

THE ENDPOINT FEFFERMAN-STEIN INEQUALITY FOR THE STRONG MAXIMAL FUNCTION

TERESA LUQUE AND IOANNIS PARISSIS

ABSTRACT. Let $M_n f$ denote the strong maximal function of f on \mathbb{R}^n , that is the maximal average of f with respect to n -dimensional rectangles with sides parallel to the coordinate axes. For any dimension $n \geq 2$ we prove the natural endpoint Fefferman-Stein inequality for M_n and any strong Muckenhoupt weight w :

$$w(\{x \in \mathbb{R}^n : M_n f(x) > \lambda\}) \lesssim_{w,n} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \left(\log^+ \frac{|f(x)|}{\lambda}\right)^{n-1}\right) M_n w(x) dx.$$

This extends the corresponding two-dimensional result of T. Mitsis.

1. INTRODUCTION

The strong maximal function. Let \mathfrak{R}_n denote the family of all rectangles in \mathbb{R}^n with sides parallel to the coordinate axes. For a locally integrable function f on \mathbb{R}^n we will denote by $M_n f$ the *strong maximal function*:

$$M_n f(x) := \sup_{\substack{R \in \mathfrak{R}_n \\ R \ni x}} \frac{1}{|R|} \int_R |f(y)| dy, \quad x \in \mathbb{R}^n.$$

Here $|S|$ denotes the n -dimensional Lebesgue measure of a set $S \subset \mathbb{R}^n$. We will sometimes use the same notation for the $(n-1)$ -dimensional Lebesgue measure, but this will be clear from the context.

The endpoint behavior of M_n close to L^1 is given by the classical theorem of Marcinkiewicz, Jessen and Zygmund, [8]:

$$(1.1) \quad |\{x \in \mathbb{R}^n : M_n f(x) > \lambda\}| \lesssim_n \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \left(\log^+ \frac{|f(x)|}{\lambda}\right)^{n-1}\right) dx,$$

where $\log^+ t := \max(0, \log t)$. Inequality (1.1) implies the strong differentiation of the integral of all functions $f \in L(1 + (\log^+ L)^{n-1})(\mathbb{R}^n)$, that is, all functions f on \mathbb{R}^n such that:

$$\int_{\mathbb{R}^n} |f(x)| \left(1 + \left(\log^+ |f(x)|\right)^{n-1}\right) dx < +\infty.$$

The strong maximal theorem cannot be improved, that is, the function $t(1 + (\log^+ t)^{n-1})$ on the right hand side of (1.1) cannot be replaced by any slower increasing function. Remember

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that the usual Hardy-Littlewood maximal function is the maximal average of f with respect to all n -dimensional Euclidean cubes, or balls and it maps $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. The important difference to be noted here is that the strong maximal function is an n -parameter maximal average, in contrast to the usual one-parameter Hardy-Littlewood maximal function, and this difference is reflected in the strong maximal theorem (1.1) which requires extra logarithmic scales of integrability. The original proof from [8] relies on the observation that M_n can be viewed as a composition of n one-dimensional Hardy-Littlewood maximal operators. One then appeals to the one-dimensional theory to get (1.1) together with its strong $L^p(\mathbb{R}^n)$ counterparts. A more geometric point of view was introduced by the work of Córdoba and R. Fefferman, [3], who gave a proof of (1.1) by means of a geometric covering argument. This is in a sense a dual point of view where the n -parameter composition of operators is replaced by induction on the dimension. The importance of the Córdoba-Fefferman geometric proof of the strong maximal theorem is highlighted by the fact that the usual Besicovitch covering argument fails when applied to families of rectangles having arbitrary eccentricities.

Strong weights. A *weight* w will be a locally integrable, non-negative function on \mathbb{R}^n . We will say that w belongs to the class A_p^* , $1 < p < \infty$, whenever

$$[w]_{A_p^*} := \sup_{R \in \mathfrak{R}_n} \left(\frac{1}{|R|} \int_R w \right) \left(\frac{1}{|R|} \int_R w^{1-p'} \right)^{p-1} < +\infty.$$

In this case we will say that w is a *strong A_p -weight*. For $p = 1$ the class A_1^* is defined by the condition

$$\frac{1}{|R|} \int_R w \leq C \operatorname{ess\,inf}_{x \in R} w(x), \quad \text{for almost every } x \in R, \quad R \in \mathfrak{R}_n,$$

which is equivalent to saying that $M_n w \leq Cw$ almost everywhere in \mathbb{R}^n . The smallest constant $C > 0$ in the previous inequality is the A_1^* -constant of the weight, denoted by $[w]_{A_1^*}$.

