thickness $\varepsilon > 0$, when ε tends to zero. The elasticity tensor also varies with ε, and it is assumed to be uniformly elliptic but non-uniformly bounded. Namely, we just impose that its norm in L^∞ is an infinitesimal of $1/\varepsilon$ and its norm in L^1 is bounded. We obtain an homogenized problem corresponding to a linear system in one dimension. It gives an approximation of the solution of the problem in the thin beam which consists in the sum of a Bernouilli-Navier's deformation plus a torsion term. This limit system provides a general asymptotic model for the behavior of an elastic beam composed by the mixture, at a mesoscopic level, of several materials, and therefore, which does not satisfy any homogeneity and/or isotropy conditions.

3.1 Introduction

Obtaining an asymptotic model for the behavior of an elastic beam of thickness $\varepsilon > 0$ is a very classical problem due to its huge interest in engineering. The idea is to approximate the deformation by the solution of a differential system in dimension one, which is much simpler to deal with from a numerical point of view. Such an ordinary differential system is usually composed by two uncoupled linear equations of fourth order, which describe the asymptotic behavior of the deformations in the orthogonal directions to the axis of the beam. From a mathematical point of view this system can be obtained by passing to the limit when ε tends to zero in the elasticity system (see e.g. [19], [29])

$$\begin{cases} -\operatorname{div}\big(\lambda\, \mathrm{trace}(e(u_\varepsilon))I + 2\mu\, e(u_\varepsilon)\big) = f_\varepsilon \ \text{ in } (0,1)\times(\varepsilon\omega), \\ \big(\lambda\, \mathrm{trace}(e(u_\varepsilon))I + 2\mu\, e(u_\varepsilon)\big)\nu = 0 \ \text{ on } (0,1)\times(\varepsilon\partial\omega), \end{cases} \tag{3.1}$$

where ω is a smooth, connected, bounded domain in $\mathbb{R}^{N-1}$ (usually $N = 2,3$), ν is the unitary outward normal vector to ω on $\partial\omega$, $\lambda, \mu > 0$ are the Lamé constans, u_ε is the deformation of the beam, $e(u_\varepsilon) := (Du + Du^T)/2$ is the strain tensor and f_ε is the exterior force which is usually supposed of the form

$$f_{\varepsilon,1}(x) = f_1(x_1), \quad f_{\varepsilon,j}(x) = \varepsilon f_j(x_1), \quad j \in \{2, \cdots, N\}. \tag{3.2}$$

By also adding certain boundaries conditions on the extremities of the beam (depending for example on whether the corresponding base is fixed or not) the classical model provides the following approximation for the deformation on the orthogonal directions to the axis of the beam:

$$u_{\varepsilon,j}(x) \sim \frac{1}{\varepsilon} u_j(x_1), \quad j \in \{2, \cdots, N\}, \tag{3.3}$$

with u_j solution to the ordinary differential equation

$$\frac{2\mu(\lambda N + 2\mu)}{\lambda(N-1) + 2\mu} I_j \frac{d^4 u_j}{dx_1^4} = f_j \ \text{ in } (0,1), \tag{3.4}$$

where I_j is the inertial momentum of ω in the j-th direction divided by $|\omega|$ (it is assumed that the center of mass of ω is zero and that the axes are inertial). These equations are usually known as the beam equations. It is also possible to get the following approximation for the deformation in the direction x_1,

$$u_{\varepsilon,1}(x) \sim u_1(x_1) - \sum_{j=2}^{N} \frac{du_j}{dx_1} \frac{x_j}{\varepsilon}, \tag{3.5}$$

with u_1 solution to

$$-\frac{2\mu(\lambda N + 2\mu)}{\lambda(N-1) + 2\mu} \frac{d^2 u_1}{dx_1^2} = f_1 \ \text{ in } (0,1). \tag{3.6}$$

We see that, with assumption (3.2), the deformation is of order one in the direction x_1, whereas it is of order $1/\varepsilon$ in the other ones. For this reason just equations for u_j, $2 \leq j \leq N$, are taken as the beam equations. A deformation of the type

$$\left(u_1 - \sum_{j=2}^{N} \frac{du_j}{dx_1} \frac{x_j}{\varepsilon}, \frac{1}{\varepsilon} u_2, \cdots, \frac{1}{\varepsilon} u_N \right),$$

is usually known as a Bernouill-Navier's deformation.

More generally, in reference [23], it has been considered the case where the elasticity tensor does not satisfy any homogeneity and/or isotropy conditions. Namely, the authors replace in (3.1) the tensor $\xi \in \mathbb{R}_s^{N\times N} \mapsto \lambda\, \mathrm{trace}(\xi)\, I + 2\mu\xi \in \mathbb{R}_s^{N\times N}$ ($\mathbb{R}_s^{N\times N}$ the space of symmetric matrices of dimension $N \times N$) by a general tensor function $\xi \to A(x_1, x'/\varepsilon)\xi$ with $A \in L^\infty(\Omega; \mathcal{L}(\mathbb{R}_s^{N\times N}))$ uniformly elliptic. A more general right-hand side is also considered. In this case, it is obtained an approximation of u_ε more intricate than (3.3), (3.5), which is given by

$$\begin{cases} u_{\varepsilon,1}(x) \sim u_1(x_1) - \displaystyle\sum_{j=2}^{N} \frac{du_j}{dx_1}(x_1) \frac{x_j}{\varepsilon} + \varepsilon z_1 \left(x_1, \frac{x'}{\varepsilon} \right), \\ u_{\varepsilon,j}(x) \sim \dfrac{1}{\varepsilon} u_j(x_1) + \displaystyle\sum_{i=2}^{N} Z_{ji}(x_1) \frac{x_i}{\varepsilon} + \varepsilon z_j \left(x_1, \frac{x'}{\varepsilon} \right), \quad j \in \{2, \cdots, N\}, \end{cases} \tag{3.7}$$

where we are denoting $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1}$, and where the matrix function $Z = (Z_{ji})$ is skew-symmetric. The functions on the right-hand side are solutions of a system in $(0,1) \times \omega$, i.e. in the macroscopic variable $y_1 = x_1$ and in the microscopic variables $y_j = x_j/\varepsilon$. From this problem, one can obtain a one-dimensional linear system for the functions u and Z. Contrary to (3.4), (3.6), the system is no longer uncoupled in the different variables. The deformation $(0, Z(x_1)\frac{x'}{\varepsilon})$ is known as the torsion term and corresponds to a (linearized) rotation around the axis of the beam. It does not appear in the classical case when only isotropic materials are considered. In [23] only the case $N = 3$ is considered. The general expression (3.7) can be obtained from the results in [10].

In the present paper we are interested in obtaining an approximation of the solutions of the linear elasticity system in a beam of thickness ε, when the tensor coefficient also depend on ε. Namely, we consider the problem

$$\begin{cases} -\mathrm{div}(A_\varepsilon e(u_\varepsilon)) = f_\varepsilon \ \text{ in } (0,1) \times (\varepsilon\omega), \\ A_\varepsilon e(u_\varepsilon)\nu = 0 \ \text{ on } (0,1) \times (\varepsilon\partial\omega), \end{cases} \tag{3.8}$$

where A_ε is a sequence in $L^1((0,1) \times (\varepsilon\omega); \mathcal{L}(\mathbb{R}_s^{N\times N}))$ and where, as in (3.1), it would be necessary to add some boundary conditions on $\{0,1\} \times (\varepsilon\omega)$ in order to have uniqueness of solution.

The study of the asymptotic behavior of elliptic problems in a thin domain where the coefficients also vary has been considered by other authors. We refer for example to [2], [9], [26] for the case where the coefficients vary periodically with a small period.

When no periodicity is assumed we give the following references. For the case of a linear diffusion equation in a plate in dimension 3, (the case of a beam would be very similar) the problem has been considered in [15] by assuming the sequence of coefficients matrices uniformly elliptic and bounded. The authors show that the solutions can be approximated by those of a partial differential equation in dimension 2. Some expressions for this limit equation have been obtained in [17] under special assumptions on the coefficients. An extension to non-linear diffusion equations has been obtained in [13]. The case of a nonlinear monotone equation in a beam $(0,1)\times(\varepsilon\omega)$, where the coefficients also depend on ε, has been considered in [12], assuming the coefficients uniformly elliptic and bounded. In [12]a right-hand side of the form $f_\varepsilon(x)=f(x_1,x'/\varepsilon)+\operatorname{div} G(x_1,x'/\varepsilon)$ is considered, and due to the presence of the function G, the limit problem is no more a one-dimensional problem.

For the linear elasticity system, the problem has been considered in [14] for a plate $\omega\times(-\varepsilon,\varepsilon)$ in dimension 3 (here ω is a smooth, connected, bounded domain in $\mathbb{R}^2$), assuming certain isotropy conditions of the coefficients and also that they are uniformly elliptic and bounded. Thanks to these isotropy conditions, the authors show that the deformation of the plate along the directions of the plane $x_3=0$ can be approximated by the solution of a fourth order equation in dimension two. This is similar to the case of an isotropic beam described at the beginning of the introduction. The problem has also been studied in [18] without assuming isotropic conditions but supposing that the coefficients only depend on the variable x_3. Now, the approximation of the solutions is of the form

$$u_{\varepsilon,1}(x)\sim u_1(x_1,x_2)-\partial_{x_1}u_3(x_1,x_2)\frac{x_3}{\varepsilon},\quad u_{\varepsilon,2}(x)\sim u_2(x_1,x_2)-\partial_{x_2}u_3(x_1,x_2)\frac{x_3}{\varepsilon},$$
$$u_{\varepsilon,3}(x)\sim\frac{1}{\varepsilon}u_3(x_1,x_2).$$

A deformation with the form of this approximation is called a Kirchhoff-Love deformation. It is the analogous for a plate to the Bernouilli-Navier deformation for a beam. Now the authors find a linear system for u_1,u_2,u_3, which is no longer uncoupled as in [14].

In our case our aim is to obtain a limit system in dimension one which approximates the solutions of (3.8) without imposing any isotropy and/or homogeneity conditions on the tensor function A_ε. We assume the ellipticity condition (3.16) below but for the upper bound we just assume that the norm in L^∞ of A_ε is an infinitesimal with respect to $1/\varepsilon$, (3.15), and that the coefficients are bounded in L^1, (3.14). However, in our knowledge the results are new even in the case of uniformly bounded coefficients. We obtain an approximation of the solutions similar to (3.7), but eliminating the term corresponding to the function z, which is of order ε. The functions u and Z are the solutions to a linear system in dimension one.

As it is well known (see e.g. [1], [28]) the interest of taking A_ε depending on ε (homogenization problem) is to describe the behavior of beams composed by mixtures of different materials at a microscopic (or more exactly mesoscopic) level. The homogenization process gives an approximation of these mixtures by a generalized material represented by the homogenized tensor. In our case, the coefficient tensor corresponding to the limit system in dimension one. Therefore, our results provide a general model for the behavior of a beam composed by a general mixture of materials. It can be used to study optimal design problems in a beam. The fact that the coefficients are not uniformly bounded means that we are considering high-contrast homogenization problems. We recall that if there is not reduction of dimension, then, contrary to our result, by assuming the coefficients just bounded in L^1 we get non-local terms in the limit problem for $N \geq 3$, [3], [16] (but not for $N < 2$, see e.g. [4], [5], [25]). Some local homogenization results where there is not reduction of dimension, but assuming the coefficients bounded in a certain L^p with $p > 1$ are obtained in [6], [7], [8].

To finish we also observe that although no equi-integrability for the coefficients is assumed, the limit tensor we find is in L^1, i.e. it does not contain any measure supported on sets with null Lebesgue measure.

Notations

- We denote by $e_1, \cdots, e_N$ the usual basis in $\mathbb{R}^N$.
- For any vector $u \in \mathbb{R}^N$, we will use the following decomposition

$$u = \begin{pmatrix} u_1 \\ u' \end{pmatrix},$$

  where $u_1 \in \mathbb{R}$ and $u' \in \mathbb{R}^{N-1}$. We will also denote by u' a vector in $\mathbb{R}^N$ whose first component vanishes. In this way, the above decomposition can be also written as $u = u_1 e_1 + u'$.
- For any matrix M, we denote by M^T the transposed matrix of M.
- $:$ denotes the euclidean inner product in $\mathbb{R}^{N\times N}$, i.e. $M_1 : M_2 = \text{trace}(M_1^T M_2)$.
- $\mathbb{R}_s^{N\times N}$ denotes the space of symmetric matrices of dimension $N \times N$.
- $\mathbb{R}_{sk}^{N\times N}$ denotes the space of skew-symmetric matrices of dimension $N \times N$.
- $\mathbb{R}_{s_1 sk'}^{N\times N}$ denotes the space of matrices $M \in \mathbb{R}^{N\times N}$ such that

$$\begin{cases} M_{1i} = M_{i1}, & \text{for } i = 1, \ldots, N, \\ M_{ij} = -M_{ji}, & \text{for } i, j = 2, \ldots, N. \end{cases}$$

- $e(v)$ denotes the symmetric part of the derivative of a function v, i.e.

$$e(v) = \frac{1}{2}(Dv + Dv^T).$$

- For a set $U \subset \mathbb{R}^N$, $\mathcal{M}(U)$ denotes the space of Radon measures on U with bounded total variation. If U is bounded and open, it agrees with the dual space of $C_0^0(U)$. If U is compact, it agrees with the dual space of $C^0(U)$.

- For any measure $\mathfrak{a} \in \mathcal{M}(U)$, we define $\mathfrak{a}^L \in L^1(U)$ as the derivative of $\mathfrak{a}$ with respect to the Lebesgue measure.
- For a Lipschitz open set $\mathcal{O} \subset \mathbb{R}^N$ and a set $F \subset \partial\mathcal{O}$, we denote by $H^k_F(\mathcal{O})$ the space of functions in $H^k(\mathcal{O})$, such that their derivatives of order less or equal than $k-1$ vanish on F.
- We denote by C a generic constant which can change from line to line.
- We denote by O_ε an arbitrary sequence of real numbers which tends to zero when ε tends to zero. It can change from line to line.

3.2 The homogenization result

Let $\omega \subset \mathbb{R}^{N-1}$ be a Lipschitz connected bounded open set, with $N \geq 2$. Then, for $\varepsilon > 0$ we define the thin beam Ω_ε by

$$\Omega_\varepsilon = (0,1) \times (\varepsilon\omega). \tag{3.9}$$

The extremities of Ω_ε are denoted by Γ_ε, i.e.

$$\Gamma_\varepsilon = \{0,1\} \times (\varepsilon\omega). \tag{3.10}$$

When $\varepsilon = 1$, we will just write Ω and Γ instead of Ω_1 and Γ_1 respectively.

The coordinate system is chosen in such way that the origin is the center of mass of ω and the coordinates axes in the x' variables coincide with the inertial axes of ω, i.e. such that ω satisfies

$$\int_\omega y' dy' = 0, \tag{3.11}$$

$$\int_\omega y_i y_j \, dy' = 0, \quad 2 \leq i,j \leq N, \; i \neq j. \tag{3.12}$$

We define the diagonal matrix $\mathcal{I}$ (it corresponds to the inertia matrix of ω divided by $|\omega|$) by

$$\mathcal{I} = \begin{pmatrix} I_2 & & \\ & \ddots & \\ & & I_N \end{pmatrix}, \quad \text{with } I_i = \frac{1}{|\omega|}\int_\omega y_i^2 \, dy', \quad 2 \leq i \leq N. \tag{3.13}$$

In the domain Ω_ε we will consider a linear elastic problem where the coefficients also depend on ε. Our purpose is to approximate its solutions for those of a one-dimensional problem.

We will assume that the coefficients of the elasticity system are given by a sequence of tensors $A_\varepsilon \in L^\infty(\Omega_\varepsilon; \mathcal{L}(\mathbb{R}^{N\times N}_s))$ which satisfies the following three properties:

$$\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} |A_\varepsilon| dx \leq C, \tag{3.14}$$

$$\varepsilon\|A_\varepsilon\|_{L^\infty(\Omega_\varepsilon;\mathcal{L}(\mathbb{R}^{N\times N}_s))} \to 0, \tag{3.15}$$

$$\exists\alpha>0, \quad A_\varepsilon\xi:\xi \geq \alpha|\xi|^2, \quad \forall\,\xi\in\mathbb{R}^{N\times N}_s, \ \text{ a.e. in } \Omega_\varepsilon. \tag{3.16}$$

Then, we will deal with a sequence $u_\varepsilon \in H^1(\Omega_\varepsilon)^N$, which satisfies the linear elasticity system

$$\begin{cases} -\mathrm{div}(A_\varepsilon e(u_\varepsilon)) = h_\varepsilon \ \text{ in } \Omega_\varepsilon, \\ A_\varepsilon e(u_\varepsilon)\nu_\varepsilon = 0 \ \text{ on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon. \end{cases} \tag{3.17}$$

Here ν_ε denotes the unit outward normal to Ω_ε on $\partial\Omega_\varepsilon$ and $h_\varepsilon = (h_{\varepsilon,1}, h'_\varepsilon)$ is defined by

$$h_{\varepsilon,1}(x) = f_{\varepsilon,1}\left(x_1, \frac{x'}{\varepsilon}\right), \ h'_\varepsilon(x) = \varepsilon f'_\varepsilon\left(x_1, \frac{x'}{\varepsilon}\right) + g'_\varepsilon\left(x_1, \frac{x'}{\varepsilon}\right), \ \text{ a.e. } x \in \Omega_\varepsilon, \tag{3.18}$$

with $f_\varepsilon \in L^2(\Omega)^N$ and $g'_\varepsilon \in L^2(\Omega)^{N-1}$ such that

$$\int_\omega g'_\varepsilon dy' = 0, \quad \text{a.e. } y_1 \in (0,1), \tag{3.19}$$

$$\exists\, f \in L^2(\Omega)^N \ \text{ with } \ f_\varepsilon \rightharpoonup f \ \text{ in } L^2(\Omega)^N, \tag{3.20}$$

$$\exists\, g' \in L^2(\Omega)^{N-1} \text{ with } \ g'_\varepsilon \rightharpoonup g' \ \text{ in } L^2(\Omega)^{N-1}. \tag{3.21}$$

Since we have not imposed any boundary condition on Γ_ε, we will also need to assume some bounds for $u_\varepsilon = (u_{\varepsilon,1}, u'_\varepsilon)$. Namely, we suppose there exists $C > 0$ such that

$$\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} A_\varepsilon e(u_\varepsilon) : e(u_\varepsilon)\, dx \leq C, \tag{3.22}$$

$$\min_{a\in[0,1]}\left\{ \|(u_{\varepsilon,1}, \varepsilon u'_\varepsilon)\|_{L^2(\{a\}\times\varepsilon\omega)^N} + \left\| u'_\varepsilon - \frac{1}{|\varepsilon\omega|}\int_{\{a\}\times\varepsilon\omega} u'_\varepsilon\, dx' \right\|_{L^2(\{a\}\times\varepsilon\omega)^{N-1}} \right\}\Bigg) \leq C|\varepsilon\omega|^{\frac{1}{2}}. \tag{3.23}$$

Our main result is given by Theorem 3.1 below. Before stating it, we introduce the following notation:
For $u = (u_1, u') \in H^1(0,1) \times H^2(0,1)^{N-1}$ and $Z \in H^1(0,1;\mathbb{R}^{(N-1)\times(N-1)}_{sk})$, we denote

$$e_0(u,Z) := \begin{pmatrix} \dfrac{du_1}{dx_1} & \left(\dfrac{d^2u'}{dx_1^2}\right)^T \\ \dfrac{d^2u'}{dx_1^2} & \dfrac{dZ}{dx_1} \end{pmatrix} \in L^2(0,1;\mathbb{R}^{N\times N}_{s_1sk'}). \tag{3.24}$$

Theorem 3.1. *Let A_ε be a sequence of tensor functions in $L^\infty(\Omega_\varepsilon;\mathcal{L}(\mathbb{R}^{N\times N}_s))$ which satisfy* (3.14), (3.15) *and* (3.16). *Then there exist $\beta > 0$, which only depends on ω, a constant γ, which only depends on α and ω, a subsequence of ε, still denoted by ε, $\mathfrak{a} \in \mathcal{M}(0,1)$ and $A \in L^1(0,1;\mathcal{L}(\mathbb{R}^{N\times N}_{s_1sk'}))$ with*

$$\frac{1}{|\varepsilon\omega|}\int_{\varepsilon\omega} |A_\varepsilon|\, dx' \overset{*}{\rightharpoonup} \mathfrak{a} \ \ \textit{in } \mathcal{M}(0,1), \tag{3.25}$$

$$|AE| \leq \beta\big(AE : E\big)^{\frac{1}{2}}\big(\mathfrak{a}^L\big)^{\frac{1}{2}}, \quad \forall\, E \in \mathbb{R}^{N\times N}_{s_1 sk'},\ \textit{a.e. in } (0,1), \tag{3.26}$$

$$|E|^2 \leq \gamma AE : E, \quad \forall\, E \in \mathbb{R}^{N\times N}_{s_1 sk'},\ \textit{a.e. in } (0,1), \tag{3.27}$$

such that the following homogenization result holds:
Let h_ε be a sequence given by (3.18) *with $f_\varepsilon \in L^2(\Omega)^N$, $g'_\varepsilon \in L^2(\Omega)^{N-1}$ satisfying* (3.19), (3.20) *and* (3.21). *If $u_\varepsilon \in H^1(\Omega_\varepsilon)^N$ satisfies (3.17), (3.22) and (3.23), then, for a subsequence of ε, there exist $u \in H^1(0,1) \times H^2(0,1)^{N-1}$ and $Z \in H^1(0,1;\mathbb{R}^{(N-1)\times(N-1)}_{sk})$, with*

$$\int_0^1 Ae_0(u,Z) : e_0(u,Z)\, dx_1 < \infty, \tag{3.28}$$

which satisfy the variational equation

$$\begin{cases} \displaystyle\int_0^1 Ae_0(u,Z) : e_0(\tilde u, \tilde Z)\, dy_1 = \frac{1}{|\omega|}\int_\Omega \left(f_1\left(\tilde u_1 - \frac{d\tilde u'}{dy_1}\cdot y'\right) + f'\cdot \tilde u' + g'\cdot(\tilde Z\, y')\right) dy, \\ \forall (\tilde u, \tilde Z) \in H^1_0(0,1) \times H^2_0(0,1)^{N-1} \times H^1_0(0,1;\mathbb{R}^{(N-1)\times(N-1)}_{sk}), \\ \qquad \textit{with } \displaystyle\int_0^1 Ae_0(\tilde u,\tilde Z) : e_0(\tilde u,\tilde Z)\, dx_1 < \infty, \end{cases} \tag{3.29}$$

and provide the following approximation of u_ε

$$\lim_{\varepsilon\to 0}\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}\left(\left|u_{\varepsilon,1} - u_1 + \frac{du'}{dx_1}\cdot\frac{x'}{\varepsilon}\right|^2 + \left|\varepsilon u'_\varepsilon - u'\right|^2 + \left|\varepsilon\partial_{x_1}u'_\varepsilon - \frac{du'}{dx_1}\right|^2 \right.$$
$$\left. + \left|\varepsilon D_{x'}u'_\varepsilon - Z\right|^2 + \left|u'_\varepsilon - \frac{1}{|\varepsilon\omega|}\int_{\varepsilon\omega} u'_\varepsilon dz' - Z\frac{x'}{\varepsilon}\right|^2\right) dx = 0. \tag{3.30}$$

Remark 3.2. *Theorem 3.1 provides the approximation of u_ε*

$$u_{\varepsilon,1}(x) \sim u_1(x_1) - \frac{du'}{dx_1}\cdot\frac{x'}{\varepsilon}, \quad u'_\varepsilon(x) \sim \frac{1}{\varepsilon}u'(x_1) + Z(x_1)\frac{x'}{\varepsilon}, \qquad \textit{a.e. in } \Omega_\varepsilon,$$

in the sense that (3.30) holds. The right-hand side is the sum of the two deformations $(u_1 - \frac{du'}{dx_1}\cdot\frac{x'}{\varepsilon}, \frac{1}{\varepsilon}u')$ and $(0, Z\frac{x'}{\varepsilon})$. The first one corresponds to a Bernouilli-Navier's deformation, which usually appears in the asymptotic description of the deformation of a beam. The second one is known as the torsion term and corresponds to an infinitesimal rotation around the axis of the beam.

Statement (3.30) can be improved by adding some weak convergences in Sobolev spaces which are interesting for example in order to deduce boundary conditions for the functions u and Z. However, to do this we need to write the corresponding convergences in a fixed domain. As usual this can be carried out by using the changes of variables $y_1 = x_1$, $y' = x'/\varepsilon$. Namely, for the sequence u_ε in Theorem 3.1, we define $U_\varepsilon \in H^1(\Omega)^N$ as

$$U_\varepsilon(y) = u_\varepsilon(y_1, \varepsilon y') \quad \textit{a.e. } y \in \Omega.$$

Then, we have

$$\begin{cases} U_{\varepsilon,1} \rightharpoonup u_1 \quad \text{in } H^1(\Omega), \\ \varepsilon U'_\varepsilon \to u' \quad \text{in } H^1(\Omega)^{N-1}, \\ U'_\varepsilon - \dfrac{1}{|\omega|}\displaystyle\int_\omega U'_\varepsilon \, d\tau' \rightharpoonup Zy' \quad \text{in } H^1(\Omega)^{N-1}. \end{cases} \tag{3.31}$$

Remark 3.3. *In the proof of Theorem 3.1 (see (3.68) and (3.89)), we will also prove that if u_ε is in the conditions of the theorem and $\tilde{u}_\varepsilon$ is another sequence which satisfies*

$$\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} |e(\tilde{u}_\varepsilon)|^2 dx \leq C,$$

and (3.30) with u and Z replaced by some other functions $\tilde{u}$, $\tilde{Z}$ then

$$\exists \lim_{\varepsilon\to 0} \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} A_\varepsilon e(u_\varepsilon) : e(\tilde{u}_\varepsilon)\, \varphi \, dx = \int_0^1 Ae_0(u,Z) : e_0(\tilde{u},\tilde{Z})\, \varphi \, dx_1, \quad \forall \varphi \in C_0^\infty(0,1). \tag{3.32}$$

In this assertion, we have used that (3.22) and (3.30) also imply (3.23), which is easy to check by using Theorem 3.9 below. In particular, we can take $\tilde{u}_\varepsilon = u_\varepsilon$ to get the convergence of the energies.

Remark 3.4. *We observe that although (3.14) only provides an estimate for A_ε in L^1 (it just implies (3.25)), the coefficient tensor A in the limit problem (3.29) is in L^1, i.e. it does not contain any measure which is not absolutely continuous with respect to the Lebesgue measure. Indeed, inequality (3.26) provides the estimate*

$$|A| \leq \beta^2 \mathfrak{a}^L \quad \text{a.e. in } (0,1).$$

In particular, if $\mathfrak{a}^L$ belongs to $L^\infty(0,1)$, we have that A is in $L^\infty(0,1;\mathcal{L}(\mathbb{R}^{N\times N}_{s_1 s k'}))$.

Remark 3.5. *Variational equation (3.29) can be written as the partial differential system*

$$\begin{cases} -\dfrac{d}{dx_1}\big[Ae_0(u,Z)\big]_{11} = \dfrac{1}{|\omega|}\displaystyle\int_\omega f_1\, dy' \ \text{in } (0,1), \\ \dfrac{d^2}{dx_1^2}\big[Ae_0(u,Z)\big]_{1j} = \dfrac{1}{|\omega|}\displaystyle\int_\omega (f_1 y_j + f'_j)\, dy' \ \text{in } (0,1), \ \forall j \in \{2,\cdots,N\}, \\ -\dfrac{d}{dx_1}\big[Ae_0(u,Z)\big]_{ij} = \dfrac{1}{2|\omega|}\displaystyle\int_\omega \big(g_i y_j - g_j y_i\big)\, dy' \ \text{in } (0,1), \ \forall i,j \in \{2,\cdots,N\}, \ i<j, \end{cases} \tag{3.33}$$

where we recall that $e_0(u,Z)$ contains derivatives of first orden in u_1 and Z and derivatives of second order in u'.

