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STUDY OF A LOGISTIC EQUATION WITH LOCAL AND NON-LOCAL REACTIONS TERMS

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Abstract. In this work we examine a logistic equation with local and non-local reaction terms both for time dependent and steady-state problems. Mainly, we use bifurcation and monotonicity methods to prove the existence of positive solutions for the steady-state equation and sub-supersolution method for the long time behavior for the time dependent problem. The results depend strongly on the size and sign of the parameters on the local and non-local terms.

1. Introduction

In this paper we study the non-local parabolic problem

$$(1.1)\qquad \begin{cases} u_t - \Delta u = u(\lambda + b\displaystyle\int_\Omega u^r dx - u) & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \ge 0 & \text{in } \Omega, \end{cases}$$

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and its corresponding steady-state problem

$$\begin{cases} -\Delta u = u(\lambda + b \displaystyle\int_\Omega u^r dx - u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded and smooth domain, $\lambda, b \in \mathbb{R}$, $r > 0$ and u_0 is a regular positive function. In (1.1), $u(x,t)$ represents the density of a species in time $t > 0$ and at the point $x \in \Omega$, a habitat surrounded by inhospitable areas. Here, λ is the growth rate of the species, the term $-u$ describes the limiting effect of crowding in the population, that is, the competition of the individuals of the species for the resources of the environment. In (1.1) we have included a non-local term with different meanings. When $b < 0$ we are assuming that this limiting effects not only depends on the value of u in the point x, but the value of u in the whole domain. However, when $b > 0$ the individuals cooperate globally to survive. When $b = 0$, (1.1) is the classical logistic equation.

Observe that when $b > 0$, the problem (1.1) can be regarded as a superlinear indefinite problem with nonlocal superlinear term, similar to the classical superlinear problem

$$\begin{cases} u_t - \Delta u = u(\lambda + ba^+ u^r - a^- u^r) & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0 & \text{in } \Omega, \end{cases} \tag{1.3}$$

where $a \in C^1(\overline{\Omega})$, $a^+ := \max\{a(x), 0\}$, $a^- := \max\{-a(x), 0\}$, studied in detail in [14], [15], [17] and references therein. This class of local problems has been considered with other boundary conditions, for example, non-homogeneous Dirichlet boundary conditions, see [9] and [18], where multiplicity results are shown. We do not consider the non-local counterpart in this paper.

The introduction of nonlocal terms, as much in the equation as in the boundary conditions, has shown to be useful modelling a number of processes in different fields such as the equations of the Mathematical Physics, the mechanic of deformable solids, the systems of the Mathematical Biology and many others. In population dynamics, it is used regularly; see, for instance, [8], [7] and [11].

We summarize our main results. In order to show them, denote by λ_1 the principal eigenvalue of the laplacian subject to homogeneous Dirichlet boundary conditions and by φ_1 the positive eigenfunction associated to λ_1 such that $\|\varphi_1\|_\infty = 1$.

Regarding the parabolic problem (1.1), first we prove the existence and uniqueness of positive local in time solution. Then, we analyze the long time behaviour of the solution. Among the main findings in this work, we mention:

(1) If $b < 0$ the solution of (1.1) is global in time and bounded. Moreover, the solution goes to zero as $\lambda < \lambda_1$.

(2) Assume now $b > 0$.
   (a) The trivial solution is locally exponentially stable for $\lambda < \lambda_1$, that is, for small u_0 the solution goes to zero if $t \to \infty$.
   (b) If $r < 1$ or $r = 1$ and b small, the solution of (1.1) is global in time and bounded. Moreover, the solution goes to zero if λ is small.
   (c) If $r > 1$ or $r = 1$ and b large, the solution of (1.1) blows up in finite time for λ or u_0 large.

We refer to Section 6 for more specific results. We would like to remark that similar results for related problems have been obtained in [21], [22] and [19] and references therein for the problem

$$u_t - \Delta u = \int_\Omega u^r(x,t)dx - ku^p,$$

for $r, p \geq 1$ and $k \geq 0$.

Now we present the results concerning to (1.2). The case $b < 0$ and $r = 1$ was analyzed in [23], showing the existence and uniqueness of positive solution of (1.2). We improve these results considering all the cases for $r > 0$. The case $b = 0$ (the pure local model) is well-known, see Proposition 2.1. The equation

$$-\Delta u = u(\lambda + b\int_\Omega u dx + u)$$

with $b < 0$ was analyzed in [8].

In order to prove our results, we use mainly the bifurcation method, used previously in this context by [1], [5] and [12].

First, we show that from the trivial solution $u = 0$ emanates at $\lambda = \lambda_1$ an unbounded continuum of positive solutions of (1.2), and we determine the local and global behaviour of this continuum. Hence, when $b \leq 0$ the behaviour does not depend on r and we can show (see Figure 1 a)):

THEOREM 1.1. *Assume $b \leq 0$. Then there exists a positive solution if and only if $\lambda > \lambda_1$. Moreover, it is unique if it exists, and it will be denoted by $\omega_{\lambda,b}$. Furthermore,*

$$\lim_{b \to -\infty} \|\omega_{\lambda,b}\|_\infty = 0.$$

When $b > 0$ the behaviour depends on the size of r. When $r < 1$ we obtain (see Figure 1 c)):

THEOREM 1.2. *Assume $b > 0$ and $r < 1$. There exists $\lambda_* < \lambda_1$ such that (1.2) possesses at least one positive solution if and only if $\lambda \geq \lambda_*$. Moreover,*

$$\lim_{b \to 0^+} \lambda_*(b) = \lambda_1 \quad \text{and} \quad \lim_{b \to +\infty} \lambda_*(b) = -\infty. \tag{1.4}$$

This behaviour is completely different to the case $r > 1$ (see Figure 1 d)):

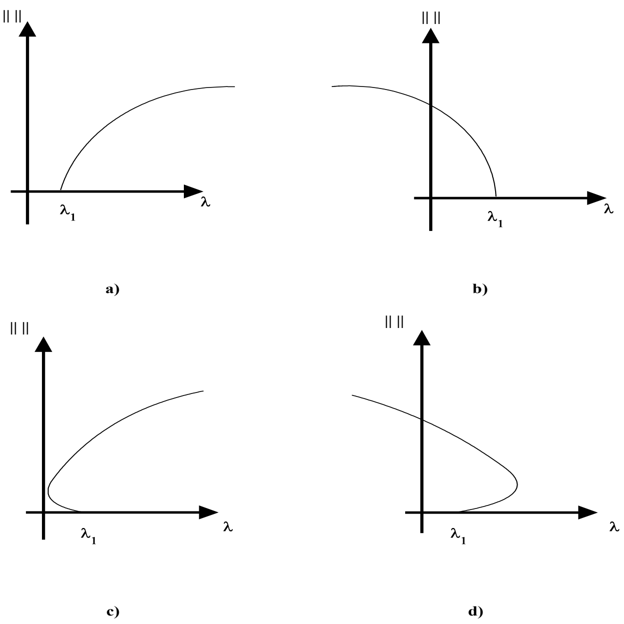

FIGURE 1. Bifurcation diagrams for equation (1.2) .