It follows by Hölder's inequality that the A_p^* classes are increasing, that is, for $1 \leq p \leq q < \infty$ we have $A_p^* \subset A_q^*$. We define the class A_∞^* by means of

$$A_\infty^* := \bigcup_{p>1} A_p^*.$$

It is equivalent to define the class A_∞^* by the following property: there exist constants $\delta, c > 0$ such that, given any rectangle $R \in \mathfrak{R}_n$ and a measurable subset $S \subset R$, then

$$(1.2) \quad \frac{w(S)}{w(R)} \leq c \left(\frac{|S|}{|R|} \right)^\delta.$$

An important feature of strong A_∞ -weights is that if we fix any $t \in \mathbb{R}$ then the weight

$$w^t(x') := w(x', t), \quad x' \in \mathbb{R}^{n-1},$$

is an A_∞^* -weight on \mathbb{R}^{n-1} , *uniformly* in $t \in \mathbb{R}$. In practice, uniformly means that all the constants connected with the properties of the A_∞^* -weight w^t can be taken to be independent of t . For these and other properties of strong Muckenhoupt weights see for example [2] or [6, Chapter IV].

Remark 1.3. Let $w \in A_\infty^*$. By the previous discussion we see that there exists some $\epsilon = \epsilon(w) > 0$ such that, for every rectangle $R \in \mathfrak{R}_n$ and all measurable sets $F \subset \mathbb{R}^n$, we have

$$|R \cap F| \leq \epsilon |R| \Rightarrow w(R \cap F) \leq \frac{1}{2} w(R) \Rightarrow w(R \setminus F) \geq \frac{1}{2} w(R).$$

In fact, it suffices to choose $\epsilon > 0$ so that $c\epsilon^\delta \leq \frac{1}{2}$, where c, δ are the constants associated to $w \in A_\infty^*$ from (1.2). Since for any $t \in \mathbb{R}$ the weight $w^t := w(\cdot, t)$ is an A_∞^* -weight on \mathbb{R}^{n-1} , uniformly in t , the $\epsilon > 0$ can be chosen sufficiently small so that we have the previous property also for w^t , rectangles $R' \in \mathfrak{R}_{n-1}$ and sets $F' \subset \mathbb{R}^{n-1}$, uniformly in t . We will use this remark several times in what follows.

It is known that M_n is bounded on $L^p(w)$, $1 < p < \infty$ if and only if $w \in A_p^*$. This result follows again by an appeal to the one dimensional theory. For the necessity of the A_p^* condition, one argues as in the case of the usual A_p weights. The corresponding endpoint bound is also true: the operator M_n satisfies the distributional estimate

$$w(\{x \in \mathbb{R}^n : M_n f(x) > \lambda\}) \lesssim_{w,n} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \left(\log^+ \frac{|f(x)|}{\lambda}\right)^{n-1}\right) w(x) dx$$

whenever $w \in A_1^*$. For these results see [2, Theorems 2.1 and 2.3].

Weighted strong maximal function. For $w \in A_\infty^*$ we will also consider the weighted strong maximal function M_n^w , defined with respect to w :

$$M_n^w f(x) := \sup_{\substack{R \in \mathfrak{R}_n \\ R \ni x}} \frac{1}{w(R)} \int_R |f(y)| w(y) dy.$$

For the weighted strong maximal function M_n^w , R. Fefferman showed in [5] that it maps $L^p(w)$ to $L^p(w)$ whenever $w \in A_\infty^*$:

$$(1.4) \quad \|M_n^w f\|_{L^p(w)} \leq_{w,n} c_{p,n} \|f\|_{L^p(w)}, \quad 1 < p \leq \infty.$$

Furthermore we have the asymptotic estimate

$$(1.5) \quad c_{p,n} = O_n((p-1)^{-n}) \quad \text{as } p \rightarrow 1^+.$$

The behavior of the constants is not explicitly studied in [5] but follows by a close examination of the proof and the standard norm bounds from Marcinkiewicz interpolation. See also [10, Lemma 4]. The endpoint bound for M_n^w is also true, namely M_n^w satisfies

$$(1.6) \quad w(\{x \in \mathbb{R}^n : M_n^w f(x) > \lambda\}) \lesssim_{w,n} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \left(\log^+ \frac{|f(x)|}{\lambda}\right)^{n-1}\right) w(x) dx$$

whenever $w \in A_\infty^*$. This was proved by Jawerth and Torchinsky, [7], and independently by Long and Shen [10].

Fefferman-Stein inequality. By this we mean in general an inequality of the form

$$\int_{\mathbb{R}^n} (\mathcal{M}f)^p w \lesssim_{w,n} \int_{\mathbb{R}^n} |f|^p \mathcal{M}w dx, \quad 1 < p < \infty,$$

where \mathcal{M} denotes some maximal operator. Inequalities of this type are important since, among other things, they can be used to derive the boundedness of vector-valued maximal operators. In fact this inequality was first proved for the Hardy-Littlewood maximal function by C. Fefferman and Stein, [4], for every non-negative, locally integrable weight w . The main application in [4] was exactly the vector-valued extension of the classical Hardy-Littlewood maximal theorem. For the strong maximal function the same inequality is true provided that $w \in A_\infty^*$; see [9] for a direct proof of this result and also [12], where the Fefferman-Stein inequality is obtained as a corollary of a more general two weight-norm inequality. Observe that, as in the case of (1.4), we need some extra assumption on the weight in order to prove the Fefferman-Stein inequality for the strong maximal function. This should be contrasted to the corresponding result for the Hardy-Littlewood weighted maximal function, as well as to the Fefferman-Stein inequality for the Hardy-Littlewood maximal function, where no assumption on the weight is needed.