It is worth comparing system (3.33) to the classical system for a beam, which is composed by $N-1$ ordinary differential equations of fourth order (see e.g. ([29]). Indeed, it corresponds to taking

$$A_\varepsilon \xi = \lambda \, \text{trace}(\xi) I + 2\mu\xi, \quad \forall \xi \in \mathbb{R}^{N\times N}_s. \tag{3.34}$$

where λ and μ are two positive constants (the Lamé constants). In this case, we can prove that system (3.33) reduces to

$$\begin{cases} -E\dfrac{d^2u_1}{dx_1^2} = \dfrac{1}{|\omega|}\displaystyle\int_\omega f_1\, dy' \quad in\ (0,1), \\ E\mathcal{I}\dfrac{d^4u'}{dx_1^4} = \dfrac{1}{|\omega|}\displaystyle\int_\omega (f_1y' + f')\, dy' \quad in\ (0,1), \\ -B\dfrac{d^2Z}{dx_1^2} = \dfrac{1}{|\omega|}\displaystyle\int_\omega \dfrac{g'\otimes y' - y'\otimes g'}{2}\, dy' \quad in\ (0,1), \end{cases} \tag{3.35}$$

where E is the Young modulus

$$E = \frac{2\mu(\lambda N + 2\mu)}{\lambda(N-1) + 2\mu},$$

and B is an elliptic tensor in $\mathcal{L}(\mathbb{R}^{(N-1)\times(N-1)}_{sk})$ which depends on α, β and ω. In particular, in this case, system (3.33) is uncoupled in the variables u_1, u' and Z. Taking the functions f_1 and g' as the null functions and choosing appropriate boundary conditions on $\{0,1\}$ in (3.35), we have that the first and third equation just give $u_1 = 0$, $Z = 0$ and then we recuperate the classical equation for a beam

$$E\mathcal{I}\frac{d^4u'}{dx_1^4} = \frac{1}{|\omega|}\int_\omega f'\, dy',$$

where usually f' is also chosen independent of the variable y'. However, we remark that even if the tensor functions A_ε are taken independent of ε, the limit problem written in the variables u and Z has the general form provided by (3.33). This result can be deduced from [23], where it is studied the asymptotic behavior of a beam with fixed coefficients but without assuming any homogeneity or isotropy condition.

Remark 3.6. *System (3.33) implies that the elements $\big[Ae_0(u,Z)\big]_{11}$ and $\big[Ae_0(u,Z)\big]_{ij}$ with $i, j \in \{2, \cdots, N\}$, $i < j$ are in $H^1(0,1)$ while the elements $\big[Ae_0(u,Z)\big]_{1j}$, with $j \in \{2, \cdots, N\}$ are in $H^2(0,1)$. Taking into account (3.27), this also proves that $e_0(u,Z)$ is in $L^\infty(0,1;\mathbb{R}^{N\times N}_{s_1sk'})$ and then that (u,Z) belongs to $W^{1,\infty}(0,1)\times W^{2,\infty}(0,1)^{N-1}\times W^{1,\infty}(0,1;\mathbb{R}^{(N-1)\times(N-1)}_{sk})$.*

In Theorem 3.1 we have not assumed any symmetry condition for the tensor matrices A_ε. However, from the physical point of view it is known that in order to have the conservation of the angular momentum, it is necessary to have A_ε symmetric, i.e. such that

$$A_\varepsilon E_1 : E_2 = A_\varepsilon E_2 : E_1, \quad \forall\, E_1, E_2 \in \mathbb{R}^{N\times N}_s.$$

In this case it is possible to show that the tensor A which appears in Theorem 3.1 also satisfies the symmetry condition

$$AE_1 : E_2 = AE_2 : E_1, \quad \forall\, E_1, E_2 \in \mathbb{R}^{N\times N}_{s_1sk'}.$$

More generally, we have the following result.

Proposition 3.7. *Let A_ε be in the conditions of Theorem 3.1 and consider the subsequence of ε and the functions A, $\mathfrak{a}$ which appear in the thesis of the theorem. Then, Theorem 3.1 also holds by replacing A_ε by A_ε^T and A by A^T.*

In Theorem 3.1 we have preferred to not impose any boundary condition on Γ_ε to show that the equation satisfied by the functions u and Z does not depend on them. As a consequence, it is now possible to get a homogenization result for different boundary conditions such as Dirichlet, Neumann or Robin conditions on Γ_ε. As an example we state in the following corollary a result corresponding to homogeneous Dirichlet boundary conditions.

Corollary 3.8. *Let A_ε be in the conditions of Theorem 3.1 and consider the subsequence of ε and the functions A, $\mathfrak{a}$ which appear in the thesis of the theorem. Then, for every sequence h_ε given by* (3.18) *with $f_\varepsilon \in L^2(\Omega)^N$, $g'_\varepsilon \in L^2(\Omega)^{N-1}$ satisfying* (3.19), (3.20) *and (3.21), the unique solution u_ε to*

$$\begin{cases} -\mathrm{div}(A_\varepsilon e(u_\varepsilon)) = h_\varepsilon \quad \text{in } \Omega_\varepsilon, \\ A_\varepsilon e(u_\varepsilon)\nu_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon, \\ u_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon, \end{cases} \tag{3.36}$$

satisfies (3.30) with (u, Z) the unique solution to the variational problem

$$\begin{cases} (u,Z) \in H_0^1(0,1) \times H_0^2(0,1)^{N-1} \times H_0^1(0,1;\mathbb{R}^{(N-1)\times(N-1)}_{s_1 sk'}), \\ \qquad \text{with } \displaystyle\int_0^1 Ae_0(u,Z) : e_0(u,Z)\,dx_1 < \infty \\ \displaystyle\int_0^1 Ae_0(u,Z) : e_0(\tilde u,\tilde Z)\,dy_1 = \frac{1}{|\omega|}\int_\Omega \left(f_1\left(\tilde u_1 - \frac{d\tilde u'}{dy_1}\cdot y'\right) + f'\cdot\tilde u' + g'\cdot(\tilde Z\, y')\right) dy, \\ \forall(\tilde u,\tilde Z) \in H_0^1(0,1) \times H_0^2(0,1)^{N-1} \times H_0^1(0,1;\mathbb{R}^{(N-1)\times(N-1)}_{s_1 sk'}), \\ \qquad \text{with } \displaystyle\int_0^1 Ae_0(\tilde u,\tilde Z) : e_0(\tilde u,\tilde Z)\,dx_1 < \infty. \end{cases} \tag{3.37}$$

3.3 Proof of the results

The present section is devoted to proving the different results stated in the previous one. An important result to do this is the following theorem. It is a particular case of a decomposition result for a sequence of deformations in a thin domain, which has been proved in [11].

Theorem 3.9. *We consider a Lipschitz connected bounded open set $\omega \subset \mathbb{R}^{N-1}$, and define Ω_ε by (3.9), then, there exists a constant $C > 0$ independent of ε, such that for every $u_\varepsilon \in H^1(\Omega_\varepsilon)^N$ there exist $q_\varepsilon \in \mathbb{R}^N$, $Q_\varepsilon \in \mathbb{R}^{N\times N}_{sk}$, $b'_\varepsilon \in H^2(0,1)^{N-1}$,*

$Z_\varepsilon \in H^1(0,1;\mathbb{R}_{sk}^{(N-1)\times(N-1)})$ *and* $w_\varepsilon \in H^1(\Omega_\varepsilon)^N$*, satisfying*

$$u_\varepsilon = q_\varepsilon + Q_\varepsilon x + \frac{1}{\varepsilon}\begin{pmatrix} 0 \\ b'_\varepsilon \end{pmatrix} + \begin{pmatrix} 0 & -\dfrac{db'_\varepsilon}{dx_1} \\ 0 & Z_\varepsilon \end{pmatrix}\begin{pmatrix} 0 \\ \dfrac{x'}{\varepsilon} \end{pmatrix} + w_\varepsilon \quad \text{in } \Omega_\varepsilon, \tag{3.38}$$

with

$$\|b'_\varepsilon\|_{H^2(0,1)^{N-1}} + \|Z_\varepsilon\|_{H^1(0,1;\mathbb{R}_{sk}^{(N-1)\times(N-1)})} + \frac{1}{|\Omega_\varepsilon|^{\frac{1}{2}}}\|w_\varepsilon\|_{H^1(\Omega_\varepsilon)^N} \leq \frac{C}{|\Omega_\varepsilon|^{\frac{1}{2}}}\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon;\mathbb{R}_s^{N\times N})}. \tag{3.39}$$

Theorem 3.9 is an improvement of Korn's inequality in a thin beam. It provides a decomposition of u_ε as the sum of a "linearized" rigid movement given by the two first terms on the right-hand side of (3.38), a sequence w_ε whose norm in $H^1(\Omega_\varepsilon)^N$ is bounded by the norm of $e(u_\varepsilon)$ in $L^2(\Omega_\varepsilon;\mathbb{R}_s^{N\times N})$ and a term (sum of the third and fourth terms in (3.38)) whose norm in $H^1(\Omega_\varepsilon)^N$ is bounded by the norm of $e(u_\varepsilon)$ in $L^2(\Omega_\varepsilon;\mathbb{R}_s^{N\times N})$ divided by ε, which has a very particular structure. Clearly it implies the following classical estimate from Korn's inequality in a beam.

Corollary 3.10. *We consider a Lipschitz connected bounded open set* $\omega \subset \mathbb{R}^{N-1}$*, and define* Ω_ε *by (3.9), then, there exists* $C > 0$ *independent of* ε*, such that for every* $u_\varepsilon \in H^1(\Omega_\varepsilon)^N$ *there exist* $q_\varepsilon \in \mathbb{R}^N$ *and* $Q_\varepsilon \in \mathbb{R}_{sk}^{N\times N}$*, which satisfy*

$$\|u_\varepsilon - q_\varepsilon - Q_\varepsilon x\|_{H^1(\Omega_\varepsilon)^N} \leq \frac{C}{\varepsilon}\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon;\mathbb{R}_s^{N\times N})}. \tag{3.40}$$

As usual, since every sequence u_ε of the form $u_\varepsilon = q_\varepsilon + Q_\varepsilon x$, with $q_\varepsilon \in \mathbb{R}^N$ and $Q_\varepsilon \in \mathbb{R}_{sk}^{N\times N}$ satisfies that $e(u_\varepsilon) = 0$, Theorem 3.9 and Corollary 3.10 do not provide any bound for the corresponding "linearized" rigid movement. In order to eliminate this term we need to get some extra information about u_ε. In this way, we have the following result.

Theorem 3.11. *We consider a Lipschitz connected bounded open set* $\omega \subset \mathbb{R}^{N-1}$ *which satisfies (3.11) and (3.12), and define* Ω_ε *by (3.9), then, there exists a constant* $C > 0$ *independent of* ε*, such that for every* $u_\varepsilon \in H^1(\Omega_\varepsilon)^N$ *there exist* $b'_\varepsilon \in H^2(0,1)^{N-1}$*,* $Z_\varepsilon \in H^1(0,1;\mathbb{R}_{sk}^{(N-1)\times(N-1)})$ *and* $w_\varepsilon \in H^1(\Omega_\varepsilon)^N$*, satisfying*

$$u_\varepsilon = \frac{1}{\varepsilon}\begin{pmatrix} 0 \\ b'_\varepsilon \end{pmatrix} + \begin{pmatrix} 0 & -\dfrac{db'_\varepsilon}{dx_1} \\ 0 & Z_\varepsilon \end{pmatrix}\begin{pmatrix} 0 \\ \dfrac{x'}{\varepsilon} \end{pmatrix} + w_\varepsilon \quad \text{in } \Omega_\varepsilon, \tag{3.41}$$

with

$$\|b'_\varepsilon\|_{H^2(0,1)^N} + \|Z_\varepsilon\|_{H^1(0,1;\mathbb{R}_{sk}^{(N-1)\times(N-1)})} + \frac{1}{|\Omega_\varepsilon|^{\frac{1}{2}}}\|w_\varepsilon\|_{H^1(\Omega_\varepsilon)^N} \leq$$
$$\frac{C}{|\Omega_\varepsilon|^{\frac{1}{2}}}\Bigg(\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon;\mathbb{R}_s^{N\times N})}$$
$$+\min_{a\in[0,1]}\left\{\|(u_{\varepsilon,1},\varepsilon u'_\varepsilon)\|_{L^2(\{a\}\times\varepsilon\omega)^N} + \left\|u'_\varepsilon - \frac{1}{|\varepsilon\omega|}\int_{\{a\}\times\varepsilon\omega} u'_\varepsilon\,dx'\right\|_{L^2(\{a\}\times\varepsilon\omega)^{N-1}}\right\}\Bigg). \tag{3.42}$$

Moreover, the following Korn's type inequality holds

$$
\begin{aligned}
&\|u_{\varepsilon,1}\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|u'_\varepsilon\|_{L^2(\Omega_\varepsilon)^{N-1}} + \varepsilon\|Du_\varepsilon\|_{L^2(\Omega_\varepsilon)^{N\times N}} \leq C\Bigg(\|e(u_\varepsilon)\|_{L^2(\Omega_\varepsilon)^{N\times N}} \\
&\quad + \min_{a\in[0,1]}\left\{ \|(u_{\varepsilon,1}, \varepsilon u'_\varepsilon)\|_{L^2(\{a\}\times\varepsilon\omega)^N} + \left\| u'_\varepsilon - \frac{1}{|\varepsilon\omega|}\int_{\{a\}\times\varepsilon\omega} u'_\varepsilon\, dx' \right\|_{L^2(\{a\}\times\varepsilon\omega)^{N-1}} \right\}\Bigg).
\end{aligned}
\tag{3.43}
$$

Proof. It is enough to prove (3.42) because (3.43) follows immediately from it. Applying Theorem 3.9, we can find $\tilde{q}_\varepsilon \in \mathbb{R}^N$, $\tilde{Q}_\varepsilon \in \mathbb{R}^{N\times N}_{sk}$, $\tilde{b}'_\varepsilon \in H^2(0,1)^{N-1}$, $\tilde{Z}_\varepsilon \in H^1(0,1;\mathbb{R}^{(N-1)\times(N-1)}_{sk})$ and $\tilde{w}_\varepsilon \in H^1(\Omega_\varepsilon)^N$, such that (3.38) and (3.39) hold with q_ε, Q_ε, b'_ε, Z_ε and w_ε replaced by $\tilde{q}_\varepsilon$, $\tilde{Q}_\varepsilon$, $\tilde{b}'_\varepsilon$, $\tilde{Z}_\varepsilon$ and $\tilde{w}_\varepsilon$ respectively. In particular, taking into account properties (3.11) and (3.12) of ω, for every $a \in [0,1]$, we have

$$
\begin{aligned}
\tilde{q}_{\varepsilon,1} &= \frac{1}{|\varepsilon\omega|}\int_{\{a\}\times\varepsilon\omega} \big(u_{\varepsilon,1} - \tilde{w}_{\varepsilon,1}\big)\, dz', \qquad \text{a.e in } \Omega_\varepsilon,\\
\tilde{q}'_\varepsilon + \big(\tilde{Q}_\varepsilon e_1\big)' a &= \frac{1}{|\varepsilon\omega|}\int_{\{a\}\times\varepsilon\omega} \left(u'_\varepsilon - \frac{1}{\varepsilon}\tilde{b}'_\varepsilon - \tilde{w}'_\varepsilon\right) dz', \qquad \text{a.e in } \Omega_\varepsilon,\\
\varepsilon(\tilde{Q}_\varepsilon)_{1j} I_j &= \frac{1}{|\varepsilon\omega|}\int_{\{a\}\times\varepsilon\omega} (u_{\varepsilon,1} - \tilde{w}_{\varepsilon,1})\,\frac{x_j}{\varepsilon} dx' + \frac{d\tilde{b}'_{\varepsilon,j}}{dx_1}(a) I_j, \quad \forall\, j \in \{2,\cdots,N\},\\
\varepsilon(\tilde{Q}_\varepsilon e_j)' I_j &= \frac{1}{|\varepsilon\omega|}\int_{\{a\}\times\varepsilon\omega} (u'_\varepsilon - \tilde{w}'_\varepsilon)\,\frac{x_j}{\varepsilon} dx' - \tilde{Z}_\varepsilon(a) e_j I_j, \quad \forall\, j \in \{2,\cdots,N\}.
\end{aligned}
$$

Recalling here that $\tilde{Q}_\varepsilon$ is skew-symmetric (and then $(\tilde{Q}_\varepsilon)_{1j} = -(\tilde{Q}_\varepsilon)_{j1}$) and using

$$
\int_{\{a\}\times\varepsilon\omega} \varphi\, x_j\, dx' = \int_{\{a\}\times\varepsilon\omega}\left(\varphi - \frac{1}{|\varepsilon\omega|}\int_{\{a\}\times\varepsilon\omega}\varphi\, dz'\right) x_j\, dx', \quad \forall\, j \in \{2,\cdots,N\},
$$

$$
\frac{1}{|\varepsilon\omega|}\left|\int_{\{a\}\times\varepsilon\omega}\varphi\, dx'\right| \leq \frac{1}{|\Omega_\varepsilon|^{\frac{1}{2}}}\|\varphi\|_{H^1(\Omega_\varepsilon)}, \quad \forall\, \varphi \in H^1(\Omega_\varepsilon),
$$

we easily deduce the result by taking

$$
b'_\varepsilon = \varepsilon\tilde{q}'_\varepsilon + \varepsilon(\tilde{Q}_\varepsilon e_1)' x_1 + \tilde{b}'_\varepsilon, \quad Z_\varepsilon = \varepsilon\tilde{Q}'_\varepsilon + \tilde{Z}_\varepsilon, \quad w_\varepsilon = \tilde{q}_{\varepsilon,1} e_1 + \tilde{w}_\varepsilon.
$$

□

We also recall the following result.

Lemma 3.12. *Let w_ε be a sequence in $H^1(\Omega_\varepsilon)$ such that there exists $C > 0$ which satisfies*

$$
\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}\big(|w_\varepsilon|^2 + |\nabla w_\varepsilon|^2\big)\, dx \leq C, \quad \forall\, \varepsilon > 0. \tag{3.44}
$$

Then, there exist a subsequence of ε, *still denoted by* ε, $w \in H^1(0,1)$ *and* $z \in L^2(0,1;H^1(\omega))$ *such that the sequence* $\hat{w}_\varepsilon \in H^1(\Omega)$ *defined by*

$$\hat{w}_\varepsilon(y) = w_\varepsilon(y_1, \varepsilon y'), \quad a.e.\ y \in \Omega, \tag{3.45}$$

satisfies

$$\hat{w}_\varepsilon \rightharpoonup w \quad in\ H^1(\Omega), \quad \frac{1}{\varepsilon}\nabla_{y'}\hat{w}_\varepsilon \rightharpoonup \nabla_{y'}z \quad in\ L^2(\Omega)^{N-1}. \tag{3.46}$$

Proof. The result is proved in [22], but not explicitly stated, it has also been used in other works such as [12]. Therefore, we just give a sketch of the proof. It is enough to use the decomposition $w_\varepsilon = \bar{w}_\varepsilon + z_\varepsilon$, with

$$\bar{w}_\varepsilon(x) = \frac{1}{|\varepsilon\omega|}\int_{\varepsilon\omega} w_\varepsilon(x_1, z')\,dz' \quad \text{a.e. } x \in \Omega_\varepsilon,$$

and $z_\varepsilon = w_\varepsilon - \bar{w}_\varepsilon$, where we observe that by Poincaré-Wirtinger's inequality, the second term satisfies

$$\int_{\Omega_\varepsilon} |z_\varepsilon|^2 dx \leq C\varepsilon^2 \int_{\Omega_\varepsilon} |\nabla_{x'} w_\varepsilon|^2 dx.$$

Then use the change of variables $y_1 = x_1$, $y' = x'/\varepsilon$ which transforms Ω_ε in Ω and take the weak limit in $H^1(\Omega)$ and $L^2(0,1;H^1(\omega))$ respectively, of each of the two sequences (which exist for a subsequence of ε). □

We are now in position to prove Theorem 3.1. The proof is an adaptation of the classical proof of the Murat-Tartar H-convergence theorem ([20], [28]) combined with decomposition (3.41).

Proof of Theorem 3.1. Let us divide the proof into several steps. Step 1 is devoted to proving (3.30) and obtaining a convergence result for $A_\varepsilon e(u_\varepsilon)$. In Step 2 we show that the weak limit of $A_\varepsilon e(u_\varepsilon)$ satisfies a limit differential problem. In particular this is used in Step 3 to prove that it satisfies better smoothness properties. Following the ideas of the proof of the classical H-convergence compactness result, in Step 4 we adapt the div-curl lemma to our problem. In Steps 5, 6 we introduce the tensor A and prove estimates (3.26) and (3.27), whereas in Step 7 we conclude that the limit problem can be formulated as (3.29).

Step 1. We consider a sequence $h_\varepsilon \in L^2(\Omega_\varepsilon)^N$ defined through (3.18), with $f_\varepsilon \in L^2(\Omega)^N$ and $g'_\varepsilon \in L^2(\Omega)^{N-1}$ satisfying (3.19), (3.20) and (3.21). Then, we take a sequence u_ε, which satisfies (3.17), (3.22) and (3.23).

By Theorem 3.11, there exist $b'_\varepsilon \in H^2(0,1)^{N-1}$, $Z_\varepsilon \in H^1(0,1;\mathbb{R}^{(N-1)\times(N-1)}_{sk})$ and $w_\varepsilon \in H^1(\Omega_\varepsilon)^N$, satisfying (3.41), with

$$\|b'_\varepsilon\|_{H^2(0,1)^N} + \|Z_\varepsilon\|_{H^1(0,1;\mathbb{R}^{(N-1)\times(N-1)}_{sk})} + \frac{1}{|\Omega_\varepsilon|^{\frac{1}{2}}}\|w_\varepsilon\|_{H^1(\Omega_\varepsilon)^N} \leq C. \tag{3.47}$$

Then, taking into account Lemma 3.12 for the third term, we deduce the existence of $u' \in H^2(0,1)^{N-1}$, $Z \in H^1(0,1;\mathbb{R}^{(N-1)\times(N-1)}_{sk})$, $w \in H^1(0,1)^N$ and $z \in L^2(0,1;H^1(\omega))^N$ such that defining $\hat{w}_\varepsilon$ by (3.45), we have

$$b'_\varepsilon \rightharpoonup u' \ \text{ weakly in } H^2(0,1)^{N-1}, \tag{3.48}$$

$$Z_\varepsilon \rightharpoonup Z \text{ weakly in } H^1(0,1;\mathbb{R}_{sk}^{(N-1)\times(N-1)}), \tag{3.49}$$

$$\hat{w}_\varepsilon \rightharpoonup w \text{ weakly in } H^1(\Omega)^N, \tag{3.50}$$

$$\frac{1}{\varepsilon} D_{y'}\hat{w}_\varepsilon \rightharpoonup D_{y'}z \text{ weakly in } L^2(\Omega)^{N\times(N-1)}. \tag{3.51}$$

We will denote

$$u_1 := w_1. \tag{3.52}$$

Let us prove that these convergences imply (3.30). Using the change of variables

$$y_1 = x_1, \quad y' = \frac{x'}{\varepsilon}, \tag{3.53}$$

and denoting $U_\varepsilon(y) = u_\varepsilon(y_1, \varepsilon y')$, a.e. $y \in \Omega$, we can write (3.41) as

$$U_{\varepsilon,1} = -\frac{db'_\varepsilon}{dy_1}\cdot y' + \hat{w}_{\varepsilon,1}, \quad U'_\varepsilon = \frac{1}{\varepsilon}b'_\varepsilon + Z_\varepsilon y' + \hat{w}'_\varepsilon \qquad \text{a.e. in } \Omega.$$

From (3.48), (3.49), (3.50), (3.51) and the fact that w only depends on the first variable, we deduce (3.31). Then, thanks to the Rellich-Kondrachov's compactness theorem, we have the following strong convergences

$$U_{\varepsilon,1} \to -\frac{du'}{dy_1}\cdot y' + u_1 \text{ in } L^2(\Omega), \quad \varepsilon U'_\varepsilon \to u' \text{ in } H^1(\Omega)^{N-1},$$

$$D_{y'}U'_\varepsilon \to Z \text{ in } L^2(\Omega)^{(N-1)\times(N-1)}, \quad U'_\varepsilon - \frac{1}{|\omega|}\int_\omega U'_\varepsilon(y_1,\tau')\,d\tau' \to Zy' \text{ in } L^2(\Omega)^{N-1}.$$

Using again the change of variables (3.53) to return to Ω_ε, we get (3.30).

