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INDUCED STRUCTURES ON SLANT SUBMANIFOLDS OF METRIC f-MANIFOLDS

L. M. FERNÁNDEZ, M. B. HANS-UBER*[1]

**Department of Geometry and Topology, Faculty of Mathematics*
University of Sevilla, Apartado de Correos 1160, 41080-*Sevilla, Spain*
e-mail : lmfer@us.es

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We study whether it is possible to obtain an induced structure on a slant submanifold of a metric f-manifold. Moreover, we give conditions for any isometric immersion between two metric f-manifolds to be slant and we prove a characterization theorem when the submanifold has the smallest possible dimension to be proper slant.

Key words : Slant immersion, metric f-structure, metric f-contact manifolds, S-manifolds

1. INTRODUCTION

Slant immersions in complex geometry were defined by Chen as a natural generalization of both holomorphic and totally real immersions [5, 6]. Recently, Lotta has introduced the notion of slant immersion in contact geometry [9] which also generalizes invariant and anti-invariant immersions. In this context, slant submanifolds of Sasakian manifolds have been studied in [2].

Moreover, for more general metric f-manifolds, that is, for Riemannian manifolds endowed with an f-structure in the sense of Yano [11], compatible with the Riemannian metric, it is also

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possible to define slant immersions [4] and some interesting examples of such slant submanifolds are given in [4, 7]. A general view about slant immersions can be found in [3].

On the other hand, it is well known that any holomorphic submanifold (resp., any invariant submanifold tangent to the structure vector field) of a Kaehlerian (resp., a Sasakian) manifold inherits the structure from the ambient space. The same result can be proved for invariant submanifolds tangent to the structure vector fields of S-manifolds, which are a particular case of metric f-manifolds. They were introduced by Blair [1] as the analogue of Kaehler manifolds in the almost complex case and of Sasakian manifolds in the almost contact case. Since those invariant submanifolds are slant ones, it is interesting to study the possibility of obtaining an induced structure on slant submanifolds of metric f-manifolds and this is the purpose of present paper.

To that end, we begin by reviewing, in Section 2, formulas and definitions for later use. In Section 3, we prove that a slant submanifold of an S-manifold inherits the ambient S-structure if and only if it is an invariant submanifold. Moreover, in the case of dimension $2+s$, which is the smallest possible dimension for the existence of proper slant submanifolds (neither invariant nor anti-invariant slant submanifolds), we show that the submanifold also needs to be invariant for having an induced metric f-contact structure. The used tools allow us to prove that any slant isometric immersions from a $(2+s)$-dimensional metric f-contact manifold in another metric f-contact manifold with compatible structure vector fields is invariant.

Finally, in Section 4, we study the conditions for an isometric immersion between two general metric f-manifold with compatible structure vector fields to be slant and we prove a characterization theorem when the submanifold has the possible smallest dimension $2+s$.

2. Slant Submanifolds of Metric f-Manifolds

A Riemannian manifold $\widetilde{M}$ is said to be a metric f-manifold if there exist on $\widetilde{M}$ an f-structure f, that is, a tensor field f of type (1,1) satisfying $f^3+f=0$ (see [11]) and s global vector fields $\xi_1,\dots,\xi_s$ (called structure vector fields) such that, if $\eta_1,\dots,\eta_s$ are the dual 1-forms of $\xi_1,\dots,\xi_s$, then

$$f\xi_\alpha=0;\ \eta_\alpha\circ f=0;\ f^2=-I+\sum_{\alpha=1}^{s}\eta_\alpha\otimes\xi_\alpha;$$

$$g(X,Y)=g(fX,fY)+\sum_{\alpha=1}^{s}\eta_\alpha(X)\eta_\alpha(Y), \tag{2.1}$$

for any $X,Y\in\mathcal{X}(\widetilde{\mathcal{M}})$ and $\alpha=1,\dots,s$, where g is denoting the Riemannian metric. Observe that, in the above conditions

$$g(X,fY)=-g(fX,Y) \tag{2.2}$$

and so, we can consider the 2-form F defined by $F(X,Y) = g(X, fY)$, for any $X, Y \in \mathcal{X}(\widetilde{\mathcal{M}})$, called the fundamental 2-form. Then, $\widetilde{M}$ is said to be a metric f-contact manifold (or to have a metric f-contact structure) if $F = \mathrm{d}\eta_\alpha$, for any $\alpha = 1, \dots, s$. When $s = 1$, metric f-contact manifolds correspond to metric contact manifolds.