THEOREM 1.3. *Assume $b > 0$ and $r > 1$. There exists $\lambda^* > \lambda_1$ such that (1.2) possesses at least one positive solution if and only if $\lambda \le \lambda^*$. Moreover,*

$$\lim_{b \to 0^+} \lambda^*(b) = +\infty \quad \textit{and} \quad \lim_{b \to +\infty} \lambda^*(b) = \lambda_1. \tag{1.5}$$

Let us remark that, unlike the local case, we do not need to impose any restriction to r in order to get the a priori bounds. Indeed, if we were considering the local case

$$-\Delta u = u(\lambda + bu^r - u),$$

then in order to obtain a priori bounds, we need to impose $r+1 < (N+2)/(N-2)$, see [10].

Finally, in the case $r = 1$, the behaviour depends of the size of b:

THEOREM 1.4. *Assume $b > 0$ and $r = 1$.*

(1) *Assume that $b < 1/|\Omega|$. Then, there exists a positive solution for $\lambda > \lambda_1$.*
(2) *Assume that $b > 1/\int_\Omega \varphi_1 dx$. Then, there exists a positive solution if and only if $\lambda < \lambda_1$.*

Here $|\Omega|$ stands for the measure of Ω. When b is small, the bifurcation is similar to the case $b \leq 0$ (see Figure 1 a)) whereas when b is large we have positive solution for $\lambda < \lambda_1$ (see Figure 1 b)).

It is evident that there exists a gap in our results for $b \in (1/|\Omega|, 1/\int_\Omega \varphi_1)$. In this case, we know that there exists an unbounded continuum of positive solutions bifurcating from $(\lambda, u) = (\lambda_1, 0)$, even we know its local bifurcation direction (see Theorem 2.2), but we are not able to assure the global behaviour of the continuum. Observe, that this does not occur in the homogeneous Neumann case. Indeed, in this case $\lambda_1 = 0$ and $\varphi_1 = 1$. Hence, $1/|\Omega| = 1/\int_\Omega \varphi_1$ and for $b = 1/|\Omega|$ there exist infinite positive solutions for $\lambda = \lambda_1 = 0$.

Observe now Figure 2, where we have represented the different bifurcation diagrams moving the parameter b. In Figure 2 a) and b) we have drawn the bifurcation diagrams when $b \to 0$ in the cases $r < 1$ and $r > 1$, respectively. In the cases c) and d) we present the case $b \to +\infty$ for $r < 1$ and $r > 1$.

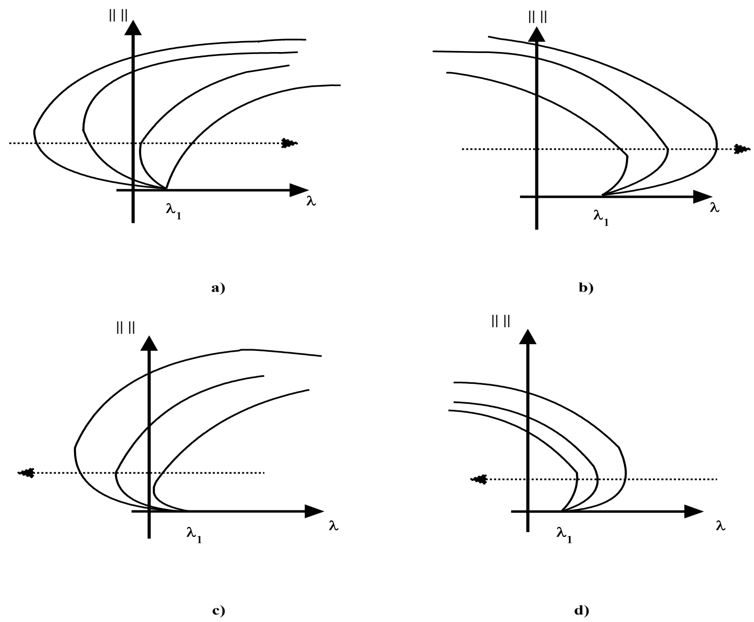

FIGURE 2. Bifurcation diagrams for equation (1.2) moving b.

The paper is organized as follows. In Section 2 we prove the existence of an unbounded continuum of positive solutions of (1.2). Section 3 is devoted to prove the non-existence results and a priori bounds of positive solutions of (1.2). In Section 4 we show the stability of the solutions in some cases. Section 5 is dedicated to prove Theorems 1.1, 1.2, 1.3 and 1.4. Finally, in Section 6 we study the parabolic problem (1.1).

2. Bifurcation results

We are going to prove that from the trivial solution $u \equiv 0$ an unbounded continuum of positive solution of (1.2) bifurcates at $\lambda = \lambda_1$. For that, we need to introduce some results.

The first one plays an important role along the paper and it will be used many times throughout the work. Consider the classical logistic equation

$$\begin{cases} -\Delta u = u(\mu - u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\mu \in \mathbb{R}$. In the next result, we summarize the principal well-known results concerning to (2.1) (see Lemma 7.8 in [6] for (2.2)).

Proposition 2.1. *There exists a positive solution of (2.1) if and only if $\mu > \lambda_1$. Moreover, it is unique if it exists. We denote it by θ_μ. Furthermore:*

(1) *The following inequalities hold:*

$$(\mu - \lambda_1)\varphi_1 \leq \theta_\mu \leq \min\{\mu, K(\mu - \lambda_1)\} \tag{2.2}$$

for some $K \geq \int_\Omega \varphi_1^2 dx / \int_\Omega \varphi_1^3 dx$ independent of μ.

(2) *If $\underline{u} > 0$ is a sub-solution of (2.1), then $\underline{u} \leq \theta_\mu$.*

(3) *If $\overline{u} > 0$ is a super-solution of (2.1), then $\theta_\mu \leq \overline{u}$.*

We consider the Banach space $X := C_0(\overline{\Omega})$ and denote $B_\rho := \{u \in X : \|u\|_\infty < \rho\}$. Define

$$f(u) := u^+(\lambda + b \int_\Omega (u^+)^r dx - u),$$

and the map

$$\mathcal{K}_\lambda : X \mapsto X; \qquad \mathcal{K}_\lambda(u) := u - (-\Delta)^{-1}(f(u))$$

where $u^+ := \max\{u, 0\}$ and $(-\Delta)^{-1}$ is the inverse of the operator $-\Delta$ under homogeneous Dirichlet boundary conditions. Now, it is clear that u is a non-negative solution of (1.2) if, and only if, u is a zero of the map $\mathcal{K}_\lambda$.

