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# ON $\xi$-CONFORMALLY FLAT CONTACT METRIC MANIFOLDS

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In this paper, the notion of $\xi$-conformally flat on a contact metric structure is introduced and it is proved that any $K$-contact metric manifold is $\xi$-conformally flat if and only if it is an $\eta$-Einstein Sasakian manifold. Finally, some applications are given.

## INTRODUCTION

Let $M$ be a Riemannian manifold with metric $g$ and let $T(M)$ be the Lie Algebra of differentiable vector fields in $M$. The Ricci operator $Q$ of $(M, g)$ is defined by $g(QX, Y) = S(X, Y)$, where $S$ denotes the Ricci tensor of type (0, 2) on $M$ and $X, Y \in T(M)$. Weyl[7, 8] constructed a generalized curvature tensor on a Riemannian manifold which vanishes whenever the metric is (locally) conformally equivalent to a flat metric; for this reason he called it the conformal curvature tensor of the metric. The Weyl conformal curvature tensor is defined as a map

$$C : T(M) \times T(M) \times T(M) \rightarrow T(M)$$

such that

$$C(X, Y)Z = R(X, Y)Z - \frac{1}{m-2}[g(QY, Z)X + g(Y, Z)QX - g(QX, Z)Y - g(X, Z)QY] + \frac{r}{(m-1)(m-2)}[g(Y, Z)X - g(X, Z)Y],$$

for any $X, Y, Z \in T(M)$, where $R$, $r$ are denoting the Riemann curvature tensor and the scalar curvature of $M$, respectively.

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In the case of contact metric manifolds, to characterize them via Weyl conformal curvature tensor, Okumura[6] proved that a conformally flat Sasakian manifold is locally isometric to the unit sphere. Later, Miyazzawa and Yamagushi[5] proved that a conformally symmetric Sasakian manifold is also locally isometric to the unit sphere. Chaki and Taraflar[3] obtained the same result for a Sasakian manifold satisfying the condition $R(X, Y)C = 0$, for any $X, Y \in T(M)$.

On the other hand, it is well known that any Sasakian manifold is a $K$-contact metric manifold, but the converse holds only if the manifold is 3-dimensional. $K$-contact metric manifolds are not too well known, because there is not such a simple expression for the curvature tensor as in the case of Sasakian manifold. In this paper we continue to investigate them.

If $\phi$ and $\xi$ denote the (1, 1)-structure tensor and the contact vector field of a contact metric manifold $M$, respectively, then $T(M)$ can be decomposed into the direct sum $T(M) = \phi(T(M)) \oplus \mathcal{L}$, where $\mathcal{L}$ is the 1-dimensional distribution generated by $\xi$. Thus, we have a map :

$$C : T(M) \times T(M) \times T(M) \to \phi(T(M)) \oplus \mathcal{L}.$$

The case of being the projection of the image of $C$ in $\phi(T_p(M))$ zero was studied by the first author Zhen[4], proving that $M$ is locally isometric to the unit sphere. In this paper, we study the case of being the projection of the image of $C$ in $\mathcal{L}$ zero, introducing $\xi$-conformally flat contact metric manifolds. At last, we prove the main theorem : "A $K$-contact metric manifold is $\xi$-conformally flat if and only if it is an $\eta$-Einstein Sasakian manifold" and we give some applications. In particular, if the manifold $M$ is of dimension 3, a $K$-contact metric structure is $\xi$-conformally flat and Sasakian and, therefore, it is $\eta$-Einstein, which was obtained by Blair, Koufogiorgos and Sharma[2].

## 1. K-Contact Metric Manifolds

A contact manifold is a $(2n + 1)$-dimensional differentiable manifold $M^{2n+1}$ equipped with a global 1-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on $M^{2n+1}$. Given a contact form $\eta$, there exists an unique vector field $\xi$ on $M^{2n+1}$ that satisfies $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$, for any vector field $X$ on $M^{2n+1}$. Furthermore, given the contact form $\eta$, there exist a tensor field $\phi$ of type (1, 1) and a Riemannian metric $g$ such that $g(X, \phi Y) = d\eta(X, Y)$, $\eta(X) = g(X, \xi)$ and $\phi^2 = -I + \eta \otimes \xi$, for any vector fields $X$, $Y$ on $M^{2n+1}$. The structure $(\phi, \xi, \eta, g)$ on $M^{2n+1}$ is called a contact metric structure and $M^{2n+1}$ equipped with this structure is said to be a contact metric manifold. If $\xi$ is a Killing vector field, then $(M^{2n+1}, \phi, \xi, \eta, g)$ is called a $K$-contact metric manifold. We refer the reader to Blair[1] and Yano and Kon[9] for the backgrounds of contact structures.

Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a $K$-contact metric manifold. Then we have :

$$d\eta(X, Y) = g(X, \phi Y) \qquad \text{... (1.1)}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y), \qquad \text{... (1.2)}$$

$$g(X, \nabla_Y\xi) + g(Y, \nabla_X\xi) = (L_\xi g)(X, Y) = 0 \qquad \text{... (1.3)}$$

and
$$R(X, \xi)Y = \nabla_X \nabla_Y\xi - \nabla_{\nabla_X Y}\xi. \qquad \text{... (1.4)}$$

Then, (1.1) and (1.3) imply

$$\nabla_X\xi = -\phi X; \quad \nabla_\xi\xi = 0. \qquad \text{... (1.5)}$$

Now, from (1.3) and (1.4) we also have

$$(\nabla_X\phi)Y = -R(X, \xi)Y \qquad \text{... (1.6)}$$

and
$$(\nabla_X\phi)\phi Y + \phi(\nabla_X\phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X. \qquad \text{... (1.7)}$$

Thus,

$$\phi R(X, \xi)Y + R(X, \xi)\phi Y = g(Y, \phi X)\xi + \eta(Y)\phi X \qquad \text{... (1.8)}$$

and, in particular,

$$R(X, \xi)\xi = X - \eta(X)\xi \qquad \text{... (1.9)}$$

and
$$g(Q\xi, \xi) = 2n, \qquad \text{... (1.10)}$$

where $Q$ is the Ricci operator, defined by $QX = \Sigma_i R(X, e_i)e_i$, for any local orthonormal basis of vector fields in $M$, $\{e_i\}_{1 \leq i \leq 2n+1}$. Notice that if we take this local basis in such a way that $e_{2n+1} = \xi$, then $\{\phi e_i, \xi\}_{1 \leq i \leq 2n}$ is another local orthonormal basis.

To study $K$-contact metric manifolds, we need the following lemmas.

*Lemma* 1.1 — Let $(M^{2n+1}, \phi, \xi, \eta, g)$ a $K$-contact metric manifold. Then

$$g((\nabla_\xi Q)X - (\nabla_X Q)\xi - 3Q\phi X, \xi) = 0,$$

for any vector field $X \in T(M)$.

PROOF : Derivating (1.9) and using (1.5) we get :

$$(\nabla_Y R)(X, \xi)\xi = R(X, \phi Y)\xi + R(X, \xi)\phi Y + g(X, \phi Y)\xi + \eta(X)\phi Y.$$

Let $\{e_i\}_{1 \leq i \leq 2n+1}$ be any local orthonormal basis of vector fields in $M$. Then,

$$\sum_i g((\nabla_{e_i} R)(X, \xi)\xi, e_i) = \sum_i g(\phi R(e_i, \xi)X + R(X, \xi)\phi e_i, e_i)$$

$$= \sum_i g(\phi R(e_i, \xi)X, e_i) + Tr\phi R(X, \xi), \qquad \text{... (1.11)}$$

where $Tr\phi R(X, Y) = \Sigma_i g(\phi R(X, Y)e_i, e_i)$, for any vector fields $X, Y \in T(M)$. From (1.8) we have :

$$\sum_i g(\phi R(e_i, \xi)X, e_i) = -\sum_i g(R(e_i, \xi)\phi X, e_i) = -g(Q\phi X, \xi). \qquad \text{... (1.12)}$$

From the second Bianchi identity, we see that :

$$g((\nabla_\xi Q)X - (\nabla_X Q)\xi, \xi) = -\sum_i g((\nabla_{e_i} R)(X, \xi)\xi, e_i). \qquad \text{... (1.13)}$$

