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ON PSEUDO-EINSTEIN HYPERSURFACES OF $\mathcal{H}^{2n+s}$

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In this paper, some properties of the geometry of pseudo-Einstein hypersurfaces of the S-manifold $\mathcal{H}^{2n+s}$ are studied and a theorem concerning their principal curvatures is obtained.

INTRODUCTION

Yano and Kon[14] have studied the geometry of a class of submanifolds of Sasakian manifolds, which they called generic submanifolds. On the other hand, Blair[1, 2] has initiated the study of S-manifolds, which reduce, in particular cases, to Sasakian manifolds. Thus, the bundle space of a principal toroidal bundle over a Kaehler manifold with certain conditions is an S-manifold. In this way, Blair[1] introduced a generalization of the Hopf fibration $\bar{\pi} : S^{2n+1} \to PC^n$ as a canonical example of an S-manifold playing the role of complex projective space in Kaehler geometry and the odd-dimensional sphere in Sasakian geometry. This space is denoted by $\mathcal{H}^{2n+s}$.

The purpose of the present paper is to study a special kind of submanifolds of $\mathcal{H}^{2n+s}$, namely the pseudo-Einstein hypersufaces. These submanifolds correspond to pseudo-Einstein real hypersurfaces of PC^n (Kon[8]). In the case $s = 1$, pseudo-Einstein hypersurfaces of S^{2n+1} are obtained (Yano and Kon[14]).

1. PRELIMINARIES

Let $(\overline{M}, g)$ be a $(2n + s)$-dimensional Riemannian manifold and denote by $T(\overline{M})$ the Lie algebra of vector fields in $\overline{M}$. Then, it is said to be an S-manifold if there exist on $\overline{M}$ an f-structure f (Yano[12]) of rank $2n$ and s global vector fields $\xi_1, ..., \xi_s$ (structure vector fields) such that (Blair[1]) :

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(i) If $\eta_1, ..., \eta_s$ are the dual 1-forms of $\xi_1, ..., \xi_s$, then

$$f\xi_\alpha = 0; \quad \eta_\alpha \circ f = 0; \quad f^2 = -I + \Sigma\, \xi_\alpha \otimes \eta_\alpha,$$

$$g(X, Y) = g(fX, fY) + \Sigma\, \eta_\alpha(X)\, \eta_\alpha(Y),$$

for any $X, Y \in T(\overline{M})$, $\alpha = 1, ..., s$.

(ii) The f-structure f is normal, that is $[f, f] + 2\, \Sigma\, \xi_\alpha \otimes d\eta_\alpha = 0$, where $[f, f]$ is the Nijenhuis torsion of f.

(iii) $\eta_1 \wedge \ldots \wedge \eta_s \wedge (d\eta_\alpha)^n \neq 0$ for any α and $d\eta_1 = \ldots = d\eta_s = F$, where F is the fundamental 2-form defined by $F(X, Y) = g(X, fY)$, $X, Y \in T(\overline{M})$.

In the case $s = 1$, an S-manifold is a Sasakian manifold. For $s \geq 2$, examples of S-manifolds are given in Blair[1, 2], Blair *et al.*[3] and Hasegawa *et al.*[6]. Thus, the bundle space of a principal toroidal bundle over a Kaehler manifold with certain conditions is an S-manifold. In this way, a generalization of the Hopf fibration $\overline{\pi} : S^{2n+1} \to PC^n$ is introduced in Blair[1] as a canonical example of an S-manifold playing the role of the complex projective space in Kaehler geometry and the odd-dimensional sphere in Sasakian geometry. This space is given by (see Blair[1, 2] for more details) :

$$\mathcal{H}^{2n+s} = \{(x_1, ..., x_s) \in S^{2n+1} \times \ldots \times S^{2n+1} / \overline{\pi}(x_1) = \ldots = \overline{\pi}(x_s)\}.$$

For the Riemannian connection $\overline{\nabla}$ of g on an S-manifold $\overline{M}$, the following were also proved in Blair[1] :

$$\overline{\nabla}_X \xi_\alpha = -fX, \; X \in T(\overline{M}), \; \alpha = 1, ..., s, \qquad \ldots (1.1)$$

$$(\overline{\nabla}_X f)Y = \Sigma\, \{g(fX, fY)\, \xi_\alpha + \eta_\alpha(Y) f^2 X\}, \; X, Y \in T(\overline{M}). \qquad \ldots (1.2)$$

Let $\mathcal{L}$ denote the distribution determined by $-f^2$ and $\mathcal{M}$ the complementary distribution. $\mathcal{M}$ is determined by $f^2 + I$ and spanned by $\xi_1, ..., \xi_s$. If $X \in \mathcal{L}$, then $\eta_\alpha(X) = 0$, for any α and if $X \in \mathcal{M}$, then $fX = 0$.