The main result of this section is:

Theorem 2.2. *The value $\lambda = \lambda_1$ is the only bifurcation point from the trivial solution for* (1.2). *Moreover, there exists a continuum $\mathcal{C}_0$ of nonnegative solutions of* (1.2) *unbounded in $\mathbb{R} \times X$ emanating from* $(\lambda_1, 0)$. *Furthermore,*

(1) *If $b \leq 0$, the direction of bifurcation is supercritical.*
(2) *Assume $b > 0$.*
   (a) *If $r < 1$, the direction of bifurcation is subcritical.*
   (b) *If $r > 1$, the direction of bifurcation is supercritical.*
   (c) *Assume that $r = 1$ and denote by*

$$b_0 := \frac{\displaystyle\int_\Omega \varphi_1^3 dx}{\displaystyle\int_\Omega \varphi_1 dx \int_\Omega \varphi_1^2 dx}.$$

   If $b > b_0$ (resp. $b < b_0$) the direction of bifurcation is subcritical (resp. supercritical).

Recall that we say that the direction of bifurcation is subcritical (resp. supercritical) if there exists a neighborhood V of $(\lambda_1, 0)$ such that every solution $(\lambda, u) \in V$ satisfies $\lambda < \lambda_1$ (resp. $\lambda > \lambda_1$).

In order to prove this result we use the Leray-Schauder degree of $\mathcal{K}_\lambda$ on B_ρ with respect to zero, denoted by $\deg(\mathcal{K}_\lambda, B_\rho)$, and the index of the isolated zero u of $\mathcal{K}_\lambda$, denoted by $i(\mathcal{K}_\lambda, u)$.

LEMMA 2.3. *If $\lambda < \lambda_1$, then $i(\mathcal{K}_\lambda, 0) = 1$.*

PROOF. Fix $\lambda < \lambda_1$. Define the map

$$\mathcal{H}_1 : [0, 1] \times X \mapsto X; \qquad \mathcal{H}_1(t, u) := (-\Delta)^{-1}(tf(u)).$$

We claim that there exists $\delta > 0$ such that

$$u \neq \mathcal{H}_1(t, u) \quad \text{for all } u \in \overline{B}_\delta \setminus \{0\}, \text{ and } t \in [0, 1].$$

Indeed, suppose that there exist sequences $u_n \in X \setminus \{0\}$ with $\|u_n\|_\infty \to 0$ and $t_n \in [0, 1]$ such that

$$u_n = \mathcal{H}_1(t_n, u_n),$$

that is

$$-\Delta u_n = t_n f(u_n),$$

and so $u_n \geq 0$. Define

$$w_n := \frac{u_n}{\|u_n\|_\infty}.$$

Then,

$$-\Delta w_n = t_n w_n(\lambda + b \int_\Omega u_n^r dx - u_n),$$

and then passing to the limit

$$-\Delta w = t_0 \lambda w,$$

for some $w \geq 0$, $\|w\|_\infty = 1$, $t_0 \in [0, 1]$. Then, $t_0 \lambda = \lambda_1$, a contradiction because $\lambda < \lambda_1$.

Taking now $\varepsilon \in (0, \delta]$, the homotopy defined by $\mathcal{H}_1$ is admissible and so,

$$\begin{aligned} i(\mathcal{K}_\lambda, 0) = & \deg(\mathcal{K}_\lambda, B_\varepsilon) = \deg(I - \mathcal{H}_1(1, \cdot), B_\varepsilon) = \deg(I - \mathcal{H}_1(0, \cdot), B_\varepsilon) = \\ = & \deg(I, B_\varepsilon) = 1. \end{aligned}$$

Lemma 2.4. *If $\lambda > \lambda_1$, then $i(\mathcal{K}_\lambda, 0) = 0$.*

Proof. Fix $\lambda > \lambda_1$ and $\phi \in X$, $\phi > 0$. First, it is clear that there exists $\varepsilon > 0$ such that

$$\lambda - \varepsilon > \lambda_1. \tag{2.3}$$

We define the map

$$\mathcal{H}_2 : [0, 1] \times X \mapsto X; \qquad \mathcal{H}_2(t, u) := (-\Delta)^{-1}(f(u) + t\phi).$$

We will show that there exists $\delta > 0$ such that $u \neq \mathcal{H}_2(t, u)$ for all $u \in \overline{B}_\delta \setminus \{0\}$, and $t \in [0, 1]$. Indeed, on the contrary it would exist sequences $u_n \in X \setminus \{0\}$ with $\|u_n\|_\infty \to 0$ and $t_n \in [0, 1]$ such that

$$u_n = \mathcal{H}_2(t_n, u_n).$$

Since $t_n\phi \geq 0$, we have that $u_n > 0$ and so

$$-\Delta u_n = u_n(\lambda + b \int_\Omega u_n^r dx - u_n) + t_n\phi > u_n(\lambda - \varepsilon) + t_n\phi \geq u_n(\lambda - \varepsilon),$$

hence, $\lambda_1 \geq \lambda - \varepsilon$, a contradiction with (2.3).

This proves that the homotopy defined by $\mathcal{H}_2$ is admissible. Then, if we take $\varepsilon \in (0, \delta]$ we have

$$i(\mathcal{K}_\lambda, 0) = \deg(\mathcal{K}_\lambda, B_\varepsilon) = \deg(I - \mathcal{H}_2(0, \cdot), B_\varepsilon) = \deg(I - \mathcal{H}_2(1, \cdot), B_\varepsilon) = 0.$$

Proof of Theorem 2.2: The fact that $\lambda = \lambda_1$ is a bifurcation point follows by Lemmas 2.3 and 2.4. Moreover, if there exists a sequence (λ_n, u_n) of positive solutions of (1.2) such that $\|u_n\|_\infty \to 0$, then, with a similar argument to the used in Lemma 2.3, we can easily conclude that $\lambda_n \to \lambda_1$. This proves that λ_1 is the only bifurcation point from the trivial solution. Hence, we can assure the existence of an unbounded continuum of solutions of (1.2), see [13].

Now, we study the bifurcation direction. Assume that $b \leq 0$, then

$$-\Delta u \leq \lambda u,$$

that is, $\lambda \geq \lambda_1$.

Assume now that $b > 0$, $r < 1$ and the existence of a sequence (λ_n, u_n) of positive solutions of (1.2) such that $\lambda_n \geq \lambda_1$ and $\|u_n\|_\infty \to 0$ as $n \to \infty$. Take $M > 0$ such that

$$1 - bM \int_\Omega \varphi_1 dx < 0.$$

For n large we have that $u_n^r > Mu_n$, and then

$$-\Delta u_n \geq u_n(\lambda_n + bM \int_\Omega u_n dx - u_n),$$

and so u_n is supersolution of

$$-\Delta u = u(\lambda_n + bM \int_\Omega u_n dx - u).$$

Using Proposition 2.1 and (2.2), we get

$$u_n \geq (\lambda_n + bM \int_\Omega u_n dx - \lambda_1)\varphi_1$$

and in consequence

$$(1 - bM \int_\Omega \varphi_1 dx) \int_\Omega u_n dx \geq (\lambda_n - \lambda_1) \int_\Omega \varphi_1 dx,$$

a contradiction.