But (1.11), (1.12) and (1.13) give :

$$g((\nabla_\xi Q)X - (\nabla_X Q)\xi, \xi) = g(Q\phi X, \xi) - Tr\phi R(X, \xi). \qquad \text{... (1.14)}$$

On the other hand, if we choose the local orthonormal basis such that $e_{2n+1} = \xi$, thus, since $\{\phi e_i, \xi\}_{1 \le i \le 2n}$ is another local orthonormal basis and using (1.8), (1.9) and the first Bianchi identity, we have :

$$g(QX, \xi) = \sum_{i=1}^{2n} g(R(\phi e_i, X)\xi, \phi e_i) = \sum_{i=1}^{2n} g(R(\phi e_i, \xi)\phi X, e_i) + 2n\eta(X)$$

$$= \sum_{i=1}^{2n} g(R(e_i, \xi)\phi X, \phi e_i) + Tr\phi R(\phi X, \xi) + 2n\eta(X).$$

But, from (1.8) again, we see that

$$\sum_{i=1}^{2n} g(R(e_i, \xi)\phi X, \phi e_i) = 2n\eta(X) - g(QX, \xi).$$

So we obtain :

$$Tr\phi R(\phi X, \xi) = 2g(QX, \xi) - 4n\eta(X). \qquad \text{... (1.15)}$$

Replacing $X$ by $\phi X$ in (1.15), we have $Tr\phi R(X, \xi) = -2g(Q\phi X, \xi)$. This equation and (1.14) show that the lemma holds.

*Lemma* 1.2 — Let $M^{2n+1}$ be a $K$-contact metric manifold. If there exists on $M^{2n+1}$ a function $u$ such that

$$g(Q\phi X, \phi Y) = ug(\phi X, \phi Y), \qquad \text{... (1.16)}$$

for any vector fields $X, Y \in T(M)$, then

$$Q\xi = 2n\xi + \frac{n-1}{12n}\phi\nabla r, \qquad \text{... (1.17)}$$

where $\nabla r$ is the gradient field of scalar curvature $r$.

PROOF : Taking a local orthonormal basis for vector fields in $M$, $\{e_i, \xi\}_{1 \le i \le 2n}$, since $\{\phi e_i, \xi\}_{1 \le i \le 2n}$ is also a local orthonormal basis, the scalar curvature is given by :

$$r = g(Q\xi, \xi) + \sum_i g(Q\phi e_i, \phi e_i).$$

Now, from (1.10) and (1.16), we obtain

$$u = -1 + \frac{r}{2n} \qquad \text{... (1.18)}$$

and, replacing $X$ by $\phi X$ in (1.16) we have :

$$g(QX, \phi Y) = ug(X, \phi Y) + \eta(X)\, g(Q\xi, \phi Y). \qquad \text{... (1.19)}$$

Derivating (1.16) and then using (1.19), we get :

$$g((\nabla_Z Q)\phi X, \phi Y) = (Zu)\, g(\phi X, \phi Y)$$
$$- g((\nabla_Z \phi)X, \xi)\, g(Q\xi, \phi Y) - g((\nabla_Z \phi)Y, \xi)\, g(Q\xi, \phi X). \qquad \text{... (1.20)}$$

Next, replacing (1.6) and (1.9) into (1.20), we obtain

$$\sum_i g((\nabla_{\phi e_i} Q)\, \phi e_i, \phi Y) = \nabla_{\phi Y} u + g(Q\xi, \phi^2 Y). \qquad \text{... (1.21)}$$

Now, a straightforward computation gives

$$\frac{1}{2} \nabla_{\phi Y} r = \sum_i g((\nabla_{e_i} Q) e_i, \phi Y) = \sum_i g((\nabla_{\phi e_i} Q)\, \phi e_i, \phi Y) + g((\nabla_\xi Q)\xi, \phi Y),$$

and so, from (1.18) and (1.21), we have :

$$\frac{n-1}{2n} \nabla_{\phi Y} r = g((Q\phi^2 + (\nabla_\xi Q)\phi)Y, \xi). \qquad \text{... (1.22)}$$

On the other hand, (1.5) and (1.10) show that $g((\nabla_Y Q)\xi, \xi) = 2g(Q\phi Y, \xi)$, so, Lemma 1.1 implies :

$$g((\nabla_\xi Q)Y, \xi) = 5g(Q\phi Y, \xi). \qquad \text{... (1.23)}$$

Finally, replacing (1.23) into (1.22) we get (1.17).