On the other hand, the f-structure f is normal if

$$[f, f] + 2\sum_{\alpha=1}^{s} \xi_\alpha \otimes \mathrm{d}\eta_\alpha = 0,$$

where $[f, f]$ is the Nijenhuis tensor of f. Thus, $\widetilde{M}$ is said to be an S-manifold if the f-structure is normal and

$$\eta_1 \wedge \cdots \wedge \eta_s \wedge (\mathrm{d}\eta_\alpha)^n \neq 0,\ F = \mathrm{d}\eta_a,$$

for any $\alpha = 1, \dots, s$. In this case, the structure vector fields are Killing vector fields. Obviously, S-manifolds are metric f-contact manifolds. When $s = 1$, S-manifolds are Sasakian manifolds. Interesting examples of S-manifolds with $s \geq 2$ can be found in [1, 8].

It is known that the Riemannian connection $\widetilde{\nabla}$ of an S-manifold satisfies [1].

$$\widetilde{\nabla}_X \xi_\alpha = -fX, \tag{2.3}$$

and

$$(\widetilde{\nabla}_X f)Y = \sum_{\alpha=1}^{s} (g(fX, fY)\xi_\alpha + \eta_\alpha(Y) f^2 X), \tag{2.4}$$

for any $X, Y \in \mathcal{X}(\widetilde{\mathcal{M}})$ and any $\alpha = 1, \dots, s$.

Now, let M be a Riemannian manifold isometrically immersed in a metric f-manifold $\widetilde{M}$ and let g denote the induced metric tensor on M too. Let $T^\perp M$ be the set of vector fields on $\widetilde{M}$ which are normal to M, that is, $\mathcal{X}(\widetilde{\mathcal{M}}) = \mathcal{X}(\mathcal{M}) \oplus \mathcal{T}^\perp\mathcal{M}$.

If ∇ denotes the Riemannian connection of M, the well known Gauss formula is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{2.5}$$

for any $X, Y \in \mathcal{X}(\mathcal{M})$, where σ is representing the second fundamental form of the immersion and so, $\sigma(X, Y) \in T^\perp M$. The curvature tensor fields, denoted by $\widetilde{R}$ and R, associated with $\widetilde{\nabla}$ and ∇, respectively, are related by the following Gauss equation:

$$\begin{aligned}\widetilde{R}(X, Y; Z, W) &= R(X, Y; Z, W) + g(\sigma(X, Z), \sigma(Y, W)) \\ &\quad - g(\sigma(X, W), \sigma(Y, Z)),\ X, Y, Z, W \in \mathcal{X}(\mathcal{M}).\end{aligned} \tag{2.6}$$

Next, for any $X \in \mathcal{X}(\mathcal{M})$ we can write

$$fX = TX + NX, \tag{2.7}$$

where TX and NX are the tangential and normal components of fX, respectively. The submanifold M is said to be invariant if N is identically zero, that is, if $fX \in \mathcal{X}(\mathcal{M})$, for any $X \in \mathcal{X}(\mathcal{M})$. On the other hand, M is said to be an anti-invariant submanifold if T is identically zero, that is, if $fX \in T^{\perp}M$, for any $X \in \mathcal{X}(\mathcal{M})$. The covariant derivative of T is given by

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y,$$

for any $X, Y \in \mathcal{X}(\mathcal{M})$. We have the following specific result for invariant submanifolds of S-manifolds:

Lemma 2.1 — Let $\widetilde{M}$ be an S-manifold and M be an invariant submanifold tangent to the structure vector fields. Then,

$$(\nabla_X T)Y = \sum_{\alpha=1}^{s}(g(TX, TY)\xi_\alpha + \eta_\alpha(Y)T^2X), \tag{2.8}$$

for any $X, Y \in \mathcal{X}(\mathcal{M})$ and any $\alpha = 1, \dots, s$,

PROOF : Since M is an invariant submanifold, we have $TX = fX$, for any $X \in \mathcal{X}(\mathcal{M})$. Then, by using (2.5),

$$(\nabla_X T)Y = \nabla_X fY - f\nabla_X Y = (\widetilde{\nabla}_X f)Y - \sigma(X, fY) + f\sigma(X, Y),$$

for any $X, Y \in \mathcal{X}(\mathcal{M})$. Now, from (2.2), we get

$$g(f\sigma(X, Y), Z) = -g(\sigma(X, Y), fZ) = 0,$$

for any $Z \in \mathcal{X}(\mathcal{M})$, because M in an invariant submanifold and so, $f\sigma(X, Y) \in T^{\perp}M$. Taking into account that $(\nabla_X T)Y$ is also a tangent vector field, from (2.4) we obtain (2.8). □