Assume now that $b > 0$, $r > 1$ and the existence of a sequence (λ_n, u_n) of positive solutions of (1.2) such that $\lambda_n \leq \lambda_1$ and $\|u_n\|_\infty \to 0$ as $n \to \infty$. Take $\varepsilon > 0$ such that

$$1 - bK\varepsilon|\Omega| > 0,$$

where K is defined in (2.2). For n large we have that $u_n^r < \varepsilon u_n$, and then

$$-\Delta u_n \leq u_n(\lambda_n + b\varepsilon \int_\Omega u_n dx - u_n),$$

and so, using again (2.2)

$$u_n \leq (\lambda_n + b\varepsilon \int_\Omega u_n dx - \lambda_1)K$$

and in consequence

$$(1 - bK\varepsilon|\Omega|) \int_\Omega u_n dx \leq (\lambda_n - \lambda_1)K|\Omega|,$$

again a contradiction.

Finally, assume that $b > 0$ and $r = 1$. In this case, we apply the Crandall-Rabinowitz Theorem, [2]. Then, there exist $\varepsilon > 0$ and two regular functions $\lambda(s)$, $u(s)$, $s \in (-\varepsilon, \varepsilon)$, such that in a neighborhood of $(\lambda_1, 0)$ the unique positive solutions of (1.2) are $(\lambda(s), u(s))$, $s \in (0, \varepsilon)$. We can write

$$u(s) = s\varphi_1 + s^2\varphi_2 + o(s^2), \qquad \lambda(s) = \lambda_1 + s\lambda_2 + o(s),$$

where $\lambda_2 \in \mathbb{R}$, $\varphi_2 \in C^2(\overline{\Omega})$. It is evident that the sign of λ_2 determines the bifurcation direction. Substituting these expansions into (1.2) and identifying the terms of order one in s yields

$$-\Delta\varphi_2 - \lambda_1\varphi_2 = \lambda_2\varphi_1 - \varphi_1^2 + b\varphi_1 \int_\Omega \varphi_1 dx.$$

Multiplying by φ_1, we conclude that

$$\lambda_2 = \frac{\displaystyle\int_\Omega \varphi_1^3 dx - b\int_\Omega \varphi_1^2 dx \int_\Omega \varphi_1 dx}{\displaystyle\int_\Omega \varphi_1^2 dx}.$$

This finishes the proof. □

3. A priori bounds and non-existence results of (1.2)

In this section we obtain a priori bounds of the solutions for $b > 0$ as well as non-existence results of (1.2).

Proposition 3.1. *Assume that $b > 0$, $r < 1$. Let (λ, u_λ) be a positive solution of (1.2) such that $\lambda \in \mathcal{K} \subset \mathbb{R}$ a compact set. Then,*

$$\|u_\lambda\|_\infty \leq C \quad \textit{for a constant independent to } \lambda.$$

Moreover, if

$$\lambda \leq \underline{\lambda} := \left(\frac{1}{|\Omega| b r}\right)^{1/(r-1)} \left(1 - \frac{1}{r}\right),$$

(1.2) does not possess any positive solution.

Proof. Since u_λ is a positive solution of (1.2) we have, using Proposition 2.1, that

$$u_\lambda \leq \lambda + b\int_\Omega u_\lambda^r dx. \tag{3.1}$$

Using now that $\|u\|_r \leq |\Omega|^{(1-r)/r}\|u\|_1$, we have that

$$\int_\Omega u_\lambda dx - b|\Omega|^{2-r}\left(\int_\Omega u_\lambda dx\right)^r \leq \lambda|\Omega|. \tag{3.2}$$

From (3.2) we get that if $\lambda \in \mathcal{K}$, then $\int_\Omega u_\lambda dx \leq C$, and so by (3.1) we get that $\|u_\lambda\|_\infty \leq C$, where by C we denote different positive constants. On the other hand, the function

$$f(s) := As - Bs^q, \qquad A, B > 0,\ 0 < q < 1,\ s \geq 0, \tag{3.3}$$

has a minimum at $s = s_m := (A/(qB))^{1/(q-1)}$ and

$$f(s_m) = A^{q/(q-1)} \left(\frac{1}{Bq}\right)^{1/(q-1)} \left(1 - \frac{1}{q}\right).$$

Hence, if

$$\lambda|\Omega| \leq \left(\frac{1}{b|\Omega|^{2-r} r}\right)^{1/(r-1)} \left(1 - \frac{1}{r}\right)$$

then by (3.2), equation (1.2) does not have positive solution.

PROPOSITION 3.2. *Assume that $b > 0$, $r > 1$. Let (λ, u_λ) be a positive solution of (1.2) such that $\lambda \in \mathcal{K} \subset \mathbb{R}$ a compact set. Then,*

$$\|u_\lambda\|_\infty \leq C \quad \textit{for a constant independent of } \lambda.$$

Moreover, if

$$\lambda \geq \overline{\lambda} := \lambda_1 + b^{1/(1-r)} \left(\int_\Omega \varphi_1 dx \right)^{r/(1-r)} |\Omega| r^{r/(1-r)} (r-1).$$

(1.2) does not possess any positive solution.

PROOF. Using now the lower bound in Proposition 2.1 we get that

$$(\lambda - \lambda_1 + b \int_\Omega u_\lambda^r dx)\varphi_1 \leq u_\lambda$$

and then

$$(\lambda - \lambda_1 + b \int_\Omega u_\lambda^r dx) \int_\Omega \varphi_1 dx \leq \int_\Omega u_\lambda dx \leq |\Omega|^{(r-1)/r} \left(\int_\Omega u_\lambda^r dx \right)^{1/r},$$

and hence

$$b \int_\Omega \varphi_1 dx \int_\Omega u_\lambda^r dx - |\Omega|^{(r-1)/r} \left(\int_\Omega u_\lambda^r dx \right)^{1/r} \leq (\lambda_1 - \lambda) \int_\Omega \varphi_1 dx. \tag{3.4}$$

From (3.4) we get that if $\lambda \in \mathcal{K}$, then $\int_\Omega u_\lambda^r \leq C$, and hence by (3.1), $\|u_\lambda\|_\infty \leq C$. On the other hand, applying again the results of (3.3) with

$$A = b \int_\Omega \varphi_1, \quad B = |\Omega|^{(r-1)/r}, \quad q = 1/r,$$

we get that if

$$(\lambda_1 - \lambda) \int_\Omega \varphi_1 dx \leq b^{1/(1-r)} \left(\int_\Omega \varphi_1 dx \right)^{1/(1-r)} |\Omega| r^{r/(1-r)} (1-r)$$

then by (3.4), equation (1.2) does not have positive solution.

For the case $r = 1$, the bounds depend on the size of b.