The form of the endpoint Fefferman-Stein inequality depends on the corresponding unweighted endpoint properties of the maximal operator under study. For the usual Hardy-Littlewood maximal function M_Q the right statement is

$$w(\{x \in \mathbb{R}^n : M_Q f(x) > \lambda\}) \lesssim_n \frac{1}{\lambda} \int_{\mathbb{R}^n} |f(x)| M_Q w(x) dx.$$

The natural endpoint Fefferman-Stein inequality for the strong maximal function was proved by Mitsis, [11], in dimension $n = 2$. In particular Mitsis showed that

$$w(\{x \in \mathbb{R}^2 : M_n f(x) > \lambda\}) \lesssim_w \int_{\mathbb{R}^2} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \frac{|f(x)|}{\lambda}\right) M_n w(x) dx.$$

The main result of the current paper is the extension of the endpoint Fefferman-Stein inequality for the strong maximal function to all dimensions:

Theorem. *Let $w \in A_\infty^*$. For all dimensions $n \geq 1$ we have*

$$w(\{x \in \mathbb{R}^n : M_n f(x) > \lambda\}) \lesssim_{n,w} \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \left(\log^+ \frac{|f(x)|}{\lambda}\right)^{n-1}\right) M_n w(x) dx.$$

By interpolation, the Fefferman-Stein inequality of our main theorem above implies the strong L^p -version of the Fefferman-Stein inequality from [9], [12]. Furthermore, since every A_1^* -weight is an A_∞^* -weight, we recover the endpoint inequality (1.6) for A_1^* -weights.

It should be noted that the proof of Mitsis in [11] uses the combinatorics of two-dimensional rectangles, which allow one to get favorable estimates for the measures

$$\{x \in \mathbb{R}^k : \sum_{k=1}^N \mathbf{1}_{R_k}(x) = \ell\};$$

here $\{R_k\}_{1 \leq k \leq N}$ is a sequence of rectangles which satisfy a certain sparseness property and ℓ is any integer in $\{1, 2, \dots, N\}$. These combinatorics do not seem to be readily available in higher dimensions and so we adopt a different approach, which relies on the boundedness of the weighted strong maximal function M_n^w and the precise estimate for its norm, (1.5). In particular, our approach is inspired by the arguments in [10], a paper which seems to have been overlooked by most of the works on the weighted inequalities for the strong maximal function.

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2. NOTATION

We write $A \lesssim B$ if $A \leq CB$ for some numerical constant $C > 0$. In order to indicate the dependence of the constant on some parameter n (say), we write $A \lesssim_n B$. Similarly, $A \simeq B$ means that $A \lesssim B$ and $B \lesssim A$.

3. SOME GEOMETRY OF n -DIMENSIONAL RECTANGLES

In this section we recall some sparseness properties of n -dimensional rectangles, introduced in [3]. Here we adopt the slightly different approach from [10]. In fact, both Lemmas in this section are mentioned in [10]. However, we present the proofs for the sake of completeness.

For $t \in \mathbb{R}$ and $E \subset \mathbb{R}^n$ we introduce the slice operator

$$P_t(E) := \{x' \in \mathbb{R}^{n-1} : (x', t) \in E\}.$$

Thus P_t is the ‘slice’ of E by a hyperplane perpendicular to the n -th coordinate axis at level $t \in \mathbb{R}$. The $(n - 1)$ -dimensional projection is

$$P_{\parallel}(E) := \{x' \in \mathbb{R}^{n-1} : (x', t) \in E \text{ for some } t \in \mathbb{R}\}.$$

We will also use the one-dimensional projection P^{\perp} defined for $E \subset \mathbb{R}^n$ as

$$P^{\perp}(E) := \{t \in \mathbb{R} : (x', t) \in E \text{ for some } x' \in \mathbb{R}^{n-1}\}.$$

If $R \in \mathfrak{R}_n$ observe that we have

$$R = P_{\parallel}(R) \times P^{\perp}(R) = P_t(R) \times P^{\perp}(R), \quad \text{for all } t \in P^{\perp}(R).$$

For any interval $I \subset \mathbb{R}$, let I^* be the interval with the same center and three times the length of I , $|I^*| = 3|I|$. For $R \in \mathfrak{R}_n$ we then use the notation

$$R^* := P_{\parallel}(R) \times (P^{\perp}(R))^*.$$

Thus R^* is the rectangle with the same center as R and whose sides parallel to the first $n - 1$ coordinate axes have the same lengths as the corresponding sides of R ; the side of R which is parallel to the n -th coordinate axis has length equal to three times the length of the corresponding side of R .

