UNIVERSAL FUNCTIONS WITH SMALL DERIVATIVES
AND EXTREMELY FAST GROWTH

M. C. Calderón-Moreno[1]

Abstract. We prove that if $\alpha \in (0, \frac{1}{2})$ and T is an infinite order differential operator there exists a dense linear manifold $\mathcal{M}$ of entire functions such that

$$\lim_{\substack{z \to \infty \\ z \in S}} \exp(|z|^\alpha) T f(z) = 0$$

for every $f \in \mathcal{M}$ and any plane strip S. Moreover, every non-null function in $\mathcal{M}$ exhibits some translation-universality property with respect to T and its growth index with respect to any prefixed sequence of non-constant entire functions is infinite.

1   Introduction

Throughout the last decades several authors have given "counterexamples" to the well-known Liouville's theorem. For instance, there exists a non-null entire function which tends to zero on every line (see [11, 15, 16]) or such that it and even all its derivatives have also vanishing integrals on every line (see [2, 19]). In 1997 Bernal [5] (see also [4]) got many functions which not only "violated" Liouville's theorem in both senses but also possessed an extremely fast growth and "sharp" asymptotic behaviour at infinite. In order to specify exactly this result, and with it the framework of this paper, let us introduce some notation.

The symbol Σ will stand for the family consisting of all strips in $\mathbb{C}$ (i.e., plane regions between two parallel straight lines) and all sectors

$$s_\beta := \{z : \ 0 \le \arg z \le \beta\} \qquad (\beta \in (0, 2\pi))$$

---

and L will stand for the family of all straight lines. We denote by $H(\mathbb{C})$ the space of all entire functions endowed with the compact-open topology, so $H(\mathbb{C})$ is a separable Fréchet space. If $r > 0$ and $f \in H(\mathbb{C})$ we denote $M_f(r) := \max\{|f(z)| : |z| = r\}$ and for any non-constant function $h \in H(\mathbb{C})$ the relative growth order of f with respect to h (see [3]) is defined as

$$\rho_h(f) = \limsup_{r \to \infty} \frac{\log M_h^{-1}(M_f(r))}{\log r}.$$

Given any sequence $\mathcal{F} = \{h_n\}_1^\infty$ of non-constant entire functions, the growth index of f with respect to $\mathcal{F}$ is $i_\mathcal{F}(f) = \min\{n \in \mathbb{N} : \rho_{h_n}(f) < \infty\}$. We set $i_\mathcal{F}(f) = \infty$ if $\rho_{h_n}(f) = \infty$ for all n. Observe that these concepts extend the older one of relative growth order [18].

With this in mind, Bernal's result [5, Theorem 3] reads as follows:

THEOREM 1.1. *(Bernal [5]) Assume that $\alpha \in (0, 1/2)$ and that $\varphi : [0, +\infty) \to (0, +\infty)$ is a continuous function which is integrable on $(1, +\infty)$. Assume, in addition, that $\mathcal{F} = \{h_n\}_1^\infty$ is a sequence of nonconstant entire functions. Then there is a linear manifold $\mathcal{M} = \mathcal{M}(\alpha, \varphi, \mathcal{F}) \subset H(\mathbb{C})$ satisfying the following seven properties:*

(a) $\mathcal{M}$ is dense in $H(\mathbb{C})$.

(b) $\displaystyle\lim_{\substack{z \to \infty \\ z \in S}} \exp(|z|^{3/2}\varphi(z))f(z) = 0$ for all $S \in \Sigma$ and all $f \in \mathcal{M}$.

(c) $\displaystyle\lim_{\substack{z \to \infty \\ z \in S}} \exp(|z|^\alpha)f^{(j)}(z) = 0$ for all $S \in \Sigma$, all $f \in \mathcal{M}$ and all $j \geq 0$.

(d) $f^{(j)}$ is bounded on S for all $S \in \Sigma$, all $f \in \mathcal{M}$ and all $j \geq 0$.