PROPOSITION 3.3. *Assume that $b > 0$, $r = 1$. Assume that $b < 1/|\Omega|$ or $b \int_\Omega \varphi_1 dx > 1$, then there exists a priori bounds of the solution of (1.2). Moreover, if $b < 1/|\Omega|$ and $\lambda \leq 0$ or $b \int_\Omega \varphi_1 dx > 1$ and $\lambda \geq \lambda_1$, then (1.2) does not possess positive solution.*

PROOF. In this case, by (2.2) we get

$$(\lambda + b \int_\Omega u dx - \lambda_1)\varphi_1 \leq u \leq \lambda + b \int_\Omega u dx, \tag{3.5}$$

and so

$$(1 - b|\Omega|) \int_\Omega u dx \leq \lambda |\Omega|, \qquad \left(b \int_\Omega \varphi_1 dx - 1 \right) \int_\Omega u dx \leq (\lambda_1 - \lambda) \int_\Omega \varphi_1 dx. \tag{3.6}$$

From these inequalities we obtain the result.

4. Stability and uniqueness results

In this section we study the stability of a positive solution u of (1.2) when $b > 0$. In order to ascertain its stability we have to calculate the sign of the principal eigenvalue of the linearized problem around u, that is,

$$
(4.1) \qquad \begin{cases} -\Delta\xi + (2u - \lambda - b\displaystyle\int_\Omega u^r dx)\xi - bru\displaystyle\int_\Omega u^{r-1}\xi dx = \sigma\xi & \text{in } \Omega, \\ \xi = 0 & \text{on } \partial\Omega. \end{cases}
$$

This problem is a nonlocal and singular (when $r < 1$) eigenvalue problem which was analyzed in other papers (see [4] and Section 5 in [12]) and it is included in the general problem

$$
(4.2) \qquad \begin{cases} -\Delta\xi + m(x)\xi - a_1(x)\displaystyle\int_\Omega a_2(x)\xi dx = \sigma\xi & \text{in } \Omega, \\ \xi = 0 & \text{on } \partial\Omega, \end{cases}
$$

where $m, a_1 \in C^1(\overline{\Omega})$, $a_2 \in C(\Omega)$, $a_1, a_2 > 0$ and $a_2(x) \leq Kd(x, \partial\Omega)^{-\beta}$, $\beta < 1$, $K > 0$. The existence of a principal eigenvalue of (4.2), denoted by $\lambda_1(-\Delta + m; a_1; a_2)$, was proved. If no confusion arises, we write $\lambda_1(-\Delta + m)$ when a_1 or a_2 vanishes (observe that $\lambda_1(-\Delta + m)$ is the classical principal eigenvalue of a local eigenvalue problem).

In the following result we give a criteria to ascertain the sign of $\lambda_1(-\Delta + m; a_1; a_2)$. The proof, adaptation of the characterization theorem of the maximum principle established in Theorem 7.10 of [16], can be found in [4].

Proposition 4.1. (1) *Assume that there exists a positive function* $\overline{u} \in C^2(\Omega) \cap C_0^{1,\delta}(\overline{\Omega})$, $\delta \in (0, 1)$, *such that*

$$
-\Delta\overline{u} + m(x)\overline{u} - a_1(x)\int_\Omega a_2(x)\overline{u}dx > 0 \quad \textit{in } \Omega,
$$

(we say that $\overline{u}$ *is a supersolution of (4.2)). Then,*

$$
\lambda_1(-\Delta + m; a_1; a_2) > 0.
$$

(2) *Assume that there exists a positive function* $\underline{u} \in C^2(\Omega) \cap C_0^{1,\delta}(\overline{\Omega})$, $\delta \in (0, 1)$, *such that*

$$
-\Delta\underline{u} + m(x)\underline{u} - a_1(x)\int_\Omega a_2(x)\underline{u}dx < 0 \quad \textit{in } \Omega.
$$

(we say that $\underline{u}$ *is a subsolution of (4.2)). Then,*

$$
\lambda_1(-\Delta + m; a_1; a_2) < 0.
$$

In the following result, we show the sign of the principal eigenvalue in some specific cases.

Proposition 4.2. *Assume that* $b > 0$.

(1) *Assume* $r \leq 1$ *and* $\lambda > \lambda_1$. *There exists* $b_1 > 0$ *such that for* $0 < b < b_1$, *any positive solution is stable.*

(2) *Assume* $r \geq 1$ *and* $\lambda \leq 0$. *Then, any positive solution is unstable.*

PROOF. 1. Observe that in our case

$$m(x) = 2u_b - \lambda - b\int_\Omega u_b^r dx, \quad a_1 = bru_b, \quad a_2 = u_b^{r-1},$$

where u_b is a positive solution of (1.2). By the strong maximum principle u_b is strongly positive. Hence, there exist $0 < k_1 < k_2$ such that $k_1 d(x,\partial\Omega) \leq u_b \leq k_2 d(x,\partial\Omega)$, and then a_2 verifies the hypothesis $a_2(x) \leq K d(x,\partial\Omega)^{-\beta}$ for $\beta = 1 - r$.

On the other hand, since u_b is a positive solution of (1.2), then

$$\lambda_1(-\Delta + u_b - \lambda - b\int_\Omega u_b^r dx) = 0,$$

and so, by the monotonicity of the principal eigenvalue with respect to the zero order term, $\lambda_1(-\Delta + 2u_b - \lambda - b\int_\Omega u_b^r dx) > 0$. Consider $e_b > 0$ the unique positive solution of the linear equation

$$\begin{cases} -\Delta e_b + (2u_b - \lambda - b\int_\Omega u_b^r dx)e_b = ru_b & \text{in } \Omega, \\ e_b = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

Now, we apply Proposition 4.1. It is clear that e_b is a supersolution of (4.1) if

$$\frac{1}{b} > \int_\Omega u_b^{r-1} e_b dx. \tag{4.4}$$

We claim that

$$u_b \to \theta_\lambda \quad \text{in } C^2(\overline{\Omega}) \text{ as } b \to 0. \tag{4.5}$$

Observe that (4.5) implies that $e_b \to e_\lambda$ in $C^2(\overline{\Omega})$ as $b \to 0$, where e_λ is the unique positive solution of

$$-\Delta e_\lambda + (2\theta_\lambda - \lambda)e_\lambda = r\theta_\lambda \quad \text{in } \Omega, \quad e_\lambda = 0 \quad \text{on } \partial\Omega.$$

Hence, we conclude that (4.4) holds for b small, then u_b is stable.

We prove (4.5). Assume $r < 1$, then using (3.2) and (3.1) we get that

$$\|u_b\|_\infty \leq C(\lambda, b) \tag{4.6}$$

where C is a constant bounded when $b \to 0$. Hence,

$$b\int_\Omega u_b^r dx \to 0 \quad \text{as } b \to 0,$$

and we conclude (4.5).

Assume now that $r = 1$, in this case by (3.5) and (3.6) we conclude that

$$\|u_b\|_\infty \leq \frac{\lambda}{1 - b|\Omega|}. \tag{4.7}$$

We can repeat the above reasoning to conclude (4.5).

2. In a similar way, u_b is subsolution of (4.1) provided that

$$u_b \le br \int_\Omega u_b^r dx. \tag{4.8}$$

Since $u_b \le \lambda + b\int_\Omega u_b^r dx$, it follows that (4.8) holds for $\lambda \le 0$ and $r \ge 1$.