## 2. ξ-Conformally Flat Contact Manifolds

Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a contact metric manifold. Then,

$$\eta(\phi T(M)) = d\eta(\xi, T(M)) = 0.$$

Conversely, if $\eta(X) = 0$, then $X = -\phi^2 X \in \phi T(M)$. The Weyl conformal curvature tensor with respect to the metric $g$ is the tensor field of type (1, 3) defined by :

$$C(X, Y)Z = R(X, Y)Z - \frac{1}{2n-1} \{g(QY, Z)X + g(Y, Z)QX$$
$$- g(QX, Z)Y - g(X, Z)QY\} + \frac{r}{2n(2n-1)} \{g(Y, Z)X - g(X, Z)Y\}, \qquad \text{... (2.1)}$$

for any $X, Y, Z \in T(M)$.

On the other hand, the Lie algebra $T(M)$ can be decomposed in a direct sum

$$T(M) = \phi T(M) \oplus \mathcal{L},$$

where $\mathcal{L}$ is the 1-dimensional distribution on $M$ generated by the structure vector field $\xi$.

*Definition* 2.1 — A contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is said to be $\xi$-conformally flat if the linear operator $C(X, Y)$ is an endomorphism of $\phi T(M)$, that is, if :

$$C(X, Y)\, \phi T(M) \subset \phi T(M).$$

Equivalently, $\xi$-conformally flat means that the projection of $C(X, Y)\, \phi T(M)$ onto $\mathcal{L}$ is zero.

We can see that any 3-dimensional contact metric manifold is $\xi$-conformally flat. One can prove that if $C(X, Y)Z \in \mathcal{L}$, for any $X$, $Y$, $Z$, then $C = 0$. In this case, a $K$-contact metric manifold is locally isometric to the unit sphere[4].

It is easy to prove the following proposition.

*Proposition* 2.2 — On a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, the following conditions are equivalent :

(i) $M$ is $\xi$-conformally flat;

(ii) $\eta(C(X, Y)Z) = 0$;

(iii) $\phi^2 C(X, Y)Z = - C(X, Y)Z$;

and (iv) $C(X, Y)\xi = 0$,

where $X, Y, Z \in T(M)$.

From (iv) in Proposition 2.2 we see that a contact metric manifold is $\xi$-conformally flat if and only if :

$$R(X, Y)\xi = \frac{1}{2n-1} \{g(QY, \xi)X + \eta(Y)QX - g(QX, \xi)Y$$

$$- \eta(X)QY\} + \frac{r}{2n(2n-1)} \{\eta(X)Y - \eta(Y)X\}. \quad \text{... (2.2)}$$

*Proposition* 2.3 — Let $M^{2n+1}$ be an $\eta$-Einstein Sasakian manifold. Then $M^{2n+1}$ is $\xi$-conformally flat.

PROOF : It is well known that the structure $(\phi, \xi, \eta, g)$ is a Sasakian structure if and only if the curvature tensor $R$ satisfies

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y \quad \text{... (2.3)}$$

and so, we have

$$Q\xi = 2n\xi. \quad \text{... (2.4)}$$

Since $(\phi, \xi, \eta, g)$ is $\eta$-Einstein, there exist functions $a$ and $b$ such that

$$g(QX, Y) = ag(X, Y) + b\eta(X)\,\eta(Y). \quad \text{... (2.5)}$$

But, from (2.4) and (2.5) we also have

$$a + b = 2n. \qquad \text{... (2.6)}$$

On the other hand, the scalar curvature satisfies :

$$r = Tr(Q) = (2n + 1)a + b. \qquad \text{... (2.7)}$$

Now, if we replace (2.3), (2.4), (2.5), (2.6) and (2.7) in formula (2.1), we get :

$$C(X, Y)\xi = R(X, Y)\xi - \frac{1}{2n-1}\left(2a + b - \frac{r}{2n}\right)(\eta(Y)X - \eta(X)Y)$$
$$= R(X, Y)\xi - (\eta(Y)X - \eta(X)Y) = 0,$$

and this completes the proof.