(e) $f^{(j)}$ is integrable on S with respect to the plane Lebesgue measure for all $S \in \Sigma$, all $f \in \mathcal{M}$ and all $j \geq 0$.

(f) $f^{(j)}$ is integrable on S with respect to the length measure for all $l \in L$, all $f \in \mathcal{M}$ and all $j \geq 0$.

(g) $\displaystyle\int_l f^{(j)}ds = 0$ for all $l \in L$, all $f \in \mathcal{M}$ and all $j \geq 1$.

(h) $i_\mathcal{F}(f) = \infty$ for all $f \in \mathcal{M} \setminus \{0\}$.

Our aim in this paper is to show that not only the derivative operator of order j, $D^j f = f^{(j)}$, can be replaced by infinite order differential operators in the seven above properties, but also that we can provide an eighth property whose feature is totally different from the others, namely, a property about wild behaviour near the infinity point.

2 Definitions and statement of the main result

An entire function $\phi(z) = \sum_{k=0}^{\infty} a_k z^k$ is said to be of exponential type if there are constants $A, B > 0$ such that $|\phi(z)| \leq Ae^{B|z|}$ for all $z \in \mathbb{C}$. The function ϕ is of subexponential type if given $\varepsilon > 0$, then there is a constant $A = A(\varepsilon) > 0$ such that $|\phi(z)| \leq Ae^{\varepsilon|z|}$ for all $z \in \mathbb{C}$. This happens if and only if $\limsup_{k \to \infty} (k!|a_k|)^{1/k} = 0$ (see, e.g., [8, 2.2.9–11]). Each entire function of subexponential type is also of exponential type and every entire function ϕ of exponential type defines a (linear, continuous) infinite order differential operator $\phi(D) = \sum_{k=0}^{\infty} a_k D^k$ on $H(\mathbb{C})$, which is onto (see [10, 14]). Here $D^0 = I =$ the identity operator.

On the other hand, given a continuous selfmapping T on $H(\mathbb{C})$, we say that an entire function f is *T-universal* whenever for each $g \in H(\mathbb{C})$ there exists a sequence $(a_n) \subset \mathbb{C}$ satisfying

$$(Tf)(z + a_n) \to g(z) \quad (n \to \infty) \quad \text{in } H(\mathbb{C}).$$

Now, we are able to establish the main result, which will be proved in the next section.

THEOREM 2.1. *Let be given an $\alpha \in (0, \frac{1}{2})$, a continuous function $\varphi : [0, +\infty) \to (0, +\infty)$ which is integrable on $(1, +\infty)$ and a sequence $\mathcal{F} = \{h_n\}_1^{\infty}$ of non-constant entire functions. Assume, in addition, that $\{\psi_{i,m}(z)\}_{m=0}^{\infty}$ $(i = 1, 2)$ are two sequences of entire functions of subexponential type. Then there is a linear manifold $\mathcal{M} = \mathcal{M}(\alpha, \varphi, \mathcal{F}, (\psi_{1,m}), (\psi_{2,m}))$ of entire functions satisfying the following properties:*

(a) *$\mathcal{M}$ is dense in $H(\mathbb{C})$.*

(b) *$\lim_{\substack{z \to \infty \\ z \in S}} \exp(|z|^{3/2}\varphi(z))f(z) = 0$ for all $S \in \Sigma$ and all $f \in \mathcal{M}$.*

(c) *$\lim_{\substack{z \to \infty \\ z \in S}} \exp(|z|^{\alpha})(\psi_{1,m}(D)f)(z) = 0$ for all $S \in \Sigma$, all $f \in \mathcal{M}$ and all $m \geq 0$.*

(d) *$\psi_{1,m}(D)f$ is bounded on S for all $S \in \Sigma$, all $f \in \mathcal{M}$ and all $m \geq 0$.*