A plane section Π on $\overline{M}$ is called an invariant f-section if there is a vector $X \in \mathcal{L}(x)$, $x \in \overline{M}$, such that $\{X, fX\}$ is an orthonormal pair spanning the section. The sectional curvature of Π is called an f-sectional curvature. If $\overline{M}$ is an S-manifold whose invariant f-sectional curvature is a constant k, then its curvature tensor, denoted by $\overline{R}$, has the form[7]

$$\begin{aligned}\overline{R}(X, Y, Z, W) = \sum_{\alpha, \beta} \; & \{g(fX, fW)\, \eta_\alpha(Y)\, \eta_\beta(Z) - g(fX, fZ)\, \eta_\alpha(Y)\, \eta_\beta(W) \\ & + g(fY, fZ)\, \eta_\alpha(X)\, \eta_\beta(W) - g(fY, fW)\, \eta_\alpha(X)\, \eta_\beta(Z)\} \\ & + \frac{k + 3s}{4} \{g(fX, fW)\, g(fY, fZ) - g(fX, fZ)\, g(fY, fW)\} \\ & + \frac{k - s}{4} \{F(X, W)\, F(Y, Z) - F(X, Z)\, F(Y, W) - 2F(X, Y)\, F(Z, W)\},\end{aligned} \qquad \ldots (1.3)$$

for any $X, Y, Z, W \in T(\overline{M})$. Then, the S-manifold is denoted by $\overline{M}(k)$ and it is said to be an S-space form. Examples of S-space forms can be found in Blair[1] and Hasegawa *et al.*[6]. Specially, $\mathcal{H}^{2n+s}$ is an S-space form with f-sectional curvature $k = 4 - 3s$ (Blair[1]).

Now, let M be a submanifold immersed in $\overline{M}$. We also denote by g the induced metric on M. Let ∇ be the covariant differentiation in M determined by the induced metric. Let $T(M)$ be the Lie algebra of vector fields in M and $T(M)^{\perp}$ be the set of all vector fields normal to M. Then, the Gauss-Weingarten formulas are given by :

$$\overline{\nabla}_X Y = \nabla_X Y + \sigma(X, Y); \quad \overline{\nabla}_X V = -A_V X + D_X V,$$

$X, Y \in T(M)$, $V \in T(M)^{\perp}$, where D is the connection in the normal bundle, σ is the second fundamental form of M and A_V is the Weingarten endomorphism associated with V. A_V and σ are related by

$$g(A_V X, Y) = g(\sigma(X, Y), V); \quad X, Y \in T(M), \quad V \in T(M)^{\perp}. \qquad \ldots (1.4)$$

Moreover, we have the following Codazzi equation :

$$\overline{R}(X, Y, Z, V) = g((\nabla_X A)_V Y, Z) - g((\nabla_Y A)_V X, Z), \qquad \ldots (1.5)$$

for any $X, Y, Z \in T(M)$ and $V \in T(M)^{\perp}$, where

$$(\nabla_X A)_V Y = \nabla_X (A_V Y) - A_{D_X V} Y - A_V \nabla_X Y.$$

This equation has the equivalent form

$$\overline{R}(X, Y, Z, V) = g((\nabla'_X \sigma)(Y, Z), V) - g((\nabla'_Y \sigma)(X, Z), V), \qquad \ldots (1.6)$$

for any $X, Y, Z \in T(M)$ and $V \in T(M)^{\perp}$, where $\nabla'\sigma$ is the covariant derivative of the second fundamental form given by

$$(\nabla'_X \sigma)(Y, Z) = D_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z). \qquad \ldots (1.7)$$

2. Previous Results

From now on, we assume that the submanifold is tangent to the structure vector fields and so, we can write dim $(M) = m + s$. For any $X \in T(M)$, we put

$$fX = TX + NX, \qquad \ldots (2.1)$$

where TX is the tangential component of fX and NX is the normal component of fX. Then, T is an endomorphism of the tangent bundle and N is a normal-bundle valued 1-form on the tangent bundle. From (1.1) and (2.1), we easily get :

$$\nabla_X \xi_\alpha = -TX; \quad \sigma(X, \xi_\alpha) = -NX, \quad X \in T(M), \ \alpha = 1, \ldots, s. \qquad \ldots (2.2)$$