To finish this step, let us also get a convergence result for $A_\varepsilon e(u_\varepsilon)$. Using

$$\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} |A_\varepsilon e(u_\varepsilon)|dx \le \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} \big(A_\varepsilon e(u_\varepsilon) : e(u_\varepsilon)\big)^{\frac{1}{2}}|A_\varepsilon|^{\frac{1}{2}}dx$$

$$\le \left(\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} A_\varepsilon e(u_\varepsilon) : e(u_\varepsilon)dx\right)^{\frac{1}{2}} \left(\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} |A_\varepsilon|dx\right)^{\frac{1}{2}},$$

and taking into account (3.14) and (3.22), we have

$$\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} |A_\varepsilon e(u_\varepsilon)|dx \le C. \tag{3.54}$$

Using the change of variables (3.53), this implies that $\sigma_\varepsilon \in L^2(\Omega;\mathbb{R}_s^{N\times N})$ defined by

$$\sigma_\varepsilon(y) = \big(A_\varepsilon e(u_\varepsilon)\big)(y_1, \varepsilon y'), \quad \text{a.e. } y \in \Omega, \tag{3.55}$$

is bounded in $L^1(\Omega;\mathbb{R}_s^{N\times N})$ and therefore we can also take the subsequence of ε such that there exists $\sigma \in \mathcal{M}(\Omega;\mathbb{R}_s^{N\times N})$ satisfying

$$\sigma_\varepsilon \overset{*}{\rightharpoonup} \sigma \text{ weakly-}* \text{ in } \mathcal{M}((0,1)\times\bar{\omega};\mathbb{R}_s^{N\times N}), \tag{3.56}$$

(the dual of $C_0^0(0,1;C^0(\bar\omega;\mathbb{R}_s^{N\times N}))$). Moreover, taking into account (3.15) and (3.22), we also have

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^2}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}|A_\varepsilon e(u_\varepsilon)|^2dx=0. \tag{3.57}$$

Step 2. Let us obtain a first differential equation for the function σ defined by (3.56). For this purpose, given $\tilde u\in C_0^\infty(0,1)^N$, $\tilde Z\in C_0^\infty(0,1;\mathbb{R}_{sk}^{(N-1)\times(N-1)})$ and $\tilde z\in C_0^\infty(0,1;C^\infty(\bar\omega))^N$, we define $\zeta_\varepsilon\in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$ as

$$\begin{cases}\zeta_{\varepsilon,1}(x)=\tilde u_1(x_1)-\dfrac{d\tilde u'}{dx_1}(x_1)\cdot\dfrac{x'}{\varepsilon}+\varepsilon\tilde z_1\left(x_1,\dfrac{x'}{\varepsilon}\right),\\[2mm] \zeta'_\varepsilon(x)=\dfrac{1}{\varepsilon}\tilde u'(x_1)+\tilde Z(x_1)\dfrac{x'}{\varepsilon}+\varepsilon\tilde z'\left(x_1,\dfrac{x'}{\varepsilon}\right),\end{cases}\quad\text{a.e. } x\in\Omega_\varepsilon. \tag{3.58}$$

We observe that we can write

$$\zeta_{\varepsilon,1}=\tilde u_1(x_1)-\frac{d\tilde u'}{dx_1}(x_1)\cdot\frac{x'}{\varepsilon}+r_{\varepsilon,1},$$

$$\varepsilon\zeta'_\varepsilon=\tilde u'(x_1)+r'_\varepsilon,$$

$$e(\zeta_\varepsilon)=\begin{pmatrix}\dfrac{d\tilde u_1}{dx_1}-\dfrac{d^2\tilde u'}{dx_1^2}\cdot\dfrac{x'}{\varepsilon} & \dfrac12\left(\nabla_{y'}\tilde z_1\left(x_1,\dfrac{x'}{\varepsilon}\right)+\dfrac{d\tilde Z}{dx_1}\dfrac{x'}{\varepsilon}\right)^T\\ \dfrac12\left(\nabla_{y'}\tilde z_1\left(x_1,\dfrac{x'}{\varepsilon}\right)+\dfrac{d\tilde Z}{dx_1}\dfrac{x'}{\varepsilon}\right) & e_{y'}(\tilde z')\left(x_1,\dfrac{x'}{\varepsilon}\right)\end{pmatrix}+R_\varepsilon,$$

where

$$\|r_\varepsilon\|_{L^\infty(\Omega_\varepsilon)^N}+\|R_\varepsilon\|_{L^\infty(\Omega_\varepsilon;\mathbb{R}_s^{N\times N})}\le C\varepsilon.$$

Taking ζ_ε as test function in (3.17), dividing by $|\Omega_\varepsilon|$, using the change of variables (3.53), recalling the definition (3.55) of σ_ε, and taking into account (3.19), we get

$$\frac{1}{|\omega|}\int_\Omega\sigma_\varepsilon:\begin{pmatrix}\dfrac{d\tilde u_1}{dy_1}-\dfrac{d^2\tilde u'}{dy_1^2}\cdot y' & \dfrac12\left(\nabla_{y'}\tilde z_1+\dfrac{d\tilde Z}{dy_1}y'\right)^T\\ \dfrac12\left(\nabla_{y'}\tilde z_1+\dfrac{d\tilde Z}{dy_1}y'\right) & e_{y'}(\tilde z')\end{pmatrix}dy$$
$$=\frac{1}{|\omega|}\int_\Omega\left(f_{1,\varepsilon}\left(\tilde u_1-\frac{d\tilde u'}{dy_1}\cdot y'\right)+f'_\varepsilon\cdot\tilde u'+g'_\varepsilon\cdot(\tilde Zy')\right)dy+O_\varepsilon.$$

Thanks to (3.56), (3.20) and (3.21), we can pass to the limit in ε in this equality to deduce

$$\frac{1}{|\omega|}\int_{\bar\Omega}\begin{pmatrix} \dfrac{d\tilde u_1}{dy_1}-\dfrac{d^2\tilde u'}{dy_1^2}\cdot y' & \dfrac12\left(\nabla_{y'}\tilde z_1+\dfrac{d\tilde Z}{dy_1}\,y'\right)^T \\ \dfrac12\left(\nabla_{y'}\tilde z_1+\dfrac{d\tilde Z}{dy_1}\,y'\right) & e_{y'}(\tilde z')\end{pmatrix}: d\sigma$$
$$= \frac{1}{|\omega|}\int_{\Omega}\left(f_1\left(\tilde u_1-\frac{d\tilde u'}{dy_1}\cdot y'\right)+f'\cdot\tilde u'+g'\cdot(\tilde Z\,y')\right)dy. \tag{3.59}$$

By density, this equality holds for every $\tilde u'\in C^2_0(0,1)^{N-1}$, $\tilde Z\in C^1_0(0,1;\mathbb{R}^{(N-1)\times(N-1)}_{sk})$, $\tilde u_1\in C^1_0(0,1)$ and $\tilde z\in C^0_0(0,1;C^1(\bar\omega))^N$.

Step 3. Let us use (3.59) to get some differential equations for the components of σ. They will be used in particular to get some regularity results for σ.

Taking in (3.59) $\tilde u=\tilde z=0$, and recalling that σ is symmetric and $\tilde Z$ skew-symmetric, we get

$$\sum_{2\le i,j\le N}\frac{1}{|\omega|}\int_0^1\left(\int_{\bar\omega}y_j d\sigma_{1i}\right)\frac{d\tilde Z_{ij}}{dy_1}\,dy_1=\frac{1}{|\omega|}\sum_{2\le i,j\le N}\int_0^1\tilde Z_{ij}\left(\int_\omega g_iy_jdy'\right)dy_1,$$

for every $\tilde Z\in C^1_0(0,1;\mathbb{R}^{(N-1)\times(N-1)}_{sk})$, which proves

$$-\frac{1}{|\omega|}\frac{d}{dy_1}\int_{\bar\omega}(y_jd\sigma_{1i}-y_id\sigma_{1j})=\int_\omega(g_iy_j-g_jy_i)dy'\ \text{ in }(0,1),\ \forall\, i,j\in\{2,\cdots,N\}. \tag{3.60}$$

In particular

$$R_{ij}:=\frac{1}{|\omega|}\int_{\bar\omega}(y_id\sigma_{ij}-y_jd\sigma_{ji})\in H^1(0,1),\quad \forall\, i,j\in\{2,\cdots,N\}, \tag{3.61}$$

and therefore, in (3.59) we can take $\tilde Z\in W^{1,1}_0(0,1;\mathbb{R}^{(N-1)\times(N-1)}_{sk})$.

Analogously, taking $\tilde u_1=0$, $\tilde z=0$, $\tilde Z=0$ in (3.59), we get

$$-\frac{d^2}{dy_1^2}\int_{\bar\omega}y'd\sigma_{11}=\frac{d}{dy_1}\int_\omega(f_1y')dy'+\int_\omega f'\,dy'\ \text{ in }(0,1), \tag{3.62}$$

which implies that

$$q':=-\frac{1}{|\omega|}\int_{\bar\omega}y'd\sigma_{11}\in H^1(0,1)^{N-1}, \tag{3.63}$$

and then that (3.59) holds true with $\tilde u'\in W^{2,1}_0(0,1)^{N-1}$. Finally, taking $\tilde u'=0$, $\tilde z'=0$, $\tilde Z=0$, we get

$$-\frac{d}{dy_1}\int_{\bar\omega}d\sigma_{11}=\int_\omega f_1\,dy'\ \text{ in }(0,1), \tag{3.64}$$

which proves

$$p:=\frac{1}{|\omega|}\int_{\bar\omega}d\sigma_{11}\in H^1(0,1), \tag{3.65}$$

and then that (3.59) holds true with $\tilde{u}_1$ just in $W^{1,1}_0(0,1)$. From now on, we denote

$$\Lambda := \begin{pmatrix} p & \frac{1}{2}(q')^T \\ \frac{1}{2}q' & R \end{pmatrix} \in H^1(0,1;\mathbb{R}^{N\times N}_{s_1 sk'}). \tag{3.66}$$

With this notation, taking into account definitions (3.61), (3.63) and (3.65) of R, q' and p, we can write (3.59) as

$$\begin{aligned}\int_0^1 \Lambda : e_0(\tilde{u},\tilde{Z})\,dy_1 + \frac{1}{|\omega|}\int_{\bar\Omega} \begin{pmatrix} 0 & \frac{1}{2}(\nabla_{y'}\tilde{z}_1)^T \\ \frac{1}{2}\nabla_{y'}\tilde{z}_1 & e_{y'}(\tilde{z}') \end{pmatrix} : d\sigma \\ = \frac{1}{|\omega|}\int_\Omega \left(f_1\left(\tilde{u}_1 - \frac{d\tilde{u}'}{dy_1}\cdot y'\right) + f'\cdot\tilde{u}' + g'\cdot(\tilde{Z}\,y')\right) dy,\end{aligned} \tag{3.67}$$

for every $\tilde{u}_1 \in W^{1,1}_0(0,1)$, $\tilde{u}' \in W^{2,1}_0(0,1)^{N-1}$, $\tilde{Z} \in W^{1,1}_0(0,1;\mathbb{R}^{(N-1)\times(N-1)}_{sk})$ and $\tilde{z} \in C^0_0(0,1;C^1(\bar\omega))^N$.

Step 4. Let us now obtain the analogous of the div-curl lemma for our framework:

We consider another sequence $\tilde{u}_\varepsilon$ which satisfies

$$\frac{1}{|\Omega_\varepsilon|}\int_\Omega |e(\tilde{u}_\varepsilon)|^2\,dx \le C,$$

and (3.23) (but it is not necessarily the solution of any differential system) and it is such that there exist $\tilde{u} \in H^1(0,1)\times H^2(0,1)^{N-1}$ and $\tilde{Z} \in H^1(0,1;\mathbb{R}^{(N-1)\times(N-1)}_{sk})$ which satisfy (3.30) with u and Z replaced by $\tilde{u}$ and $\tilde{Z}$ respectively. Let us prove that we have

$$\lim_{\varepsilon\to 0}\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} A_\varepsilon e(u_\varepsilon) : e(\tilde{u}_\varepsilon)\,\varphi\,dx = \int_0^1 \Lambda : e_0(\tilde{u},\tilde{Z})\varphi\,dx_1, \quad \forall\varphi\in C^\infty_0(0,1). \tag{3.68}$$

Reasoning as in Step 1, we know that $\tilde{u}_\varepsilon$ satisfy (3.41) for certain functions $\tilde{b}'_\varepsilon$, $\tilde{Z}_\varepsilon$ and $\tilde{w}_\varepsilon$. Extracting a subsequence if necessary, and defining

$$\breve{w}_\varepsilon(y) = \tilde{w}_\varepsilon(y_1,\varepsilon y'), \tag{3.69}$$

(it is the analogous to (3.45)), we also know that there exist $\tilde{u}' \in H^2(0,1)^{N-1}$, $\tilde{Z} \in H^1(0,1;\mathbb{R}^{(N-1)\times(N-1)}_{sk})$, $\tilde{w} \in H^1(0,1)^N$ and $\tilde{z} \in L^2(0,1;H^1(\omega))^N$, such that the analogous to (3.48), (3.49), (3.50), (3.51) and (3.52) are satisfied. We denote $\tilde{u}_1 := \tilde{w}_1$.

For $\varphi \in C^\infty_0(0,1)$, we define $\hat{u}_\varepsilon \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$ by

$$\begin{cases} \hat{u}_{\varepsilon,1} = -\dfrac{d}{dx_1}\big(\varphi\tilde{b}'_\varepsilon\big)\cdot\dfrac{x'}{\varepsilon} + \varphi\tilde{w}_{\varepsilon,1}, \\ \hat{u}'_\varepsilon = \varphi\,\tilde{u}'_\varepsilon, \end{cases} \quad \text{a.e. in } \Omega_\varepsilon.$$

We observe

$$e(\hat{u}_\varepsilon) = \varphi e(\tilde{u}_\varepsilon) + S_\varepsilon, \tag{3.70}$$

with

$$S_\varepsilon = \begin{pmatrix} \dfrac{d\varphi}{dx_1}\tilde{w}_{\varepsilon,1} - \left(2\dfrac{d\varphi}{dx_1}\dfrac{d\tilde{b}'_\varepsilon}{dx_1} + \dfrac{d^2\varphi}{dx_1^2}\tilde{b}'_\varepsilon\right)\cdot\dfrac{x'}{\varepsilon} & \dfrac{1}{2}\dfrac{d\varphi}{dx_1}\left(\tilde{Z}_\varepsilon\dfrac{x'}{\varepsilon} + \tilde{w}'_\varepsilon\right)^T \\ \dfrac{1}{2}\dfrac{d\varphi}{dx_1}\left(\tilde{Z}_\varepsilon\dfrac{x'}{\varepsilon} + \tilde{w}'_\varepsilon\right) & 0 \end{pmatrix}.$$

Let us study the asymptotic behavior of S_ε. For this purpose, we use the change of variables (3.53), namely, we introduce $\Xi_\varepsilon \in L^2(\Omega;\mathbb{R}^{N\times N}_s)$ by

$$\Xi_\varepsilon(y) = S_\varepsilon(y_1, \varepsilon y'),$$

which can be decomposed as

$$\Xi_\varepsilon = \Xi^1_\varepsilon + \Xi^2_\varepsilon,$$

with (see (3.69) for the definition of $\breve{w}_\varepsilon$)

$$\Xi^1_\varepsilon = \begin{pmatrix} \dfrac{d\varphi}{dy_1}\dfrac{1}{|\omega|}\displaystyle\int_\omega \breve{w}_{\varepsilon,1}\,d\eta' - \left(2\dfrac{d\varphi}{dy_1}\dfrac{d\tilde{b}'_\varepsilon}{dy_1} + \dfrac{d^2\varphi}{dy_1^2}\tilde{b}'_\varepsilon\right)\cdot y' & \dfrac{1}{2}\dfrac{d\varphi}{dy_1}\left(\tilde{Z}_\varepsilon y' + \dfrac{1}{|\omega|}\displaystyle\int_\omega \breve{w}'_\varepsilon\,d\eta'\right)^T \\ \dfrac{1}{2}\dfrac{d\varphi}{dy_1}\left(\tilde{Z}_\varepsilon y' + \dfrac{1}{|\omega|}\displaystyle\int_\omega \breve{w}'_\varepsilon\,d\eta'\right) & 0 \end{pmatrix},$$

$$\Xi^2_\varepsilon = \frac{d\varphi}{dy_1}\begin{pmatrix} \left(\breve{w}_{\varepsilon,1} - \dfrac{1}{|\omega|}\displaystyle\int_\omega \breve{w}_{\varepsilon,1}\,d\eta'\right) & \dfrac{1}{2}\left(\breve{w}'_\varepsilon - \dfrac{1}{|\omega|}\displaystyle\int_\omega \breve{w}'_\varepsilon\,d\eta'\right)^T \\ \dfrac{1}{2}\left(\breve{w}'_\varepsilon - \dfrac{1}{|\omega|}\displaystyle\int_\omega \breve{w}'_\varepsilon\,d\eta'\right) & 0 \end{pmatrix}.$$

For Ξ^1_ε, we use (3.48), (3.49), (3.50), $\tilde{w}$ depending only on the first variable, and the compact embedding of $H^1(0,1)$ into $C^0([0,1])$ to prove

$$\Xi^1_\varepsilon \to \Xi^1 := \begin{pmatrix} \dfrac{d\varphi}{dy_1}\tilde{u}_1 - \left(2\dfrac{d\varphi}{dy_1}\dfrac{d\tilde{u}'}{dy_1} + \dfrac{d^2\varphi}{dy_1^2}\tilde{u}'\right)\cdot y' & \dfrac{1}{2}\dfrac{d\varphi}{dy_1}\left(\tilde{Z}\,y' + \tilde{w}'\right)^T \\ \dfrac{1}{2}\dfrac{d\varphi}{dy_1}\left(\tilde{Z}\,y' + \tilde{w}'\right) & 0 \end{pmatrix}, \tag{3.71}$$

in $C^0_0(0,1;C^0(\bar{\omega}))$. For Ξ^2_ε, we use Poincaré-Wirtinger's inequality which gives

$$\frac{1}{|\Omega|}\int_\Omega \left|\breve{w}_\varepsilon - \frac{1}{|\omega|}\int_\omega \breve{w}_\varepsilon\,d\eta'\right|^2 dy \le \frac{C}{|\Omega|}\int_\Omega |D_{y'}\breve{w}_\varepsilon|^2\,dy = C\frac{\varepsilon^2}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}|D_{x'}\tilde{w}_\varepsilon|^2 dx,$$

and then, thanks to (3.47), we get

$$\|\Xi^2_\varepsilon\|_{L^2(\Omega;\mathbb{R}^{N\times N}_s)} \le C\varepsilon. \tag{3.72}$$

Now, we take $\hat{u}_\varepsilon$ as test function in (3.17), we divide by $|\Omega_\varepsilon|$ and then we use the change of variables (3.53). Taking into account (3.70), (3.56), (3.57), (3.71) and (3.72), we can then pass to the limit in ε to get

$$\lim_{\varepsilon\to 0}\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} A_\varepsilon e(u_\varepsilon):e(\tilde{u}_\varepsilon)\,\varphi\,dx = \frac{1}{|\omega|}\int_{\Omega}\left(f_1\left(-\frac{d}{dy_1}(\varphi\,\tilde{u}')\cdot y' + \varphi\tilde{u}_1\right) + \varphi f'\cdot\tilde{u}'\right)dy$$
$$+\frac{1}{|\omega|}\int_{\Omega}\varphi\, g'\cdot(\tilde{Z}y')\,dy' - \frac{1}{|\omega|}\int_{\bar{\Omega}}\Xi^1 : d\sigma.$$

In the first and second terms of this equality, we use (3.59) with $\tilde{u}$ replaced by $\varphi\tilde{u}$, $\tilde{z}_1$ replaced by $\frac{d\varphi}{dy_1}\tilde{w}'\cdot y'$, $\tilde{Z}$ replaced by $\varphi\tilde{Z}$ and $\tilde{z}'$ replaced by the null function. This gives

$$\lim_{\varepsilon\to 0}\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} A_\varepsilon e(u_\varepsilon):e(\tilde{u}_\varepsilon)\,\varphi\,dx = \frac{1}{|\omega|}\int_{\bar{\Omega}}\varphi\begin{pmatrix} \dfrac{d\tilde{u}_1}{dy_1} - \dfrac{d^2\tilde{u}'}{dy_1^2}\cdot y' & \dfrac{1}{2}\left(\dfrac{d\tilde{Z}}{dy_1}y'\right)^T \\ \dfrac{1}{2}\dfrac{d\tilde{Z}}{dy_1}y' & 0\end{pmatrix} : d\sigma,$$

which, using the definitions (4.1) and (3.24) of Λ and the operator e_0, is equivalent to (3.68).

Step 5. Let us now obtain some estimates for Λ.

For every $\varphi\in C_0^\infty(0,1)$, $\varphi\geq 0$, recalling the definition (3.25) of $\mathfrak{a}$ and (3.68), we have

$$\begin{aligned}
&\frac{1}{|\omega|}\int_{\Omega}|\sigma_\varepsilon|\,\varphi\,dy \\
&\quad\leq\left(\frac{1}{|\omega|}\int_{\Omega}\big(A_\varepsilon e(u_\varepsilon):e(u_\varepsilon)\big)(y_1,\varepsilon y')\,\varphi\,dy\right)^{\frac12}\left(\frac{1}{|\omega|}\int_0^1\int_\omega |A_\varepsilon|(y_1,\varepsilon y')\varphi\,dy'dy_1\right)^{\frac12} \\
&\quad=\left(\frac{1}{|\varepsilon\omega|}\int_{\Omega_\varepsilon}\big(A_\varepsilon e(u_\varepsilon):e(u_\varepsilon)\big)\,\varphi\,dx\right)^{\frac12}\left(\frac{1}{|\varepsilon\omega|}\int_0^1\int_{\varepsilon\omega}|A_\varepsilon|\varphi\,dx'dx_1\right)^{\frac12} \\
&\quad=\left(\int_0^1\Lambda : e_0(u,Z)\varphi\,dx_1\right)^{\frac12}\left(\int_0^1\varphi\,d\mathfrak{a}\right)^{\frac12}+O_\varepsilon,
\end{aligned}$$

and therefore, using the definition (3.56) of σ, we get

$$\frac{1}{|\omega|}\int_{\bar{\Omega}}\varphi\,d|\sigma|\leq\left(\int_0^1\Lambda : e_0(u,Z)\varphi\,dx_1\right)^{\frac12}\left(\int_0^1\varphi\,d\mathfrak{a}\right)^{\frac12},\quad\forall\,\varphi\in C_0^\infty(0,1),\ \varphi\geq 0,$$

which using definitions (3.61), (3.63) and (3.65) of the compontents R, q and p of Λ also proves the existence of a constant β which only depends on ω such that

$$\int_0^1|\Lambda|\varphi\,dx_1\leq\beta\left(\int_0^1\Lambda : e_0(u,Z)\varphi\,dx_1\right)^{\frac12}\left(\int_0^1\varphi\,d\mathfrak{a}\right)^{\frac12},\quad\forall\,\varphi\in C_0^\infty(0,1),\ \varphi\geq 0.$$

Using the measures derivation theorem and recalling that the components of Λ belong to $H^1(0,1)$ we get

$$|\Lambda| \leq \beta\,(\Lambda : e_0(u,Z))^{\frac{1}{2}}\left(\mathfrak{a}^L\right)^{\frac{1}{2}}, \quad \text{a.e. in } (0,1). \tag{3.73}$$

On the other hand, using (3.16) combined with (3.68), we have

$$\lim_{\varepsilon\to 0}\frac{\alpha}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}|e(u_\varepsilon)|^2\varphi\,dx \leq \int_0^1 \Lambda : e_0(u,Z)\varphi\,dx_1, \quad \forall\,\varphi\in C_0^\infty(0,1),\ \varphi\geq 0,$$

which taking into account (3.41), (3.48), (3.49), (3.50), (3.51), and using the semi-continuity properties of the weak convergence, implies

$$\begin{aligned}&\frac{\alpha}{|\omega|}\int_\Omega\left(\left|\frac{du_1}{dx_1}-\frac{d^2u'}{dx_1^2}\cdot y'\right|^2+\frac{1}{2}\left|\nabla_{y'}z_1+\frac{dZ}{dx_1}\,y'\right|^2+|e_{y'}(z')|^2\right)\varphi\,dy\\&\leq\int_0^1\Lambda : e_0(u,Z)\varphi\,dx_1, \quad \forall\,\varphi\in C_0^\infty(0,1),\ \varphi\geq 0,\end{aligned}$$

which gives

$$\begin{aligned}&\frac{\alpha}{|\omega|}\int_\omega\left(\left|\frac{du_1}{dx_1}-\frac{d^2u'}{dx_1^2}\cdot y'\right|^2+\frac{1}{2}\left|\nabla_{y'}z_1+\frac{dZ}{dy_1}\,y'\right|^2+|e_{y'}(z')|^2\right)dy'\\&\qquad\leq\Lambda : e_0(u,Z), \quad \text{a.e. in } (0,1).\end{aligned} \tag{3.74}$$

The first term on the left-hand side satisfies, thanks to (3.11), (3.12) and definition (3.13) of $\mathcal{I}$,

$$\frac{1}{|\omega|}\int_\omega\left|\frac{du_1}{dx_1}-\frac{d^2u'}{dx_1^2}\cdot y'\right|^2dy'=\left|\frac{du_1}{dx_1}\right|^2+\sum_{j=2}^N I_j\left|\frac{d^2u_j}{dx_1^2}\right|^2. \tag{3.75}$$

For the second term, we take a function $\psi\in C_0^\infty(\omega)$ such that

$$\int_\omega\psi\,dy'=1.$$

Then, we observe that thanks to Z valued in $\mathbb{R}_{sk}^{(N-1)\times(N-1)}$, we have

$$2\frac{dZ_{ik}}{dx_1}=\int_\omega\big(\nabla_{y'}z_1+\frac{dZ}{dy_1}\,y'\big)\cdot\big(\partial_{y_i}\psi\,e_k-\partial_{y_k}\psi\,e_i\big)\,dy' \;\text{ a.e. in } (0,1), \quad \forall i,k\in\{2,\cdots,N\},$$

which proves the existence of a constant C depending only on ω such that

$$\left|\frac{dZ_{ik}}{dx_1}\right|^2\leq C\int_\omega\left|\nabla_{y'}z_1+\frac{dZ}{dy_1}\,y'\right|^2dy', \quad \text{a.e. in } (0,1), \quad \forall i,k\in\{2,\cdots,N\}. \tag{3.76}$$

Using (3.75) and (3.76) in (3.74) and recalling the definition (3.24) of the operator e_0, we then deduce the existence of a constant γ depending only on α and ω such that

$$|e_0(u,Z)|^2\leq\gamma\big(\Lambda : e_0(u,Z)\big), \quad \text{a.e. in } (0,1). \tag{3.77}$$

Recalling that Λ belongs to $H^1(0,1;\mathbb{R}^{N\times N}_{s_1sk'})$, we deduce from this inequality that

$$e_0(u,Z)\in L^\infty(0,1;\mathbb{R}^{N\times N}_{s_1sk'}). \tag{3.78}$$

Step 6. We consider $E\in\mathbb{R}^{N\times N}_{s_1sk'}$, which we decompose as

$$E=\begin{pmatrix} E_{11} & (E_1')^T \\ E_1' & E' \end{pmatrix},$$

with $E_{11}\in\mathbb{R}$, $E_1'\in\mathbb{R}^{N-1}$, $E'\in\mathbb{R}^{(N-1)\times(N-1)}_{sk}$. For $m\in\mathbb{N}$, we define $u_\varepsilon^{E,m}$ as the unique solution to

$$\begin{cases} -\operatorname{div}(A_\varepsilon e(u_\varepsilon^{E,m}))+m\left(u_{\varepsilon,1}^{E,m}-E_{11}x_1+x_1E_1'\cdot\dfrac{x'}{\varepsilon}\right)e_1 \\ \qquad\qquad +\dfrac{m}{|\varepsilon\omega|}\displaystyle\sum_{l=2}^N\int_{\varepsilon\omega}\left((u_\varepsilon^{E,m})'-x_1E'\dfrac{\eta'}{\varepsilon}\right)\dfrac{\eta_l}{\varepsilon}d\eta'\,\dfrac{x_l}{\varepsilon}=0\ \text{ in }\Omega_\varepsilon, \\ u_\varepsilon^{E,m}=0\ \text{ on }\{0\}\times\varepsilon\omega,\quad A_\varepsilon e(u_\varepsilon^{E,m})\nu_\varepsilon=0\ \text{ on }\partial\Omega_\varepsilon\setminus\big(\{0\}\times\varepsilon\omega\big). \end{cases} \tag{3.79}$$

The existence and uniqueness of solution for this equation is a simple application of Lax-Milgram's theorem combined with Korn's inequality.