(e) *$\psi_{1,m}(D)f$ is integrable on S with respect to the plane Lebesgue measure for all $S \in \Sigma$, all $f \in \mathcal{M}$ and all $m \geq 0$.*

(f) *$\psi_{1,m}(D)f$ is integrable on S with respect to the length measure for all $l \in L$, all $f \in \mathcal{M}$ and all $m \geq 0$.*

(g) *If $\psi_{1,m}(0) = 0$, then $\int_l \psi_{1,m}(D)f ds = 0$ for all $l \in L$ and all $f \in \mathcal{M}$.*

(h) *$i_{\mathcal{F}}(f) = \infty$ for all $f \in \mathcal{M} \setminus \{0\}$.*

(i) Every non–null function in $\mathcal{M}$ is $\psi_{2,m}(D)$-universal for all $m \geq 0$.

Observe that any polynomial $p(z)$ is an entire function of subexponential type. In particular, taking $\psi_{1,m}(z) = z^m$ for all $m \geq 0$, we obtain the conditions (a)–(h) of Theorem 1.1 together with an additional universal property for a sequence of infinite order differential operators. It is noteworthy the case in which also $\psi_{2,m}(z) = z^m$ for all $m \geq 0$. Then Theorem 2.1 provides a linear dense manifold of entire functions such that each of them and all its derivatives are universal functions in Birkhoff's sense [7] (see also [6, 12, 13] for the related concept of holomorphic monster in $\mathbb{C}$) with growth conditions.

Finally, we mention that in 2000 A. Bonilla [9] studied an analogous problem in the space of harmonic functions in $\mathbb{R}^N$, providing similar conditions (c)–(f) for any derivative operator D^α and the universal condition (i) for the identity operator.

3   An auxiliary result and proof of the main result

We will use the following theorem about tangential approximation due to Arakelian [1, p. 1189]. From now on $\mathbb{C}_\infty$ is the extended plane. If $F \subset \mathbb{C}$ is a closed set, then $A(F)$ is the space of all continuous functions on F which are holomorphic in the interior of F. A closed subset $F \subset \mathbb{C}$ is said to be an Arakelian set [17] whenever $\mathbb{C}_\infty \setminus F$ is both connected and locally connected at infinity.

THEOREM 3.1. *(Arakelian [1]) Assume that $F \subset \mathbb{C}$ is an Arakelian set and that $\varepsilon(t)$ is continuous and positive for $t \geq 0$. In addition, suppose that*

$$\int_1^\infty t^{-3/2} \log \varepsilon(t) dt > -\infty. \tag{1}$$

Then for every $g \in A(K)$ there exists an entire function f such that

$$|f(z) - g(z)| < \varepsilon(|z|) \qquad (z \in F).$$

The statement does not remain valid for every F is (1) is violated.

Proof. (of Theorem 2.1) Suppose that α, φ, $\mathcal{F} = \{h_n\}_1^\infty$ and

$$\{\psi_{i,m}(z) = \sum_{k=0}^\infty b_k^{i,m} z^k\}_m \qquad (i = 1, 2)$$

are as in the hypotheses. Thus for each $m \geq 0$, $i = 1, 2$, there exists a positive constant $A_{i,m}$ such that

$$|b_k^{i,m}| \leq A_{i,m} \frac{(1/2)^k}{k!} \qquad (\forall k \geq 0). \tag{2}$$