From now on, we suppose that all the structure vector fields are tangent to the submanifold and we denote by $\mathcal{M}$ the distribution of $\mathcal{X}(\mathcal{M})$ spanned by the structure vector fields and by $\mathcal{L}$ the orthogonal complementary distribution to $\mathcal{M}$ in $\mathcal{X}(\mathcal{M})$. Then, we have the orthogonal direct decomposition $\mathcal{X}(\mathcal{M}) = \mathcal{L} \oplus \mathcal{M}$. Moreover, if $\widetilde{M}$ is an S-manifold, by using (2.3), (2.5) and (2.7) it is easy to show that

$$\sigma(X, \xi_\alpha) = -NX, \tag{2.9}$$

for any $X \in \mathcal{X}(\mathcal{M})$ and $\alpha = 1, \dots, s$. Consequently $\sigma(\xi_\alpha, \xi_\beta) = 0$, for any $\alpha, \beta = 1, \dots, s$.

The submanifold M is said to be a slant submanifold if, given any point $x \in M$ and any nonzero vector $X \in \mathcal{L}_x$, the angle between fX and T_xM is a constant $\theta \in [0, \pi/2]$, called the slant angle of M in $\widetilde{M}$, which is independent on the choice of the point x and the vector X. Note that this definition generalizes that one given by Chen [6] for complex geometry and that one given by Lotta [9] for contact geometry. Moreover, invariant and anti-invariant submanifolds tangent to the structure vector fields are slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A slant immersion which is neither invariant nor anti-invariant is called a proper slant immersion. Observe that, for invariant submanifolds, $T = f$ on TM and, so

$$T^2 = -I + \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha, \tag{2.10}$$

while, for anti-invariant submanifolds, $T^2 = 0$. In fact, we have the following general result whose proof can be obtained by following the same steps as in the case $s = 1$ (see [2]):

***Theorem* 2.1** — *Let M be a submanifold of a metric f-manifold $\widetilde{M}$, tangent to the structure vector fields. Then, M is a slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that*:

$$T^2 = -\lambda I + \lambda \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha = \lambda f^2.$$

Furthermore, in such case, if θ is the slant angle of M, it satisfies that $\lambda = \cos^2\theta$.

Using (2.1), (2.7) and Theorem 2.1, a direct computation gives:

Corollary 2.1 — Let M be a slant submanifold of a metric f-manifold $\widetilde{M}$, with slant angle θ. Then, for any $X, Y \in \mathcal{X}(\mathcal{M})$, we have:

$$g(TX, TY) = \cos^2\theta(g(X, Y) - \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y)),$$

$$g(NX, NY) = \sin^2\theta(g(X, Y) - \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y)).$$

3. Structure on a Slant Submanifold

Let M be an invariant submanifold of a metric f-manifold $\widetilde{M}$, tangent to the structure vector fields. Then, from (2.10), we see that such submanifold is a metric f-manifold too, with the f-structure T (induced on M by f). In this section, we want to study if it is possible to obtain an induced structure on non-invariant slant (isometrically immersed) submanifolds of a metric f-manifold. This problem is suggested by the similar situation on slant submanifolds in complex geometry [5, 6].

First, we are going to prove that there always exists an induced f-structure on any non-anti-invariant slant submanifold.