*Lemma* 2.4 — Let $C$ be the Weyl conformal curvature tensor on a Riemannian manifold $(M^m, g)$, $m > 3$ and let $V$ be a vector field on $M^m$. If $C(X, Y)V = 0$, for any vector fields $X, Y \in T(M)$, then

$$g((\nabla_X Q)V - (\nabla_V Q)X, V) = \frac{1}{2(m-1)}(g(V, V)Xr - g(V, X)Vr). \qquad \text{... (2.8)}$$

PROOF : Equation $C(X, Y)V = 0$ is equivalent to

$$R(X, Y)V = \frac{1}{m-2}\{g(QY, V)X + g(V, Y)QX - g(QX, V)Y$$
$$- g(V, X)QY\} - \frac{r}{(m-1)(m-2)}\{g(Y, V)X - g(X, V)Y\}.$$
$$\text{... (2.9)}$$

Using the properties of the curvature tensor $R$ and symmetry of $Q$ with respect to $g$, we also have

$$R(X, V)Y = \frac{1}{m-2}\{g(QV, Y)X + g(V, Y)QX - g(QX, Y)V$$
$$- g(X, Y)QV\} - \frac{r}{(m-1)(m-2)}\{g(V, Y)X - g(X, Y)V\},$$
$$\text{... (2.10)}$$

for any $X, Y \in T(M)$. Replacing $Y$ by $V$ in (2.9), derivating this equation and taking account of (2.9) and (2.10), we get :

$$(\nabla_W R)(X, V)V = \frac{1}{m-2}\{g((\nabla_W Q)V, V)X + g(V, V)(\nabla_W Q)X$$
$$- g((\nabla_W Q)X, V)V - g(V, X)(\nabla_W Q)V\}$$
$$- \frac{Wr}{(m-1)(m-2)}\{g(V, V)X - g(X, V)V\}.$$

Therefore

$$\sum_i g((\nabla_{e_i}R)(e_i, V)V, X) = \frac{1}{m-2}\{g((\nabla_X Q)V - (\nabla_V Q)X, V)\}$$

$$+ \frac{m-3}{2(m-2)(m-1)}\{g(V, V)Xr - g(X, V)Vr\}. \qquad \text{... (2.11)}$$

On the other hand, from the second Bianchi identity, we know :

$$g((\nabla_X Q)V - (\nabla_V Q)X, V) = \sum_i g((\nabla_{e_i}R)(X, V)V, e_i). \qquad \text{... (2.12)}$$

Thus, (2.11) and (2.12) yield equation (2.8).

***Theorem*** 1 — *A K-contact metric manifold* $M^{2n+1}$ *is* $\xi$*-conformally flat if and only if it is an* $\eta$*-Einstein Sasakian manifold.*

PROOF : We only have to prove that a $\xi$-conformally flat $K$-contact metric manifold is an $\eta$-Einstein Sasakian manifold. The converse follows from Proposition 2.3.

On a $\xi$-conformally flat $K$-contact metric manifold, (1.9) and (2.2) yield

$$QX = \left\{2n - 1 - g(Q\xi, \xi) + \frac{r}{2n}\right\}X$$

$$+ \left\{g(Q\xi, X) - \left(2n - 1 + \frac{r}{2n}\eta(X)\right)\right\}\xi + \eta(X)\,Q\xi, \qquad \text{... (2.13)}$$

for any vector field $X$. Since $g(Q\xi, \xi) = 2n$, we have

$$g(Q\phi X, \phi Y) = \left(-1 + \frac{r}{2n}\right)g(\phi X, \phi Y)$$

and so, Lemma 1.2 shows that :

$$Q\xi = 2n\xi + \frac{n-1}{12n}\phi\,\nabla r. \qquad \text{... (2.14)}$$

Replacing (2.14) into (2.13) we get

$$QX = aX + \left\{b\eta(X) + \frac{n-1}{12n}g(\phi\nabla r, X)\right\}\xi + \frac{n-1}{12n}\eta(X)\,\phi\nabla r, \qquad \text{... (2.15)}$$

where $a = -1 + \frac{r}{2n}$ and $b = 2n + 1 - \frac{r}{2n}$.