Fix a sequence $\{p_n\}_{n=1}^{\infty}$ which is dense in $H(\mathbb{C})$ and a number $\beta \in (\alpha, \frac{1}{2})$. For every $n \in \mathbb{N}$, the function

$$\varepsilon(t) = \varepsilon_n(t) := \min\left\{\frac{1}{n} \cdot \frac{1}{1+t}, \exp(-t^{3/2}\varphi(t) - t^{\beta})\right\}$$

is positive and continuous for $t \geq 0$ and satisfies (1). Let P be the parabolic curve

$$P = \{x - ix^{1/2} : \ x \geq 0\}.$$

For each $n \in \mathbb{N}$, we define the sets

$$E_n = \{z \in \mathbb{C} : \ |z| \geq n+1 \text{ and } \operatorname{dist}(z, P) \geq 1 + |z|\},$$

$$B_n = \{z : \ |z| \leq n\}.$$

Consider a sequence of closed balls $D_j = B(a_j, 1 + 2^j)$ such that

$$D_j \subset \{z : \ \operatorname{dist}(z, P) < 1 + |z|\} \setminus P$$

and

$$|a_j| + 2^{j+2} < |a_{j+1}| \qquad (j \geq 1).$$

In particular the balls D_j are pairwise disjoint and

$$|z| > 2^j \qquad (z \in D_j). \tag{3}$$

Let $H = \{z_k\}_{k=1}^{\infty}$ be a sequence of pairwise different complex numbers in P with $z_k \to \infty$ $(k \to \infty)$. For each $n \in \mathbb{N}$ we define

$$F_n = B_n \cup E_n \cup H \cup \left(\bigcup_{j>j_0} D_j\right),$$

where $j_0 = j_0(n)$ is the largest index such that $D_j \cap B_{n+1} \neq \emptyset$. Then F_n is an Arakelian set.

Divide $\{a_j\}$ into infinitely many disjoint subsequences $\{a_{i(m,l,j)}\}$ by setting

$$i(m, l, j) = \frac{[(m+l)(m+l+1) + 2j][(m+l)(m+l+1) + 2(j+1)]}{8} + j$$

for all $m \geq 0$, all $l \geq 1$ and all $j \geq 1$. Define inductively a sequence $\{f_n\}_{n=1}^{\infty}$ of entire functions as follows. Denote $r_k = |z_k|$ for all $k \geq 1$ and for each $m \geq 0$ consider a sequence $\{q_{m,n}\}_n$ of entire functions such that

$$\psi_{2,m}(D)q_{m,n} = p_n \qquad (n \geq 1).$$

Recall that $\psi_{2,m}(D)$ is an onto operator on $H(\mathbb{C})$. Let $g_1 : F_1 \to \mathbb{C}$ denote the function

$$g_1(z) = \begin{cases} p_1(z) & (z \in B_1) \\ 0 & (z \in E_1) \\ 1 + \max_{1 \le j \le k} M_{h_j}(\exp r_k) & (z = z_k \text{ and } |z_k| > 1) \\ q_{m,j}(z - a_{i(m,1,j)}) & (z \in D_{i(m,1,j)}) \\ 0 & (z \in D_{i(m,l,j)} \quad l \ne 1). \end{cases}$$

Then $g_1 \in A(F_1)$ and by Theorem 3.1 there exists an entire function f_1 such that

$$|f_1(z) - g_1(z)| < \varepsilon_1(|z|) \qquad (z \in F_1).$$

Assume that $n \in \{2, 3, \ldots\}$ and that we have constructed $2n - 2$ functions $g_1, f_1, \ldots, g_{n-1}$, f_{n-1} in such a way that $g_i \in A(F_i)$, $f_i \in H(\mathbb{C})$ and

$$|f_i(z) - g_i(z)| < \varepsilon_i(|z|) \qquad (z \in F_i)$$

for all $i \in \{1, 2, \ldots, n - 1\}$. Now, we define the function $g_n : F_n \to \mathbb{C}$ by

$$g_n(z) = \begin{cases} p_n(z) & (z \in B_n) \\ 0 & (z \in E_n) \\ 1 + \max_{1 \le j \le k} M_{h_j}(\exp r_k) + k \sum_{i=1}^{n-1} M_{f_i}(r_k) & (z = z_k \text{ and } |z_k| > n) \\ q_{m,j}(z - a_{i(m,n,j)}) & (z \in D_{i(m,n,j)}) \\ 0 & (z \in D_{i(m,l,j)} \quad l \ne n). \end{cases}$$