On the other hand, M is said to be totally f-geodesic (Ornea[10]) if

$$\sigma(X, Y) = -\sum_{\alpha} \{\eta_\alpha(X) NY + \eta_\alpha(Y) NX\}, \qquad \text{... (2.3)}$$

for any $X, Y \in T(M)$ and it is said to be totally f-umbilical (Ornea[10]) if there exists a normal vector field V such that

$$\sigma(X, Y) = g(fX, fY)V - \sum_{\alpha} \{\eta_\alpha(X) NY + \eta_\alpha(Y) NX\}, \qquad \text{... (2.4)}$$

for any $X, Y \in T(M)$.

Now, suppose that M is an $(m + s)$-dimensional submanifold of an S-manifold, tangent to the structure vector fields and such that $fT_x(M)^\perp \subseteq T_x(M)$, for any $x \in M$. For example, any hypersurface satisfies the last condition. On the other hand, notice that when $s = 1$, a generic submanifold of a Sasakian manifold, in the sense of Yano and Kon[14], is obtained†. Then, the tangent space $T_x(M)$ of M can be decomposed as

$$T_x(M) = \mathcal{D}_x(M) \oplus fT_x(M)^\perp \oplus \mathcal{M},$$

where $\mathcal{D}_x(M)$ is the orthogonal complement of $fT_x(M)^\perp$ in $T_x(M) - \mathcal{M}$. Thus, $f\mathcal{D}_x(M) = \mathcal{D}_x(M)$. Now, from (2.1), if $X \in T(M)$ we get :

$$T^2X = -X + \sum \eta_\alpha(X)\xi_\alpha - fNX; \; NTX = 0. \qquad \text{... (2.5)}$$

On the other hand, from (1.2), (2.1) and the Gauss-Weingarten formulas, if $X, Y \in T(M)$ and $V \in T(M)^\perp$, we obtain

$$\sigma(X, TY) = -(\nabla_X N)Y \qquad \text{... (2.6)}$$

and

$$\sigma(X, fV) = -NA_V X. \qquad \text{... (2.7)}$$

Now, let p be the codimension of M, that is, $p = 2n - m$. Then :

Proposition 2.1 — Let M be a submanifold of an S-manifold, tangent to the structure vector fields and such that $fT_x(M)^\perp \subseteq T_x(M)$, for any $x \in M$. If $p \geq 2$ and M is totally f-umbilical, then M is totally f-geodesic.

PROOF : Let $X \in fT(M)^\perp$. From (1.4) and (2.7),

$$g(A_{NX} fV, X) = g(\sigma(X, fV), NX) = -g(NA_V X, NX) = -g(A_V X, X),$$

where V is a normal vector field. Now, from (1.4) and (2.4),

$$g(X, X)\, g(V, V) = -g(fV, X)\, f(V, NX) = g(V, NX)\, g(V, NX),$$

for any $X \in fT(M)^\perp$. Since $p \geq 2$, we can take X such that $g(V, NX) = 0$. Consequently, $V = 0$ and M is totally f-geodesic. □

†Definitions of generic submanifolds can be found in Chen[4], Okumura[9], Wells[11] and Yano and Kon[13] in the context of Kaehlerian manifolds.

Proposition 2.2 — Let M be a $(m + s)$-dimensional ($m \geq 3$) submanifold of a $(2n + s)$-dimensional S-space form $\overline{M}(k)$, tangent to the structure vector fields and such that $fT_x(M)^{\perp} \subseteq T_x(M)$, for any $x \in M$. If M is totally f-umbilical and $m > n$, then $k = -3s$.

PROOF : If $p \geq 2$, from the above Proposition, M is totally f-geodesic. Then, from (2.3), (2.5) and (2.6), we have that $(\nabla_X N)Y = 0$, for any $X, Y \in T(M)$. Then, by using (1.7) and (2.3) again, we obtain

$$(\nabla'_X \sigma)(Y, Z) = s\{g(Y, TX)NZ + g(Z, TX)\, NY\},$$

for any X, Y, $Z \in T(M)$. Thus, the Codazzi equation (1.6) and (1.3) give :

$$(k + 3s)\, \{g(TY, Z)NX - g(TX, Z)\, NY + 2g(X, TY)\, NZ\} = 0.$$

Since $m > n$, we know that $\dim(\mathcal{D}(M)) > 0$ and so, $\dim(\mathcal{D}(M)) \geq 2$. Then, we can take $Y, Z \in \mathcal{D}(M)$ such that $Z = TY$. Therefore $NY = NZ = 0$ and, from the above equation, $(k + 3s)g\,(TY, TY)\, NX = 0$, for any $X \in T(M)$, that is $k = -3s$.