Now, if $n = 1$ then $QX = aX + b\eta(X)\xi$.

If $n > 1$, then $\phi\nabla r = 0$. In fact, since $n > 1$, we can use Lemma 2.4. From Lemmas 1.1 and 1.2, for a $\xi$-conformally flat $K$-contact metric structure, we have :

$$3g(Q\phi X, \xi) = g((\nabla_\xi Q)X - (\nabla_X Q)\xi, \xi) = \frac{1}{4n}(\eta(X)\,\xi r - Xr) = \frac{1}{4n}(\phi^2 X)r. \quad \ldots (2.16)$$

Since $\phi^3 = -\phi$, if we replace $X$ by $\phi X$ in (2.16), we obtain

$$g(QX, \xi) = \eta(X)\, g(Q\xi, \xi) + \frac{1}{12n}(\phi X)r$$

and, by using (1.2) and (1.10) :

$$Q\xi = 2n\xi - \frac{1}{12n}\phi \nabla r. \quad \ldots (2.17)$$

Now, comparing (2.17) with (2.14), we have $\phi\nabla r = 0$ and then, (2.15) gives :

$$QX = aX + b\eta(X)\,\xi. \quad \ldots (2.18)$$

So, equation (2.18) holds for $n \geq 1$ and hence (2.2) turns to

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

which means that the manifold is also a Sasakian manifold.

*Corollary* 2.6 — Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a $\xi$-conformally flat $K$-contact metric manifold. If there exist functions $\lambda$ and $\mu$ on $M^{2n+1}$ such that

$$(\nabla_X Q)Y - (\nabla_Y Q)X = \lambda X + \mu Y, \quad \ldots (2.19)$$

then,

$$QX = 2nX.$$

PROOF : From Theorem 1 we have $QX = aX + b\xi$, where $a = -1 + \frac{r}{2n}$ and $b = 2n + 1 - \frac{r}{2n}$. Thus, we have :

$$(\nabla_X Q)Y - (\nabla_Y Q)X = (Xa)Y - (Ya)X + (Xb)\,\eta(Y)\xi$$
$$- (Yb)\,\eta(X)\xi - b\{2g(\phi X, Y)\xi + \eta(Y)\,\phi X - \eta(X)\,\phi(Y)\}. \quad \ldots (2.20)$$

Replacing $X$ and $Y$ by $\phi X$ and $\phi Y$ in (2.20) we get :

$$(\nabla_{\phi X} Q)\,\phi Y - (\nabla_{\phi Y} Q)\,\phi X = (\phi Xa)\,\phi Y - (\phi Ya)\,\phi X - 2bg(\phi^2 X, \phi Y)\xi. \quad \ldots (2.21)$$

From (2.19) and (2.21) we obtain $(\lambda + (\phi Ya))\,\phi X + (\mu - (\phi Xa))\phi Y = -2bg(\phi^2 X, \phi Y)\xi$, which implies $-2bg(\phi^2 X, \phi Y) = 0$. But replacing here $X$ by $\phi Y$, we obtain $bg(\phi Y, \phi Y) = 0$ and hence $b = 0$.

From Corollary 2.6 we easily obtain the following applications :

*Corollary* 2.7 — Any conformally flat $K$-contact metric manifold is locally isometric to the unit sphere.

PROOF : It is well known that on a conformally flat Riemannian manifold the following equation holds, for $n > 1$ (Weyl[7, 8]) :

$$(\nabla_X Q)Y - (\nabla_Y Q)X = \frac{1}{4n}\{(Xr)Y - (Yr)X\}.$$

Then, Corollary 2.6 shows that $QX = 2nX$ and, therefore, equation $C(X, Y)X = 0$ yields :

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y.$$

This completes the proof.

*Corollary* 2.8 — Let $M^{2n+1}$ be a $\xi$-conformally flat $K$-contact metric manifold. If the curvature tensor is harmonic, then $M^{2n+1}$ is $\eta$-Einstein.

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