In order to obtain a previous estimate for $u_\varepsilon^{E,m}$, we multiply the equation by

$$\left(u_{\varepsilon,1}^{E,m}-E_{11}x_1+x_1E_1'\cdot\frac{x'}{\varepsilon}\right)e_1+\left((u_\varepsilon^{E,m})'-\frac{x_1^2}{2\varepsilon}E_1'-x_1E'\frac{x'}{\varepsilon}\right).$$

Thanks to (3.11), we get

$$\begin{aligned} &\int_{\Omega_\varepsilon}A_\varepsilon e(u_\varepsilon^{E,m}):e(u_\varepsilon^{E,m})\,dx+m\int_{\Omega_\varepsilon}\left|u_{\varepsilon,1}^{E,m}-E_{11}x_1+x_1E_1'\cdot\frac{x'}{\varepsilon}\right|^2dx \\ &+\frac{m}{|\varepsilon\omega|}\sum_{l=2}^N\int_0^1\left|\int_{\varepsilon\omega}\left((u_\varepsilon^{E,m})'-x_1E'\frac{\eta'}{\varepsilon}\right)\frac{\eta_l}{\varepsilon}\,d\eta'\right|^2dx_1 \\ &=\textstyle\int_{\Omega_\varepsilon}A_\varepsilon e(u_\varepsilon^{E,m}):\begin{pmatrix} E_{11}-E_1'\cdot\dfrac{x'}{\varepsilon} & \dfrac12\left(E'\dfrac{x'}{\varepsilon}\right)^T \\ \dfrac12E'\dfrac{x'}{\varepsilon} & 0\end{pmatrix}dx, \end{aligned}$$

which, using Young's inequality, gives

$$\begin{aligned} &\frac12\int_{\Omega_\varepsilon}A_\varepsilon e(u_\varepsilon^{E,m}):e(u_\varepsilon^{E,m})\,dx+m\int_{\Omega_\varepsilon}\left|u_{\varepsilon,1}^{E,m}-E_{11}x_1+x_1E_1'\cdot\frac{x'}{\varepsilon}\right|^2dx \\ &+\frac{m}{|\varepsilon\omega|}\sum_{l=2}^N\int_0^1\left|\int_{\varepsilon\omega}\left((u_\varepsilon^{E,m})'-x_1E'\frac{\eta'}{\varepsilon}\right)\frac{\eta_l}{\varepsilon}d\eta'\right|^2dx_1 \\ &\le\frac12\int_{\Omega_\varepsilon}A_\varepsilon\begin{pmatrix} E_{11}-E_1'\cdot\dfrac{x'}{\varepsilon} & \dfrac12\left(E'\dfrac{x'}{\varepsilon}\right)^T \\ \dfrac12E'\dfrac{x'}{\varepsilon} & 0\end{pmatrix}:\begin{pmatrix} E_{11}-E_1'\cdot\dfrac{x'}{\varepsilon} & \dfrac12\left(E'\dfrac{x'}{\varepsilon}\right)^T \\ \dfrac12E'\dfrac{x'}{\varepsilon} & 0\end{pmatrix}dx. \end{aligned}$$

Taking into account (3.14), we then deduce

$$\frac{1}{2|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} A_\varepsilon e(u_\varepsilon^{E,m}):e(u_\varepsilon^{E,m})\,dx+\frac{m}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}\left|u_{\varepsilon,1}^{E,m}-E_{11}x_1+x_1E_1'\cdot\frac{x'}{\varepsilon}\right|^2 dx$$
$$+m\sum_{l=2}^N\int_0^1\left|\frac{1}{|\varepsilon\omega|}\int_{\varepsilon\omega}\left((u_\varepsilon^{E,m})'-x_1E'\frac{\eta'}{\varepsilon}\right)\frac{\eta_l}{\varepsilon}d\eta'\right|^2 dx_1\le C|E|^2. \tag{3.80}$$

In particular, $u_\varepsilon^{E,m}$ satifies (3.22) and since it vanishes on $x_1=0$, it also satisfies (3.23). This allows us to decompose $u_\varepsilon^{E,m}$ as in (3.41), with b_ε', w_ε and Z_ε replaced by $(b_\varepsilon^{E,m})'$, $w_\varepsilon^{E,m}$ and $Z_\varepsilon^{E,m}$. Up to a subsequence of ε, still denoted by ε, we can assume the existence of $(u^{E,m})'$, $Z^{E,m}$, $w^{E,m}$, $z^{E,m}$, and $\sigma^{E,m}$ such that the analogous to (3.48), (3.49), (3.50), (3.51) hold. As above, we will denote $w_1^{E,m}$ as $u_1^{E,m}$. Moreover, by linearity we can take the subsequence of ε independent of E.

Using the decomposition of $u_\varepsilon^{E,m}$ and taking into account (3.11), (3.12) and (3.13), we observe that the sequence $u_\varepsilon^{E,m}$ satisfies (3.17), with h_ε replaced by $h_\varepsilon^{E,m}$, defined by

$$h_{\varepsilon,1}^{E,m}(x)=f_{\varepsilon,1}^{E,m}\left(x_1,\frac{x'}{\varepsilon}\right),\quad (h_\varepsilon^{E,m})'=(g_\varepsilon^{E,m})'\left(x_1,\frac{x'}{\varepsilon}\right),$$

with

$$f_{\varepsilon,1}^{E,m}(y):=-m\left(-\left(\frac{d(b_\varepsilon^{E,m})'}{dy_1}-y_1E_1'\right)\cdot y'+w_{\varepsilon,1}^{E,m}(y_1,\varepsilon y')-E_{11}y_1\right)\quad \text{a.e. } y\in\Omega,$$

and

$$(g_\varepsilon^{E,m})'(y):=-m\sum_{l=2}^N\big((Z_\varepsilon^{E,m})'-y_1E'\big)e_lI_ly_l-m\sum_{l=2}^N\frac{1}{|\varepsilon\omega|}\int_{\varepsilon\omega}\left(w_\varepsilon^{E,m}\right)'\frac{\eta_l}{\varepsilon}\,d\eta' y_l,$$

which, taking into account (3.11) and Poincaré-Wirtinger's inequality, imply

$$\left|\int_{\varepsilon\omega}\left(w_\varepsilon^{E,m}\right)'\frac{\eta_l}{\varepsilon}\,d\eta'\right|=\left|\int_{\varepsilon\omega}\left(\left(w_\varepsilon^{E,m}\right)'-\frac{1}{|\varepsilon\omega|}\int_{\varepsilon\omega}\left(w_\varepsilon^{E,m}\right)'d\mu'\right)\frac{\eta_l}{\varepsilon}\,d\eta'\right|$$
$$\le C\varepsilon^3\left(\int_{\varepsilon\omega}\left|D_{\eta'}\left(w_\varepsilon^{E,m}\right)'\right|^2 d\eta'\right)^{\frac12}.$$

Thus, we have

$$f_{\varepsilon,1}^{E,m}\to f_1^{E,m}:=-m\left(-\left(\frac{d(u^{E,m})'}{dy_1}-y_1E_1'\right)\cdot y'+u_1^{E,m}-E_{11}y_1\right)\ \text{ in } L^2(\Omega),$$

$$(g_\varepsilon^{E,m})'\to -m\sum_{l=2}^N\big((Z^{E,m})'-y_1E'\big)e_lI_ly_l\ \text{ in } L^2(\Omega)^{N-1}.$$

This allows us to apply Steps 3 and 4 to $u_\varepsilon^{E,m}$ and, taking into account the boundary conditions imposed to $u_\varepsilon^{E,m}$, to deduce that the corresponding function $\Lambda^{E,m}\in$

$H^1(0,1;\mathbb{R}^{N\times N}_{s_1sk'})$ given by (4.1) with $p^{E,m}$, $(q^{E,m})'$ and $R^{E,m}$ given by (3.65), (3.63) and (3.61) respectively, satisfies

$$|\Lambda^{E,m}| \leq \beta\big(\Lambda^{E,m} : e_0(u^{E,m}, Z^{E,m})\big)^{\frac{1}{2}} \left(\mathfrak{a}^L\right)^{\frac{1}{2}}, \quad \text{a.e. in } (0,1), \tag{3.81}$$

$$\big|e_0(u^{E,m}, Z^{E,m})\big|^2 \leq \gamma\big(\Lambda^{E,m} : e_0(u^{E,m}, Z^{E,m})\big), \quad \text{a.e. in } (0,1), \tag{3.82}$$

$$\begin{cases} \displaystyle\int_0^1 \Lambda^{E,m} : e_0(\tilde{u}, \tilde{Z})\, dx_1 + m\int_0^1 \left(u_1^{E,m} - E_{11}x_1\right)\tilde{u}_1 dx_1 \\ \displaystyle +m\int_0^1 \mathcal{I}\left(\frac{d(u^{E,m})'}{dx_1} - x_1E_1'\right)\cdot\frac{d\tilde{u}'}{dx_1}dx_1 + m\int_0^1 \big((Z^{E,m} - E'x_1)\mathcal{I}\big) : \tilde{Z}\, dy_1 = 0, \\ \forall\, \tilde{u}_1 \in W^{1,1}_{\{0\}}(0,1),\ \tilde{u}' \in W^{2,1}_{\{0\}}(0,1)^{N-1},\ \tilde{Z} \in W^{1,1}_{\{0\}}(0,1;\mathbb{R}^{(N-1)\times(N-1)}_{sk}). \end{cases} \tag{3.83}$$

Moreover, passing to the limit in (3.80), thanks to (3.68), we have

$$\frac{1}{2}\int_0^1 \Lambda^{E,m} : e_0(u^{E,m}, Z^{E,m})\, dx_1 + \frac{m}{|\Omega|}\int_\Omega \left|u_1^{E,m} - E_{11}y_1 - \left(\frac{d\left(u^{E,m}\right)'}{dy_1} - y_1E'\right)y'\right|^2 dy$$
$$+m\sum_{l=2}^N \int_0^1 \left|\frac{1}{|\omega|}\int_\omega \left(Z^{E,m} - x_1E'\right)y'\, y_l\, dy'\right|^2 dx_1 \leq C|E|^2,$$

which taking into account (3.11) and (3.12) can also be written as

$$\int_0^1 \Lambda^{E,m} : e_0(u^{E,m}, Z^{E,m})\, dx_1 + m\int_0^1 \left|u_1^{E,m} - E_{11}x_1\right|^2 dx_1 + m\int_0^1 \left|\frac{d(u^{E,m})'}{dx_1} - E_1'x_1\right|^2 dx_1$$
$$+m\int_0^1 \left|Z^{E,m} - x_1E'\right|^2 dx_1 \leq C|E|^2. \tag{3.84}$$

From (3.82) and (3.84), taking m converging to ∞, we deduce

$$\begin{cases} u_1^{E,m} \rightharpoonup E_{11}x_1 \ \text{ weakly in } H^2_{\{0\}}(0,1), \\ (u^{E,m})' \rightharpoonup \dfrac{1}{2}E_1'x_1^2 \ \text{ weakly in } H^2_{\{0\}}(0,1)^{N-1}, \\ Z^{E,m} \rightharpoonup E'x_1 \ \text{ weakly in } H^1_{\{0\}}(0,1;\mathbb{R}^{(N-1)\times(N-1)}_{sk}). \end{cases} \tag{3.85}$$

By (3.81), $\mathfrak{a}^L \in L^1(0,1)$ and (3.84), we also deduce that $\Lambda^{E,m}$ is bounded in $L^1(0,1;\mathbb{R}^{N\times N}_{s_1sk'})$ and is equi-integrable. Therefore, by linearity, we can extract a subsequence of m, still denoted by m, such that there exist $A \in L^1(0,1;\mathcal{L}(\mathbb{R}^{N\times N}_{s_1sk'}))$ satisfying

$$\Lambda^{E,m} \rightharpoonup AE \ \text{ weakly in } L^1(0,1;\mathbb{R}^{N\times N}_{s_1sk'}), \quad \forall\, E \in \mathbb{R}^{N\times N}_{s_1sk'}. \tag{3.86}$$

Let us also show the inequality

$$\limsup_{m\to\infty}\int_0^1 \Lambda^{E,m} : e_0(u^{E,m}, Z^{E,m})\, \varphi\, dx_1 \leq \int_0^1 AE : E\, \varphi\, dx_1, \quad \forall\, \varphi \in C^\infty([0,1]),\ \varphi \geq 0. \tag{3.87}$$

For this purpose, given $\varphi \in C^\infty([0,1])$, $\varphi \geq 0$, we take as test function in (3.83)

$$\begin{cases} \tilde{u}_1 = \varphi\big(u_1^{E,m} - E_{11}x_1\big), \\ \tilde{u}' = \displaystyle\int_0^{x_1} \varphi(t) \left(\frac{d(u^{E,m})'}{dt}(t) - E_1' t\right) dt, \\ \tilde{Z} = \varphi(Z^{E,m} - E'x_1). \end{cases}$$

We get

$$\int_0^1 \Lambda^{E,m} : e_0(u^{E,m}, Z^{E,m})\, \varphi\, dx_1 - \int_0^1 \Lambda^{E,m} : E\, \varphi\, dx_1$$
$$+ \int_0^1 \Lambda^{E,m} : \begin{pmatrix} u_1^{E,m} - E_{11}x_1 & \left(\dfrac{d(u^{E,m})'}{dx_1} - E_1' x_1\right)^T \\ \dfrac{d(u^{E,m})'}{dx_1} - E_1' x_1 & Z^{E,m} - E'x_1 \end{pmatrix} \frac{d\varphi}{dx_1} dx_1$$
$$+ m\int_0^1 \Big(u_1^{E,m} - E_{11}x_1\Big)^2 \varphi\, dx_1 + m\int_0^1 \mathcal{I}\left(\frac{d(u^{E,m})'}{dx_1} - x_1 E_1'\right) \cdot \left(\frac{d(u^{E,m})'}{dx_1} - x_1 E_1'\right) dx_1$$
$$+ m\int_0^1 \big((Z^{E,m} - E'x_1)\mathcal{I}\big) : (Z^{E,m} - E'x_1)\, dx_1 = 0.$$

Thanks to (3.85), (3.86) and the compact embedding of $H^1(0,1)$ into $C^0([0,1])$, we can pass to the limit in the first and second terms of this equality. By also using that the three last terms are non-negative, we conclude(3.87).

By (3.86), (3.87), (3.81), (3.82) and the semicontiuity of the norm for the weak convergence we deduce that A satisfies (3.26) and (3.27).

Step 7. Let us now finish the proof of the theorem by showing that if u_ε is a sequence which satisfies (3.17) and (3.23), with $h_\varepsilon \in L^2(\Omega_\varepsilon)^N$ defined by (3.18), and $f_\varepsilon \in L^2(\Omega)^N$ and $g_\varepsilon' \in L^2(\Omega)^{N-1}$ satisfying (3.19), (3.20) and (3.21), then the matrix function Λ defined by (4.1) is given by

$$\Lambda = Ae_0(u, Z),$$

with u, Z defined by (3.48), (3.49), (3.50) and (3.52), which combined with (3.67) with $\tilde{z} = 0$ shows that (u, Z) satisfies (3.37). For this purpose, we observe that by linearity, for every $E \in \mathbb{R}^{N\times N}_{s_1 sk'}$, and every $m \in \mathbb{N}$, the sequence $u_\varepsilon - u_\varepsilon^{m,E}$, with $u_\varepsilon^{m,E}$ defined by (3.79), is also the solution to a problem similar to (3.17) and satisfies properties (3.22) and (3.23). Applying then (3.73) to this sequence, we deduce the inequality

$$\left|\Lambda - \Lambda^{E,m}\right| \leq C\Big(\big(\Lambda - \Lambda^{E,m}\big) : e_0(u - u^{E,m}, Z - Z^{E,m})\Big)^{\frac{1}{2}} (\mathfrak{a}^L)^{\frac{1}{2}}, \quad \text{a.e. in } (0,1).$$

Multiplying this inequality by $\varphi \in C^\infty([0,1])$, $\varphi \geq 0$, integrating in $(0,1)$, using the

Cauchy-Schwarz inequality and developing the factors, we get

$$\int_0^1 \left|\Lambda - \Lambda^{E,m}\right| \varphi\, dx_1 \le C \left(\int_0^1 \mathfrak{a}^L \varphi\, dx_1\right)^{\frac{1}{2}}$$

$$\cdot\left(\int_0^1 \Big(\Lambda\!:\!e_0(u,Z)-\Lambda\!:\!e_0(u^{E,m}\!,Z^{E,m})-\Lambda^{E,m}\!:\!e_0(u,Z)+\Lambda^{E,m}\!:\!e_0(u^{E,m}\!,Z^{E,m})\Big)\varphi dx_1\right)^{\frac{1}{2}}. \tag{3.88}$$

Let us pass to the limit when m tends to infintiy, in the different terms of the last factor. For the second term we use that $\Lambda \in H^1(0,1;\mathbb{R}^{N\times N}_{s_1sk'}) \subset L^2(0,1;\mathbb{R}^{N\times N}_{s_1sk'})$ and (3.85), which imply

$$\int_0^1 \Lambda : e_0(u^{E,m}, Z^{E,m})\varphi\, dx_1 \to \int_0^1 \Lambda : E\,\varphi\, dx_1.$$

In the third term we use (3.86) and (3.78) to get

$$\int_0^1 \Lambda^{E,m} : e_0(u,Z)\varphi\, dx_1 \to \int_0^1 (AE) : e_0(u,Z)\,\varphi\, dx_1.$$

In the fourth term, we use (3.87). Therefore, using also the semicontinuity of the norm for the weak convergence in $L^1(0,1)$ in the left-hand side of (3.88) we have proved

$$\int_0^1 |\Lambda - AE|\,\varphi\, dx_1 \le C\left(\int_0^1 \mathfrak{a}^L\varphi\, dx_1\right)^{\frac{1}{2}}\left(\int_0^1 (\Lambda - AE) : (e_0(u,Z) - E)\varphi\, dx_1\right)^{\frac{1}{2}},$$

for all $\varphi \in C^\infty([0,1]),\ \varphi \ge 0$, which implies

$$|\Lambda - AE| \le C\big(\mathfrak{a}^L\big)^{\frac{1}{2}}\big((\Lambda - AE) : (e_0(u,Z) - E)\big)^{\frac{1}{2}}, \quad \forall\, E \in \mathbb{R}^{N\times N}_{s_1sk'}, \ \text{ a e. in } (0,1).$$

This proves

$$\Lambda = Ae_0(u,Z), \ \text{ a.e in } (0,1). \tag{3.89}$$

□

Proof of Proposition 3.7. For every $E \in \mathbb{R}^{N\times N}_{s_1sk'}$, $m \in \mathbb{N}$ and $\varepsilon > 0$, we consider the function $u^{E,m}_\varepsilon$ defined by (3.79), which satisfies (3.30), with u, Z replaced by $u^{E,m}$, $Z^{E,m}$, solution to (3.83), where thanks to (3.89), we now know that

$$\Lambda^{E,m} = Ae_0(u^{E,m}, Z^{E,m}) \tag{3.90}$$

(and then (3.83) has a unique solution). On the other hand, we define $\tilde{u}^{E,m}_\varepsilon$ as the solution to (3.79) when A_ε is replaced by A^T_ε. By applying Theorem 3.1 to $\tilde{A}^T_\varepsilon$, we can also assume the existence of functions $\tilde{u}^{E,m}$, $\tilde{Z}^{E,m}$ $\tilde{A}$, which are the analogous to $u^{E,m}$, $Z^{E,m}$ and A. By (3.32) applied to the two sequences $u^{E,m}_\varepsilon$ and $\tilde{u}^{\tilde{E},\tilde{m}}_\varepsilon$, with $E, \tilde{E} \in \mathbb{R}^{N\times N}_{s_1sk'}$, $m, \tilde{m} \in \mathbb{N}$, we have

$$\int_0^1 Ae_0(u^{E,m}, Z^{E,m}) : e_0(\tilde{u}^{\tilde{E},\tilde{m}}, \tilde{Z}^{\tilde{E},\tilde{m}})\,\varphi\, dx = \lim_{\varepsilon\to 0}\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} A_\varepsilon e(u^{E,m}_\varepsilon) : e(\tilde{u}^{\tilde{E},\tilde{m}}_\varepsilon)\varphi\, dx$$

$$= \lim_{\varepsilon\to 0}\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} \tilde{A}_\varepsilon e(\tilde{u}^{\tilde{E},\tilde{m}}_\varepsilon) : e(u^{E,m}_\varepsilon)\varphi\, dx = \int_0^1 \tilde{A}e_0(\tilde{u}^{\tilde{E},\tilde{m}}, \tilde{Z}^{\tilde{E},\tilde{m}}) : e_0(u^{E,m}, Z^{E,m})\,\varphi\, dx,$$

for every $\varphi \in C_0^\infty(0,1)$, which proves

$$Ae_0(u^{E,m}, Z^{E,m}) : e_0(\tilde{u}^{\tilde{E},\tilde{m}}, \tilde{Z}^{\tilde{E},\tilde{m}}) = \tilde{A}e_0(\tilde{u}^{\tilde{E},\tilde{m}}, \tilde{Z}^{\tilde{E},\tilde{m}}) : e_0(u^{E,m}, Z^{E,m}) \text{ a.e. in } (0,1).$$

Taking into account Remark 3.6, we know that for every $\tilde{E} \in \mathbb{R}^{N\times N}_{s_1 sk'}$ and $\tilde{m} \in \mathbb{N}$, the functions $\tilde{A}e_0(\tilde{u}^{\tilde{E},\tilde{m}}, \tilde{Z}^{\tilde{E},\tilde{m}})$ and $e_0(\tilde{u}^{\tilde{E},\tilde{m}}, \tilde{Z}^{\tilde{E},\tilde{m}})$ are in $L^\infty(0,1;\mathbb{R}^{N\times N}_{s_1 sk'})$. Therefore, taking into account (3.85) and (3.86), applied to the sequence $\tilde{u}^{\tilde{E},\tilde{m}}$, combined with (3.90) we can pass to the limit when $\tilde{m}$ tends to infinity in the above equality to deduce

$$AE : e_0(\tilde{u}^{\tilde{E}}, \tilde{Z}^{\tilde{E}}) = \tilde{A}e_0(\tilde{u}^{\tilde{E},\tilde{m}}, \tilde{Z}^{\tilde{E},\tilde{m}}) : E \text{ a.e. in } (0,1). \tag{3.91}$$

Now, for $K > 0$ we take

$$A_K = \{x_1 \in (0,1) : \ |A(x_1)\tilde{E}| \le K\}.$$

Using again (3.85) and (3.86) but now applied to $\tilde{u}^{E,m}$, we can pass to the limit in m in (3.91) restricted to A_K to deduce

$$AE : \tilde{E} = \tilde{A}\tilde{E} : E \text{ a.e. in } A_K, \quad \forall\, K > 0,$$

and then, passing to the limit when K tends to infinity

$$AE : \tilde{E} = \tilde{A}\tilde{E} : E \text{ a.e. in } (0,1), \quad \forall\, E, \tilde{E} \in \mathbb{R}^{N\times N}_{s_1 s'_k},$$

which gives

$$\tilde{A}\tilde{E} = A^T\tilde{E} \text{ a.e. in } (0,1), \quad \forall\, \tilde{E} \in \mathbb{R}^{N\times N}_{s_1 s'_k},$$

and then proves the equality $\tilde{A} = A^T$. □

Proof of Corollary 3.8. Since u_ε vanishes on $x_1 = 0$, the second term on the right-hand side of (3.43) vanishes. Therefore, taking u_ε as test function in (3.36), we get

$$\begin{aligned} &\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} A_\varepsilon e(u_\varepsilon) : e(u_\varepsilon)\,dx \\ &\le \frac{C}{|\Omega_\varepsilon|}\left(\int_{\Omega_\varepsilon}\left(\left|f_\varepsilon\left(x_1, \frac{x'}{\varepsilon}\right)\right|^2 + \left|g'_\varepsilon\left(x_1, \frac{x'}{\varepsilon}\right)\right|^2\right)dx\right)^{\frac12}\left(\int_{\Omega_\varepsilon}|e(u_\varepsilon)|^2dx\right)^{\frac12} \\ &= C\left(\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}\left(|f_\varepsilon|^2 + |g'_\varepsilon|^2\right)dx\right)^{\frac12}\left(\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}|e(u_\varepsilon)|^2dx\right)^{\frac12}. \end{aligned} \tag{3.92}$$

By (3.16), this proves

$$\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} A_\varepsilon e(u_\varepsilon) : e(u_\varepsilon)\,dx \le C,$$

which proves that u_ε satisfies (3.22). Since u_ε vanishes on $x_1 = 0$, it also satisfies (3.23). Therefore, u_ε is in the conditions of Theorem 3.1. Applying this theorem and taking into account (3.31), which gives the boundary conditions for u and Z, we conclude the thesis of the corollary. □

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Chapter 4

Homogenization of weakly equicoercive integral functionals in three-dimensional elasticity

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Abstract.
This paper deals with the homogenization through Γ-convergence of weakly coercive integral energies with the oscillating density $\mathbb{L}(x/\varepsilon)\nabla v : \nabla v$ in three-dimensional elasticity. The energies are weakly coercive in the sense where the classical functional coercivity satisfied by the periodic tensor $\mathbb{L}$ (using smooth test functions v with compact support in $\mathbb{R}^3$) which reads as $\Lambda(\mathbb{L}) > 0$, is replaced by the relaxed condition $\Lambda(\mathbb{L}) \geq 0$. Surprisingly, we prove that contrary to the two-dimensional case of [2] which seems *a priori* more constrained, the homogenized tensor $\mathbb{L}^0$ remains strongly elliptic, or equivalently $\Lambda(\mathbb{L}^0) > 0$, for any tensor $\mathbb{L} = \mathbb{L}(y_1)$ satisfying $\mathbb{L}(y)M : M + D : \mathrm{Cof}(M) \geq 0$, a.e. $y \in \mathbb{R}^3$, $\forall\, M \in \mathbb{R}^{3\times 3}$, for some matrix $D \in \mathbb{R}^{3\times 3}$ (which implies $\Lambda(\mathbb{L}) \geq 0$), and the periodic functional coercivity (using smooth test

functions v with periodic gradients) which reads as $\Lambda_{\rm per}(\mathbb{L}) > 0$. Moreover, we derive the loss of strong ellipticity for the homogenized tensor using a rank-two lamination, which justifies by Γ-convergence the formal procedure of [8].

4.1 Introduction

In this paper, for a bounded domain Ω of $\mathbb{R}^3$ and for a periodic symmetric tensor-valued function $\mathbb{L} = \mathbb{L}(y)$, we study the homogenization of the elasticity energy

$$v \in H_0^1(\Omega;\mathbb{R}^3) \mapsto \int_\Omega \mathbb{L}(x/\varepsilon)\nabla v \cdot \nabla v\, dx \quad \text{as } \varepsilon \to 0, \tag{4.1}$$

especially when the tensor $\mathbb{L}$ is weakly coercive (see below). It is shown in [10, 4] that for any periodic symmetric tensor-valued function $\mathbb{L} = \mathbb{L}(y)$ satisfying the functional coercivity, *i.e.*

$$\Lambda(\mathbb{L}) := \inf\left\{\int_{\mathbb{R}^3} \mathbb{L}\nabla v : \nabla v\, dy,\ v \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3),\ \int_{\mathbb{R}^3} |\nabla v|^2\, dy = 1\right\} > 0, \tag{4.2}$$

and for any $f \in H^{-1}(\Omega;\mathbb{R}^3)$, the elasticity system

$$\begin{cases} -\operatorname{div}\big(\mathbb{L}(x/\varepsilon)\nabla u^\varepsilon\big) = f & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

H-converges as $\varepsilon \to 0$ in the sense of Murat-Tartar [3] to the elasticity system with the so-called homogenized tensor $\mathbb{L}^0$ defined by

$$\mathbb{L}^0 M : M := \inf\left\{\int_{Y_3} \mathbb{L}(M + \nabla v) : (M + \nabla v)\, dy,\ v \in H^1_{\rm per}(Y_3;\mathbb{R}^3)\right\} \quad \text{for } M \in \mathbb{R}^{3\times 3}. \tag{4.4}$$

Equivalently, under the functional coercivity (4.2) the energy (4.1) Γ-converges for the weak topology of $H_0^1(\Omega;\mathbb{R}^3)$ (see Definition 4.2) to the functional

$$v \in H_0^1(\Omega;\mathbb{R}^3) \mapsto \int_\Omega \mathbb{L}^0\nabla v : \nabla v\, dx. \tag{4.5}$$

The functional coercivity (4.2), which is a nonlocal condition satisfied by the symmetric tensor $\mathbb{L}$, is implied by the very strong ellipticity, *i.e.* the local condition

$$\alpha_{\rm vse}(\mathbb{L}) := \operatorname*{ess-inf}_{y\in\mathbb{R}^3} \big(\min\{\mathbb{L}(y)M : M,\ M \in \mathbb{R}^{3\times 3}_s,\ |M| = 1\}\big) > 0, \tag{4.6}$$

and the converse is not true in general. Moreover, condition (4.2) implies the strong ellipticity, *i.e.*

$$\alpha_{\rm se}(\mathbb{L}) := \operatorname*{ess-inf}_{y\in\mathbb{R}^3} \big(\min\{\mathbb{L}(y)(a\otimes b) : (a\otimes b),\ a, b \in \mathbb{R}^3,\ |a| = |b| = 1\}\big) > 0, \tag{4.7}$$

but contrary to the scalar case, the converse is not true in general.