Now, we assume that $p = 1$. Then M is a hypersurface of $\overline{M}$. Denote by C the unit normal of M in $\overline{M}$ and let $U = -fC$. Then, $\eta_\alpha(U) = 0$, for any α. If we take $X, Y \in \mathcal{L}$ such that $g(U, X) = g(U, Y) = 0$ and tangent to M, from (1.7), (2.2) and (2.4), we get $(\nabla'_X \sigma)(Y, U) = sg(TX, Y)C$. Then, the Codazzi equation (1.6) and (1.3) give $(k + 3s)\, g(TX, Y) = 0$. Since $m \geq 3$, we can put $TX = Y$. Thus, $k = -3s$. □

3. PSEUDO-EINSTEIN HYPERSURFACES OF $\mathcal{H}^{2n+s}$

Let M be a hypersurface of an S-manifold $\overline{M}$, tangent to the structure vector fields (so, $\dim(M) = 2n - 1 + s$). As above, denote by C the unit normal of M in $\overline{M}$ and let $U = -fC$. Then, $\eta_\alpha(U) = 0$, for any α. Moreover, if $X \in T(M)$, we have

$$fX = TX + u(X)C, \qquad \text{... (3.1)}$$

where $u(X) = g(U, X)$. From this, we find that $TU = 0$ and $u(U) = 1$.

From now on, we denote the Weingarten endomorphism by A in place of A_C. Then, the Gauss-Weingarten formulas are given by :

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)C;\ \overline{\nabla}_X C = -AX;\ X, Y \in T(M). \qquad \text{... (3.2)}$$

From (1.1) and (3.2), we get $\eta_\alpha(AX) = -u(X)$, $A\xi_\alpha = -U$, for any $X \in T(M)$ and any α. Moreover, since, from (1.2), $(\overline{\nabla}_X f)C = 0$, we have by using (3.1) and (3.2) :

$$\nabla_X U = TAX,\ X \in T(M). \qquad \text{... (3.3)}$$

In the following, we assume that the ambient S-manifold is $\mathcal{H}^{2n+s}(4 - 3s)$. Then, the Ricci tensor S of a hypersurface M is given by

$$S(X, Y) = (2n + 1 - 2s)\, g(fX, fY) + 3(s - 1)\, u(X)\, u(Y) + (2n - 1) \sum_{\alpha, \beta} \eta_\alpha (X)\, \eta_\beta (Y) + Hg(AX, Y) - g(AX, AY), \quad \text{... (3.4)}$$

for any $X, Y \in T(M)$, where H denotes the mean curvature of M. The hypersurface M is said to be pseudo-Einstein if its Ricci tensor satisfies

$$S(f^2X, f^2Y) = ag(f^2X, f^2Y) + bu(f^2X)\, u(f^2Y),$$

for any $X, Y \in T(M)$, a and b being constant. This is equivalent to :

$$\begin{aligned} S(X, Y) = {} & a\{g(X, Y) - \sum \eta_\alpha (X)\, \eta_\alpha (Y)\} + bu(X)\, u(Y) \\ & + \sum [\eta_\alpha(X)\, S(X, \xi_\alpha) + \eta_\alpha (Y)\, S(\xi_\alpha, Y)] \\ & + \sum_{\alpha, \beta} \eta_\alpha(X)\, \eta_\beta (Y)\, S(\xi_\alpha, \xi_\beta), \end{aligned} \quad \text{... (3.5)}$$

for any $X, Y \in T(M)$.

Now, we suppose that M is a pseudo-Einstein hypersurface of $\mathcal{H}^{2n+s}$ and $n \geq 3$. We can choose a local field of orthonormal frames $\{e_1, ..., e_{2n+s}\}$ in $\mathcal{H}^{2n+s}$ such that, restricted to M, $e_1, ..., e_{2n-1+s}$ are tangent to M, $e_{2n-1} = U$, $e_{2n-1+\alpha} = \xi_\alpha$ $(1 \leq \alpha \leq s)$ and $e_{2n+s} = C$. Moreover, choosing $e_1, ..., e_{2n-2}$ suitably, the Weingarten endomorphism A is represented by a matrix of the form