COROLLARY 4.3. *Assume $0 < b < b_1$, $r \le 1$ and $\lambda > \lambda_1$, where b_1 is from Proposition 4.2. Then, there exists a unique positive solution of (1.2).*

PROOF. We use the fixed point index in cones. Define

$$P := \{u \in X : u \ge 0 \quad \text{in } \Omega\}.$$

Assume that $r \le 1$ and $b < b_1$, then using (4.6) and (4.7) there exists R_1 independent of b such that

$$\|u\|_{L^\infty(\Omega)} \le R_1,$$

for all u positive solution of (1.2).

Finally, take $M > 0$ large enough and consider the operator $\mathcal{K} : X \mapsto X$ defined by

$$\mathcal{K}(u) := (-\Delta + M)^{-1}(u(\lambda + M - u + b\int_\Omega u^r dx)).$$

It is clear that $\mathcal{K}$ is a positive operator whose fixed points are nonnegative solutions of (1.2).

Hence, the fixed point index of $\mathcal{K}$ over $\mathcal{B}$ with respect to the cone P is well defined, where

$$\mathcal{B} := \{u \in P : \|u\|_{L^\infty(\Omega)} \le R_1 + 1\}.$$

Now, we are going to compute this index in some cases. We claim that if $\lambda > \lambda_1$ then

$(I.1)$ $i_P(\mathcal{K}, \mathcal{B}) = 1$;
$(I.2)$ $i_P(\mathcal{K}, 0) = 0$;
$(I.3)$ $i_P(\mathcal{K}, u_b) = 1$,

for any positive solution u_b of (1.2). Of course, we conclude the uniqueness of positive solution of (1.2).

(I.2) follows by a similar argument to the used in the proof of Lemma 2.4. Proposition 4.2 implies (I.3). Finally, we show (I.1). Consider the operator $\mathcal{H}_1 : [0, 1] \times X \mapsto X$ defined by

$$\mathcal{H}_1(t, u) := (-\Delta + M)^{-1}(u(\lambda + M - u + tb\int_\Omega u^r dx)).$$

By the a priori bounds, $\mathcal{H}_1$ has no fixed points on $\partial\mathcal{B}$ for $t \in [0, 1]$. Thus, it follows by homotopy invariance that

$$i_P(\mathcal{K}, \mathcal{B}) = \quad i_P(\mathcal{H}_1(1, \cdot), \mathcal{B}) = i_P(\mathcal{H}_1(0, \cdot), \mathcal{B}) = 1.$$

This last inequality follows because $u = \mathcal{H}_1(0, u)$ is equivalent to the classical equation (2.1), and for this equation it is well-known that the fixed point index is equal to one.

5. Proofs of Theorems 1.1, 1.2, 1.3 and 1.4

5.1. Proof of Theorem 1.1: By Theorem 2.2 we know the existence of an unbounded continuum of positive solutions bifurcating from the trivial solution at $\lambda = \lambda_1$. Since $-\Delta u \leq u(\lambda - u)$, we know that positive solutions do not exist for $\lambda \leq \lambda_1$ and that for any solution $u \leq \theta_\lambda$. Hence we conclude the existence of positive solution for $\lambda > \lambda_1$.

We show now the uniqueness. Assume that there exist two positive solutions $u \neq v$. If $\int_\Omega u^r dx = \int_\Omega v^r dx$ then we conclude easily that $u = v$. So, assume that for instance $\int_\Omega u^r dx > \int_\Omega v^r dx$. Then,

$$-\Delta u = u(\lambda + b\int_\Omega u^r dx - u) < u(\lambda + b\int_\Omega v^r dx - u),$$

and then by Proposition 2.1 we get $u < v$, a contradiction.

On the other hand, we have that

$$u \leq \lambda + b\int_\Omega u^r dx,$$

and then $u \leq \lambda$. So, as $b \to -\infty$ we get

$$\int_\Omega u^r dx \to 0.$$

Moreover as

$$(\lambda + b\int_\Omega u^r dx - \lambda_1)\varphi_1 \leq u \quad \text{and} \quad \lambda + b\int_\Omega u^r dx - \lambda_1 > 0$$

we conclude that $b\int_\Omega u^r dx \to \lambda_1 - \lambda$. This implies that $\|u\|_\infty \to 0$.

5.2. Proof of Theorem 1.2: Assume $b > 0$ and $r < 1$. Define

$$\lambda_* := \inf\{\lambda \in \mathbb{R} : (1.2) \quad \text{possesses at least a positive solution}\}.$$

We know by Theorem 2.2 and Proposition 3.1 that $-\infty < \lambda_* < \lambda_1$. We prove now that there exists positive solution for all $\lambda > \lambda_*$, for which we are going to use the sub-supersolution method, see for instance [3]. Indeed, take $\lambda > \lambda_*$, then there exists $\mu \in [\lambda_*, \lambda)$ such that (1.2) possesses at least a positive solution, denoted by u_μ. Now, it is clear that $(\underline{u}, \overline{u}) = (u_\mu, K)$ is a sub-supersolution of (1.2) for K large, specifically for K verifying

$$K - bK^r|\Omega| \geq \lambda.$$

Enlarging if necessary K such that $u_\mu \leq K$ we conclude the existence of a positive solution for λ.

Finally, take a sequence of positive solutions (λ_n, u_n) of (1.2) such that $\lambda_n \geq \lambda_*$ and $\lambda_n \to \lambda_*$. Thanks to the bounds of Proposition 3.1 we have that $u_n \to u_* \geq 0$, u_* a solution for $\lambda = \lambda_*$. Since $\lambda_* < \lambda_1$ and λ_1 is the unique bifurcation point from the trivial solution, we conclude that $u_* > 0$.

On the other hand, since u is bounded and

$$\lambda + b\int_\Omega u^r dx > \lambda_1,$$

and then taking $b \to 0$ we have that $\lambda \geq \lambda_1$, that is $\lim_{b\to 0}\lambda_*(b) = \lambda_1$.

Finally, we prove that $\lim_{b\to\infty}\lambda_*(b) = -\infty$, for that it suffices to show that for any $\lambda < \lambda_1$ there exists $b > 0$ large such that (1.2) possesses at least one positive solution. Fixed $\lambda < \lambda_1$ there exists $b > 0$ large enough (see (3.3)) such that the function $f(s) = s - s^r b\int_\Omega \varphi_1^r dx$ has a minimum s_m such that $f(s_m) < \lambda - \lambda_1$. Fixed such b, take $\varepsilon > 0$ such that $f(\varepsilon) < \lambda - \lambda_1$. Then, $(\underline{u}, \overline{u}) = (\varepsilon\varphi_1, K)$ is sub-supersolution of (1.2) for K large. Indeed, $\underline{u}$ is subsolution if

$$\varepsilon\varphi_1 - b\varepsilon^r \int_\Omega \varphi_1^r dx \leq \lambda - \lambda_1,$$

that is, taking into account that $\|\varphi_1\|_\infty = 1$, $f(\varepsilon) < \lambda - \lambda_1$.