Let $\mathcal{R} = \{R_k\}_{1 \leq k \leq N}$ be a finite sequence of rectangles from \mathfrak{R}_n . We will say that \mathcal{R} satisfies the sparseness property (P_2) if

$$(P_2) \quad \begin{cases} P^\perp(R_1) \geq P^\perp(R_2) \geq \dots \geq P^\perp(R_N), \\ |R_k \cap \bigcup_{j < k} R_j^*| \leq \epsilon |R_k|, \quad k = 1, 2, \dots, N. \end{cases}$$

Here $0 < \epsilon < 1$ will be assumed sufficiently small in various parts of the arguments below.

For $t \in \mathbb{R}$ we now consider the collection $\mathcal{T}(t) = \mathcal{T} = \{P_t(R_k)\}_{1 \leq k \leq N} \subset \mathfrak{R}_{n-1}$ which is produced by slicing all the n -dimensional rectangles of \mathcal{R} by a hyperplane perpendicular to the n -th coordinate axis, at the level t . The collection \mathcal{T} depends on t but we will many times suppress this fact in what follows. The main point about the collections \mathcal{R} and \mathcal{T} is contained in the following standard fact.

Lemma 3.1. *Suppose that the sequence $\mathcal{R} = \{R_k\}_{1 \leq k \leq N}$ has the sparseness property (P_2) . Then, for all $t \in \mathbb{R}$, the $(n-1)$ -dimensional collection of rectangles $\mathcal{T}(t) = \{P_t(R_k)\}_{1 \leq k \leq N}$ has the sparseness property (P_1) , uniformly in t :*

$$(P_1) \quad |P_t(R_k) \cap \bigcup_{j < k} P_t(R_j)| \leq \epsilon |P_t(R_k)|, \quad k = 1, 2, \dots, N.$$

Proof. We fix some $1 \leq k \leq N$ and $t \in P^\perp(R_k)$. Denoting $J := \{j < k : P_t(R_k) \cap P_t(R_j) \neq \emptyset\}$ we have by the second condition in (P_2) that

$$(3.2) \quad \epsilon |R_k| \geq |R_k \cap \bigcup_{j < k} R_j^*| = \left| \bigcup_{j < k} (R_k \cap R_j^*) \right| \geq \left| \bigcup_{j \in J} (R_k \cap R_j^*) \right|$$

Observe that for $j \in J$ we have that $\emptyset \neq P^\perp(R_k) \cap P^\perp(R_j) \ni t$ and by the first condition in (P_2) we have $|P^\perp(R_j)| \geq |P^\perp(R_k)|$. A moment's reflection shows that if I_1, I_2 are intervals in \mathbb{R} , $|I_2| \geq |I_1|$ and $I_1 \cap I_2 \neq \emptyset$ then $I_1 \subseteq I_2^*$. We conclude that $P^\perp(R_k) \subseteq P^\perp(R_j^*)$. Thus the n -dimensional rectangle $R_k \cap R_j^*$ is of the form $P^\perp(R_k) \times P_\parallel(R_k \cap R_j^*)$. However, $j \in J$ implies that $P_t(R_k \cap R_j) = P_t(R_k) \cap P_t(R_j) \neq \emptyset$, so we conclude that $P_\parallel(R_k \cap R_j^*) = P_t(R_k \cap R_j)$ and thus

$$(3.3) \quad R_k \cap R_j^* = P^\perp(R_k) \times P_t(R_k \cap R_j).$$

Now estimate (3.2) and identity (3.3) give

$$\begin{aligned} \epsilon |P^\perp(R_k)| \times |P_t(R_k)| &= \epsilon |R_k| \geq \left| \bigcup_{j \in J} P^\perp(R_k) \times P_t(R_k \cap R_j) \right| \\ &= |P^\perp(R_k)| \times \left| \bigcup_{j \in J} P_t(R_k \cap R_j) \right| \\ &= |P^\perp(R_k)| \times \left| P_t(R_k) \cap \bigcup_{j \in J} P_t(R_j) \right| \\ &= |P^\perp(R_k)| \times \left| P_t(R_k) \cap \bigcup_{j < k} P_t(R_j) \right|. \end{aligned}$$

This proves the lemma for $t \in P^\perp(R_k)$ while for $t \notin P^\perp(R_k)$ the conclusion follows trivially. \square

The next lemma gives a precise quantitative bound on the overlap of the rectangles in \mathcal{R} under the sparseness property (P_2) .

Lemma 3.4. *Let $w \in A_\infty^*$ and suppose that the finite sequence $\mathcal{R} = \{R_k\}_{1 \leq k \leq N} \subset \mathfrak{R}_n$ satisfies property (P_2) with ϵ sufficiently small, depending on the weight w . We set $\Omega := \cup_{k=1}^N R_k$. For $1 < p < \infty$ we have*

$$\left(\int_{\Omega} \left| \sum_{k=1}^N \mathbf{1}_{R_k} \right|^p w(x) dx \right)^{\frac{1}{p}} \lesssim_{w,n} c_{p,n} w(\Omega)^{\frac{1}{p}}$$

with $c_{p,n} = O_n(p^{n-1})$ as $p \rightarrow +\infty$.