$$A = \begin{pmatrix} \lambda_1 & & & h_1 & & & \\ & \ddots & & \vdots & & 0 & \\ & & \lambda_{2n-2} & h_{2n-2} & & & \\ h_1 & \cdots & h_{2n-2} & \gamma & -1 & \cdots & -1 \\ & & & -1 & & & \\ & 0 & & \vdots & & 0 & \\ & & & -1 & & & \end{pmatrix}, \quad \text{... (3.6)}$$

where $h_i = g(AU, e_i)$ $(1 \leq i \leq 2n - 2)$ and $\gamma = g(AU, U)$. Since $g(Ae_i, e_j) = 0$, from (3.4)-(3.6), we get $g(Ae_i, Ae_j) = 0$, $i \neq j$, $i, j = 1, ..., 2n - 2$. But $g(Ae_i, Ae_j) = h_i h_j$. Then :

$$h_i h_j = 0, \ i \neq j, \ \ i, j = 1, ..., 2n - 2. \quad \text{... (3.7)}$$

On the other hand, from (3.4)-(3.6) again, if $j = 1, ..., 2n - 2$, we have $Hg(AU, e_j) - g(AU, Ae_j) = 0$. But $g(Ae_j, AU) = \lambda_j h_j + h_j \gamma$. Thus :

$$h_j (H - \lambda_j - \gamma) = 0, \ j = 1, ..., 2n - 2. \quad \text{... (3.8)}$$

Since from (3.7) we observe that at most one h_i does not vanish, we can assume that $h_i = 0$ for $i = 2, ..., 2n - 2$. Then, from (3.8), we get that either $H = \lambda_1 + \gamma$ or $h_1 = 0$.

Moreover, if $H = \lambda_1 + \gamma$, from (3.4) and (3.5), we obtain :

$$a = 2n + 1 - 2s + \gamma\lambda_1 - h_1^2; \quad a + b = 2n - 2 + \gamma\lambda_1 - h_1^2.$$

Then, these formulas imply that $b = 2s - 3$. Consequently, for any $X, Y \in T(M) \cap \mathcal{L}$, we get, by using (3.4) and (3.5) :

$$ag(X, Y) - su(X)\, u(Y)$$
$$= (2n + 1 - 2s)\, g(X, Y) + Hg(AX, Y) - g(AX, AY). \qquad \text{... (3.9)}$$

Now, let $\{e_1, ..., e_{2n-1}, \xi_1, ..., \xi_s\}$ be a new local field of orthonormal frames on M, for which the Weingarten endomorphism A is represented by a matrix of the form

$$A = \begin{pmatrix} L & U \\ U & 0 \end{pmatrix}, \qquad \text{... (3.10)}$$

where L is the $(2n - 1) \times (2n - 1)$ matrix

$$\mathbf{L} = \begin{pmatrix} \beta_1 & & \\ & \ddots & \\ & & \beta_{2n-1} \end{pmatrix}$$

and U is the $(2n - 1) \times s$ matrix such that $u_{i\alpha} = g(A\xi_\alpha, e_i) = -u(e_i)$, for $i = 1, ..., 2n - 1$ and $\alpha = 1, ..., s$. From (3.9) and (3.10), we can obtain $a = (2n + 1 - 2s) + H\beta_i - \beta_i^2$, $i = 1, ..., 2n - 1$. Therefore, we find that each β_i satisfies the quadratic equation :

$$t^2 - Ht + a - 2n - 1 + 2s = 0. \qquad \text{... (3.11)}$$

Next, we have the following lemma :

Lemma 3.1 — Let M be a hypersurface of $\mathcal{H}^{2n+s}$, tangent to the structure vector fields. If M is pseudo-Einstein, then :

(i) If $\beta_1 = \ldots = \beta_{2n-1}$ at every point of M, then M is totally f-umbilical.

(ii) If $AU = \gamma U - \Sigma\, \xi_\alpha$, then γ is constant.

(iii) rank $(\mathbf{L}) > 1$ at some point of M.

The proof of this lemma is similar to that one in case $s = 1$ for pseudo-Einstein hypersurfaces of S^{2n+1} (see Yano and Kon[14]), so it is omitted here.

From (3.11) we see that at most two β_i can be distinct at each point of M. We denote them by λ and μ. Suppose that p is the miltiplicity of λ. Then, the multiplicity of μ is $q = 2n-1-p$. Now, by using a lengthy computation, we can prove that if $H = \lambda_1 + \gamma$ and λ and μ are constant with p, $q \geq 2$, then either $\lambda\mu > 0$ or $h_1 = 0$. In fact, we have :

Lemma 3.2 — Let M be a hypersurface of $\mathcal{H}^{2n+s}$ ($n \geq 3$), tangent to the structure vector fields. If M is pseudo-Einstein, then $h_1 = 0$.