5.3. Proof of Theorem 1.3: Assume that $b > 0$ and $r > 1$. Define now

$$\lambda^* := \sup\{\lambda \in \mathbb{R}; (1.2) \quad \text{possesses at least one positive solution}\}.$$

We know by Theorem 2.2 and Proposition 3.2 that $\lambda_1 < \lambda^* < +\infty$. We prove now that there exists positive solution for all $\lambda \in [\lambda_1, \lambda^*)$ and observe that for $\lambda \leq \lambda_1$ positive solutions exist. Indeed, take $\lambda < \lambda^*$, then there exists $\mu \in (\lambda, \lambda^*]$ such that (1.2) possesses at least one positive solution, denoted by u_μ. Now, it is clear that $(\underline{u}, \overline{u}) = (\varepsilon\varphi_1, u_\mu)$ is a sub-supersolution of (1.2) for $\varepsilon > 0$ small, specifically for ε verifying

$$\varepsilon - b\varepsilon^r \int_\Omega \varphi_1^r dx \leq \lambda - \lambda_1 \quad \text{and} \quad \varepsilon\varphi_1 \leq u_\mu.$$

Finally, taking a sequence of solutions (λ_n, u_n) with $\lambda_n \leq \lambda^*$, $\lambda_n \to \lambda^*$ and thanks to the bounds of Proposition 3.2, we have that $u_n \to u^* > 0$, where u^* is one positive solution for $\lambda = \lambda^*$.

Observe that since $\lambda_1 < \lambda^* \leq \overline{\lambda}$, where $\overline{\lambda}$ is defined in Proposition 3.2 and $\lim_{b\to\infty}\overline{\lambda}(b) = \lambda_1$ we conclude that

$$\lim_{b\to\infty}\lambda^*(b) = \lambda_1.$$

Finally, we prove that $\lim_{b\to 0}\lambda^*(b) = +\infty$, for that it suffices to show that for any $\lambda > \lambda_1$, there exists $b > 0$ small enough such that (1.2) possesses at least one positive solution. Let us fix $\lambda > \lambda_1$, take $\tilde{\Omega} \supset \Omega$ and consider $\tilde{\varphi}_1$ and $\tilde{\lambda}_1$ the positive eigenfunction and eigenvalue associated to $\tilde{\Omega}$. Consider the function

$$g(s) := s(\tilde{\varphi}_1)_L - bs^r \int_\Omega \varphi_1^r dx,$$

where $(\tilde{\varphi}_1)_L := \min_{x\in\overline{\Omega}} \varphi_1(x)$. This function attains a maximum at

$$s = s_M = ((\tilde{\varphi}_1)_L/(b\int_\Omega \varphi_1^r dx))^{1/(r-1)}$$

and

$$g(s_M) = (\tilde{\varphi}_1)_L^{r/(r-1)} \left(\frac{1}{b\int_\Omega \varphi_1^r dx}\right)^{1/(r-1)} (1 - \frac{1}{r}).$$

Hence for b small enough we get

$$g(s_M) > \lambda - \tilde{\lambda}_1.$$

Take $K > 0$ such that $g(K) > \lambda - \tilde{\lambda}_1$. Fixing such b and K, we have that $(\underline{u}, \overline{u}) = (\varepsilon\varphi_1, K\tilde{\varphi}_1)$ is a sub-supersolution of (1.2) for ε small.

5.4. Proof of Theorem 1.4: By Theorem 2.2 there exists an unbounded continuum of positive solutions bifurcating from the trivial solution at $\lambda = \lambda_1$.

Assume $b < 1/|\Omega|$, then by Proposition 3.3 there does not exist positive solution for $\lambda \leq 0$ and the positive solutions are bounded. Hence the existence of positive solution for $\lambda > \lambda_1$ is obtained.

Assume now that $b > 1/\int_\Omega \varphi_1 dx$. In this case by Proposition 3.3 we know that for $\lambda \geq \lambda_1$ there does not exist positive solution and that the positive solutions are bounded for $\lambda \leq \lambda_1$. Hence the existence of positive solution for $\lambda < \lambda_1$ is obtained.

6. The parabolic problem

Consider now the time dependent problem (1.1). The existence and uniqueness of the local positive solution follows by classical theory, see for instance Example 51.13 in [19]. Moreover, the solution can be extended in time if the L^∞-norm remains finite.

First, we show that in the case $b < 0$ the solution is global and bounded.

LEMMA 6.1. *Assume $b \leq 0$. Then, the positive solution u of (1.1) is global in time and bounded. Moreover, if $\lambda < \lambda_1$ we get $\|u(x,t)\|_\infty \to 0$ as $t \to \infty$.*

PROOF. If $b < 0$, the solution u of (1.1) is a sub-solution of the local logistic equation

$$U_t - \Delta U = U(\lambda - U), \quad U(x,0) = u_0(x).$$

It is well known, see for instance [20], that the above equation is global and bounded and that $u \leq U$. Finally, $\|U(x,t)\|_\infty \to 0$ as $t \to \infty$ for $\lambda < \lambda_1$ and this completes the proof.

Now, we consider the case $b > 0$. In this case, thanks to the maximum principle (see again [20] or [22]) we can assume that $u_0(x) > 0$ for $x \in \Omega$ and $u_0(x) = 0$ on $\partial\Omega$.

Theorem 6.2. *(Global existence results.) Assume $b > 0$.*

(1) *If $r < 1$, the solution exists globally in time $\forall \lambda \in \mathbb{R}$.*
(2) *If $r = 1$ and $b|\Omega| < 1$, the solution exists globally in time for all $\lambda \in \mathbb{R}$.*
(3) *If $r = 1$ and $b|\Omega| \geq 1$, the solution exists globally in time for all $\lambda < 0$ if*

$$u_0(x) \leq \frac{\lambda}{1 - b|\Omega|} \ , \forall x \in \Omega.$$

(4) *Assume $r = 1$. Let e be the unique positive solution of*

$$\left\{ \begin{array}{ll} -\Delta e = 1 & \text{in } \Omega, \\ e = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{6.1}$$

Then, there exists a small number $a_1 > 0$ such that if $u_0(x) \leq a_1 e(x)$ $x \in \Omega$, the solution exists globally in time $\forall \lambda \in \left(-\infty, \dfrac{1}{\max_{x \in \bar{\Omega}} e(x)} \right)$.

(5) *Assume $r > 1$, then, there exists $a_2 > 0$ (which can be computed explicitly) such that the solution exists globally in time, for all*

$$\lambda \in \left(-\infty, \left(\frac{1}{b|\Omega| r} \right)^{\frac{1}{r-1}} \frac{r-1}{r} \right)$$

provided that $u_0(x) \leq a_2$.