Proof. For a sequence $\{R_k\}_{1 \leq k \leq N}$ as before, consider the sequence $\mathcal{T}(t)$ of $(n-1)$ -dimensional rectangles, by slicing the collection \mathcal{R} with a hyperplane perpendicular to the n -th coordinate axis, at level $t \in \mathbb{R}$. Let $\Omega_t := P_t(\Omega)$ denote the corresponding slice of Ω at level t and set $T_k := P_t(R_k)$ in order to simplify the notation. By Lemma 3.1 the collection $\mathcal{T}(t) = \{T_k\}_{1 \leq k \leq N}$ has the property (P_1) . We set $E_k := T_k \setminus \cup_{j < k} T_j$. For fixed $t \in \mathbb{R}$, the function $w^t(x') = w(x', t)$, $x' \in \mathbb{R}^{n-1}$, is an A_∞^* -weight in \mathbb{R}^{n-1} , uniformly in $t \in \mathbb{R}$; see [5]. By the property (P_1) and the fact that $w^t \in A_\infty^*$ uniformly in t , we will have that $w^t(T_k) \geq w^t(E_k) \geq \frac{1}{2} w^t(T_k)$ if $\epsilon > 0$ was selected sufficiently small in property (P_2) , and thus also in (P_1) , according to Remark 1.3.

Define the linear operator

$$L_{w^t} f(x') := \sum_{k=1}^N \frac{1}{w^t(T_k)} \left(\int_{T_k} f(y') w^t(y') dy' \right) \mathbf{1}_{E_k}(x'), \quad x' \in \mathbb{R}^{n-1}.$$

For any locally integrable function f on \mathbb{R}^{n-1} we have that $L_{w^t} f(x') \leq M_{w^t} f(x')$, $x' \in \mathbb{R}^{n-1}$. Also observe that for f, g locally integrable we have

$$\begin{aligned} \int_{\Omega_t} L_{w^t} f(x') g(x') w^t(x') dx' &= \int_{\Omega_t} \sum_{k=1}^N \frac{1}{w^t(T_k)} \left(\int_{E_k} g(y') w^t(y') dy' \right) \mathbf{1}_{R_k}(x') f(x') w^t(x') dx' \\ &=: \int_{\Omega_t} L_{w^t}^* g(x') f(x') w^t(x') dx'. \end{aligned}$$

Furthermore $L_{w^t}^*(\mathbf{1}_{\Omega_t}) = \sum_{k=1}^N \frac{w^t(E_k)}{w^t(T_k)} \mathbf{1}_{T_k} \geq \frac{1}{2} \sum_{k=1}^N \mathbf{1}_{T_k}$. For any locally integrable function g on \mathbb{R}^{n-1} we thus have

$$\begin{aligned} \int_{\Omega_t} g(x') \sum_{k=1}^N \mathbf{1}_{T_k}(x') w^t(x') dx' &\lesssim \int_{\Omega_t} g(x') L_{w^t}^*(\mathbf{1}_{\Omega_t})(x') w^t(x') dx' \\ &= \int_{\Omega_t} L_{w^t} g(x') w^t(x') dx' \leq \|M_{w^t} g\|_{L^{p'}(w^t, \mathbb{R}^{n-1})} w^t(\Omega_t)^{\frac{1}{p}} \\ &\lesssim_{w,n} (p' - 1)^{-(n-1)} \|g\|_{L^{p'}(w^t, \mathbb{R}^{n-1})} w^t(\Omega_t)^{\frac{1}{p}}, \end{aligned}$$

by (1.5). Taking the supremum over $g \in L^{p'}(\mathbb{R}^{n-1})$ with $\|g\|_{L^{p'}(w^t, \mathbb{R}^{n-1})} \leq 1$ gives the estimate

$$\int_{\Omega_t} \left| \sum_{k=1}^N \mathbf{1}_{T_k}(x') \right|^p w^t(x') dx \lesssim_{w,n} p^{(n-1)p} w^t(\Omega_t),$$

as $p \rightarrow \infty$. It is essential to note here that this estimate is uniform in $t \in \mathbb{R}$. Thus integrating over $t \in P^\perp(\Omega)$, gives the claim. \square

4. PROOF OF THE ENDPOINT FEFFERMAN-STEIN INEQUALITY

We begin with a simple lemma.