Here, we focus on the case where the tensor $\mathbb{L}$ is weakly coercive, *i.e.* relaxing the condition $\Lambda(\mathbb{L}) > 0$ by $\Lambda(\mathbb{L}) \geq 0$. In this case the homogenization of the elasticity system (4.3) associated with the energy (4.1) is badly posed in general, since one has no *a priori* L^2-bound on the stress tensor ∇u^ε (assuming the existence of a solution u^ε to the elasticity system (4.3)) due to the loss of coercivity. However, it was shown by Geymonat *et al.* [7] that the previous Γ-convergence result still holds when $\Lambda(\mathbb{L}) \geq 0$, under the extra condition of periodic functional coercivity, *i.e.*

$$\Lambda_{\rm per}(\mathbb{L}) := \inf\left\{\int_{Y_3} \mathbb{L}\nabla v : \nabla v\, dy,\ v \in H^1_{\rm per}(Y_3;\mathbb{R}^3),\ \int_{Y_3} |\nabla v|^2\, dy = 1\right\} > 0. \quad (4.8)$$

Furthermore, using the Murat-Tartar 1^*-convergence for tensors which depend only on one direction (see [3] in the conductivity case, see [8, Section 3] and [2, Lemma 3.1] in the elasticity case) Gutiérrez [8, Proposition 1] derived in two and three dimensions a 1-periodic rank-one laminate with two isotropic phases whose tensor is

$$\mathbb{L}_1(y_1) = \chi_1(y_1)\,\mathbb{L}_a + \big(1 - \chi(y_1)\big)\,\mathbb{L}_b \quad \text{for } y_1 \in \mathbb{R}, \quad (4.9)$$

which is strongly elliptic, *i.e.* $\alpha_{\rm se}(\mathbb{L}) > 0$, and weakly coercive, *i.e.* $\Lambda(\mathbb{L}) \geq 0$, but such that the homogenized tensor $\mathbb{L}^0$ (in fact the homogenized tensor induced by 1^*-convergence which is shown to agree with $\mathbb{L}^0$ in the step 4 of the proof of Theorem 4.14) is not strongly elliptic, *i.e.* $\alpha_{\rm se}(\mathbb{L}^0) = 0$. However, the 1^*-convergence process used by Gutiérrez in [8] needs to have *a priori* L^2-bounds for the sequence of deformations, which is not compatible with the weak coercivity assumption. Therefore, Gutiérrez' approach is not a H-convergence process applied to the elasticity system (4.3). Francfort and the first author [2] obtained in dimension two a similar loss of ellipticity through a homogenization process using the Γ-convergence approach of [7] from a more generic (with respect to (4.9)) 1-periodic isotropic tensor $\mathbb{L} = \mathbb{L}(y_1)$ satisfying

$$\Lambda(\mathbb{L}) = 0, \quad \Lambda_{\rm per}(\mathbb{L}) > 0 \quad \text{and} \quad \alpha_{\rm se}(\mathbb{L}^0) = 0. \quad (4.10)$$

They also showed that Gutiérrez' lamination is the only one among rank-one laminates which implies such a loss of strong ellipticity.

The aim of the paper is to extend the result of [2] to dimension three, namely justifying the loss of ellipticity of [8] by a homogenization process. The natural idea is to find as in [2] a 1-periodic isotropic tensor $\mathbb{L} = \mathbb{L}(y_1)$ satisfying (4.10). Firstly, in order to check the relaxed functional coercivity $\Lambda(\mathbb{L}) \geq 0$, we apply the translation method used in [2], which consists in adding to the elastic energy density a suitable null lagrangian such that the following pointwise inequality holds for some matrix $D \in \mathbb{R}^{3\times3}$:

$$\mathbb{L}M : M + D : \mathrm{Cof}(M) \geq 0, \quad \forall\, M \in \mathbb{R}^{3\times3}. \quad (4.11)$$

Note that in dimension two the translation method reduces to adding the term $d\det(M)$ with one coefficient d, rather than a (3×3)-matrix D in dimension three. But surprisingly, and contrary to the two-dimensional case of [2], we prove (see Theorem 4.8) that for any 1-periodic tensor $\mathbb{L} = \mathbb{L}(y_1)$, condition (4.11) combined with

$\Lambda_{\rm per}(\mathbb{L}) > 0$ actually implies that $\alpha_{\rm se}(\mathbb{L}^0) > 0$, making impossible the loss of ellipticity through homogenization. This specificity was already observed by Gutiérrez [8] in the particular case of isotropic two-phase rank-one laminates (4.9), where certain regimes satisfied by the Lamé coefficients of the isotropic phases $\mathbb{L}_a, \mathbb{L}_b$ are not compatible with the desired equality $\alpha_{\rm se}(\mathbb{L}^0) = 0$.

To overcome this difficulty Gutiérrez [8] considered a rank-two laminate obtained by mixing in the direction y_2 the homogenized tensor $\mathbb{L}_1^*$ of $\mathbb{L}_1(y_1)$ defined by (4.9), with a very strongly elliptic isotropic tensor $\mathbb{L}_c$. In the present context we derive a similar loss of ellipticity by rank-two lamination, but justifying it through homogenization still using a Γ-convergence procedure (see Theorem 4.14). However, the proof is rather delicate, since we have to choose the isotropic materials a, b, c so that the 1-periodic rank-one laminate tensor $\mathbb{L}_2$ in the direction y_2 obtained after the first rank-one lamination of $\mathbb{L}_a, \mathbb{L}_b$ in the direction y_1, namely

$$\mathbb{L}_2(y_2) = \chi_2(y_2)\,\mathbb{L}_1^* + \big(1 - \chi_2(y_2)\big)\,\mathbb{L}_c \quad \text{for } y_2 \in \mathbb{R}, \tag{4.12}$$

satisfies

$$\Lambda(\mathbb{L}_2) \geq 0 \quad \text{and} \quad \alpha_{\rm se}(\mathbb{L}_2^0) = 0, \tag{4.13}$$

where $\mathbb{L}_2^0$ is the homogenized tensor defined by formula (4.4) with $\mathbb{L} = \mathbb{L}_2$. Moreover, the condition $\Lambda(\mathbb{L}_2) \geq 0$ without $\Lambda_{\rm per}(\mathbb{L}_2) > 0$ (which seems very intricate to check) needs to extend the Γ-convergence result of [7, Theorem 3.1(i)]. However, Braides and the first author have proved (see Theorem 4.5) that the Γ-convergence result for the energy (4.1) holds true under the sole condition $\Lambda(\mathbb{L}) \geq 0$.

The paper is divided in two sections. In the first section we prove the Γ-convergence result for (4.1) under the assumption $\Lambda(\mathbb{L}) \geq 0$, and without the condition $\Lambda_{\rm per}(\mathbb{L}) > 0$. The second section is devoted to the main results of the paper: In Section 4.3.1 we prove the strong ellipticity of the homogenized tensor $\mathbb{L}^0$ for any isotropic tensor $\mathbb{L} = \mathbb{L}(y_1)$ satisfying both the two conditions (4.11) (which implies $\Lambda(\mathbb{L}) \geq 0$) and $\Lambda_{\rm per}(\mathbb{L}) > 0$. In Section 4.3.2 we show the loss ellipticity by homogenization using a suitable rank-two laminate tensor $\mathbb{L}_2$ of type (4.12), and the Γ-convergence result under the sole condition $\Lambda(\mathbb{L}_2) \geq 0$. Finally, the Appendix is devoted to the proof of Theorem 4.4.

Notations

- The space dimension is denoted by $N \geq 2$, but most of the time it will be $N = 3$.
- $\mathbb{R}_s^{N\times N}$ denotes the set of the symmetric matrices in $\mathbb{R}^{N\times N}$.
- I_N denotes the identity matrix of $\mathbb{R}^{N\times N}$.
- For any $M \in \mathbb{R}^{N\times N}$, M^T denotes the transposed of M, and M^s denotes the symmetrized matrix of M.
- $:$ denotes the Frobenius inner product in $\mathbb{R}^{N\times N}$, *i.e.* $M : M' := \mathrm{tr}(M^T M')$ for $M, M' \in \mathbb{R}^{N\times N}$.

- $\mathscr{L}_s(\mathbb{R}^{N\times N})$ denotes the space of the symmetric tensors $\mathbb{L}$ on $\mathbb{R}^{N\times N}$ satisfying

$$\mathbb{L}M = \mathbb{L}M^s \in \mathbb{R}^{N\times N}_s \quad \text{and} \quad \mathbb{L}M : M' = \mathbb{L}M' : M, \quad \forall\, M, M' \in \mathbb{R}^{N\times N}_s.$$

  In terms of the entries $\mathbb{L}_{ijkl}$ of $\mathbb{L}$, this is equivalent to $\mathbb{L}_{ijkl} = \mathbb{L}_{jikl} = \mathbb{L}_{klij}$ for any $i, j, k, l \in \{1, \ldots, N\}$.

- $\mathbb{I}_s$ denotes the unit tensor of $\mathscr{L}_s(\mathbb{R}^{N\times N})$ defined by $\mathbb{I}_s M := M^s$ for $M \in \mathbb{R}^{N\times N}$.

- M_{ij} denotes the (i, j) entry of the matrix $M \in \mathbb{R}^{N\times N}$.

- $\tilde{M}^{ij}$ denotes the $(N-1) \times (N-1)$-matrix resulting from deleting the i-th row and the j-th column of the matrix $M \in \mathbb{R}^{N\times N}$ for $i, j \in \{1, \ldots, N\}$.

- $\mathrm{Cof}(M)$ denotes the cofactors matrix of $M \in \mathbb{R}^{N\times N}$, *i.e.* the matrix with entries $(\mathrm{Cof}M)_{ij} = (-1)^{i+j} \det(\tilde{M}^{ij})$ for $i, j \in \{1, \ldots, N\}$.

- $\mathrm{adj}(M)$ denotes the adjugate matrix of $M \in \mathbb{R}^{N\times N}$, *i.e.* $\mathrm{adj}(M) = (\mathrm{Cof}\, M)^T$.

- $Y_N := [0, 1)^N$ denotes the unit cube of $\mathbb{R}^N$.

- $e(u)$ denotes the symmetric part of the gradient of u, ∇u, for $u \in W^{1,p}(\mathbb{R}^N; \mathbb{R}^N)$.

Let $\mathbb{L} \in L^\infty_{\mathrm{per}}\big(Y_N; \mathscr{L}_s(\mathbb{R}^{N\times N})\big)$ be a Y_N-periodic symmetric tensor-valued function. In the whole paper we will use the following ellipticity constants related to the tensor $\mathbb{L}$ (see [7, Section 3] for further details):

- $\alpha_{\mathrm{se}}(\mathbb{L})$ denotes the best ellipticity constant for $\mathbb{L}$, *i.e.*

$$\alpha_{\mathrm{se}}(\mathbb{L}) := \operatorname*{ess\text{-}inf}_{y \in Y_N} \big(\min\{\mathbb{L}(y)(a \otimes b) : (a \otimes b),\ a, b \in \mathbb{R}^N,\ |a| = |b| = 1\}\big).$$

- $\alpha_{\mathrm{vse}}(\mathbb{L})$ denotes the best constant of very strong ellipticity of $\mathbb{L}$, *i.e.*

$$\alpha_{\mathrm{vse}}(\mathbb{L}) := \operatorname*{ess\text{-}inf}_{y \in Y_N} \big(\min\{\mathbb{L}(y)M : M,\ M \in \mathbb{R}^{N\times N}_s, |M| = 1\}\big).$$

- $\Lambda(\mathbb{L})$ denotes the global functional coercivity constant for $\mathbb{L}$, *i.e.*

$$\Lambda(\mathbb{L}) := \inf\left\{ \int_{\mathbb{R}^N} \mathbb{L}\nabla v : \nabla v \, dy,\ v \in C^\infty_c(\mathbb{R}^N; \mathbb{R}^N),\ \int_{\mathbb{R}^N} |\nabla v|^2 \, dy = 1 \right\}.$$

- $\Lambda_{\mathrm{per}}(\mathbb{L})$ denotes the functional coercivity constant of $\mathbb{L}$ with respect to Y_N-periodic deformations, *i.e.*

$$\Lambda_{\mathrm{per}}(\mathbb{L}) := \inf\left\{ \int_{Y_N} \mathbb{L}\nabla v : \nabla v \, dy,\ v \in H^1_{\mathrm{per}}(Y_N; \mathbb{R}^N),\ \int_{Y_N} |\nabla v|^2 \, dy = 1 \right\}.$$

Remark 4.1.

- *The very strong ellipticity implies the strong ellipticity,* i.e. *for any tensor* $\mathbb{L}$,

$$\alpha_{\rm vse}(\mathbb{L}) > 0 \;\Rightarrow\; \alpha_{\rm se}(\mathbb{L}) > 0.$$

- *According to [7, Theorem 3.3(i)], if* $\alpha_{\rm se}(\mathbb{L}) > 0$*, then the following inequalities hold:*

$$\Lambda(\mathbb{L}) \leq \Lambda_{\rm per}(\mathbb{L}) \leq \alpha_{\rm se}(\mathbb{L}). \tag{4.14}$$

- *Using a Fourier transform we get that for any constant tensor* $\mathbb{L}_0$,

$$\alpha_{\rm se}(\mathbb{L}_0) > 0 \;\Leftrightarrow\; \Lambda(\mathbb{L}_0) > 0.$$

In the sequel will always assume the strong ellipticity of the tensor $\mathbb{L}$, *i.e.* $\alpha_{\rm se}(\mathbb{L}) > 0$.

We conclude this section with the definition of Γ-convergence of a sequence of functionals (see, *e.g.*, [6, 1]):

Definition 4.2. *Let* X *be a reflexive Banach space endowed with the metrizable weak topology on bounded sets of* X*, and let* $\mathscr{F}^\varepsilon : X \to \mathbb{R}$ *be a* ε*-indexed sequence of functionals. The sequence* $\mathscr{F}^\varepsilon$ *is said to* Γ*-converge to the functional* $\mathscr{F}^0 : X \to \mathbb{R}$ *for the weak topology of* X*, and we denote* $\mathscr{F}^\varepsilon \overset{\Gamma-X}{\rightharpoonup} \mathscr{F}^0$*, if for any* $u \in X$*,*

- $\forall\, u_\varepsilon \rightharpoonup u,\ \mathscr{F}^0(u) \leq \liminf_{\varepsilon\to 0} \mathscr{F}^\varepsilon(u_\varepsilon),$
- $\exists\, \bar{u}_\varepsilon \rightharpoonup u,\ \mathscr{F}^0(u) = \lim_{\varepsilon\to 0} \mathscr{F}^\varepsilon(\bar{u}_\varepsilon).$

Such a sequence $\bar{u}_\varepsilon$ *is called a recovery sequence.*

4.2 The Γ-convergence results

It is stated in [10, Ch. 6, Sect. 11] that the first homogenization result in linear elasticity can be found in the Duvaut work (unavailable reference). It claims that if the tensor $\mathbb{L}$ is very strongly elliptic, *i.e.* $\alpha_{\rm vse}(\mathbb{L}) > 0$, then the solution $u^\varepsilon \in H^1_0(\Omega;\mathbb{R}^3)$ to the elasticity system (4.3) satisfies

$$\begin{cases} u^\varepsilon \rightharpoonup u \quad \text{weakly in } H^1_0(\Omega;\mathbb{R}^3), \\ \mathbb{L}^\varepsilon\nabla u^\varepsilon \rightharpoonup \mathbb{L}^0\nabla u \quad \text{weakly in } L^2(\Omega;\mathbb{R}^{3\times 3}), \\ -\operatorname{div}(\mathbb{L}^0\nabla u) = f, \end{cases} \tag{4.15}$$

where $\mathbb{L}^0$ is given by

$$\mathbb{L}^0 M : M := \inf\left\{ \int_{Y_3} \mathbb{L}(M+\nabla v):(M+\nabla v)\,dy,\ v \in H^1_{\rm per}(Y_3;\mathbb{R}^3)\right\} \quad \text{for } M \in \mathbb{R}^{3\times 3}, \tag{4.16}$$

which is attained when $\Lambda_{\rm per}(\mathbb{L}) > 0$. The previous homogenization result actually holds under the weaker assumption of functional coercivity, *i.e.* $\Lambda(\mathbb{L}) > 0$, as shown in [4].

Otherwise, from the point of view of the elastic energy consider the functionals

$$\mathscr{F}^{\varepsilon}(v) := \int_{\Omega} \mathbb{L}(x/\varepsilon)\nabla v : \nabla v \, dx, \tag{4.17}$$

$$\mathscr{F}^{0}(v) := \int_{\Omega} \mathbb{L}^{0}\nabla v : \nabla v \, dx \quad \text{for } v \in H^1(\Omega, \mathbb{R}^3). \tag{4.18}$$

Then, the following homogenization result [7, Theorem 3.4(i)] through the Γ-convergence of energy (4.17), allows us to relax the very strong ellipticity of $\mathbb{L}$.

Theorem 4.3 (Geymonat *et al.* [7])**.** *Under the conditions*

$$\Lambda(\mathbb{L}) \geq 0 \quad \textit{and} \quad \Lambda_{\text{per}}(\mathbb{L}) > 0,$$

one has

$$\mathscr{F}^{\varepsilon} \overset{\Gamma - H^1_0(\Omega;\mathbb{R}^3)}{\rightharpoonup} \mathscr{F}^{0},$$

for the weak topology of $H^1_0(\Omega;\mathbb{R}^3)$*, where* $\mathbb{L}^0$ *is given by* (4.16).

4.2.1 Generic examples of tensors satisfying $\Lambda(\mathbb{L}) \geq 0$ and $\Lambda_{\text{per}}(\mathbb{L}) > 0$

Reference [2] provides a class of isotropic strongly elliptic tensors for which Theorem 4.3 applies. However, this work is restricted to dimension two. We are going to extend the result [2, Theorem 2.2] to dimension three.

Let us assume that there exist $p > 0$ phases Z_i, $i = 1, \ldots, p$ satisfying

$$\begin{cases} Z_i \text{ is open, connected and Lipschitz for any } i \in \{1, \ldots, p\}, \\ Z_i \cap Z_j = \emptyset \quad \forall\, i \neq j \in \{1, \ldots, p\}, \\ \overline{Y}_3 = \displaystyle\bigcup_{i=1}^{p} \overline{Z}_i, \end{cases} \tag{4.19}$$

such that the tensor $\mathbb{L}$ satisfies

$$\begin{cases} \mathbb{L}(y)M = \lambda(y)\,\text{tr}(M)I_3 + 2\mu(y)M, \quad \forall\, y \in Y_3, \forall\, M \in \mathbb{R}^{3\times 3}_s, \\ \lambda(y) = \lambda_i,\ \mu(y) = \mu_i \text{ in } Z_i, \quad \forall\, i \in \{1, \ldots, p\}, \\ \mu_i > 0,\ 2\mu_i + \lambda_i > 0, \quad \forall\, i \in \{1, \ldots, p\}. \end{cases} \tag{4.20}$$

We further assume the existence of $d > 0$ such that

$$-\min_{i=1,\ldots,p}\{2\mu_i + 3\lambda_i\} \leq d \leq 4 \min_{i=1,\ldots,p}\{\mu_i\}. \tag{4.21}$$

Now, we define the following subsets of indexes

$$\begin{cases} I := \{i \in \{1, \ldots, p\} :\ d = 4\mu_i\}, \\ J := \{j \in \{1, \ldots, p\} :\ 2\mu_j + 3\lambda_j = -d\}, \\ K := \{1, \ldots, p\} \setminus (I \cup J). \end{cases} \tag{4.22}$$

Note that the three previous sets are disjoint. This is true, since we have $4\mu_i > -(2\mu_i + 3\lambda_i)$ due to $2\mu_i + \lambda_i > 0$.

In this framework, we are able to prove the following theorem which is an easy extension of the two-dimensional result of [2, Theorem 2.2]. For the reader convenience the proof is given in the Appendix.

Theorem 4.4. *Let $\mathbb{L}$ be the tensor defined by* (4.20) *and* (4.21). *Then we have $\Lambda(\mathbb{L}) \geq 0$. We also have $\Lambda_{\rm per}(\mathbb{L}) > 0$ provided that one of the two following conditions is fulfilled by the sets defined in* (4.22)*:*

Case 1. For each $j \in J$, there exist intervals $(a_j^-, a_j^+), (b_j^-, b_j^+) \subset [0,1]$ such that

$$\begin{aligned} &(a_j^-, a_j^+) \times (b_j^-, b_j^+) \times \{0,1\} \subset \partial Z_j, \quad \text{or} \\ &(a_j^-, a_j^+) \times \{0,1\} \times (b_j^-, b_j^+) \subset \partial Z_j, \quad \text{or} \\ &\{0,1\} \times (a_j^-, a_j^+) \times (b_j^-, b_j^+) \subset \partial Z_j. \end{aligned}$$

Case 2. For each $j \in J$, there exists $k \in K$ with $\mathscr{H}^2(\partial Z_j \cap \partial Z_k) > 0$, where $\mathscr{H}^2$ denotes the 2-dimensional Hausdorff measure.

4.2.2 Relaxation of condition $\Lambda_{\rm per}(\mathbb{L}) > 0$

According to Theorem 4.3 the Γ-convergence of the functional (4.17) holds true if both $\Lambda(\mathbb{L}) \geq 0$ and $\Lambda_{\rm per}(\mathbb{L}) > 0$. However, the following theorem due to Braides and the first author shows that in N-dimensional elasticity for $N \geq 2$, the Γ-convergence result still holds under the sole assumption $\Lambda(\mathbb{L}) \geq 0$.

Theorem 4.5 (Braides & Briane)**.** *Let Ω be a bounded open subset of $\mathbb{R}^N$, and let $\mathbb{L}$ be a bounded Y_N-periodic symmetric tensor-valued function in $L^\infty_{\rm per}\big(Y_N; \mathscr{L}_s(\mathbb{R}^{N\times N})\big)$ such that*

$$\Lambda(\mathbb{L}) \geq 0. \tag{4.23}$$

Then, we have

$$\mathscr{F}^\varepsilon \overset{\Gamma - H_0^1(\Omega;\mathbb{R}^N)}{\rightharpoonup} \mathscr{F}^0, \tag{4.24}$$

for the weak toplogy of $H_0^1(\Omega;\mathbb{R}^N)$, where $\mathscr{F}^0$ is given by (4.18) *with the tensor $\mathbb{L}^0$ defined by* (4.16).

Proof. For $\delta > 0$, set $\mathbb{L}_\delta := \mathbb{L} + \delta\, \mathbb{I}_s$ where $\mathbb{I}_s$ is the unit symmetric tensor, and let $\mathscr{F}_\delta^\varepsilon$ be the functional defined by (4.17) with $\mathbb{L}_\delta$. We claim that

$$\Lambda(\mathbb{L}_\delta) > 0. \tag{4.25}$$

To prove it consider $v \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$ and take $R > 0$ such that $\operatorname{supp} v \subset B(0,R)$. Then, by (4.23) we have

$$\int_{\mathbb{R}^N} \mathbb{L}_\delta \nabla v : \nabla v \, dy = \int_{B(0,R)} \mathbb{L} \nabla v : \nabla v \, dy + \delta \int_{B(0,R)} \mathbb{I}_s \nabla v : \nabla v \, dy \geq \delta \int_{B(0,R)} |e(v)|^2 \, dy.$$

By Korn's inequality there exists a constant $\alpha > 0$ which *a priori* depends on $B(0,R)$, such that

$$\int_{B(0,R)} |e(u)|\,dy \geq \alpha \int_{B(0,R)} |\nabla v|\,dy.$$

Nevertheless, the Korn constant α is known to be invariant by homothetic transformations of the domain. Hence, the constant α actually does not depend on the radius R. Therefore, the two previous inequalities imply that $\Lambda(\mathbb{L}_\delta) \geq \delta\alpha > 0$.

Thanks to (4.25) we can apply Theorem 4.3 with the functional $\mathscr{F}_\delta^\varepsilon$. Hence, $\mathscr{F}_\delta^\varepsilon \xrightarrow{\Gamma} \mathscr{F}_\delta^0$ for the weak topology of $H_0^1(\Omega;\mathbb{R}^N)$, where

$$\mathscr{F}_\delta^0(u) := \int_\Omega \mathbb{L}_\delta^0 \nabla u : \nabla u\,dx \quad \text{for } u \in H_0^1(\Omega,\mathbb{R}^N),$$

and $\mathbb{L}_\delta^0$ is given by (4.16) with $\mathbb{L} = \mathbb{L}_\delta$.

On the one hand, since $H_0^1(\Omega;\mathbb{R}^N)$ is a separable metric space, up to subsequence there exists the Γ-limit of $\mathscr{F}^\varepsilon$ for the weak topology of $H_0^1(\Omega;\mathbb{R}^N)$ as $\varepsilon \to 0$. Fix $u \in H_0^1(\Omega;\mathbb{R}^N)$, and consider a recovery sequence u_ε for $\mathscr{F}^\varepsilon$ (see Definition 4.2) which converges weakly to u in $H_0^1(\Omega;\mathbb{R}^N)$. Since u_ε is bounded in $H_0^1(\Omega,\mathbb{R}^N)$, we have

$$\begin{aligned}
(\Gamma\text{-}\lim \mathscr{F}^\varepsilon)(u) &\leq \mathscr{F}_\delta^0(u) \\
&\leq \liminf_{\varepsilon\to 0} \int_\Omega \mathbb{L}_\delta(x/\varepsilon)\nabla u_\varepsilon : \nabla u_\varepsilon\,dx \\
&\leq \liminf_{\varepsilon\to 0} \int_\Omega \mathbb{L}(x/\varepsilon)\nabla u_\varepsilon : \nabla u_\varepsilon\,dx + O(\delta) \\
&= (\Gamma\text{-}\lim \mathscr{F}^\varepsilon)(u) + O(\delta),
\end{aligned}$$

which implies that $\mathscr{F}_\delta^0(u)$ converges to $\mathscr{F}^0(u)$ as $\delta \to 0$.

On the other hand, let $\mathbb{L}^0$ be given by (4.16). For $\eta > 0$ and for $M \in \mathbb{R}^{N\times N}$, consider a function φ_η in $H^1_{\text{per}}(Y_N;\mathbb{R}^N)$ such that

$$\int_{Y_N} \mathbb{L}(y)(M + \nabla\varphi_\eta) : (M + \nabla\varphi_\eta)\,dy \leq \mathbb{L}^0 M : M + \eta.$$

We then have

$$\begin{aligned}
\mathbb{L}^0 M : M &\leq \mathbb{L}_\delta^0 M : M \\
&\leq \int_{Y_N} \mathbb{L}_\delta(y)(M + \nabla\varphi_\eta) : (M + \nabla\varphi_\eta)\,dy \\
&\leq \int_{Y_N} \mathbb{L}(y)(M + \nabla\varphi_\eta) : (M + \nabla\varphi_\eta)\,dy + O_\eta(\delta).
\end{aligned}$$

Hence, making δ tend to 0 for a fixed η, we obtain

$$\begin{aligned}\mathbb{L}^0 M : M &\leq \liminf_{\delta\to 0}\,(\mathbb{L}^0_\delta M : M)\\ &\leq \limsup_{\delta\to 0}\,(\mathbb{L}^0_\delta M : M)\\ &\leq \int_{Y_N} \mathbb{L}(y)(M + \nabla\varphi_\eta) : (M + \nabla\varphi_\eta)\,dy\\ &\leq \mathbb{L}^0 M : M + \eta.\end{aligned}$$

Due to the arbitrariness of η, we get that $\mathbb{L}^0_\delta$ converges to $\mathbb{L}^0$ as $\delta \to 0$.