Trivially $g_n \in A(F_n)$ and by Theorem 3.1 there exists an entire function f_n such that

$$|f_n(z) - g_n(z)| < \varepsilon_n(|z|) \qquad (z \in F_n).$$

Thus, for all $n \in \mathbb{N}$,

$$|f_n(z) - p_n(z)| < \frac{1}{n} \qquad (z \in B_n), \tag{4}$$

$$|f_n(z)| < \exp(-|z|^{3/2}\varphi(z) - |z|^\beta) \qquad (z \in E_n), \tag{5}$$

$$|f_n(z) - (1 + \max_{1 \le j \le k} M_{h_j}(\exp r_k) + S(n,k))| < \frac{1}{n} \le 1 \tag{6}$$

for all k such that $|z_k| > n$, where $S(1,k) = 0$ and $S(n,k) = k \sum_{i=1}^{n-1} M_{f_i}(r_k)$ if $n \ge 2$,

$$|f_n(z) - q_{m,j}(z - a_{i(m,n,j)})| < \frac{1}{n} \cdot \frac{1}{1 + |z|} \tag{7}$$

for all $z \in D_{i(m,n,j)}$, all $j \ge 1$ and all $m \ge 0$, and

$$|f_n(z)| < \frac{1}{n} \cdot \frac{1}{1 + |z|} \tag{8}$$

for all $z \in D_{i(m,l,j)}$, all $j \geq 1$, all $m \geq 0$ and all $l \geq 1$ with $l \neq n$. Although we do not mention it explicitly, it is clear that we have (8) whenever $D_{i(m,l,j)} \cap B_n = \emptyset$.

By (4), the sequence $\{f_n\}_{n=1}^{\infty}$ is dense in $H(\mathbb{C})$. Let us define $\mathcal{M}$ as the linear span of $\{f_n\}_n$. Evidently, $\mathcal{M}$ is a linear dense manifold of $H(\mathbb{C})$: this proves (a). In order to verify that (b), (c) hold for every $f \in \mathcal{M}$, it suffices to check that both properties are satisfied for every function $f = f_n$. From (5),

$$\exp(|z|^{3/2}\varphi(z))|f_n(z)| < \exp(-|z|^{\beta}) \qquad (z \to \infty, z \in E_n).$$

For any sector or strip $S \in \Sigma$, we have that $S \setminus E_n$ is a bounded set, thus

$$\exp(|z|^{3/2}\varphi(z))f_n(z) \to 0 \qquad (z \to \infty, z \in S).$$

This proves (b).

Now, we define the set $E_n^{\star}$ as

$$E_n^{\star} = \{z \in \mathbb{C} : |z| \geq n + 2 \text{ and } \text{dist}(z, P) \geq 2 + |z|\}.$$

Then, one uses the Cauchy estimates and (5) to infer that

$$|f_n^{(k)}(z)| \leq k! \max\{|f_n(w)| : |w - z| = 1\} \leq$$

$$\leq k! \max\{\exp(-|w|^{\beta}) : |w| \geq |z| - 1\} \leq k! \exp(-(|z| - 1)^{\beta})$$

for all $z \in E_n^{\star}$ and all $k \geq 0$ (remember that φ is positive). Hence, for each $m \geq 0$,

$$|\exp(|z|^{\alpha})\psi_{1,m}(D)f_n(z)| = \left|\exp(|z|^{\alpha}) \cdot \sum_{k=0}^{\infty} b_k^{1,m} f_n^{(k)}(z)\right| \leq$$

$$\leq \left|\exp(|z|^{\alpha}) \cdot A_{1,m} \sum_{k=0}^{\infty} \frac{(1/2)^k}{k!} \cdot k! \exp(-(|z| - 1)^{\beta})\right| \leq$$

$$\leq 2A_{1,m} \exp(|z|^{\alpha} - (|z| - 1)^{\beta}) \to 0 \qquad (z \to \infty, z \in E_n^{\star}).$$

If $S \in \Sigma$, we have again that $S \setminus E_n^{\star}$ is bounded, so

$$\lim_{\substack{z \to \infty \\ z \in S}} \exp(|z|^{\alpha}\psi_{1,m}(D)f_n(z)) = 0,$$

which proves (c).