Lemma 4.1. *Let $\epsilon > 0$, f be a locally integrable function and set $F := \{x \in \mathbb{R}^n : M_n f(x) > 1\}$. There exists a finite collection of rectangles $\mathcal{R} = \{R_k^s\}_{1 \leq k \leq N}$ such that:*

- (i) *The collection \mathcal{R} has the property (P₂) with parameter ϵ .*
- (ii) *For each $1 \leq k \leq N$ we have*

$$|R_k^s| < \int_{R_k^s} |f(y)| dy$$

- (iii) *We have the estimate*

$$w(F) \lesssim_{\epsilon, w, n} w(\cup_k R_k^s).$$

Proof. For every $x \in F$ let $R_x \in \mathfrak{R}_n$ be a rectangle such that

$$\frac{1}{|R_x|} \int_{R_x} |f(y)| dy > 1.$$

Without loss of generality we may assume that $\{R_x\}_{x \in F}$ is a finite sequence $\{R_j\}_{1 \leq j \leq M}$, so that (ii) is satisfied and such that $w(F) \leq w(\cup_{1 \leq j \leq M} R_j)$. From the collection $\{R_j\}_{1 \leq j \leq M}$ we will now choose a subcollection $\{R_k^s\}_{1 \leq k \leq N}$ so that (i) and (iii) are also satisfied. First we reorder the rectangles R_j so that $P^\perp(R_1) \geq P^\perp(R_2) \geq \dots \geq P^\perp(R_M)$. We choose $R_1^s := R_1$ and assume that the rectangles $R_1^s, R_2^s, \dots, R_\tau^s$, have been selected. Also let $1 \leq j_o < M$ so that $R_\tau^s = R_{j_o}$. We then choose $R_{\tau+1}^s$ to be the rectangle with the smallest index among the rectangles $S \in \{R_{j_o+1}, \dots, R_M\}$ that satisfy

$$|S \cap \bigcup_{j \leq \tau} (R_j^s)^*| \leq \epsilon |S|.$$

Since the collection $\{R_j\}_{1 \leq j \leq M}$ is finite, the selection process will end after a finite number of N steps, and the collection $\{R_k^s\}_{1 \leq k \leq N}$ will automatically satisfy (i). Of course this subcollection still satisfies (ii). Now assume that some $S \in \{R_1, \dots, R_M\}$ was not selected. We can then find some positive integer $K \in \{1, 2, \dots, N\}$ such that

$$|S \cap \bigcup_{j \leq K} (R_j^s)^*| > \epsilon |S|.$$

Thus we get for all $x \in S$

$$M_n(\mathbf{1}_{\cup_{j \leq N} (R_j^s)^*})(x) \geq M_n(\mathbf{1}_{\cup_{j \leq k} (R_j^s)^*})(x) > \epsilon,$$

which means that

$$\bigcup_{\substack{1 \leq j \leq N \\ R_j \text{ not selected}}} R_j \subseteq \{x : M_n(\mathbf{1}_{\cup_{j \leq N} (R_j^s)^*})(x) > \epsilon\}.$$

However, since $w \in A_\infty^*$ we know that $M_n : L^{p_0}(w) \rightarrow L^{p_0, \infty}(w)$ for some $p_0 > 1$. We conclude that

$$w\left(\bigcup_{\substack{1 \leq j \leq N \\ R_j \text{ not selected}}} R_j\right) \lesssim_{\epsilon, w, n} w(\cup_{j \leq N} (R_j^s)^*) \lesssim w(\cup_{j \leq N} R_j^s).$$

Thus $w(F) \leq w(\cup_{1 \leq j \leq M} R_j) \lesssim_{\epsilon, w, n} w(\cup_{1 \leq k \leq N} R_k^s)$ as we wanted. \square

We are now ready to give the proof of our main result.

Proof of the Main Theorem. We assume that $n \geq 2$ since in one dimension M_1 is the usual Hardy-Littlewood maximal function and there is nothing (new) to prove. We henceforth write M for M_n since the dimension n is fixed throughout the proof. Furthermore, it suffices to prove the theorem for $\lambda = 1$. Let $F := \{x \in \mathbb{R}^n : M(f)(x) > 1\}$ and consider the collection $\mathcal{R} := \{R_k^s\}_{k=1}^N \subset \mathcal{R}_n$ given by Lemma 4.1. By (i) of that Lemma the collection \mathcal{R} has the sparseness property (P_2) . We assume that $\epsilon > 0$ was chosen small enough in Lemma 4.1, and thus in (P_2) , so that Lemma 3.4 is valid. Observe that (P_2) also implies that

$$|R_k^s \cap \bigcup_{j < k} R_j^s| \leq \epsilon |R_k^s|.$$

By choosing $\epsilon > 0$ small enough we can also assume that $w(R_k^s \cap \cup_{j < k} R_j^s) \leq \frac{1}{2} w(R_k^s)$, according to Remark 1.3. Setting $E_k := R_k^s \setminus \cup_{j < k} R_j^s$ we will thus have

$$(4.2) \quad w(R_k^s) \geq w(E_k) \geq \frac{1}{2} w(R_k^s), \quad k = 1, 2, \dots, N,$$

and the choice of $\epsilon > 0$ depends only on the weight $w \in A_\infty^*$. Denoting $\Omega := \cup_{k=1}^N R_k^s$, we use (ii) and (iii) of Lemma 4.1 to estimate

$$\begin{aligned} w(F) &\lesssim_{\epsilon, w, n} w(\Omega) \leq \sum_{k=1}^N w(R_k^s) \leq \sum_{k=1}^N \frac{w(R_k^s)}{|R_k^s|} \int_{R_k^s} |f(y)| dy \\ &= \int_{\Omega} f(x) \sum_{k=1}^N \frac{w(R_k^s)}{|R_k^s|} \mathbf{1}_{R_k^s}(x) dx. \end{aligned}$$