Therefore, by the Lebesgue dominated convergence theorem we conclude that for any $u \in H^1_0(\Omega;\mathbb{R}^N)$,

$$\mathscr{F}^0(u) = \lim_{\delta\to 0} \mathscr{F}^0_\delta(u) = \lim_{\delta\to 0} \int_\Omega \mathbb{L}^0_\delta \nabla u : \nabla u\,dx = \int_\Omega \mathbb{L}^0 \nabla u : \nabla u\,dx.$$

□

4.3 Loss of ellipticity in three-dimensional linear elasticity through the homogenization of a laminate

In this section we will construct an example of a three-dimensional strong elliptic material $\mathbb{L}$ which is weakly coercive, *i.e.* $\Lambda(\mathbb{L}) \geq 0$, but for which the strong ellipticity is lost through homogenization. Firstly, let us recall the following result due to Gutiérrez [8].

Proposition 4.6 (Gutiérrez [8]). *For any strongly, but not semi-very strongly elliptic isotropic material, referred to as material a, there are very strongly elliptic isotropic materials such that if we laminate them with material a, in appropriately chosen proportions and directions, we generate an effective elasticity tensor that is not strongly elliptic.*

Remark 4.7 (Isotropic tensors). *The elasticity tensor* $\mathbb{L} \in L^\infty\big(Y_3;\mathscr{L}_s(\mathbb{R}^{3\times 3})\big)$ *of an isotropic material is given by*

$$\mathbb{L}(y)M = \lambda(y)\,\mathrm{tr}(M) I_3 + 2\mu(y)M, \quad \text{for } y \in Y_3 \text{ and } M \in \mathbb{R}^{3\times 3}_s,$$

where λ *and* μ *are the Lamé coefficients of* $\mathbb{L}$.
As a consequence, we have

$$\alpha_{\mathrm{se}}(\mathbb{L}) = \operatorname*{ess\text{-}inf}_{y\in Y_3}\big(\min\{\mu(y), 2\mu(y) + \lambda(y)\}\big),$$

$$\alpha_{\mathrm{vse}}(\mathbb{L}) = \operatorname*{ess\text{-}inf}_{y\in Y_3}\big(\min\{\mu(y), 2\mu(y) + 3\lambda(y)\}\big).$$

Here is a summary of the proof of Proposition 4.6. Consider two isotropic, homogeneous tensors $\mathbb{L}_a$ and $\mathbb{L}_b$ such that $\mathbb{L}_a$ is strongly elliptic, *i.e.*

$$\lambda_a + 2\mu_a > 0, \quad \mu_a > 0,$$

but not semi-very strongly elliptic, *i.e.*

$$3\lambda_a + 2\mu_a < 0.$$

and such that $\mathbb{L}_b$ is very strongly elliptic, *i.e.*

$$3\lambda_b + 2\mu_b > 0, \quad \mu_b > 0.$$

Considering the rank-one laminate in the direction y_1 mixing $\mathbb{L}_a$ with volume fraction $\theta_1 \in (0,1)$ and $\mathbb{L}_b$ with volume fraction $(1-\theta_1)$, Gutiérrez [8] proved that the effective tensor $\mathbb{L}_1^*$ in the sense of Murat-Tartar 1^*-convergence (see, *e.g.*, [8, Section 3]) satisfies the following properties:

- If $0 \leq \mu_a + \lambda_a$, then
$$\alpha_{\text{se}}(\mathbb{L}_1^*) > 0.$$
- If $-\mu_b \leq \mu_a + \lambda_a < 0$, then
$$\alpha_{\text{se}}(\mathbb{L}_1^*) \begin{cases} = 0 & \text{if } \mu_b = -\mu_a - \lambda_a, \\ \geq 0 & \text{if } -\mu_a - \lambda_a < \mu_b \leq -\frac{1}{4}(2\mu_a + 3\lambda_a), \\ > 0 & \text{if } -\frac{1}{4}(2\mu_a + 3\lambda_a) < \mu_b. \end{cases}$$
- The case $\mu_a + \lambda_a < -\mu_b$ is disposed of, since $\mathbb{L}_1^*$ does not even satisfy the Legendre-Hadamard condition.

In the case where $\alpha_{\text{se}}(\mathbb{L}_1^*) > 0$, Gutiérrez (see [8, Section 5.2]) performed a second lamination in the direction y_2 mixing the anisotropic material generated by the first lamination with volume fraction $\theta_2 \in (0,1)$, and a suitable very strongly elliptic isotropic material $(\mathbb{L}_c, \mu_c, \lambda_c)$ with volume fraction $(1-\theta_2)$. In this way he derived a rank-two laminate of effective tensor $\mathbb{L}_2^*$ which is not strongly elliptic.

In this section we will try to find a general class of periodic laminates for which the strong ellipticity is lost through homogenization. To this end we will extend to dimension three the rank-one lamination approach of [2] performed in dimension two. However, the outcome is surprisingly different from that of the two-dimensional case of [2]. Indeed, we will prove in the first subsection that it is not possible to lose strong ellipticity by a rank-one lamination through homogenization following the two-dimensional approach of [2]. This is the reason why we will perform a second lamination in the second part of the section.

4.3.1 Rank-one lamination

In this subsection we are going to focus on the rank-one lamination. As noted before, in the two-dimensional case of [2] it was proved a necessary and sufficient condition for a general rank-one laminate to lose strong ellipticity. Mimicking the same approach in dimension three we obtain the following quite different result.

Theorem 4.8. *Let* $\mathbb{L} \in L^\infty_{\text{per}}\big(Y_1; \mathscr{L}_s(\mathbb{R}^{3\times 3})\big)$ *be a* Y_1*-periodic isotropic tensor-valued function which is strongly elliptic,* i.e. $\alpha_{\text{se}}(\mathbb{L}) > 0$*. Assume that* $\Lambda_{\text{per}}(\mathbb{L}) > 0$ *and that there exists a constant matrix* $D \in \mathbb{R}^{3\times 3}$ *such that*

$$\mathbb{L}(y_1)M : M + D : \text{Cof}(M) \geq 0, \quad \textit{a.e. } y_1 \in Y_1, \quad \forall\, M \in \mathbb{R}^{3\times 3}. \tag{4.26}$$

Then, the homogenized tensor $\mathbb{L}^0$ *defined by* (4.16) *is strongly elliptic,* i.e. $\alpha_{\text{se}}(\mathbb{L}^0) > 0$.

Remark 4.9. *In dimension two for any periodic function* $\varphi \in H^1_{\text{per}}(Y_2; \mathbb{R}^2)$*, the only null lagrangian (up to a multiplicative constant) is the determinant of* $\nabla\varphi$*. Although the two-dimensional case seems* a priori *more restrictive than the three-dimensional case from an algebraic point of view, the two-dimensional Theorem 3.1 of [2] shows that for a suitable isotropic tensor* $\mathbb{L} = \mathbb{L}(y_1)$*, satisfying for some constant* $d \in \mathbb{R}$*, the condition*

$$\mathbb{L}(y_1)M : M + d\det(M) \geq 0, \quad \textit{a.e. in } Y_1,\ \forall\, M \in \mathbb{R}^{2\times 2}, \tag{4.27}$$

it is possible to lose strong ellipticity through homogenization. On the contrary, the three-dimensional Theorem 4.8 shows that it is not possible to lose strong ellipticity under condition (4.26) *which is the natural three-dimensional extension of* (4.27).

Remark 4.10. *Observe that condition* (4.26) *implies that* $\mathbb{L}$ *is weakly coercive,* i.e. $\Lambda(\mathbb{L}) \geq 0$*, but the converse is not true in general. Therefore, it might be possible to find a weakly coercive, strongly elliptic isotropic tensor* $\mathbb{L} = \mathbb{L}(y_1)$ *for which the strong ellipticity is lost. However, we have not succeeded in deriving such a tensor.*

Remark 4.11. *In the proof of Proposition 4.6 Gutiérrez implicitly proved the result of Theorem 4.8 when the matrix* D *has the form* $D = dI_3$ *and* $\mathbb{L}$ *is of the type*

$$\mathbb{L}(y_1) = \chi(y_1)\,\mathbb{L}_a + \big(1 - \chi(y_1)\big)\,\mathbb{L}_b.$$

Moreover, it is worth mentioning that the cases for which Guitiérrez obtained the loss of ellipticity with a rank-one lamination do not contradict Theorem 4.8, since in those cases condition (4.26) *does not hold.*

The rest of this subsection is devoted to the proof of Theorem 4.8. For any Y_1-periodic tensor-valued function $\mathbb{L} \in L^\infty_{\text{per}}\big(Y_1; \mathscr{L}_s(\mathbb{R}^{3\times 3})\big)$ which is strongly elliptic, *i.e.* $\alpha_{\text{se}}(\mathbb{L}) > 0$, define for a.e. $y_1 \in Y_1$, the y_1-dependent inner product

$$(\xi, \eta) \in \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{L}(y_1)(\xi \otimes e_1) : (\eta \otimes e_1).$$

It is indeed an inner product because $\alpha_{\rm se}(\mathbb{L}) > 0$. The matrix-valued function

$$L(y_1) = \begin{pmatrix} l_1(y_1) & l_{12}(y_1) & l_{13}(y_1) \\ l_{12}(y_1) & l_2(y_1) & l_{23}(y_1) \\ l_{13}(y_1) & l_{23}(y_1) & l_3(y_1) \end{pmatrix} :=$$
$$\begin{pmatrix} \mathbb{L}(y_1)(e_1\otimes e_1):(e_1\otimes e_1) & \mathbb{L}(y_1)(e_1\otimes e_1):(e_2\otimes e_1) & \mathbb{L}(y_1)(e_1\otimes e_1):(e_3\otimes e_1) \\ \mathbb{L}(y_1)(e_1\otimes e_1):(e_2\otimes e_1) & \mathbb{L}(y_1)(e_2\otimes e_1):(e_2\otimes e_1) & \mathbb{L}(y_1)(e_2\otimes e_1):(e_3\otimes e_1) \\ \mathbb{L}(y_1)(e_1\otimes e_1):(e_3\otimes e_1) & \mathbb{L}(y_1)(e_2\otimes e_1):(e_3\otimes e_1) & \mathbb{L}(y_1)(e_3\otimes e_1):(e_3\otimes e_1) \end{pmatrix} \tag{4.28}$$

is therefore symmetric positive definite.

Similarly to [2, Lemma 3.3] the next result provides an estimate which is a direct consequence of condition (4.26) with a matrix of the type $D = dI_3$. Observe that for the moment we are not assuming that the tensor $\mathbb{L}$ is isotropic.

Lemma 4.12. *Let $\mathbb{L} \in L^\infty_{\rm per}(Y_1;\mathscr{L}_s(\mathbb{R}^{3\times 3}))$ be a Y_1-periodic bounded tensor-valued function with $\Lambda_{\rm per}(\mathbb{L}) > 0$. Assume the existence of a constant $d \in \mathbb{R}$ such that $\mathbb{L}$ satisfies condition (4.26) with $D = dI_3$. Then, we have*

$$\mathbb{L}(y_1)M : M \geq Q(M), \quad \text{a.e. in } Y_1,\ \forall\, M \in \mathbb{R}^{3\times 3},\ M \text{ rank-one}, \tag{4.29}$$

where

$$\begin{aligned} Q(M) :=& \\ &\frac{\det(\tilde{L}^{11})}{\det(L)}\left(\mathbb{L}M:(e_1\otimes e_1) + \frac{d}{2}M_{33} + \frac{d}{2}M_{22}\right)^2 + \frac{\det(\tilde{L}^{22})}{\det(L)}\left(\mathbb{L}M:(e_2\otimes e_1) - \frac{d}{2}M_{12}\right)^2 \\ &+ \frac{\det(\tilde{L}^{33})}{\det(L)}\left(\mathbb{L}M:(e_3\otimes e_1) - \frac{d}{2}M_{13}\right)^2 \\ &- \frac{2\det(\tilde{L}^{12})}{\det(L)}\left(\mathbb{L}M:(e_1\otimes e_1) + \frac{d}{2}M_{33} + \frac{d}{2}M_{22}\right)\left(\mathbb{L}M:(e_2\otimes e_1) - \frac{d}{2}M_{12}\right) \\ &+ \frac{2\det(\tilde{L}^{13})}{\det(L)}\left(\mathbb{L}M:(e_1\otimes e_1) + \frac{d}{2}M_{33} + \frac{d}{2}M_{22}\right)\left(\mathbb{L}M:(e_3\otimes e_1) - \frac{d}{2}M_{13}\right) \\ &- \frac{2\det(\tilde{L}^{23})}{\det(L)}\left(\mathbb{L}M:(e_2\otimes e_1) - \frac{d}{2}M_{12}\right)\left(\mathbb{L}M:(e_3\otimes e_1) - \frac{d}{2}M_{13}\right). \end{aligned}$$

Furthermore, if $\mathbb{L}^0$ is the homogenized tensor of $\mathbb{L}$, then $\alpha_{\rm se}(\mathbb{L}^0) = 0$ if and only if there exists a rank-one matrix M such that

$$\mathbb{L}(y_1)M : M = Q(M), \quad \text{a.e. in } Y_1, \tag{4.30}$$

together with

$$
\begin{cases}
\displaystyle\int_{Y_1} \frac{\det(\tilde{L}^{13})}{\det(L)}(t)\left(\mathbb{L}(t)M:(e_1\otimes e_1)+\frac{d}{2}M_{22}+\frac{d}{2}M_{33}\right)dt \\
\displaystyle\quad=\int_{Y_1}\left[\frac{\det(\tilde{L}^{23})}{\det(L)}(t)\left(\mathbb{L}(t)M:(e_2\otimes e_1)-\frac{d}{2}M_{12}\right)\right. \\
\displaystyle\qquad\left.-\frac{\det(\tilde{L}^{33})}{\det(L)}(t)\left(\mathbb{L}(t)M:(e_3\otimes e_1)-\frac{d}{2}M_{13}\right)\right]dt, \\
\\
\displaystyle\int_{Y_1} \frac{\det(\tilde{L}^{12})}{\det(L)}(t)\left(\mathbb{L}(t)M:(e_1\otimes e_1)+\frac{d}{2}M_{22}+\frac{d}{2}M_{33}\right)dt \\
\displaystyle\quad=\int_{Y_1}\left[\frac{\det(\tilde{L}^{22})}{\det(L)}(t)\left(\mathbb{L}(t)M:(e_2\otimes e_1)-\frac{d}{2}M_{12}\right)\right. \\
\displaystyle\qquad\left.-\frac{\det(\tilde{L}^{23})}{\det(L)}(t)\left(\mathbb{L}(t)M:(e_3\otimes e_1)-\frac{d}{2}M_{13}\right)\right]dt, \\
\\
\displaystyle\int_{Y_1} \frac{\det(\tilde{L}^{11})}{\det(L)}(t)\left(\mathbb{L}(t)M:(e_1\otimes e_1)+\frac{d}{2}M_{22}+\frac{d}{2}M_{33}\right)dt \\
\displaystyle\quad=\int_{Y_1}\left[\frac{\det(\tilde{L}^{12})}{\det(L)}(t)\left(\mathbb{L}(t)M:(e_2\otimes e_1)-\frac{d}{2}M_{12}\right)\right. \\
\displaystyle\qquad\left.-\frac{\det(\tilde{L}^{13})}{\det(L)}(t)\left(\mathbb{L}(t)M:(e_3\otimes e_1)-\frac{d}{2}M_{13}\right)\right]dt.
\end{cases}
\tag{4.31}
$$

Finally, we state a corollary of the previous result in the particular case of isotropic tensors.

Lemma 4.13. *Let* $\mathbb{L}\in L^\infty_{\text{per}}(Y_1;\mathscr{L}_s(\mathbb{R}^{3\times3}))$ *be a* Y_1*-periodic bounded isotropic tensor-valued function with* $\Lambda_{\text{per}}(\mathbb{L})>0$*. Assume that there exists a constant* $d\in\mathbb{R}$ *such that the Lamé coefficients of* $\mathbb{L}(y_1)$ *satisfy*

$$\max\{0,-2\mu(y_1)-3\lambda(y_1)\}\le d\le 4\mu(y_1)\quad\textit{for a.e. } y_1 \textit{ in } Y_1. \tag{4.32}$$

Then, the homogenized tensor $\mathbb{L}^0$ *defined by* (4.16) *is strongly elliptic.*

Thanks to the previous lemmas, we are now able to demonstrate the main result of this section.

Proof of Theorem 4.8. Firstly, assume that (4.26) is satisfied with the matrix D being of the type $D=dI_3$ for some $d\in\mathbb{R}$. This is equivalent to condition (4.32), as it was proved by Gutiérrez in [8, Section 4.2]. By virtue of Lemma 4.13, $\mathbb{L}^0$ is strongly elliptic, which concludes the proof in this case.

In the sequel we will show that if there exists a constant matrix $D \in \mathbb{R}^{3\times 3}$ such that condition (4.26) is fulfilled, then there exists a constant $d \in \mathbb{R}$ such that (4.26) holds with $D = dI_3$. This combined with Lemma 4.13 implies that $\mathbb{L}^0$ is strongly elliptic.

Assume that (4.26) holds for some matrix $D \in \mathbb{R}^{3\times 3}$, namely for any $M \in \mathbb{R}^{3\times 3}$, we have a.e. in Y_1,

$$
\begin{aligned}
0 \le\, & \lambda(M_{11} + M_{22} + M_{33})^2 \\
& + 2\mu\left(M_{11}^2 + M_{22}^2 + M_{33}^2 + 2\left[\left(\frac{M_{12}+M_{21}}{2}\right)^2 + \left(\frac{M_{13}+M_{31}}{2}\right)^2 + \left(\frac{M_{23}+M_{32}}{2}\right)^2\right]\right) \\
& + D_{11}(M_{22}M_{33} - M_{23}M_{32}) - D_{12}(M_{21}M_{33} - M_{23}M_{31}) + D_{13}(M_{21}M_{32} - M_{22}M_{31}) \\
& - D_{21}(M_{12}M_{33} - M_{13}M_{32}) + D_{22}(M_{11}M_{33} - M_{13}M_{31}) - D_{23}(M_{11}M_{32} - M_{12}M_{31}) \\
& + D_{31}(M_{12}M_{23} - M_{13}M_{22}) - D_{32}(M_{11}M_{23} - M_{13}M_{21}) + D_{33}(M_{11}M_{22} - M_{12}M_{21}).
\end{aligned}
$$

The previous condition is equivalent to the following matrix being positive semi-definite a.e. in Y_1

$$
\begin{pmatrix}
\lambda+2\mu & \lambda+\frac{D_{33}}{2} & \lambda+\frac{D_{22}}{2} & 0 & 0 & 0 & 0 & -\frac{D_{32}}{2} & -\frac{D_{23}}{2} \\
\lambda+\frac{D_{33}}{2} & \lambda+2\mu & \lambda+\frac{D_{11}}{2} & 0 & 0 & -\frac{D_{31}}{2} & \frac{D_{13}}{2} & 0 & 0 \\
\lambda+\frac{D_{22}}{2} & \lambda+\frac{D_{11}}{2} & \lambda+2\mu & -\frac{D_{21}}{2} & -\frac{D_{12}}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{D_{21}}{2} & \mu & \mu-\frac{D_{33}}{2} & 0 & \frac{D_{23}}{2} & \frac{D_{31}}{2} & 0 \\
0 & 0 & -\frac{D_{12}}{2} & \mu-\frac{D_{33}}{2} & \mu & \frac{D_{32}}{2} & 0 & 0 & \frac{D_{13}}{2} \\
0 & -\frac{D_{31}}{2} & 0 & 0 & \frac{D_{32}}{2} & \mu & \mu-\frac{D_{22}}{2} & 0 & \frac{D_{21}}{2} \\
0 & -\frac{D_{13}}{2} & 0 & \frac{D_{23}}{2} & 0 & \mu-\frac{D_{22}}{2} & \mu & \frac{D_{12}}{2} & 0 \\
-\frac{D_{32}}{2} & 0 & 0 & \frac{D_{31}}{2} & 0 & 0 & \frac{D_{12}}{2} & \mu & \mu-\frac{D_{11}}{2} \\
-\frac{D_{23}}{2} & 0 & 0 & 0 & \frac{D_{13}}{2} & \frac{D_{21}}{2} & 0 & \mu-\frac{D_{11}}{2} & \mu
\end{pmatrix}.
$$

In particular, this implies that the following matrices are positive semi-definite a.e. in Y_1:

$$
\begin{pmatrix}
\mu & \mu - \frac{D_{ii}}{2} \\
\mu - \frac{Dii}{2} & \mu
\end{pmatrix} \quad \text{for } i = 1, 2, 3, \tag{4.33}
$$

$$
B := \begin{pmatrix}
\lambda+2\mu & \lambda+\frac{D_{33}}{2} & \lambda+\frac{D_{22}}{2} \\
\lambda+\frac{D_{33}}{2} & \lambda+2\mu & \lambda+\frac{D_{11}}{2} \\
\lambda+\frac{D_{22}}{2} & \lambda+\frac{D_{11}}{2} & \lambda+2\mu
\end{pmatrix}. \tag{4.34}
$$

Now, we will prove that there exists $i \in \{1,2,3\}$ such that

$$- \operatorname*{ess\text{-}inf}_{y_1 \in Y_1}\{2\mu(y_1) + 3\lambda(y_1)\} \le D_{ii} \le 4 \operatorname*{ess\text{-}inf}_{y_1 \in Y_1}\{\mu(y_1)\}. \tag{4.35}$$

Note that we can assume

$$\operatorname*{ess\text{-}inf}_{y_1 \in Y_1}\{2\mu(y_1) + 3\lambda(y_1)\} < 0. \tag{4.36}$$

Otherwise, since the matrix (4.33) is positive semi-definite, or equivalently

$$0 \le D_{ii} \le 4 \operatorname*{ess\text{-}inf}_{y_1 \in Y_1}\{\mu(y_1)\} \quad \text{for } i = 1,2,3, \tag{4.37}$$

condition (4.35) holds immediately.

We assume by contradiction that (4.35) is violated for any $i = 1,2,3$. Since the matrix B defined by (4.34) is positive semi-definite, we get for any $i = 1,2,3$,

$$\begin{vmatrix} \lambda + 2\mu & \lambda + \frac{D_{ii}}{2} \\ \lambda + \frac{D_{ii}}{2} & \lambda + 2\mu \end{vmatrix} \ge 0 \quad \text{a.e. in } Y_1,$$

which is equivalent to

$$-4 \operatorname*{ess\text{-}inf}_{y_1 \in Y_1}\{\mu(y_1) + \lambda(y_1)\} \le D_{ii} \le 4 \operatorname*{ess\text{-}inf}_{y_1 \in Y_1}\{\mu(y_1)\} \quad \text{for } i = 1,2,3.$$

Since by assumption (4.35) is not satisfied for any $i = 1,2,3$ and (4.37) holds, then the previous condition yields

$$-4 \operatorname*{ess\text{-}inf}_{y_1 \in Y_1}\{\mu(y_1)+\lambda(y_1)\} \le D_{ii} < - \operatorname*{ess\text{-}inf}_{y_1 \in Y_1}\{2\mu(y_1)+3\lambda(y_1)\} \quad \text{for } i = 1,2,3. \tag{4.38}$$

Set $d := \max_{i=1,2,3}\{D_{ii}\}$. By (4.38) there exists $\varepsilon > 0$ such that

$$d + \varepsilon < - \operatorname*{ess\text{-}inf}_{y_1 \in Y_1}\{2\mu(y_1) + 3\lambda(y_1)\}. \tag{4.39}$$

Define the set $P_\varepsilon \subset Y_1$ by

$$P_\varepsilon := \left\{ x_1 \in Y_1 : 2\mu(x_1) + 3\lambda(x_1) < \operatorname*{ess\text{-}inf}_{y_1 \in Y_1}\{2\mu(y_1) + 3\lambda(y_1)\} + \varepsilon \right\}.$$

It is clear that $|P_\varepsilon| > 0$, and from (4.39) and the definition of P_ε we obtain

$$d + \varepsilon < - \operatorname*{ess\text{-}inf}_{y_1 \in Y_1}\{2\mu(y_1) + 3\lambda(y_1)\} < -\big(2\mu(x_1) + 3\lambda(x_1)\big) + \varepsilon \quad \text{a.e. } x_1 \in P_\varepsilon,$$

which leads to

$$\lambda(x_1) + \frac{d}{2} < -\frac{1}{2}\big(\lambda(x_1) + 2\mu(x_1)\big) < 0 \quad \text{a.e. } x_1 \in P_\varepsilon. \tag{4.40}$$

Since the matrix B from (4.34) is positive semi-definite, then, its determinant is non-negative a.e. in Y_1. In particular we have

$$\begin{aligned} 0 &\le \det\big(B(x_1)\big) \\ &= \big(\lambda(x_1)+2\mu(x_1)\big)^3 + 2\left(\lambda(x_1)+\frac{D_{11}}{2}\right)\left(\lambda(x_1)+\frac{D_{22}}{2}\right)\left(\lambda(x_1)+\frac{D_{33}}{2}\right) \\ &\quad - \big(\lambda(x_1)+2\mu(x_1)\big)\left[\left(\lambda(x_1)+\frac{D_{11}}{2}\right)^2+\left(\lambda(x_1)+\frac{D_{22}}{2}\right)^2+\left(\lambda(x_1)+\frac{D_{33}}{2}\right)^2\right], \end{aligned} \tag{4.41}$$

a.e. $x_1 \in P_\varepsilon$. Then, it follows that

$$\det\big(B(x_1)\big) \le \big(\lambda(x_1)+2\mu(x_1)\big)^3 + 2\left(\lambda(x_1)+\frac{d}{2}\right)^3 - 3\big(\lambda(x_1)+2\mu(x_1)\big)\left(\lambda(x_1)+\frac{d}{2}\right)^2, \tag{4.42}$$

a.e. $x_1 \in P_\varepsilon$. To derive a contradiction let us show that the right-hand side of inequality (4.42) is negative. By (4.40) we get

$$4\left(\lambda(x_1)+\frac{d}{2}\right)^2 > \big(\lambda(x_1)+2\mu(x_1)\big)^2 \quad \text{a.e. } x_1 \in P_\varepsilon,$$

which, multiplying by $\lambda(x_1)+2\mu(x_1) > 0$, leads to

$$\big(\lambda(x_1)+2\mu(x_1)\big)^3 - 4\big(\lambda(x_1)+2\mu(x_1)\big)\left(\lambda(x_1)+\frac{d}{2}\right)^2 < 0 \quad \text{a.e. } x_1 \in P_\varepsilon.$$

Again using (4.40) we deduce that

$$2\left(\lambda(x_1)+\frac{d}{2}\right)^3 < -\big(\lambda(x_1)+2\mu(x_1)\big)\left(\lambda(x_1)+\frac{d}{2}\right)^2 \quad \text{a.e. } x_1 \in P_\varepsilon.$$

Adding the two last inequalities we obtain

$$\big(\lambda(x_1)+2\mu(x_1)\big)^3 + 2\left(\lambda(x_1)+\frac{d}{2}\right)^3 - 3\big(\lambda(x_1)+2\mu(x_1)\big)\left(\lambda(x_1)+\frac{d}{2}\right)^2 < 0,$$

a.e. $x_1 \in P_\varepsilon$, which by (4.42) implies that $\det(B) < 0$ in P_ε, a contradiction with (4.41).