Proposition 3.1 — Let M be a non-anti-invariant slant submanifold of a metric f-manifold $\widetilde{M}$. Then, $\overline{f} = (\sec\theta)T$ is an f-structure on M. With this f-structure, M becomes a metric f-manifold too, with structure vector fields $\xi_1, \ldots, \xi_s$

PROOF : By virtue of Theorem 2.1, it is easy to show that

$$\overline{f}^2 X = \sec^2\theta T^2 X = -X + \sum_{\alpha} \eta_\alpha(X)\xi_\alpha,$$

and by using Corollary 2.1,

$$g(\overline{f}X, \overline{f}Y) = \sec^2\theta g(TX, TY) = g(X,Y) - \sum_{\alpha} \eta_\alpha(X)\eta_\alpha(Y),$$

for any $X, Y \in \mathcal{X}(\mathcal{M})$. □

In particular, if $\theta = 0$, the induced structure on the invariant submanifold M is the usual one, given by $\overline{f} = T$.

On the other hand, if $\widetilde{M}$ is a metric f-contact manifold (that is, if the fundamental 2-form of $\widetilde{M}$ verifies that F=dη_α, for any α) and M is a proper slant submanifold, then the structure induced by $\overline{f}$ does not verify the same property, because, denoting by $\overline{F}$ the fundamental 2-form of M with such induced structure,

$$\overline{F}(X,Y) = g(X, \overline{f}Y) = \sec\theta g(X, TY) = \sec\theta F(X,Y) = \sec\theta d\eta_\alpha(X,Y),$$

for any $X, Y \in \mathcal{X}(\mathcal{M})$ and $\alpha = 1, \ldots, s$.

Consequently, with this induced structure, we have that a slant submanifold of a metric f-contact manifold is also a metric f-contact manifold if and only if it is an invariant submanifold. Moreover, we can prove the following proposition:

Proposition 3.2 — Let $\widetilde{M}$ be an S-manifold and M be an invariant submanifold tangent to the structure vector fields. Then, the f-structure on M defined by $\overline{f} = T$ is an S-structure on M.

PROOF : As we have noticed above, it is easy to show that $\overline{F}(X,Y) = F(X,Y)$= $d\,\eta_\alpha(X,Y)$, for any $X, Y \in \mathcal{X}(M)$ and $\alpha = 1, \ldots, s$, where $\overline{F}$ is denoting the fundamental 2-form on M associated with the f-structure $\overline{f}$. Furthermore, this f-structure is also normal because by using (2.8) we obtain that

$$[T,T](X,Y) = (\nabla_{TX}T)Y - (\nabla_{TY}T)Y + T(\nabla_Y T)X - T(\nabla_X T)Y$$

$$= -2\sum_{\alpha=1}^{s} g(X, TY)\xi_\alpha = -2\sum_{\alpha=1}^{s} \overline{F}(X,Y)\xi_\alpha = -2\sum_{\alpha=1}^{s} d\eta_\alpha(X,Y)\xi_\alpha,$$

for any $X, Y \in \mathcal{X}(\mathcal{M})$. So, M is an S-manifold too. □

We can now wonder if it is possible to obtain an S-structure on the submanifold from another way, by choosing the appropriate conditions. However, we have the following necessary intrinsic condition for slant immersions in S-manifolds:

Proposition 3.3 — Let M be a θ-slant submanifold of an S-manifold $\widetilde{M}$. Then, for any $x \in M$, the sectional curvature of any 2-plane of T_xM containing one and only one of the structure vector fields equals to $\cos^2\theta$.

PROOF : Let $X \in \mathcal{X}(\mathcal{M})$ be a unit vector field such that $\{X_x, (\xi_\alpha)_x\}$ is an orthonormal basis of a 2-plane of T_xM. Then, since $\widetilde{R}(X, \xi_\alpha; \xi_\alpha, X) = 1$ (see [1]), from Gauss equation (2.6) and from (2.9), we get that

$$R(X, \xi_\alpha; \xi_\alpha, X) = 1 - g(\sigma(X, \xi_\alpha), \sigma(X, \xi_\alpha)) = 1 - g(NX, NX)$$

and, from Corollary 2.1, we complete the proof. □

Thus, we can prove the following proposition.

Proposition 3.4 — A slant submanifold M of an S-manifold $\widetilde{M}$ is an S-manifold if and only if it is an invariant submanifold.