Define the linear operators

$$Tf(x) = \sum_{k=1}^N \frac{1}{|R_k^s|} \int_{R_k^s} f(y) dy \mathbf{1}_{E_k}(x), \quad T^*f(x) = \sum_{k=1}^N \frac{1}{|R_k^s|} \int_{E_k} f(y) dy \mathbf{1}_{R_k^s}(x), \quad x \in \mathbb{R}^n.$$

For locally integrable f, g we have

$$\int_{\Omega} Tf(x)g(x)dx = \int_{\Omega} T^*g(x)f(x)dx, \quad Tf(x) \leq Mf(x), \quad x \in \mathbb{R}^n.$$

By (4.2) we have

$$T^*w(x) = \sum_{k=1}^N \frac{w(E_k^s)}{|R_k^s|} \mathbf{1}_{R_k^s}(x) \simeq \sum_{k=1}^N \frac{w(R_k^s)}{|R_k^s|} \mathbf{1}_{R_k^s}(x)$$

thus we can estimate for any $\delta > 0$

$$\begin{aligned} w(\Omega) &\lesssim \int_{\Omega} fT^*w \leq \int_{\{\Omega: T^*w \leq \delta Mw\}} f(x)T^*w(x)dx + \int_{\{\Omega: T^*w > \delta Mw\}} f(x) \frac{T^*w(x)}{Mw(x)} Mw(x)dx \\ &\leq \delta \int_{\mathbb{R}^n} |f(x)|Mw(x)dx + \int_{\{\Omega: T^*w > \delta Mw\}} f(x) \frac{T^*w(x)}{Mw(x)} Mw(x)dx. \end{aligned}$$

We will use the following elementary estimate: For each $\theta > 0$ there exists a constant $c_{\theta} > 0$ such that for all $s, t \geq 0$ we have

$$st \leq c_{\theta}s[1 + (\log^+ s)^{n-1}] + \exp(\theta t^{\frac{1}{n-1}}) - 1, \quad n \geq 2.$$

The interested reader can find a detailed proof of this classical inequality in [1]. Applying this pointwise estimate we get for every $\theta > 0$:

$$\begin{aligned} w(\Omega) &\lesssim (\delta + c_{\theta}) \int_{\mathbb{R}^n} |f(x)| \left(1 + (\log^+ |f(x)|)^{n-1}\right) Mw(x)dx \\ &\quad + \int_{\{\Omega: T^*w > \delta Mw\}} \left(\exp \left[\theta \left(\frac{T^*w(x)}{Mw(x)}\right)^{\frac{1}{n-1}}\right] - 1\right) Mw(x)dx. \end{aligned}$$

We now estimate the last term,

$$\begin{aligned} Q &:= \int_{\{\Omega: T^*w > \delta Mw\}} \left(\exp \left[\theta \left(\frac{T^*w(x)}{Mw(x)}\right)^{\frac{1}{n-1}}\right] - 1\right) Mw(x)dx \\ &= \sum_{k=1}^{\infty} \frac{\theta^k}{k!} \int_{\{\Omega: T^*w > \delta Mw\}} \left(\frac{T^*w(x)}{Mw(x)}\right)^{\frac{k}{n-1}} Mw(x)dx \leq \sum_{1 \leq k \leq n-1} + \sum_{k > n-1} =: I + II. \end{aligned}$$

For I we just observe that since $k/(n-1) \leq 1$ and $T^*w/(\delta Mw) > 1$ we have the elementary estimate

$$\left(\frac{T^*w}{Mw}\right)^{\frac{k}{n-1}} = \left(\frac{T^*w}{\delta Mw}\right)^{\frac{k}{n-1}} \delta^{\frac{k}{n-1}}$$

$$\begin{aligned}
&= \delta^{\frac{k}{n-1}-1} (T^*w / (\delta Mw))^{\frac{k}{n-1}-1} \frac{T^*w}{Mw} \\
&\leq \delta^{\frac{k}{n-1}-1} \frac{T^*w}{Mw}.
\end{aligned}$$

So we have

$$I \leq \sum_{1 \leq k \leq n-1} \frac{\theta^k \delta^{\frac{k}{n-1}-1}}{k!} \int_{\Omega} T^*w(x) dx \leq \frac{\theta}{\delta} e^{\delta^{\frac{1}{n-1}}} \int_{\Omega} T1(x)w(x) \lesssim_{\delta, n} \theta w(\Omega).$$