Therefore, condition (4.35) is satisfied by $D_{ii} \ge 0$ (due to (4.37)) for some $i = 1, 2, 3$. Hence, condition (4.32) holds with $d = D_{ii}$, or equivalently (4.26) is satisfied by the matrix $D_{ii} I_3$, which concludes the proof. □

Now, let us prove the auxiliary results of the section.

Proof of Lemma 4.12. Let $M \in \mathbb{R}^{3\times 3}$ be a rank-one matrix. Then, $\det(M) = 0$, and

$\mathrm{adj}_{ii}(M) = 0$ for $i = 1, 2, 3$. Therefore, we get

$$\begin{aligned}
&\mathbb{L}^0 M : M \\
&= \min\left\{ \int_{Y_3} \mathbb{L}(M + \nabla\varphi) : (M + \nabla\varphi)\, dy : \varphi \in H^1_{\mathrm{per}}(Y_3; \mathbb{R}^3) \right\} \\
&= \min\left\{ \int_{Y_3} \big(\mathbb{L}(M+\nabla\varphi) : (M + \nabla\varphi) + dI_3 : \mathrm{Cof}(M+\nabla\varphi) : \varphi \in H^1_{\mathrm{per}}(Y_3; \mathbb{R}^3)\big) dy \right\} \geq 0.
\end{aligned} \tag{4.43}$$

Take $\varphi = \varphi(y_1) = (\varphi_1, \varphi_2, \varphi_3) \in C^1_{\mathrm{per}}(Y_1; \mathbb{R}^3)$. Then, the matrix

$$\nabla\varphi = \varphi' \otimes e_1 = \varphi_1'(e_1 \otimes e_1) + \varphi_2'(e_2 \otimes e_1) + \varphi_3'(e_3 \otimes e_1),$$

is a rank-one (or the null) matrix. Also, note that

$$\mathrm{adj}_{ij}(M) = (-1)^{i+j} \det(\tilde{M}^{ji}).$$

Considering the previous expressions, from (4.26) it follows that

$$\begin{aligned}
0 &\leq \mathbb{L}(M + \nabla\varphi) : (M + \nabla\varphi) + d \sum_{i=1}^{3} \mathrm{adj}_{ii}(M + \nabla\varphi) \\
&= \mathbb{L}M : M + 2\mathbb{L}M : (e_1 \otimes e_1)\varphi_1' + 2\mathbb{L}M : (e_2 \otimes e_1)\varphi_2' \\
&\quad + 2\mathbb{L}M : (e_3 \otimes e_1)\varphi_3' + l_1(\varphi_1')^2 + 2l_{12}\varphi_1'\varphi_2' \\
&\quad + 2l_{13}\varphi_1'\varphi_3' + l_2(\varphi_2')^2 + 2l_{23}\varphi_2'\varphi_3' + l_2(\varphi_3')^2 \\
&\quad + d(M_{33}\varphi_1' - M_{13}\varphi_3' + M_{22}\varphi_1' - M_{12}\varphi_2') \\
&= \mathbb{L}M : M + l_1(\varphi_1')^2 + l_2(\varphi_2')^2 + l_3(\varphi_3') + 2l_{12}\varphi_1'\varphi_2' + 2l_{13}\varphi_1'\varphi_3' + 2l_{23}\varphi_2'\varphi_3' \\
&\quad \big[2\mathbb{L}M : (e_1 \otimes e_2) + d(M_{33} + dM_{22})\big]\varphi_1' \\
&\quad + \big[2\mathbb{L}M : (e_2 \otimes e_1) - dM_{12}\big]\varphi_2' + \big[2\mathbb{L}M : (e_3 \otimes e_1) - dM_{13}\big]\varphi_3'.
\end{aligned}$$

For the previous equalities we have used that

$$\mathrm{adj}_{ii}(A + B) = \mathrm{adj}_{ii}(A) + \mathrm{adj}_{ii}(B) + \mathrm{Cof}(\tilde{A}^{ii}) : \tilde{B}^{ii}.$$

The purpose is to rewrite the last expression as the sum of squares. With that in mind, one obtains

$$
\begin{aligned}
0 \le\ & \mathbb{L}(M+\nabla\varphi):(M+\nabla\varphi)+dI_3:\mathrm{Cof}(M+\nabla\varphi)\\
=\ & \mathbb{L}M:M-Q(M)\\
& + l_1\left[\varphi_1'+\frac{l_{12}}{l_1}\varphi_2'+\frac{l_{13}}{l_1}\varphi_3'+\frac{1}{l_1}\left(\mathbb{L}M:(e_1\otimes e_1)+\frac{d}{2}M_{22}+\frac{d}{2}M_{33}\right)\right]^2\\
& + \frac{\det(\tilde{L}^{33})}{l_1}\left[\varphi_2'+\frac{\det(\tilde{L}^{23})}{\det(\tilde{L}^{33})}\varphi_3'-\frac{l_{12}}{\det(\tilde{L}^{33})}\left(\mathbb{L}M:(e_1\otimes e_1)+\frac{d}{2}M_{22}+\frac{d}{2}M_{33}\right)\right.\\
& \left. +\frac{l_1}{\det(\tilde{L}^{33})}\left(\mathbb{L}M:(e_2\otimes e_1)-\frac{d}{2}M_{12}\right)\right]^2\\
& + \frac{\det(L)}{\det(\tilde{L}^{33})}\left[\varphi_3'+\frac{\det(\tilde{L}^{13})}{\det(L)}\left(\mathbb{L}M:(e_1\otimes e_1)+\frac{d}{2}M_{22}+\frac{d}{2}M_{33}\right)\right.\\
& -\frac{\det(\tilde{L}^{23})}{\det(L)}\left(\mathbb{L}M:(e_2\otimes e_1)-\frac{d}{2}M_{12}\right)\\
& \left. +\frac{\det(\tilde{L}^{33})}{\det(L)}\left(\mathbb{L}M:(e_3\otimes e_1)-\frac{d}{2}M_{13}\right)\right]^2 .
\end{aligned}
\tag{4.44}
$$

Since φ_1', φ_2' and φ_3' can be chosen arbitrarily, the three square brackets in the previous equality can be equated to 0 at any Lebesgue point $y_1 \in Y_1$ of $\mathbb{L}$, and thus (4.29) holds. Using a density argument the previous equality also holds a.e. in Y_1, for any $\varphi \in H^1_{\mathrm{per}}(Y_1;\mathbb{R}^3)$.

Now, we are going to prove the second part of Lemma 4.12. Assume $\mathbb{L}^0$ is not strongly elliptic. Then, there exists a rank-one matrix M such that $\mathbb{L}^0 M : M = 0$. Taking into account expressions (4.43) the minimizer v_M associated with $\mathbb{L}^0 M : M$ (see [2, Lemma 3.2]) satisfies $v_M = v_M(y_1)$ and

$$
\begin{aligned}
0 = \mathbb{L}^0 M : M &= \int_{Y_1} \mathbb{L}(t)(M + v_M'(t)\otimes e_1):(M+v_M'(t)\otimes e_1)dt\\
&= \int_{Y_1}\big[\mathbb{L}(t)(M+\nabla v_M(t)):(M+\nabla v_M(t))+dI_3:\mathrm{Cof}(M+\nabla v_M)\big]dt.
\end{aligned}
$$

The first inequality in (4.44) implies that the integrand of the previous expression must be pointwisely 0, and thus the inequality in (4.44) for $\varphi = v_M$ is actually an equality. From this we deduce

$$
\mathbb{L}M : M = Q(M),
$$

and

$$\begin{cases} 0 = (v'_M)_1 + \dfrac{l_{12}}{l_1}(v'_M)_2 + \dfrac{l_{13}}{l_1}(v'_M)_3 + \dfrac{1}{l_1}\left(\mathbb{L}M : (e_1 \otimes e_1) + \dfrac{d}{2}M_{22} + \dfrac{d}{2}M_{33}\right), \\ 0 = (v'_M)_2 + \dfrac{\det(\tilde{L}^{23})}{\det(\tilde{L}^{33})}(v'_M)_3 - \dfrac{l_{12}}{\det(\tilde{L}^{33})}\left(\mathbb{L}M : (e_1 \otimes e_1) + \dfrac{d}{2}M_{22} + \dfrac{d}{2}M_{33}\right) \\ \qquad + \dfrac{l_1}{\det(\tilde{L}^{33})}\left(\mathbb{L}M : (e_2 \otimes e_1) - \dfrac{d}{2}M_{12}\right), \\ 0 = (v'_M)_3 + \dfrac{\det(\tilde{L}^{13})}{\det(L)}\left(\mathbb{L}M : (e_1 \otimes e_1) + \dfrac{d}{2}M_{22} + \dfrac{d}{2}M_{33}\right) \\ \qquad - \dfrac{\det(\tilde{L}^{23})}{\det(L)}\left(\mathbb{L}M : (e_2 \otimes e_1) - \dfrac{d}{2}M_{12}\right) \\ \qquad + \dfrac{\det(\tilde{L}^{33})}{\det(L)}\left(\mathbb{L}M : (e_3 \otimes e_1) - \dfrac{d}{2}M_{13}\right). \end{cases} \tag{4.45}$$

Since v_M is Y_1-periodic, we have

$$\int_{Y_1} (v'_M)_i \, dy_1 = 0 \quad i = 1, 2, 3.$$

Integrating the third equality in (4.45) we obtain the first equality in (4.31). Replacing $(v'_M)_3$ in the second equality of (4.45), we end up getting the second equality in (4.31). Finally, replacing $(v'_M)_2$ and $(v'_M)_3$ in the first equality of (4.45) it yields the last equality in (4.31).

Conversely, let us assume that equalities (4.30) and (4.31) hold. Considering the first equation in (4.31), taking into account that the all the integrands belong to $L^\infty(Y_1)$, there exists a function $\varphi_3 \in W^{1,\infty}_{\text{per}}(Y_1)$ such that, a.e. in Y_1, it holds

$$0 = \varphi'_3 + \frac{\det(\tilde{L}^{13})}{\det(L)}\left(\mathbb{L}M : (e_1 \otimes e_1) + \frac{d}{2}M_{22} + \frac{d}{2}M_{33}\right) - \frac{\det(\tilde{L}^{23})}{\det(L)}\left(\mathbb{L}M : (e_2 \otimes e_1) - \frac{d}{2}M_{12}\right)$$
$$+ \frac{\det(\tilde{L}^{33})}{\det(L)}\left(\mathbb{L}M : (e_3 \otimes e_1) - \frac{d}{2}M_{13}\right).$$

Repeating the argument with the second and the third equation of (4.31), we get the existence of functions φ_2 and φ_1 in $W^{1,\infty}_{\text{per}}(Y_1)$ respectively, such that

$$\varphi'_2 + \frac{\det(\tilde{L}^{23})}{\det(\tilde{L}^{33})}\varphi'_3 - \frac{l_{12}}{\det(\tilde{L}^{33})}\left(\mathbb{L}M : (e_1 \otimes e_1) + \frac{d}{2}M_{22} + \frac{d}{2}M_{33}\right) + \frac{l_1}{\det(\tilde{L}^{33})}\left(\mathbb{L}M : (e_2 \otimes e_1) - \frac{d}{2}M_{1}\right.$$
$$\varphi'_1 + \frac{l_{12}}{l_1}\varphi'_2 + \frac{l_{13}}{l_1}\varphi'_3 + \frac{1}{l_1}\left(\mathbb{L}M : (e_1 \otimes e_1) + \frac{d}{2}M_{22} + \frac{d}{2}M_{33}\right) = 0.$$

These three equalities together with (4.30) imply the equality in (4.44), and thus by (4.43) it follows that

$$0 = \int_{Y_1} \big(\mathbb{L}(M + \nabla\varphi) : (M + \nabla\varphi) + dI_3 : \text{Cof}(M + \nabla\varphi)\big)\, dy_1 \geq \mathbb{L}^0 M : M \geq 0,$$

which shows that $\mathbb{L}^0$ is not strongly elliptic.

Finally, due to the equality $\mathbb{L}^0 M : M = \mathbb{L}^0 M^T : M^T$, conditions (4.30) and (4.31) are equivalent to the similar equalities replacing M by M^T. $\square$

Proof of Lemma 4.13. Since $\mathbb{L}$ is isotropic, condition (4.32) is equivalent to the condition (4.26) with $D = dI_3$. As a consequence, (4.32) implies $\Lambda(\mathbb{L}) \geq 0$. By [7, Corollary 3.5], we have $\alpha_{\mathrm{se}}(\mathbb{L}^0) \geq \Lambda(\mathbb{L})$. Therefore, we get that $\alpha_{\mathrm{se}}(\mathbb{L}^0) \geq 0$.

Assume that $\mathbb{L}^0$ is not strongly elliptic, *i.e.* $\alpha_{\mathrm{se}}(\mathbb{L}^0) = 0$. Then, there exists a rank-one matrix $M := \xi \otimes \eta$ in $\mathbb{R}^{3\times 3}$, with $\xi, \eta \in \mathbb{R}^3 \setminus \{0\}$, such that $\mathbb{L}^0 M : M = 0$.

Since $\mathbb{L}$ is isotropic, the matrix L defined in (4.28) is

$$L = \begin{pmatrix} \lambda + 2\mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}.$$

Moreover, the following equalities hold

$$\begin{gathered} M_{ij} = \xi_i \eta_j \quad i, j \in \{1,2,3\}, \\ \mathbb{L}M : (e_1 \otimes e_1) = (\lambda + 2\mu)\xi_i \eta_1 + \lambda(\xi_2\eta_2 + \xi_3\eta_3), \\ \mathbb{L}M : (e_2 \otimes e_1) = \mu(\xi_1\eta_2 + \xi_2\eta_1), \\ \mathbb{L}M : (e_3 \otimes e_1) = \mu(\xi_1\eta_{\cdot} + \xi_3\eta_1), \\ \mathbb{L}M : M = (\lambda + \mu)(\xi : \eta)^2 + \mu|\xi|^2|\eta|^2. \end{gathered}$$

Because $\mathbb{L}^0 M : M = 0$, from equalities (4.30) and (4.31) in Lemma 4.12 we obtain a.e. in Y_1

$$\begin{aligned} &(\lambda + \mu)(\xi : \eta)^2 + \mu|\xi|^2|\eta|^2 \\ &= \frac{1}{\lambda + 2\mu}\left[(\lambda + 2\mu)\xi_1\mu_1 + \lambda(\xi_2\eta_2 + \xi_3\eta_3) + \frac{d}{2}(\xi_2\eta_2 + \xi_3\eta_3)\right]^2 \\ &+ \frac{1}{\mu}\left[\mu(\xi_1\eta_2 + \xi_2\eta_1) - \frac{d}{2}\xi_1\eta_2\right]^2 + \frac{1}{\mu}\left[\mu(\xi_1\eta_3 + \xi_3\eta_1) - \frac{d}{2}\xi_1\eta_3\right]^2, \end{aligned} \tag{4.46}$$

together with

$$0 = \xi_1\eta_3 + \xi_3\eta_1 - \frac{\xi_1\eta_3}{2}\int_{Y_1} \frac{d}{\mu}(t)\,dt, \tag{4.47}$$

$$0 = \xi_1\eta_2 + \xi_2\eta_1 - \frac{\xi_1\eta_2}{2}\int_{Y_1} \frac{d}{\mu}(t)\,dt, \tag{4.48}$$

$$0 = \xi_1\eta_1 + (\xi_2\eta_2 + \xi_3\eta_3)\int_{Y_1} \frac{\lambda + \frac{d}{2}}{\lambda + 2\mu}(t)\,dt. \tag{4.49}$$

After some calculations, from (4.46) we get

$$\frac{(\lambda + 2\mu)^2 - (\lambda + \frac{d}{2})^2}{\lambda + 2\mu}(\xi_2\eta_2 + \xi_3\eta_3)^2 + \mu(\xi_2\eta_3 - \xi_3\eta_2)^2 + \frac{d(\mu - \frac{d}{4})}{\mu}\xi_1^2(\eta_2^2 + \eta_3^2) = 0, \tag{4.50}$$

a.e. in Y_1. Observe that, since $\mathbb{L}$ is isotropic and (strictly) strongly elliptic in Y_1, we have

$$\mu > 0,\ 2\mu + \lambda > 0 \quad \text{a.e. in } Y_1,$$

which implies that

$$(\lambda + 2\mu)^2 - \left(\lambda + \frac{d}{2}\right)^2 \geq 0 \quad \text{a.e. in } Y_1.$$

Hence, taking into account assumption (4.32), equality (4.50) implies the following three conditions:

$$\left[(\lambda + 2\mu)^2 - \left(\lambda + \frac{d}{2}\right)^2\right] (\xi_2\eta_2 + \xi_3\eta_3)^2 = 0 \quad \text{a.e. in } Y_1, \tag{4.51}$$

$$\xi_2\eta_3 = \xi_3\eta_2, \tag{4.52}$$

$$d\left(\mu - \frac{d}{4}\right)\xi_1^2(\eta_2^2 + \eta_3^2) = 0 \quad \text{a.e. in } Y_1. \tag{4.53}$$

We will now prove by contradiction that we cannot have $d = 4\mu$ a.e. in Y_1. Otherwise, equalities (4.47), (4.48) and (4.49) can be written as

$$\begin{cases} 0 = \xi_1\eta_3 - \xi_3\eta_1, \\ 0 = \xi_1\eta_2 - \xi_2\eta_1, \\ 0 = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3. \end{cases} \tag{4.54}$$

Under these conditions, if $\eta_1 \neq 0$, then the first and second equalities of (4.54) lead to

$$\xi_3 = \eta_3\frac{\xi_1}{\eta_1}, \quad \xi_2 = \eta_2\frac{\xi_1}{\eta_1}.$$

Replacing ξ_2 and ξ_3 in the third equality in (4.54), we obtain

$$\xi_1(\eta_1^2 + \eta_2^2 + \eta_3^2) = 0.$$

Since $\eta \neq 0$, we get $\xi_1 = 0$. This implies that $\xi_2 = \xi_3 = 0$, a contradiction with $\xi \neq 0$. Therefore, we have necessarily $\eta_1 = 0$. Moreover, using the two first equalities of (4.54) and the fact that $\eta \neq 0$, we obtain $\xi_1 = 0$. As a consequence, (4.54) reduces to

$$\xi_2\eta_2 + \xi_3\eta_3 = 0. \tag{4.55}$$

If $\eta_2 \neq 0$, then using (4.52) we get

$$\xi_3 = \xi_2\frac{\eta_3}{\eta_2},$$

and replacing ξ_3 in the previous equality, it yields

$$\xi_2(\eta_2^2 + \eta_3^2) = 0.$$

Again, since $\eta \neq 0$, we have $\xi_2 = 0$. Using (4.52) and the assumption $\eta_2 \neq 0$, it follows that $\xi_3 = 0$, again a contradiction with $\xi, \eta \neq 0$. Thus, we have necessarily $\eta_2 = 0$. Taking into account that $\eta_1 = \eta_2 = 0$ we have $\eta_3 \neq 0$, hence from (4.55) we deduce that $\xi_3 = 0$. Now (4.52) is written as $\xi_2\eta_3 = 0$. However, recall that $\xi_1 = \xi_3 = \eta_1 = \eta_2 = 0$. This implies that either $\xi = 0$ or $\eta = 0$, a contradiction.

We have just shown that the set $\{d < 4\mu\}$ has a positive Lebesgue measure. Similarly, we can check that $d > 0$. Using (4.51) and (4.53) together with $0 < d \leq 4\mu$, we deduce that

$$\xi_2\eta_2 + \xi_3\eta_3 = \xi_1^2(\eta_2^2 + \eta_3^2) = 0,$$

which combined with (4.49) also gives $\xi_1\eta_1 = 0$. As above, using the three previous equalities, (4.47), (4.48) and (4.52), we get a contradiction with the fact that $\xi, \eta \neq 0$. Therefore, we have proved that $\mathbb{L}^0$ is strongly elliptic if (4.32) holds for some d. □

4.3.2 Rank-two lamination

In the proof of Proposition 4.6 for dimension three [8, Section 5.2], Gutiérrez performed a rank-one laminate mixing a strongly elliptic but not semi-very strongly isotropic material $\mathbb{L}_a$, and a very strongly elliptic isotropic material $\mathbb{L}_b$. However, as it was noted at the beginning of the section, there are some cases for which the strong ellipticity of the homogenized tensor is not lost after this first lamination. In fact, our Theorem 4.8 shows that for a general rank-one laminate, it is not possible to lose the strong ellipticity through homogenization if there exists a matrix $D \in \mathbb{R}^{3\times3}$ satisfying condition (4.26). As done in [8], we need to perform a second lamination with a third material $\mathbb{L}_c$ which can be very strongly elliptic, in order to lose the strong ellipticity in those cases.

Our purpose is to justify Gutiérrez' approach using formally 1*-convergence (see [8, Section 3]), by a homogenization procedure using the Γ-convergence result of Theorem 4.5.

Theorem 4.14. *For any strongly elliptic but not semi-very strongly elliptic isotropic tensor $\mathbb{L}_a$ whose Lamé coefficients satisfy*

$$4\mu_a + 3\lambda_a > 0, \tag{4.56}$$

there exist two very strongly elliptic isotropic tensors $\mathbb{L}_b, \mathbb{L}_c$ and volume fractions $\theta_1, \theta_2 \in (0,1)$ such that the tensor $\mathbb{L}_2$ obtained by laminating in the direction y_2 the effective tensor $\mathbb{L}_1^*$ – firstly obtained by laminating in the direction y_1 the tensors $\mathbb{L}_a$, $\mathbb{L}_b$ with proportions θ_1, $1-\theta_1$ – and the tensor $\mathbb{L}_c$ with proportions θ_2 and $1-\theta_2$ respectively, namely

$$\mathbb{L}_2(y_2) := \chi_2(y_2)\,\mathbb{L}_1^* + \big(1-\chi_2(y_2)\big)\,\mathbb{L}_c \quad \textit{for } y_2 \in Y_1, \tag{4.57}$$

satisfies

$$\Lambda(\mathbb{L}_2) = 0, \tag{4.58}$$

and

$$\int_\Omega \mathbb{L}_2(x_2/\varepsilon)\nabla v : \nabla v\,dx \overset{\Gamma - H_0^1(\Omega)^3}{\rightharpoonup} \int_\Omega \mathbb{L}_2^0\nabla v : \nabla v\,dx, \tag{4.59}$$

where the homogenized tensor $\mathbb{L}_2^0$ is not strongly elliptic, i.e.

$$\alpha_{\mathrm{se}}(\mathbb{L}_2^0) = 0. \tag{4.60}$$

Remark 4.15. *Theorem 4.14 shows that for certain strongly elliptic but not very strongly elliptic isotropic tensors, namely those whose Lamé parameters fulfil* (4.56)*, it is possible to find two very strongly elliptic isotropic tensors for which the homogenization process through* Γ*-convergence using a rank-two lamination leads to the loss of ellipticity of the effective tensor.*

Proof of Theorem 4.14. We divide the proof into four steps.

Step 1. Choice of $\mathbb{L}_a$, $\mathbb{L}_b$, θ_1, θ_2.
Let $\mathbb{L}_a$ be a strongly elliptic but not semi-very strongly elliptic isotropic tensor satisfying (4.56). Our aim is to find two very strongly isotropic tensors $\mathbb{L}_b, \mathbb{L}_c$ and two volume fractions θ_1, θ_2 such that the strong ellipticity is lost through homogenization using a rank-two lamination.

Let $\chi_1, \chi_2 : \mathbb{R} \to \{0,1\}$ be two 1-periodic characteristic functions such that

$$\int_{Y_1} \chi_1(y_1)\, dy_1 = \theta_1 \quad \text{and} \quad \int_{Y_1} \chi_2(y_2)\, dy_2 = \theta_2,$$

where $\theta_1, \theta_2 \in (0,1)$ will be chosen later.

The 1^*-convergence procedure of [8, Section 5.2] applied to the tensor

$$\mathbb{L}_1(y_1) := \chi_1(y_1)\,\mathbb{L}_a + \big(1 - \chi_1(y_1)\big)\,\mathbb{L}_b \quad \text{for } y_1 \in Y_1, \tag{4.61}$$

yields a non-isotropic effective tensor $\mathbb{L}_1^*$. The computations of [8, Section 5.2] lead to an explicit expression of the tensor $\mathbb{L}_1^*$ whose non-zero entries are

$$\begin{aligned}
&(\mathbb{L}_1^*)_{1111} = \frac{1}{A},\\
&(\mathbb{L}_1^*)_{1122} = (\mathbb{L}_1^*)_{2211} = (\mathbb{L}_1^*)_{1133} = (\mathbb{L}_1^*)_{3311} = \frac{B}{A},\\
&(\mathbb{L}_1^*)_{1212} = (\mathbb{L}_1^*)_{1221} = (\mathbb{L}_1^*)_{2112} = (\mathbb{L}_1^*)_{2121} = \frac{1}{E},\\
&(\mathbb{L}_1^*)_{1313} = (\mathbb{L}_1^*)_{1331} = (\mathbb{L}_1^*)_{3113} = (\mathbb{L}_1^*)_{3131} = \frac{1}{E},\\
&(\mathbb{L}_1^*)_{2222} = \frac{B^2}{A} + 2(C+D),\\
&(\mathbb{L}_1^*)_{2233} = (\mathbb{L}_1^*)_{3322} = \frac{B^2}{A} + 2D,\\
&(\mathbb{L}_1^*)_{2323} = (\mathbb{L}_1^*)_{2332} = (\mathbb{L}_1^*)_{3223} = (\mathbb{L}_1^*)_{3232} = C,\\
&(\mathbb{L}_1^*)_{3333} = \frac{B^2}{A} + 2(C+D),
\end{aligned} \tag{4.62}$$

where

$$\begin{aligned} A &= \frac{\theta_1}{2\mu_a+\lambda_a} + \frac{1-\theta_1}{2\mu_b+\lambda_b}, \\ B &= \frac{\theta_1\lambda_a}{2\mu_a+\lambda_a} + \frac{(1-\theta_1)\lambda_b}{2\mu_b+\lambda_b}, \\ C &= \theta_1\mu_a + (1-\theta_1)\mu_b, \\ D &= \frac{\theta_1\mu_a\lambda_a}{2\mu_a+\lambda_a} + \frac{(1-\theta_1)\mu_b\lambda_b}{2\mu_b+\lambda_b}, \\ E &= \frac{\theta_1}{\mu_a} + \frac{1-\theta_1}{\mu_b}. \end{aligned} \tag{4.63}$$

Now, let us specify the choice of the two very strongly elliptic isotropic tensors $\mathbb{L}_b$, $\mathbb{L}_c$, and the volume fractions θ_1, θ_2. For the Lamé parameters of material c we denote $\lambda_c = \alpha_c\mu_c$ as done in [8]. We assume that

$$-\frac{1}{4}(2\mu_a+3\lambda_a) \leq \mu_b < \frac{\mu_a(2\mu_a+3\lambda_a)}{3\lambda_a}, \tag{4.64}$$

$$\lambda_b > \frac{2\mu_b^2\lambda_a}{\mu_a(2\mu_a+3\lambda_a)-3\mu_b\lambda_a}, \tag{4.65}$$

$$\theta_1 = \frac{-\lambda_b(2\mu_a+\lambda_a)}{2(\mu_b\lambda_a-\mu_a\lambda_b)}, \tag{4.66}$$

$$\alpha_c \geq \frac{-D}{C+D}, \tag{4.67}$$

$$\mu_c = C\frac{\alpha_c(C+2D)}{D(1+\alpha_c)}, \tag{4.68}$$

and

$$\theta_2 = \frac{\alpha_c(C+D)}{\alpha_c(C+D)-D(2+\alpha_c)}. \tag{4.69}$$

Observe that, thanks to the first inequality in (4.64), the tensor $\mathbb{L}_1$ given by (4.61) satisfies $\Lambda(\mathbb{L}_1) \geq 0$ (see [8, Section 4.2]). Hence, by Theorem 4.8 the homogenized tensor $\mathbb{L}_1^*$ is strongly elliptic. This justifies the first lamination from the point of view of homogenization through Γ-convergence.