PROOF : The direct implication is an easy consequence of the above proposition and the mentioned result of [1] which establishes that the sectional curvature of any 2-plane of an S-manifold containing one and only one of the structure vector fields equals to 1. For the converse, we only have to consider Proposition 3.2. □

Hence, it is not possible to have an induced S-structure on a non-invariant slant submanifold of an S-manifold. Nevertheless, we can still wonder if it would be possible to induce a metric f-contact structure. The answer will be negative again with respect to the case of the smallest dimension for non-anti-invariant slant immersions and we shall use a different method. Let

$$\varphi : (M, \overline{f}, \overline{\xi}_1, \ldots, \overline{\xi}_s, \overline{\eta}_1, \ldots, \overline{\eta}_s, \overline{g}) \hookrightarrow (\widetilde{M}, f, \xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_s, g)$$

be an immersion between two metric f-manifolds. We suppose that $\overline{g} = \varphi^* g$ (or, in other words, that φ is an isometric immersion) and, moreover, that

$$\varphi_{*x}\mathcal{M}_x = \widetilde{\mathcal{M}}_{\varphi(x)}, \tag{3.1}$$

for any $x \in M$, where, as above, $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are denoting the distributions spanned by the structure vector fields in $\mathcal{X}(M)$ and $\mathcal{X}(\widetilde{M})$, respectively. In particular, we have that $\xi_\alpha \in \mathcal{X}(M)$, for any $\alpha = 1, \ldots, s$. Now, let $\varphi^* F$ be the 2-form on M given by

$$\varphi^* F(X, Y) = F(\varphi_* X, \varphi_* Y) = g(\varphi_* X, f\varphi_* Y),$$

for any $X, Y \in \mathcal{X}(M)$. From now on, we are going to identify X and $\varphi_* X$, for any $X \in \mathcal{X}(M)$.

Next, we consider that φ is a slant immersion with slant angle θ. Since it is easy to check that any $(1+s)$-dimensional submanifold of a metric f-manifold tangent to the structure vector fields is an anti-invariant submanifold, we study the case of $dim(M) = 2+s$. Then, we can obtain a relationship between $\overline{F}$ and φ^*F:

Proposition 3.5 — In the above conditions, $\varphi^*F = \pm(\cos\theta)\overline{F}$.

PROOF : We can suppose that φ is a non-anti-invariant immersion because, if it is anti-invariant, the result is obvious since $\varphi^*F = 0$.

So, let e_1 be a unit local vector field tangent to M and perpendicular to the structure vector fields and define $e_2 = (\sec\theta)Te_1$. By Corollary 2.1, we get that $\{e_1, e_2, \xi_1, \ldots, \xi_s\}$ is a local orthonormal basis of $\mathcal{X}(\mathcal{M})$. Then, $\overline{f}e_1 = g(\overline{f}e_1, e_2)e_2$ and, consequently, $g(\overline{f}e_1, \overline{f}e_1) = g^2(\overline{f}e_1, e_2)$. Now, since $g(\overline{f}e_1, \overline{f}e_1) = 1$, we have:

$$\overline{f}e_1 = \pm e_2;\ \overline{f}e_2 = \mp e_1. \tag{3.2}$$

Let X, Y be any two tangent vector fields to M. We can write them with respect to the above local orthonormal basis as follows:

$$X = X^1e_1 + X^2e_2 + \sum_\alpha \eta_\alpha(X)\xi_\alpha;\ Y = Y^1e_1 + Y^2e_2 + \sum_\alpha \eta_\alpha(Y)\xi_\alpha.$$

From (3.2), we obtain that:

$$\overline{F}(X,Y) = g(X, \overline{f}Y) = \mp X^1Y^2 \pm X^2Y^1. \tag{3.3}$$

On the other hand,

$$\varphi^*F(X,Y) = F(\varphi_*X, \varphi_*Y) = g(X, fY) = g(X, TY) = -\cos\theta X^1Y^2 + \cos\theta X^2Y^1, \tag{3.4}$$

since $TY = -\cos\theta Y^2e_1 + \cos\theta Y^1e_2$. Thus, from (3.3) and (3.4), we complete the proof. □

Now, by using Proposition 3.5, we can prove the following theorem:

***Theorem* 3.1** — *Let $\varphi : M \hookrightarrow \widetilde{M}$ be a slant immersion from a metric f-contact manifold*

$$(M, \overline{f}, \overline{\xi}_1, \ldots, \overline{\xi}_s, \overline{\eta}_1, \ldots, \overline{\eta}_s, \varphi^*g),$$

with dimension $2+s$ in another metric f-contact manifold

$$(\widetilde{M}, f, \xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_s, g)$$

such that $\varphi_*\overline{\xi}_\alpha = \xi_\alpha$, for any $\alpha = 1, \ldots, s$. Then, φ is an invariant immersion.