Here we abuse notation by denoting $T1, T^*1(x)$ the action of T, T^* , respectively, on the constant function 1. For II we use the fact that $T^*w \simeq \sum_{k=1}^N \frac{w(\mathbb{R}_k^s)}{|\mathbb{R}_k^s|} \mathbf{1}_{\mathbb{R}_k^s} \leq Mw \sum_{k=1}^N \mathbf{1}_{\mathbb{R}_k^s} \simeq MwT^*1$. We have

$$\begin{aligned}
II &\leq \sum_{k > n-1} \int_{\Omega} \frac{\theta^k}{k!} \left(\frac{T^*w(x)}{Mw(x)} \right)^{\frac{k}{n-1}-1} \frac{T^*w(x)}{Mw(x)} Mw(x) dx \\
&\lesssim \sum_{k > n-1} \frac{\theta^k}{k!} \int_{\Omega} (T^*1(x))^{\frac{k}{n-1}-1} T^*w(x) dx \\
&\lesssim \sum_{k > n-1} \frac{\theta^k}{k!} \int_{\Omega} (T^*1(x))^{\frac{k}{n-1}} T^*w(x) dx \quad \left(\text{because } T^*1 \gtrsim 1 \text{ on } \Omega \right) \\
&\lesssim \sum_{k > n-1} \frac{\theta^k}{k!} \int_{\Omega} T(T^*(1)^{\frac{k}{n-1}})(x)w(x) dx =: \sum_{k > n-1} \frac{\theta^k}{k!} Q_k.
\end{aligned}$$

Since $w \in A_{p_o}^*$ for some $1 < p_o < \infty$ and $Tf \leq Mf$ we have $\|T(f)\|_{L^{p_o}(w)} \lesssim_{w, n} \|f\|_{L^{p_o}(w)}$. This together with Lemma 3.4 yields

$$Q_k \lesssim_{w, n} w(\Omega)^{\frac{1}{p_o}} \left(\int_{\Omega} |T^*1(x)|^{\frac{kp_o}{n-1}} w(x) dx \right)^{\frac{1}{p_o}} \lesssim_{w, n} [kp_o / (n-1)]^k w(\Omega).$$

Overall we get

$$\begin{aligned}
II &\lesssim_{w, n} \sum_{k > n-1} \frac{\theta^k (kp_o)^k}{k! (n-1)^k} w(\Omega) \lesssim \sum_{k > n-1} \frac{(\theta ep_o / (n-1))^k}{\sqrt{k}} w(\Omega) \\
&\lesssim \frac{(\theta ep_o / (n-1))^n}{\sqrt{n}} w(\Omega),
\end{aligned}$$

if θ is small enough. Thus $Q \lesssim_{w, n} \theta w(\Omega)$. We have proved that for $\theta > 0$ small and fixing $\delta = 1$ (say) in the previous estimates we have

$$w(\Omega) \lesssim_{w, n} \theta w(\Omega) + (1 + c_{\theta}) \int_{\mathbb{R}^n} |f(x)| \left(1 + (\log^+ |f(x)|)^{n-1} \right) Mw(x) dx.$$

Choosing $\theta > 0$ sufficiently small we thus have

$$w(F) \lesssim w(\Omega) \lesssim_{w,n} \int_{\mathbb{R}^n} |f(x)| \left(1 + (\log^+ |f(x)|)^{n-1}\right) Mw(x) dx,$$

which is the desired estimate. \square

We have actually proved the following weighted analogue of the Córdoba-Fefferman covering lemma from [3].

Lemma 4.3. *Let $w \in A_\infty^*$. Suppose that $\{R_j\}_{j \in J}$ is a finite sequence of rectangles from \mathfrak{R}_n . Then there exists a subcollection $\{R_k^s\}_{1 \leq k \leq N} \subset \cup_{j \in J} R_j$ such that*

- (i) $w(\cup_{j \in J} R_j) \lesssim_{w,n} w(\cup_{k=1}^N R_k^s)$.
- (ii) For every $\delta > 0$ there exists $\theta_o = \theta_o(\delta, w, n) > 0$ such that, for every $\theta < \theta_o$ we have

$$\int_{\{\Omega: T^*w(x) > \delta Mw(x)\}} \left(\exp \left[\theta \left(\frac{T^*w(x)}{Mw(x)} \right)^{\frac{1}{n-1}} \right] - 1 \right) Mw(x) dx \lesssim_{w,n,\theta,\delta} w(\cup_{k=1}^N R_k^s).$$

Here $T^*w = \sum_{k=1}^N \frac{w(R_k^s)}{|R_k^s|} \mathbf{1}_{R_k^s}$ and M denotes the strong maximal function.

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TERESA LUQUE, DEPARTAMENTO DE ANALISIS MATEMÁTICO, FACULTAD DE MATEMÁTICAS, UNIVERSIDAD DE SEVILLA, 41080 SEVILLA, SPAIN
E-mail address: tluquem@us.es

IOANNIS PARISSIS, DEPARTMENT OF MATHEMATICS, AALTO UNIVERSITY, P.O.Box 11100, FI-00076 AALTO, FINLAND
E-mail address: ioannis.parissis@gmail.com