PROOF : From (3.8), we need to show that $H \neq \lambda_1 + \gamma$. If $H = \lambda_1 + \gamma$, from (3.10) and (3.11), we find that at most two β_i are distinct. We denote them by λ and μ. If $\lambda = \mu$ at any point of M, then Lemma 3.1 (i) gives that M is totally f-umbilical, which is a contradiction with Theorem 2.2. Thus, $\lambda \neq \mu$ at some point. Now, from (3.11) we see that :

$$H = \lambda + \mu; \quad \lambda\mu = a - 2n - 1 + 2. \qquad \text{... (3.12)}$$

Let p be the multiplicity of λ. Therefore, $H = p\lambda + (2n-1-p)\mu$ and, using (3.12) :

$$(p-1)\lambda + (2n-2-p)\mu = 0. \qquad \text{... (3.13)}$$

Now, we suppose that $a > 2n + 1 - 2s$. Then, from (3.12), λ and μ have the same sign. Consequently, (3.13) gives $p = 1$ and $n = \frac{3}{2}$, which is a contradiction. If $a < 2n + 1 - 2s$ and $\lambda = \mu$ at some point, from (3.9) and (3.10),

$$H\lambda - \lambda^2 = (2n-1)\lambda^2 - \lambda^2 = a - (2n + 1 - 2s) < 0,$$

and so, $n < 1$, which is again a contradiction. Then, there exist exactly two distinct eigenvalues λ and μ of $\mathbf{L}$ at each point of M. From (3.13) we get $1 < p < 2n - 2$. Thus, (3.12) and (3.13) give

$$\mu^2 = \frac{(p-1)(a-2n-1+2s)}{2n-2-p}; \quad \lambda^2 = -\frac{(2n-2-p)(a-2n-1+2s)}{p-1}.$$

Consequently, λ and μ are constant. So, we have either $\lambda\mu > 0$ or $h_1 = 0$. But $\lambda\mu > 0$ contradicts (3.12).

Finally, if $a = 2n + 1 - 2s$, (3.12) shows that $\lambda\mu = 0$ and then rank $(\mathbf{L}) \leq 1$, which is not possible, from Lemma 3.1 (iii). Therefore, the proof is complete. □

Theorem 3.3 — Let M be a hypersurface of $\mathcal{H}^{2n+s}$ ($n \geq 3$), tangent to the structure vector fields. If M is pseudo-Einstein, then :

(i) If $s = 2$, M has four or five constant principal curvatures.

(ii) If $s > 2$, M has three, four or five constant principal curvatures.

PROOF : Since $h_1 = 0$, for a local field of orthonormal frames

$$\{e_1, ..., e_{2n-2}, e_{2n-1} = U, \quad e_{2n} = \xi_1, ..., e_{2n-1+s} = \xi_s\}$$

of M, the Weingarten endomorphism is represented by a matrix of the form :

$$A = \begin{pmatrix} \lambda_1 & & & & & & \\ & \ddots & & 0 & & 0 & \\ & & \lambda_{2n-2} & & & & \\ & 0 & & \gamma & -1 & \dots & -1 \\ & & & -1 & & & \\ & 0 & & \vdots & & 0 & \\ & & & -1 & & & \end{pmatrix},$$

From (3.4) and (3.5) :

$$a = 2n + 1 - 2s + H\lambda_i - \lambda_i^2, \; i = 1, \dots, 2n - 2; \qquad \dots (3.14)$$

$$a + b = 2n - 2 + H\gamma - \gamma^2. \qquad \dots (3.15)$$

Moreover, since $AU = \gamma U - \Sigma\,\xi_\alpha$, γ is constant, using Lemma 3.1 (ii). If $\gamma \neq 0$, from (3.15) H is constant and from (3.14) at most there are two λ_i distinct. We denote them by λ and μ and we see that they are both constant too, from (3.14).