PROOF : Denote by θ the slant angle of the immersion φ. Observe that the condition $\varphi_*\overline{\xi}_\alpha = \xi_\alpha$, for any $\alpha = 1, \ldots, s$, implies (3.1). Moreover, since φ is an isometric immersion, such condition also implies that $\varphi^*\eta_\alpha = \overline{\eta}_\alpha$, for any α. So,

$$d(\varphi^*\eta_\alpha) = d\overline{\eta}_\alpha = \overline{F}, \tag{3.5}$$

because M is a metric f-contact manifold. Then, from Proposition 3.5, as $\widetilde{M}$ is a metric f-contact manifold too, we obtain:

$$d(\varphi^*\eta_\alpha) = \varphi^* d\eta_\alpha = \varphi^* F = \pm(\cos\theta)\overline{F}. \tag{3.6}$$

But, from (3.5) and (3.6), we deduce that $\cos\theta = 1$, that is, the immersion φ is invariant.

The following corollary gives an answer to our question for $(2+s)$-dimensional slant submanifolds.

Corollary 3.1 — Let M be a $(2+s)$-dimensional slant submanifold of an S-manifold $\widetilde{M}$. Then, the S-structure of $\widetilde{M}$ induces a metric f-contact structure on M if and only if M is an invariant submanifold.

PROOF : The direct implication is obvious from Theorem 3.1 because any S-manifold is a metric f-contact manifold. The converse is already known. □

Nevertheless, we can consider slant immersions between metric f-manifolds. In fact, it is enough to choose a local orthonormal basis

$$\{e_1, e_2, \xi_1, \ldots, \xi_s\}$$

and define $\overline{f}$ such that $\overline{f}e_1 = e_2$ and $\overline{f}e_2 = -e_1$ (in particular, the f-structure given in Proposition 3.1 verifies this property in the case of non-anti-invariant slant submanifolds) to obtain a metric f-structure on a $(2+s)$-dimensional slant submanifold.

4. Conditions for an Immersion Between Metric f-Manifolds to be Slant

In this section we consider any isometric immersion

$$\varphi : (M, \overline{g}) \hookrightarrow (\widetilde{M}, f, \xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_s, g)$$

from a Riemannian manifold into a metric f-manifold such that the structure vector fields are tangent to M. Then, we have the following proposition.

Proposition 4.1 — Under the above conditions, if $dim(M) = p$ and

$$\{e_1, \ldots, e_{p-s}, \xi_1, \ldots, \xi_s\}$$

is a local orthonormal basis of $\mathcal{X}(\mathcal{M})$, then, the immersion φ is slant if and only if there exists a constant $\lambda \in [0,1]$ such that

$$\sum_{i=1}^{p-s} g(fe_j, e_i) g(fe_k, e_i) = \lambda \delta_{jk}, \tag{4.1}$$

for any $j, k = 1, \ldots, p-s$. Furthermore, in this case, $\lambda = \cos^2\theta$, where θ is denoting the slant angle of the immersion.

PROOF : First, we suppose that φ is a slant immersion with slant angle θ. Thus, given any unit tangent vector field $X \in \mathcal{L}$, we have

$$\sum_{i=1}^{p-s} g^2(fX, e_i) = \cos^2\theta, \tag{4.2}$$

from which, writing $X = e_j$, we deduce that:

$$\sum_{i=1}^{p-s} g^2(fe_j, e_i) = \cos^2\theta. \tag{4.3}$$

Consequently, we get (4.1) in the case $j = k$, with $\lambda = \cos^2\theta$. Next, let $j \neq k$ and consider the unit local vector field

$$X = \frac{1}{\sqrt{2}}(e_j + e_k),$$

which is perpendicular to the structure vector fields. By applying (4.2), we obtain:

$$\cos^2\theta = \frac{1}{2}\sum_{i=1}^{p-s} g^2(fe_j, e_i) + \frac{1}{2}\sum_{i=1}^{p-s} g^2(fe_k, e_i) + \sum_{i=1}^{p-s} g(fe_j, e_i) g(fe_k, e_i).$$