Proof. For the first three paragraphs, use $\overline{u}(x,t) = M$ as super-solution, where M is a positive constant. Indeed, $\overline{u}$ is super-solution of (1.1) if

$$M \geq \lambda + b|\Omega| M^r, \quad M \geq u_0(x).$$

For 5., observe that the function $g(M) = \lambda + b|\Omega| M^r - M$, goes to $+\infty$ as $M \to +\infty$ if $r > 1$ and attains a minimum at $M_m = \left(\dfrac{1}{b|\Omega| r} \right)^{1/(r-1)}$. It is enough to impose that $g(M_m) \leq 0$ and a_2 will be defined by $g(a_2) = 0$. Finally, for (4), take $\overline{u}(x,t) = a_1 e(x)$. It is clear that $\overline{u}$ is super-solution if

$$1 > \lambda e + a_1 e \left(b \int_\Omega e(x)\, dx - e \right), \quad a_1 e(x) \geq u_0(x).$$

If $1 > \lambda e$ we can take a_1 small.

The next result studies the case when the solution goes to zero:

Proposition 6.3. *Assume $b > 0$.*

(1) *If $r > 0$, the trivial solution is locally exponentially stable if $\lambda < \lambda_1$.*
(2) *If $r < 1$, there exists $\underline{\lambda}$ such that for all $\lambda < \underline{\lambda}$ and initial datum u_0, we have that $\|u(x,t)\|_\infty \to 0$ as $t \to \infty$.*

(3) *If* $r = 1$ *and* b *small, then for* $\lambda < \lambda_1$ *and for all initial datum* u_0*, we have that* $\|u(x,t)\|_\infty \to 0$ *as* $t \to \infty$.

(4) *If* $r > 1$*, then for all* u_0 *there exists* $\underline{\lambda}(u_0)$ *such that for* $\lambda < \underline{\lambda}(u_0)$*, we have that* $\|u(x,t)\|_\infty \to 0$ *as* $t \to \infty$.

PROOF. First, take a domain $\Omega_1 \supset \Omega$ such that, if necessary,

$$\lambda < \mu_1 < \lambda_1 \tag{6.2}$$

where μ_1 is the principal eigenvalue associated to $-\Delta$ in Ω_1 and denote by ψ_1 the positive eigenfunction associated to μ_1 such that $\|\psi_1\|_\infty = 1$.

In all the cases, we take $\overline{u}(x,t) = Me^{-\sigma t}\psi_1$ as supersolution, with $M > 0$ and $\sigma > 0$ to be chosen. It is clear that $\overline{u}$ is supersolution of (1.1) if

$$M\psi_1(x) \geq u_0(x) \quad x \in \Omega \quad \text{and} \quad -\sigma + \mu_1 - \lambda \geq -Me^{-\sigma t}R + bM^r e^{-r\sigma t}B, \tag{6.3}$$

where $R := \min_{x\in\overline{\Omega}} \psi_1(x)$ and $B := \int_\Omega \psi_1^r dx$.

For the first paragraph, take M small, and then it suffices to take $0 < \sigma < \mu_1 - \lambda$ which is possible thanks to (6.2). For the second one ($r < 1$), observe that

$$-Me^{-\sigma t}R + bM^r e^{-r\sigma t}B \leq C$$

for some positive constant C independent of t and M. It suffices to take λ negative. When $r = 1$, then

$$-Me^{-\sigma t}R + bMe^{-\sigma t}B = Me^{-\sigma t}(-R + bB) < 0$$

for b small. Fixing this value of b, take $\lambda < \lambda_1$ and $\sigma > 0$.

For the last paragraph ($r > 1$), for a given u_0 take M such that $u_0 \leq M\psi_1$. Fixed such M, take λ small such that (6.3) is verified.

THEOREM 6.4. *(Blow-up in finite time.) Assume* $b > 0$.

(1) *Assume* $r = 1$ *and define*

$$A := \int_\Omega \varphi_1 dx.$$

If $bA = 1$ *and* $\lambda > \lambda_1$ *the solution* $\|u(x,t)\|_\infty$ *goes to* ∞ *as* $t \to \infty$*. In the case* $bA > 1$ *the solution blows up in finite time for* $\lambda > \lambda_1$ *or for any* λ *if* u_0 *is large enough.*

(2) *Assume* $r > 1$*. Then, there exists* $\overline{\lambda}$ *such that for* $\lambda > \overline{\lambda}$ *the solution blows up in finite time for any* u_0.

(3) *Assume* $r > 1$*. Then, there exists* $b_2 > 0$ *such that the solution blows up in finite time if* $u_0(x) \geq b_2\varphi_1(x)$.

PROOF. Take $\underline{u}(x,t) = q(t)\varphi_1(x)$ with $q(t)$ and $q(0) > 0$ to be chosen. Observe that $\underline{u}$ is sub-solution of (1.1) if

$$q'(t) \leq (\lambda - \lambda_1)q - q^2\varphi_1 + bq^{r+1}B \quad \text{and} \quad q(0)\varphi_1(x) \leq u_0(x),$$

with $B := \int_\Omega \varphi_1^r dx$. Since $\|\varphi_1\|_\infty = 1$, we can take q such that

$$q'(t) = (\lambda - \lambda_1)q - q^2 + bq^{r+1}B.$$

If $r = 1$ the results follow easily. Indeed, in this case the above equation can be written as

$$q'(t) = (\lambda - \lambda_1)q + q^2(-1 + bA).$$

This proves first paragraph.

Assume that $r > 1$. It can be proved that for $1 < p < r + 1$, there exists $\mu \in \mathbb{R}$ such that

$$(\lambda - \lambda_1)q - q^2 + bq^{r+1}B \geq \mu q + q^p.$$

Indeed, this is equivalent to $\lambda - \lambda_1 - \mu \geq q - bBq^r + q^{p-1}$, and observe that the function $h(q) = q - bq^r + q^{p-1}$ is bounded.

Taking $\mu = 0$, the above inequality for λ large, and hence $q' \geq q^p$ and so q blows up in finite time. This completes second paragraph.

For the third paragraph, we take $\mu < 0$ with $|\mu|$ large, and hence in this case $q' \geq \mu q + q^p$. In this case, q blows-up in finite time for $q(0) > 0$ large, that is, for u_0 large.

Remark 6.5. (1) Remember that for $r \leq 1$ and b small the steady-state problem (1.2) has a unique positive solution. Then, using arguments of [20] (see for instance Theorem 5.4.4) the solution of (1.1) converges to the unique positive solution of (1.2).

(2) The blow-up in finite time of problem (1.3) was studied in [15]. In order to compare the results of [15] with ours, let us assume that $r = 1$ and fix the function a. In [15] it was proved that that there exists a value $\lambda^* > 0$ (related with some eigenvalue problem associated to a^+) such that:

(a) If $\lambda_1 < \lambda < \lambda^*$, then the solution of (1.3) blows-up if

$$b > A(\lambda, u_0), \tag{6.4}$$

for some specific positive constant A depending on λ and u_0. Moreover, the maps $\lambda \mapsto A(\lambda, u_0)$ and $u_0 \mapsto A(\lambda, u_0)$ are decreasing.

(b) If $\lambda \geq \lambda^*$, the solution of (1.3) blows-up for any u_0.

Hence, as consequence, the solution of (1.3) blows-up for any b if λ is large or any λ and u_0 large. However, in our results we need to impose that b is large to obtain that the solution of (1.1) blows-up.

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