The proofs of (d)–(f) and (h) are analogous to those we may find in [5]. In order to prove (g), we fix $f \in \mathcal{M}$. Suppose that $\psi_{1,m}(0) = 0$, then $\psi_{1,m}(D)f = D(\sum_{k=1}^{\infty} b_k^{1,m} f^{(k-1)})$, and by the fundamental calculus theorem

$$\int_l (\psi_{1,m}(D)f)ds = \lim_{\substack{b \to \infty \\ b \in l}} (\sum_{k=1}^{\infty} b_k^{1,m} f^{(k-1)}(b)) - \lim_{\substack{a \to \infty \\ a \in l}} (\sum_{k=1}^{\infty} b_k^{1,m} f^{(k-1)}(a)).$$

Now we have only to observe that the same way followed to obtain (c) leads us to "(c)" for $\sum_{k=1}^{\infty} b_k^{1,m} D^{(k-1)}$.

It remains only to get (i). Let be $f \in \mathcal{M}$ and fix $m \geq 0$. Since every non-zero scalar multiple of a $\psi_{2,m}(D)$-universal function is again $\psi_{2,m}(D)$-universal, we may suppose that $f = \sum_{j \in I} \alpha_j f_j$ with $\alpha_{j_1} = 1$ and $I = \{j_1, \ldots, j_r\}$ finite. In order to prove that f is $\psi_{2,m}(D)$-universal it is enough to check that

$$\lim_{n \to \infty} ((\psi_{2,m}(D)f)(z + a_{i(m,j_1,n)}) - p_n(z)) = 0 \qquad (9)$$

uniformly on compact subsets.

Fix $n \in \mathbb{N}$. We have

$$|(\psi_{2,m}(D)f)(z + a_{i(m,j_1,n)}) - p_n(z)| =$$

$$= \left| \sum_{j \in I} \alpha_j \cdot (\psi_{2,m}(D)f_j)(z + a_{i(m,j_1,n)}) - \psi_{2,m}(D)q_{m,n}(z) \right| \leq$$

$$\leq |\psi_{2,m}(D)(f_{j_1}(z + a_{i(m,j_1,n)}) - q_{m,n}(z))| + \sum_{\substack{j \in I \\ j \neq j_1}} |\alpha_j| |f_j(z + a_{i(m,j_1,n)})|.$$

Now, for any $z \in \overline{B}(0, 2^n) \subset \overline{B}(0, 2^{i(m,j_1,n)})$ we have $z + a_{i(m,j_1,n)} \in D_{i(m,j_1,n)}$, thus by (3) and (8),

$$|f_j(z + a_{i(m,j_1,n)})| < \frac{1}{j} \cdot \frac{1}{(1 + |z + a_{i(m,j_1,n)}|)} < \frac{1}{2^n} \qquad (10)$$

for all $j \in I$ with $j \neq j_1$.