Next, from (4.3), we have (4.1) in the case $j \neq k$. Conversely, let $X \in \mathcal{X}(\mathcal{M})$ be any tangent vector field to M. So, we can write:

$$X = \sum_{i=1}^{p-s} g(X, e_i) e_i + \sum_{\alpha=1}^{s} \eta_\alpha(X) \xi_\alpha. \tag{4.4}$$

Moreover:

$$T^2 X = \sum_{i=1}^{p-s} g(T^2 X, e_i) e_i. \tag{4.5}$$

But, since

$$g(T^2X, e_i) = g(X, T^2 e_i) = \sum_{j=1}^{p-s} g(X, e_j) g(T^2 e_i, e_j)$$

and, from (4.1),

$$g(T^2e_i, e_j) = -g(Te_i, Te_j) = -\sum_{k=1}^{p-s} g(Te_i, e_k)g(Te_j, e_k) = -\lambda\delta_{ij},$$

then

$g(T^2X, e_i) = -\lambda g(X, e_i)$. Thus, by using (4.4) and (4.5), we get

$$T^2X = -\lambda \left(X - \sum_{\alpha=1}^{s} \eta_a(X)\xi_\alpha \right)$$

and, from Theorem 2.1, we obtain that φ is a slant immersion with slant angle $\cos^{-1}\sqrt{\lambda}$. □

We remark that the Kaehlerian version of above proposition can be found in [10].

Now, we consider a metric f-structure on M,

$$(M, \overline{f}, \overline{\xi}_1, \ldots, \overline{\xi}_s, \overline{\eta}_1, \ldots, \overline{\eta}_s, \overline{g}),$$

such that $\varphi_{*x}\mathcal{M}_x = \widetilde{\mathcal{M}}_{\varphi(x)}$, for any $x \in M$. As above, let $\overline{F}$ and F be the fundamental 2-forms of M and $\widetilde{M}$, respectively and, moreover, we consider on M the 2-form φ^*F. Then, we can prove the following theorem.

***Theorem* 4.1** — *Under the above conditions, if there exists a constant $k \in [-1, 1]$ such that $\varphi^*F = k\overline{F}$, then φ is a slant immersion with slant angle $\cos^{-1}|k|$.*

PROOF : Since M is a metric f-manifold, we can choose an orthonormal basis in M given by $\mathcal{B} = \{e_1, \ldots, e_m, \overline{f}e_1, \ldots, \overline{f}e_m, \overline{\xi}_1, \ldots, \overline{\xi}_s\}$, where we are putting $dim(M) = 2m + s$. Let $X \in \mathcal{L} \cap \mathcal{B}$. Since $\varphi^*F = k\overline{F}$, we have that

$$g(fX, e_i) = kg(\overline{f}X, e_i) = -kg(X, \overline{f}e_i), \tag{4.6}$$

meanwhile

$$g(fX, \overline{f}e_i) = kg(\overline{f}X, \overline{f}e_i) = kg(X, e_i), \tag{4.7}$$

because $\overline{\eta}_\alpha(e_i) = 0$, for any $\alpha = 1, \ldots, s$ and $i = 1, \ldots, m$. Hence, by using (4.6) and (4.7), we obtain, for any $X, Y \in \mathcal{L} \cap \mathcal{B}$:

$$\sum_{i=1}^{m} g(fX, e_i)g(fY, e_i) + \sum_{i=1}^{m} g(fX, \overline{f}e_i)g(fY, \overline{f}e_i)$$
$$= k^2 \left(\sum_{i=1}^{m} g(X, \overline{f}e_i)g(Y, \overline{f}e_i) + \sum_{i=1}^{m} g(X, e_i)g(Y, e_i) \right). \tag{4.8}$$

Now, by checking (4.8) according to the possible choices of X and Y in $\mathcal{L} \cap \mathcal{B}$, it is easy to show, from Proposition 4.1, which holds with $\lambda = k^2$, that φ is a slant immersion with slant angle $\cos^{-1}|k|$. □

Corollary 4.1 — If $dim(M) = 2 + s$, then φ is a slant immersion if and only if there exists a constant $k \in [-1, 1]$ such that $\varphi^* F = k\overline{F}$. Furthermore, in this case, $|k| = \cos\theta$, where θ is denoting the slant angle of the immersion.

PROOF : It follows directly from Proposition 3.5 and Theorem 4.1. □

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