If $\gamma = 0$, then $H = p\lambda + (2n-2-p)\mu$, where p is the multiplicity of λ. Suppose that $a > 2n + 1 - 2s$. If $\lambda \neq \mu$ at some point of M, since $H = \lambda + \mu$, we get $(p - 1)\lambda + (2n - 3 - p)\mu = 0$. But $\lambda\mu = a - 2n - 1 + 2s > 0$ and so, λ and μ have the same sign. Hence $p = 1$ and $n = 2$, which is a contradiction. Consequently, $\lambda = \mu$ at each point and, since $H = (2n - 2)\lambda = 2\lambda$, $\lambda = 0$, but this contradicts (3.14). So, this case is not possible.

Now, suppose that $a < 2n + 1 - 2s$ and $\lambda = \mu$ at some point of M. Then, from (3.14), $(2n - 3)\lambda^2 = a - 2n - 1 + 2s < 0$, which is impossible. Thus, $\lambda \neq \mu$ at each point. Moreover

$$H = p\lambda + (2n-2-p)\mu = \lambda + \mu, \quad \lambda\mu = a - 2n - 1 + 2s$$

and so, λ and μ are both constant.

Finally, suppose that $a = 2n + 1 - 2s$. Then $\lambda\mu = 0$ and, if $\lambda \neq 0$, $H = p\lambda$. From (3.14), we get $p = 1$. Thus, we can assume that $Ae_i = 0$, for $i = 1, \dots, 2n - 3$ and $Ae_{2n-2} = \lambda e_{2n-2}$. From this, if $1 \leq i, j \leq 2n-3$, we have $g((\nabla_{e_i} A)e_j, U) = sg(e_i, Te_j)$. Using (1.3) and the Codazzi equation (1.5), $g(e_i, Te_j) = 0$, $1 \leq i, j \leq 2n-3$. But, since $n \geq 3$, we can choose e_i and e_j such that $g(e_i, Te_j) \neq 0$, which is a contradiction. Consequently $a \neq 2n + 1 - 2s$.

Now, we consider the matrix :

$$B = \begin{pmatrix} \gamma & -1 & \cdots & -1 \\ -1 & & & \\ \cdot & & & \\ \cdot & & 0 & \\ \cdot & & & \\ -1 & & & \end{pmatrix}.$$

Then, its eigenvalues satisfy :

$$t^{s-1}(t^2 - \gamma t - s) = 0. \qquad \text{... (3.16)}$$

Let $\lambda = \mu$. For $1 \leq i, j \leq 2n-2$, we have

$$g((\nabla_{e_i} A)e_j, U) = (-\lambda^2 + \gamma\lambda + s)\, g(e_j, Te_i)$$

and, from (1.3) and the Codazzi equation (1.5) again :

$$\lambda^2 - \gamma\lambda - 1 = 0. \qquad \text{... (3.17)}$$

Thus, $\lambda \neq 0$ and λ does not satisfy (3.16). Consequently, M has four principal curvatures : λ, 0 (of multiplicity $s - 1$) and

$$\frac{\gamma \pm \sqrt{\gamma^2 + 4s}}{2}.$$

Let $\lambda \neq \mu$. We consider the orthonormal frame such that :

$$Ae_a = \lambda e_a,\ a = 1, \ldots, p;\ Ae_r = \mu e_r,\ r = p+1, \ldots, 2n-2.$$

Then, a similar computation gives $(\gamma\lambda + \gamma\mu - 2\lambda\mu + 2)\, g(e_a, Te_r) = 0$.

Suppose that $g(e_a, Te_r) = 0$, for any a and r. Then, $p \geq 2$ and $2n - 2 - p \geq 2$. As above, we can obtain analogous formulas to (3.17) and so $\lambda^2 - \gamma\lambda - 1 = 0 = \mu^2 - \gamma\mu - 1$. This implies that λ and μ do not satisfy (3.16). Consequently, if $\lambda, \mu \neq 0$, M has five principal curvatures and if $\lambda = 0$ (or $\mu = 0$), M has four principal curvatures.

Next, suppose that $g(e_a, Te_r) \neq 0$, for some a and r. Then, we get :

$$(\gamma\lambda + \gamma\mu - 2\lambda\mu + 2) = 0. \qquad \text{... (3.18)}$$

If $\gamma = 0$, the eingenvalues of **B** are 0 and $\pm\sqrt{s}$. Moreover, (3.18) is $\lambda\mu = 1$. Thus, λ and μ do not verify (3.16) and M has five principal curvatures : λ, μ, 0 and $\pm\sqrt{s}$.