To conclude the first step, let us check that the previous conditions satisfy the assumptions of Theorem 4.14. The tensor $\mathbb{L}_a$ is strongly elliptic but not semi-very strongly elliptic, *i.e.*

$$\mu_a > 0, \quad 2\mu_a + 3\lambda_a < 0,$$

which implies that $\mu_b > 0$. The fact that necessarily $\lambda_a < 0$ together with (4.64) implies that $\lambda_b > 0$ thanks to (4.65), and thus $\mathbb{L}_b$ is very strongly elliptic. The volume fraction θ_1 clearly belongs to $(0,1)$, since (4.66) reads as

$$\theta_1 = \frac{\lambda_b(2\mu_a+\lambda_a)}{\lambda_b(2\mu_a+\lambda_a)-\lambda_a(2\mu_b+\lambda_b)}.$$

The choice of θ_1 implies that in (4.63)

$$B = 0. \tag{4.70}$$

In addition, $C + D > 0$ as it was proved in [8, Appendix C] and $C + 2D < 0$ by (4.64), (4.65) and (4.66). This also implies that $D < 0$. Thanks to the previous inequalities we have $\theta_2 \in (0, 1)$, $\alpha_c > 0$ and $\mu_c > 0$, which implies that $\mathbb{L}_c$ is very strongly elliptic.

Step 2. $\Lambda(\mathbb{L}_2) \geq 0$.
To get $\Lambda(\mathbb{L}_2) \geq 0$ we will prove that for

$$D := \begin{pmatrix} 4\mu_c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we have

$$\mathbb{L}_2(y_2)M : M + D : \operatorname{Cof}(M) \geq 0 \quad \text{a.e. } y_2 \in Y_1, \text{ for all } M \in \mathbb{R}^{N\times N}. \tag{4.71}$$

We need to prove that the previous inequality holds in each homogeneous phase of $\mathbb{L}_2$.

Firstly, for the phase $\mathbb{L}_c$ which is isotropic and very strongly elliptic, we get for any $M \in \mathbb{R}^{3\times 3}$,

$$\begin{aligned}
&\mathbb{L}_c M : M + D : \operatorname{Cof}(M) \\
&= 2\mu_c \left[M_{11}^2 + M_{22}^2 + M_{33}^2 + 2\left(\frac{M_{12}+M_{21}}{2}\right)^2 + 2\left(\frac{M_{13}+M_{31}}{2}\right)^2 + 2\left(\frac{M_{23}+M_{32}}{2}\right)^2 \right] \\
&\quad + \lambda_c (M_{11} + M_{22} + M_{33})^2 + 4\mu_c (M_{22}M_{33} - M_{23}M_{32}) \\
&= (\lambda_c + 2\mu_c)(M_{11}^2 + M_{22}^2 + M_{33}^2) + 2\lambda_c (M_{11}M_{22} + M_{11}M_{33}) + 2(\lambda_c + 2\mu_c) M_{22}M_{33} \\
&\quad + \mu_c (M_{12} + M_{21})^2 + \mu_c (M_{31} + M_{13})^2 + \mu_c (M_{23} - M_{32})^2.
\end{aligned}$$

This quantity is non-negative for any $M \in \mathbb{R}^{3\times 3}$, since the following matrix is positive semi-definite:

$$\begin{pmatrix} \lambda_c + 2\mu_c & \lambda_c & \lambda_c \\ \lambda_c & \lambda_c + 2\mu_c & \lambda_c + 2\mu_c \\ \lambda_c & \lambda_c + 2\mu_c & \lambda_c + 2\mu_c \end{pmatrix},$$

due to the strong ellipticity of $\mathbb{L}_c$. Therefore, the desired inequality holds for the homogeneous phase $\mathbb{L}_c$.

Secondly, we need to check the same inequality for the phase with $\mathbb{L}_1^*$. By (4.62) we have for $M \in \mathbb{R}^{3\times 3}$,

$$\begin{aligned}\mathbb{L}_1^* M : M + D : \mathrm{Cof}(M) = {} & \frac{1}{A} M_{11}^2 + \left[\frac{B^2}{A} + 2(C+D)\right](M_{22}^2 + M_{33}^2) \\ & + 2\frac{B}{A}(M_{11}M_{22} + M_{11}M_{33}) \\ & + 2\left[\frac{B^2}{A} + 2D + 2\mu_c\right](M_{22}M_{33}) \\ & + \frac{1}{E}(M_{12}+M_{21})^2 + \frac{1}{E}(M_{13}+M_{31})^2 \\ & + C(M_{23}^2 + M_{32}^2) + 2(C-2\mu_c)M_{23}M_{32}.\end{aligned}$$

Since $E \geq 0$, this quantity is non-negative for any $M \in \mathbb{R}^{3\times 3}$ if the following two matrices are positive semi-definite:

$$\begin{pmatrix} \frac{1}{A} & \frac{B}{A} & \frac{B}{A} \\ \frac{B}{A} & \frac{B^2}{A} + 2(C+D) & \frac{B^2}{A} + 2D + 2\mu_c \\ \frac{B}{A} & \frac{B^2}{A} + 2D + 2\mu_c & \frac{B^2}{A} + 2(C+D) \end{pmatrix}, \tag{4.72}$$

$$\begin{pmatrix} C & C - 2\mu_c \\ C - 2\mu_c & C \end{pmatrix}. \tag{4.73}$$

Since $C \geq 0$, the matrix (4.73) is positive semi-definite if and only if $\mu_c \leq C$. Taking into account that $\mu_c \leq C$, we can check that the matrix (4.72) is positive semi-definite if $-(C+2D) \leq \mu_c$. Therefore, the matrices (4.72) and (4.73) are positive semi-definite if

$$-(C+2D) \leq \mu_c \leq C. \tag{4.74}$$

By the definition (4.68) of μ_c, we deduce that the first inequality of (4.74) holds if and only if

$$\frac{\alpha_c C}{-D(1+\alpha_c)} \geq 1,$$

which is satisfied due to inequality (4.67). For the second inequality of (4.74), we need to check that (see (4.68))

$$\frac{\alpha_c(C+2D)}{D(1+\alpha_c)} \leq 1,$$

or equivalently,

$$\alpha_c \geq \frac{D}{C+D}.$$

This is true since $\alpha_c > 0$ by (4.67) and $\frac{D}{C+D} < 0$. Therefore, condition (4.71) holds true, and consequently

$$\Lambda(\mathbb{L}_2) \geq 0. \tag{4.75}$$

Step 3. $\mathbb{L}_2$ loses the strong ellipticity through homogenization.
On the one hand, due to $\Lambda(\mathbb{L}_2) \geq 0$, by virtue of Theorem 4.5 the Γ-convergence (4.59) holds with the homogenized tensor $\mathbb{L}_2^0$ which is given by the minimization formula (4.16) replacing $\mathbb{L}$ by $\mathbb{L}_2$.

On the other hand, following Gutiérrez' 1*-convergence procedure we obtain a homogenized tensor $\mathbb{L}_2^*$ such that (see [8, Section 5.2] for the expression of $\mathbb{L}_2^*$)

$$\mathbb{L}_2^*(e_3 \otimes e_3) : (e_3 \otimes e_3) = I_1 + \frac{G_1^2}{F_1},$$

where by (4.70),

$$\begin{aligned} I_1 &= 4(1-\theta_2)\frac{1+\alpha_c}{2+\alpha_c} + 2\theta_2 C\frac{C+2D}{C+D}, \\ G_1 &= (1-\theta_2)\frac{\alpha_c}{2+\alpha_c} + \theta_2\frac{D}{C+D}, \\ F_1 &\neq 0. \end{aligned}$$

It is not difficult to check that the choice of $\mathbb{L}_b$, $\mathbb{L}_c$, θ_1, θ_2 leads to $I_1 = G_1 = 0$, which yields

$$\mathbb{L}_2^*(e_3 \otimes e_3) : (e_3 \otimes e_3) = 0. \tag{4.76}$$

To conclude the proof it is enough to show that

$$\mathbb{L}_2^* = \mathbb{L}_2^0. \tag{4.77}$$

Indeed, thanks to $\mathbb{L}_2^* = \mathbb{L}_2^0$ equality (4.76) implies the loss of ellipticity (4.60), and (4.60) implies $\Lambda(\mathbb{L}_2) \leq 0$. This combined with (4.75) finally shows the desired loss of functional coercivity (4.58).

Step 4. $\mathbb{L}_2^* = \mathbb{L}_2^0$.
By formally using 1*-convergence in terms of [2, Lemma 3.1], Gutiérrez's computations for the tensor $\mathbb{L}_2^*$ in [8, Section 5.2] can be written as

$$\begin{cases} A^{-1}[\mathbb{L}_2^*] = \displaystyle\int_0^1 A^{-1}[\mathbb{L}_2](t)\,dt, \\ A^{-1}_{im}[\mathbb{L}_2^*](\mathbb{L}_2^*)_{2mkl} = \displaystyle\int_0^1 \big(A^{-1}_{im}[\mathbb{L}_2](t)(\mathbb{L}_2)_{2mkl}(t)\big)\,dt, \\ (\mathbb{L}_2^*)_{ijkl} - (\mathbb{L}_2^*)_{ij2m}A^{-1}_{mn}[\mathbb{L}_2^*](\mathbb{L}_2^*)_{2nkl} \\ \qquad = \displaystyle\int_0^1 \big((\mathbb{L}_2)_{ijkl}(t) - (\mathbb{L}_2)_{ij2m}(t)A^{-1}_{mn}[\mathbb{L}_2](t)(\mathbb{L}_2)_{2nkl}(t)\big)\,dt, \end{cases} \tag{4.78}$$

where in the present context, for any $\mathbb{L} \in L^\infty_{\text{per}}(Y_1; \mathscr{L}_s(\mathbb{R}^{3\times 3}))$, $A[\mathbb{L}] \in L^\infty_{\text{per}}(Y_1; \mathbb{R}^{3\times 3}_s)$ is defined by

$$A[\mathbb{L}](y_2)\xi := [\mathbb{L}(y_2)(\xi \otimes e_2)]e_2 \quad \text{for } y_2 \in Y_1 \text{ and } \xi \in \mathbb{R}^3.$$

By focusing on the first equality of (4.78) we have

$$A^{-1}[\mathbb{L}_2^*] = \int_0^1 A^{-1}[\mathbb{L}_2](t)\,dt = \theta_2 A^{-1}[\mathbb{L}_1^*] + (1-\theta_2)A^{-1}[\mathbb{L}_c], \tag{4.79}$$

where all the quantities are finite. Now, similarly to the proof of Theorem 4.5 we consider the perturbation of $\mathbb{L}_2$ defined by

$$\mathbb{L}_\delta := \mathbb{L}_2 + \delta\,\mathbb{I}_s \quad \text{for } \delta > 0. \tag{4.80}$$

On the one hand, due to $\Lambda(\mathbb{L}_\delta) > 0$ (which by (4.14) implies $0 < \Lambda_{\text{per}}(\mathbb{L}_\delta) \le \alpha_{\text{se}}(\mathbb{L}_\delta)$), thanks to [2, Lemma 3.2] the 1^*-limit $\mathbb{L}_\delta^*$ of $\mathbb{L}_\delta$ and the homogenized tensor $\mathbb{L}_\delta^0$ of $\mathbb{L}_\delta$ defined by (4.16) agree. Then, applying [2, Lemma 3.1] with $\mathbb{L}_\delta$ we get that

$$A^{-1}[\mathbb{L}_\delta^*] = \int_0^1 A^{-1}[\mathbb{L}_\delta](t)dt = \theta_2 A^{-1}[\mathbb{L}_1^* + \delta\,\mathbb{I}_s] + (1-\theta_2)A^{-1}[\mathbb{L}_c + \delta\,\mathbb{I}_s]. \tag{4.81}$$

Observe that we have

$$\begin{aligned} &A[\mathbb{L}_1^* + \delta\,\mathbb{I}_s] \ge A[\mathbb{L}_1^*], \\ &A[\mathbb{L}_1^* + \delta\,\mathbb{I}_s] \to A[\mathbb{L}_1^*] \quad \text{as } \delta \to 0, \end{aligned}$$

where the previous inequality must be understood in the sense of the quadratic forms. This combined with the fact that both $\mathbb{L}_1^* + \delta\,\mathbb{I}_s$ and $\mathbb{L}_1^*$ are strongly elliptic tensors (which implies that the previous matrices are positive definite), yields

$$A^{-1}[\mathbb{L}_1^* + \delta\,\mathbb{I}_s] \le A^{-1}[\mathbb{L}_1^*],$$

and thus,

$$A^{-1}[\mathbb{L}_1^* + \delta\,\mathbb{I}_s] \to A^{-1}[\mathbb{L}_1^*] \quad \text{as } \delta \to 0.$$

Similarly, we have

$$A^{-1}[\mathbb{L}_c + \delta\,\mathbb{I}_s] \to A^{-1}[\mathbb{L}_c] \quad \text{as } \delta \to 0.$$

Hence, from the two previous convergences and taking into account (4.79), (4.81), we deduce that

$$A^{-1}[\mathbb{L}_\delta^*] \to A^{-1}[\mathbb{L}_2^*] \quad \text{as } \delta \to 0.$$

On the other hand, following the proof of Theorem 4.5 we have

$$\mathbb{L}_\delta^* = \mathbb{L}_\delta^0 \to \mathbb{L}_2^0 \quad \text{as } \delta \to 0.$$

Therefore, we obtain the equality

$$A^{-1}[\mathbb{L}_2^0] = A^{-1}[\mathbb{L}_2^*]. \tag{4.82}$$

Using similar arguments, we can prove that $\mathbb{L}_2^0$ and $\mathbb{L}_2^*$ satisfy for any $i, j, k, l \in \{1, 2, 3\}$,

$$A_{im}^{-1}[\mathbb{L}_2^*](\mathbb{L}_2^*)_{2mkl} = A_{im}^{-1}[\mathbb{L}_2^0](\mathbb{L}_2^0)_{2mkl}, \tag{4.83}$$

$$(\mathbb{L}_2^*)_{ijkl} - (\mathbb{L}_2^*)_{ij2m}A_{mn}^{-1}[\mathbb{L}_2^*](\mathbb{L}_2^*)_{2nkl} = (\mathbb{L}_2^0)_{ijkl} - (\mathbb{L}_2^0)_{ij2m}A_{mn}^{-1}[\mathbb{L}_2^0](\mathbb{L}_2^0)_{2nkl}. \tag{4.84}$$

Since the set of equalities (4.78) completely determine the tensor $\mathbb{L}_2^*$, equalities (4.82), (4.83), (4.84) thus imply the desired equality (4.77), which concludes the proof. □

Appendix

Proof of Theorem 4.4. We simply adapt the proof of [2, Theorem 2.2] to dimension 3.

Firstly, let us prove the first part of the theorem, *i.e.* $\Lambda(\mathbb{L}) \geq 0$. The quasi-affinity of the cofactors (see [5]) reads as

$$\int_{Y_3} \mathrm{adj}_{ii}(\nabla v)\, dy = 0, \quad \forall\, v \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3),\ \forall\, i \in \{1,2,3\}. \tag{4.85}$$

As a consequence, for any $d \in \mathbb{R}$, the definition of $\Lambda(\mathbb{L})$ can be rewritten as

$$\Lambda(\mathbb{L}) = \inf\left\{\int_{\mathbb{R}^3}\left[\mathbb{L}e(v) : e(v) + d\sum_{i=1}^{3}\mathrm{adj}_{ii}(\nabla v)\right] dy,\ v \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)\right\}.$$

If we compute the integrand in the previous infimum, we obtain

$$\begin{aligned}\Lambda(\mathbb{L}) = \inf\Bigg\{\int_{\mathbb{R}^3} &\big[P(y;\partial_1 v_1, \partial_2 v_2, \partial_3 v_3) + Q(y;\partial_3 v_2, \partial_2 v_3) \\ &+ Q(y;\partial_3 v_1, \partial_1 v_3) + Q(y;\partial_2 v_1, \partial_1 v_2)\big]\, dy,\ v \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)\Bigg\},\end{aligned} \tag{4.86}$$

where

$$\begin{cases} P(y;a,b,c) := \begin{pmatrix} a & b & c\end{pmatrix}\begin{pmatrix}\lambda+2\mu & \lambda+\frac{d}{2} & \lambda+\frac{d}{2}\\ \lambda+\frac{d}{2} & \lambda+2\mu & \lambda+\frac{d}{2}\\ \lambda+\frac{d}{2} & \lambda+\frac{d}{2} & \lambda+2\mu\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}, \\[2ex] Q(y;a,b) := \begin{pmatrix} a & b\end{pmatrix}\begin{pmatrix}\mu & \mu-\frac{d}{2}\\ \mu-\frac{d}{2} & \mu\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}. \end{cases}$$

We can check that condition (4.21) with $d \geq 0$ implies that the quadratic forms P and Q are non negative. Hence, the integrand in (4.86) is pointwisely non-negative, and thus $\Lambda(\mathbb{L}) \geq 0$.

Now, let us prove that $\Lambda_{\mathrm{per}}(\mathbb{L}) > 0$. By the definition of $\Lambda_{\mathrm{per}}(\mathbb{L})$ and using the same argument as before, we have

$$\Lambda_{\mathrm{per}}(\mathbb{L}) = \inf\left\{\int_{Y_3}\left[\mathbb{L}e(v) : e(v) + d\sum_{i}\mathrm{adj}_{ii}(\nabla v)\right] dy,\ v \in H^1_{\mathrm{per}}(Y_3;\mathbb{R}^3),\ \int_{Y_3}|\nabla v|^2 dy = 1\right\}.$$

Similar computations lead to

$$\begin{aligned}\Lambda_{\mathrm{per}}(\mathbb{L}) = \inf\Bigg\{\int_{Y_3} &\big[P(y;\partial_1 v_1, \partial_2 v_2, \partial_3 v_3) + Q(y;\partial_3 v_2, \partial_2 v_3) \\ &+ Q(y;\partial_3 v_1, \partial_1 v_3) + Q(\partial_2 v_1, \partial_1 v_2)\big]\, dy\Bigg\}.\end{aligned} \tag{4.87}$$

Take $y \in Z_i, i \in I$. Then, using that $4\mu_i = d$, we have

$$P(y;a,b,c) = (\lambda_i + 2\mu_i)(a+b+c)^2 \geq 0,$$

and

$$Q(y;a,b) = \mu_i(a-b)^2 \geq 0.$$

For $y \in Z_j, j \in J$, using that $2\mu_j + 3\lambda_j = -d$, we get

$$P(y;a,b,c) = \left(\mu_j + \frac{\lambda_j}{2}\right)\left[(a-b)^2 + (a-c)^2 + (b-c)^2\right] \geq 0,$$

and

$$Q(y;a,b) = d\left(\mu_j + \frac{d}{4}\right) \geq 0.$$

Finally, for $y \in Z_k$, $k \in K$, since $-(2\mu_k + 3\lambda_k) < d < 4\mu_k$, it is easy to see that the quadratic forms P and Q are positive semi-definite. Therefore, we have just proved that there exists $\alpha > 0$ such that

$$P(y;a,b,c) \geq \alpha(a+b+c)^2,\ Q(y;a,b) \geq \alpha(a-b)^2,\ y \in Z_i, i \in I, \tag{4.88}$$

$$P(y;a,b,c) \geq \alpha[(a-b)^2+(a-c)^2+(b-c)^2],\ Q(y;a,b) \geq \alpha(a^2+b^2),\ y \in Z_j, j \in J, \tag{4.89}$$

$$P(y;a,b,c) \geq \alpha(a^2+b^2+c^2),\ Q(y;a,b) \geq \alpha(a^2+b^2),\ y \in Z_k, k \in K, \tag{4.90}$$

which implies that $\Lambda_{\text{per}}(\mathbb{L}) \geq 0$.

Assume by contradiction that $\Lambda_{\text{per}}(\mathbb{L}) = 0$. In this case there exists a sequence $v^n \in H^1_{\text{per}}(Y_3;\mathbb{R}^3)$ with

$$\int_{Y_3} v^n \, dy = 0,$$

such that

$$\int_{Y_3} |\nabla v^n|^2 \, dy = 1, \quad \forall\, n \in \mathbb{N}, \tag{4.91}$$

together with

$$\int_{Y_3} \mathbb{L}(y)e(v^n) : e(v^n)\, dy \to 0.$$

By the Poincaré-Wirtinger inequality v^n is bounded in $L^2(Y_3;\mathbb{R}^3)$. Moreover, by (4.87) we have

$$\int_{Y_3} \left[P(y;\partial_1 v_1^n, \partial_2 v_2^n, \partial_3 v_3^n) + \sum_{i<j} Q(y;\partial_j v_i^n, \partial_i v_j^n)\right] dy \to 0. \tag{4.92}$$

Take $k \in K$. Using (4.90) we get

$$\int_{Z_k} \left[P(y;\partial_1 v_1^n, \partial_2 v_2^n, \partial_3 v_3^n) + \sum_{i<j} Q(y;\partial_j v_i^n, \partial_i v_j^n)\right] dy \geq \alpha \int_{Z_k} |\nabla v^n|^2 \, dy.$$

Then, using (4.92) and the fact that both P and Q are non negative, it follows that

$$\int_{Z_k} |\nabla v^n|^2\, dy \to 0 \quad \forall\, k \in K,$$

and therefore

$$\lim_{n\to\infty} \sum_{k\in K} \int_{Z_k} \sum_{q,r=1,2,3} (\partial_r v_q^n)^2\, dy = 0. \tag{4.93}$$

Next, take $j \in J$. By (4.89) we obtain

$$\begin{aligned}&\int_{Z_j} \left[P(y; \partial_1 v_1^n, \partial_2 v_2^n, \partial_3 v_3^n) + \sum_{i<k} Q(y; \partial_k v_i^n, \partial_i v_k^n) \right] dy \\ &\geq \alpha \int_{Z_j} \sum_{i<k} \left[(\partial_i v_i^n - \partial_k v_k^n)^2 + (\partial_k v_i^n)^2 + (\partial_i v_k^n)^2 \right] dy.\end{aligned}$$

Again using (4.92) and the non-negativity of P and Q we get

$$\lim_{n\to\infty} \int_{Z_j} \left[(\partial_i v_i^n - \partial_k v_k^n)^2 + (\partial_k v_i^n)^2 + (\partial_i v_k^n)^2 \right] = 0 \quad \text{for } i, k \in \{1, 2, 3\},\ i < k. \tag{4.94}$$

From (4.94) and the continuity of the operator $\partial_1 : L^2(Z_j) \to H^{-1}(Z_j)$ we deduce that

$$\begin{cases} \partial_2(\partial_1 v_1^n) = \partial_1(\partial_2 v_1^n) \to 0 \quad \text{strongly in } H^{-1}(Z_j), \\ \partial_1(\partial_1 v_1^n) = \partial_1(\partial_1 v_1^n - \partial_2 v_2^n) + \partial_2(\partial_1 v_2^n) \to 0 \quad \text{strongly in } H^{-1}(Z_j). \end{cases} \tag{4.95}$$

By (4.91) we also have

$$\partial_1 v_1^n \text{ is bounded in } L^2(Z_j). \tag{4.96}$$

However, thanks to Korn's Lemma (see, *e.g.*, [9]) the following norms are equivalent in $L^2(Z_j)$:

$$\begin{cases} \|\nabla \cdot\|_{H^{-1}(Z_j;\mathbb{R}^3)} + \|\cdot\|_{H^{-1}(Z_j)}, \\ \|\cdot\|_{L^2(Z_j)}. \end{cases}$$

Hence, from estimates (4.95), (4.96) and the compact embedding of L^2 into H^{-1}, it follows that

$$\partial_1 v_1^n \text{ is strongly convergent in } L^2(Z_j).$$

Furthermore, by (4.95) and the fact that Z_j is connected for all j, there exists $c_j \in \mathbb{R}$ such that

$$\partial_1 v_1^n \to c_j \text{ strongly in } L^2(Z_j),$$

which combined with (4.94) yields

$$\nabla v^n \to c_j I_3 \text{ strongly in } L^2(Z_j)^3.$$

Since v^n is bounded in $L^2(Y_3; \mathbb{R}^3)$, we can conclude that there exists $V_j \in \mathbb{R}^3$ such that

$$v^n \to v := c_j y + V_j \quad \text{strongly in } H^{-1}(Z_j; \mathbb{R}^3). \tag{4.97}$$

In Case 1, by the periodicity of the limit $c_j y + V_j$ it is necessary to have $c_j = 0$.

In Case 2, since Z_k is connected, by (4.93) there exists a constant c_k such that v_n converges to $\chi_{Z_j} v + \chi_{Z_k} c_k$ strongly in $H^1(Z_j \cup Z_k)$. Hence, since the sets Z_j and Z_k are regular, the trace of v must be equal to c_k a.e. on $\partial Z_j \cap \partial Z_k$. Therefore, the only way for $c_j y + V_j$ to remain constant on a set of non-null $\mathscr{H}^2$-measure is to have $c_j = 0$.

In both cases this implies that ∇v^n converges strongly to 0 in $L^2(Z_j; \mathbb{R}^{3\times 3})$, and thus

$$\lim_{n\to\infty} \sum_{j\in J} \int_{Z_j} \sum_{r,q=1,2,3} (\partial_q v_r^n)^2 dy = 0. \tag{4.98}$$

Finally, take $i \in I$. By (4.88) we have

$$\int_{Z_i} \left[P(y; \partial_1 v_1^n, \partial_2 v_2^n, \partial_3 v_3^n) + \sum_{r<q} Q(y; \partial_q v_r^n, \partial_r v_q^n) \right] dy \geq$$

$$\alpha \int_{Z_i} \left[(\partial_1 v_1^n + \partial_2 v_2^n + \partial_3 v_3^n)^2 + (\partial_2 v_1^n + \partial_1 v_2^n)^2 + (\partial_3 v_1^n + \partial_1 v_3^n)^2 + (\partial_3 v_2^n + \partial_2 v_3^n)^2 \right] dy.$$

By virtue of (4.92) we also have

$$\int_{Z_i} \left[(\partial_1 v_1^n + \partial_2 v_2^n + \partial_3 v_3^n)^2 + (\partial_2 v_1^n + \partial_1 v_2^n)^2 + (\partial_3 v_1^n + \partial_1 v_3^n)^2 + (\partial_3 v_2^n + \partial_2 v_3^n)^2 \right] dy \to 0, \tag{4.99}$$

as $n \to \infty$. Limits (4.98), (4.93) combined with (4.85) yield

$$\lim_{n\to\infty} \sum_{i\in I} \int_{Z_i} \sum_{r=1}^{3} \mathrm{adj}_{rr}(\nabla v^n) dy = 0.$$

Therefore, upon subtracting this quantity to the sum over $i \in I$ of (4.99) we conclude that

$$\lim_{n\to\infty} \sum_{i\in I} \int_{Z_i} \sum_{r,q=1}^{3} (\partial_q v_r^n)^2 dy = 0. \tag{4.100}$$

Finally, limits (4.98), (4.93) and (4.100) contradict condition (4.91). The proof is thus complete. □

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