On the other hand, because of Cauchy's formula for derivatives applied to the curve $\gamma \equiv \{|w| = 2^n + \frac{1}{2}\}$, we have by (7) that

$$|(f_{j_1}(z + a_{i(m,j_1,n)}) - q_{m,n}(z))^{(k)}| < k! \frac{1}{j_1 2^n} \qquad (k \geq 0).$$

Therefore, by (2),

$$|\psi_{2,m}(D)(f_{j_1}(z + a_{i(m,j_1,n)}) - q_{m,n}(z))| <$$

$$< \sum_{k=0}^{\infty} A_{2,m} \frac{(1/2)^k}{k!} \cdot k! \frac{1}{j_1 2^n} = \frac{2A_{2,m}}{j_1} \cdot \frac{1}{2^n}. \qquad (11)$$

Joining (10) and (11) we obtain

$$|(\psi_{2,m}(D)f)(z + a_{i(m,j_1,n)}) - p_n(z)| < \left(\frac{2A_{2,m}}{j_1} + \sum_{\substack{j \in I \\ j \neq j_1}} |\alpha_j| \right) \frac{1}{2^n}$$

for all $z \in \overline{B}(0, 2^n)$. It is clear that this implies (9). Consequently, f is $\psi_{2,m}(D)$-universal and we have (i). $\square$

References

[1] N. U. Arakelian, Uniform approximation on closed sets by entire functions (in Russian), Izv. Akad. Nauk SSSR Ser. Mat. **28**, 1187–1206 (1964).

[2] D. H. Armitage, A non–constant continuous function on the plane whose integral on every line is zero, Amer. Math. Monthly **101**, 892–894 (1994).

[3] L. Bernal–González, Orden relativo de crecimiento de funciones enteras, Collect. Math. **39**, 209–229 (1988).

[4] L. Bernal–González, A lot of "counterexamples" to Liouville's theorem, J. Math. Anal. Appl. **201**, 1002–1009 (1996).

[5] L. Bernal–González, Small entire functions with extremely fast growth, J. Math. Anal. Appl. **207**, 541–548 (1997).

[6] L. Bernal–González and M. C. Calderón–Moreno, Holomorphic T–monsters and strongly omnipresent operators, J. Approx. Theory **104**, 204–219 (2000).

[7] C. D. Birkhoff, Démonstration d'un théorème élémentaire sur les fonctions entières, C. R. Acad. Sci. Paris **189**, 473–475 (1929).

[8] R. P. Boas, Entire Functions, Academic Press, New York, 1954.

[9] A. Bonilla, "Counterexamples" to the harmonic Liouville theorem and harmonic functions with zero nontangential limits, Colloq. Math. **83**, 155–160 (2000).

[10] L. Ehrenpreis, Mean periodic functions I, Amer. J. Math. **77**, 293–328 (1995).

[11] P. M. Gauthier, Uniform approximation, in Complex Potential Theory, pp. 1–37, Kluwer Academic, Dordrecht/Norwell, MA, 1994.

[12] K.-G. Grosse–Erdmann, Holomorphe Monster und universelle Funktionen, Mitt. Math. Sem. Giessen **176** (1987).

[13] W. Luh, Holomorphic monsters, J. Approx. Theory **53**, 128–144 (1988).

[14] B. Malgrange Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution, Ann. Institut Fourier (Grenoble) **6**, 271–355 (1955/1956).

[15] G. Mittag–Leffler, Sur la représentation analytique d'une branche uniforme d'une fonction monogène (Sixième note), Acta Math. **42**, 285–308 (1920).

[16] D. J. Newman, An entire function bounded in every direction, Amer. Math. Monthly **83**, 192–193 (1976).

[17] J. P. Rosay and W. Rudin, Arakelian's approximation theorem, Amer. Math. Monthly **96**, 432–434 (1989).

[18] D. Sato, On the rate of growth of entire functions of fast growth, Bull. Amer. Math. Soc. **69**, 411–414 (1963).

[19] L. Zalcman, Uniqueness and nonuniqueness for the Radon transform, Bull. London Math. Soc. **14**, 241–245 (1982).

MARÍA DEL CARMEN CALDERÓN MORENO
DEPARTAMENTO DE ANÁLISIS MATEMÁTICO
FACULTAD DE MATEMÁTICAS, APDO. 1160
AVENIDA REINA MERCEDES
41080 SEVILLA, SPAIN
E-mail: mccm@cica.es