Now, we assume that $\gamma \neq 0$. If $s = 2$ and $\lambda \neq 0 \neq \mu$, M has at least four principal curvatures because, in other case, $\lambda\mu = -2, \lambda + \mu = \gamma$ and (3.18) is not satisfied. If $\lambda = 0$ (similarly, if $\mu = 0$), (3.18) is $\gamma\mu + 2 = 0$ and μ does not verify (3.16). So, M has four principal curvatures.

Next, we assume that $s > 2$. If $\lambda = 0$, for $\gamma = \pm\dfrac{2}{\sqrt{s-2}}$, μ satisfies (3.16). Consequently, M has three principal curvatures : 0 and ·

$$\frac{\gamma \pm \sqrt{\gamma^2 + 4s}}{2}.$$

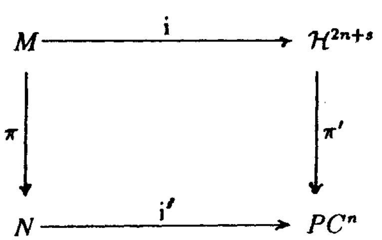
If $\gamma \neq \pm \frac{2}{\sqrt{s-2}}$, M has four principal curvatures. The case $\mu = 0$ is analogous. Finally, if $\lambda \neq 0 \neq \mu$, M has at most five principal curvatures. This completes the proof. □

This result should be compared with that one in case $s = 1$ (see Yano and Kon[14]).

Finally, we shall prove that pseudo-Einstein hypersurfaces of $\mathcal{H}^{2n+s}$ correspond to pseudo-Einstein real hypersurfaces of the complex projective space PC^n with constant holomorphic sectional curvature 4. In fact, let N be a $(2n-1)$-dimensional real hypersurface of PC^n. We denote by J the almost complex structure of PC^n, by G its metric tensor field as well as the induced metric tensor field on N and by C' the unit normal of N in PC^n. We put $JC' = -U'$ and $u'(X) = G(U', X)$, for any $X \in T(N)$. Then, N is said to be a pseudo-Einstein real hypersurface of PC^n if there exist two constant α and β such that the Ricci tensor S' of N is of the form (Kon[8]) :

$$S'(X, Y) = \alpha G(X, Y) + \beta u'(X)\, u'(Y), \quad X, Y \in T(N). \qquad \text{... (3.19)}$$

Now, we consider the following commutative diagram :



where M is a hypersurface of $\mathcal{H}^{2n+s}$, tangent to the structure vector fields, π and π' denote the Riemannian fibre bundles and the immersion i is a diffeomorphism on the fibres (see Blair[1, 2], Blair *et al.*[3] and Fernández[5] for more details). If we denote by * the horizontal lift and by S the Ricci tensor of M, then (Fernández[5])

$$S'(X, Y) = S(X^*, Y^*) + 2sg(TX^*, TY^*), \; X, Y \in T(N). \qquad \text{... (3.20)}$$

But, from (3.1), we have $g(TX^*, TY^*) = g(X^*, Y^*) - u(X^*)\, u(Y^*)$.

Consequently, from (3.20), we get

$$S'(X, Y) = S(X^*, Y^*) + 2sg(X^*, Y^*) - 2su(X^*)\, u(Y^*), \qquad \text{... (3.21)}$$

for any $X, Y \in T(N)$. Then, we can prove the following theorem :

Theorem 3.4 — Let M be a hypersurface of $\mathcal{H}^{2n+s}$, tangent to the structure vector fields. Then, M is a pseudo-Einstein hypersurface of $\mathcal{H}^{2n+s}$ if and only if N is a pseudo-Einstein real hypersurface of PC^n.

PROOF : First, if M is a pseudo-Einstein hypersurface of $\mathcal{H}^{2n+s}$, then (3.5) is satisfied, a and b being constant. Now, let $X, Y \in T(N)$. Then, from (3.5) and (3.21), we have

$$S'(X, Y) = (a + 2s)\, G(X, Y) + (b - 2s)u'(X)u'(Y)$$

and, therefore, N is a pseudo-Einstein real hypersurface of PC^n.

Conversely, there exist two constant α and β such that (3.19) is satisfied. Let $X, Y \in T(M)$. Then, if $X' = \pi_* X$ and $Y' = \pi_* Y$, it is known that $f^2X = -X'^*$ and $f^2Y = -Y'^*$. Thus, from (3.21) :

$$S(f^2X, f^2Y) = (\alpha - 2s)\, g(f^2X, f^2Y) + (\beta + 2s)\, u(f^2X)\, u(f^2Y).$$

Consequently, M is pseudo-Einstein hypersurface of $\mathcal{H}^{2n+s}$ and the proof is complete. □

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