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Effective results on nonlinear ergodic averages in CAT(κ) spaces

Laurenţiu Leuştean[1,2], Adriana Nicolae[3,4]

[1] Faculty of Mathematics and Computer Science, University of Bucharest, Academiei 14, P.O. Box 010014, Bucharest, Romania

[2] Simion Stoilow Institute of Mathematics of the Romanian Academy, P. O. Box 1-764, RO-014700 Bucharest, Romania

[3] Department of Mathematics, Babeş-Bolyai University, Kogălniceanu 1, 400084 Cluj-Napoca, Romania

[4] Simion Stoilow Institute of Mathematics of the Romanian Academy, Research group of the project PD-3-0152, P. O. Box 1-764, RO-014700 Bucharest, Romania

E-mails: Laurentiu.Leustean@imar.ro, anicolae@math.ubbcluj.ro

Abstract

In this paper we apply proof mining techniques to compute, in the setting of CAT(κ) spaces (with $\kappa > 0$), effective and highly uniform rates of asymptotic regularity and metastability for a nonlinear generalization of the ergodic averages, known as the Halpern iteration. In this way, we obtain a uniform quantitative version of a nonlinear extension of the classical von Neumann mean ergodic theorem.

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1 Introduction

In this paper we apply methods from mathematical logic to obtain a uniform quantitative version of a generalization of the classical von Neumann mean ergodic theorem, giving effective rates of metastability for the so-called Halpern iteration, a nonlinear generalization of the ergodic averages. Our results are a contribution to the line of research known as *proof mining*, initiated in the 50's by Kreisel under the name of *unwinding of proofs* and extensively developed by Kohlenbach, beginning with the 90's. The idea of this research direction is to extract new, effective information from mathematical proofs making use of ineffective principles. Hence, it can be related to Terence Tao's proposal [32] of *hard analysis*, based on finitary arguments, instead of the infinitary ones from *soft analysis*. Proof mining has already been applied in approximation theory, nonlinear analysis, ergodic theory, topological dynamics and Ramsey theory. Related to these applications, general logical metatheorems were proved, having the following form: if certain statements satisfying general logical conditions (e.g. $\forall\exists$-sentences) are proved in some formal system associated to an abstract space, then uniform finitary versions of these statements are guaranteed to hold and, furthermore, one can transform the initial proof into a quantitative one for the finitary version and, in this way, extract effective uniform bounds. We refer to Kohlenbach's book [11] for an introduction to proof mining.

Our theorems guarantee under general logical conditions such strong uniform versions of non-uniform existence statements. Moreover, they provide algorithms for actually extracting effective uniform bounds and transforming the original proof into one for the stronger uniformity result.

Let us recall the Hilbert space formulation of the celebrated von Neumann mean ergodic theorem.

Theorem 1.1. *Let H be a Hilbert space and $U : H \to H$ be a unitary operator. Then for all $x \in H$, the Cesàro mean $x_n = \frac{1}{n}\sum_{i=0}^{n-1} U^i x$ converges strongly to the projection of x onto the set of fixed points of U.*

If $\mathcal{X} = (X, \mathcal{B}, \mu, T)$ is a probability measure-preserving system, $H = L^2(\mathcal{X})$ and $U = U_T : L^2(\mathcal{X}) \to L^2(\mathcal{X})$, $f \mapsto f \circ T$ is the induced operator, the Cesàro mean starting with $f \in L^2(\mathcal{X})$ becomes the ergodic average $A_n f = \frac{1}{n}\sum_{i=0}^{n-1} f \circ T^i$.

The convergence of the ergodic averages can be arbitrarily slow, as shown by Krengel [22]. Furthermore, one cannot expect, in general, to get effective rates of convergence for the ergodic averages. Avigad, Gerhardy and Towsner [1] applied methods of computable analysis on Hilbert spaces to obtain an example of a computable Lebesgue measure-preserving transformation T on $[0, 1]$ and a computable characteristic function χ_A such that the limit of the sequence $A_n\chi_A$ is not a computable element of $L^2([0, 1])$, which implies that there is no computable bound on the rate of convergence of $(A_n\chi_A)$.

However, one can consider the following equivalent reformulation of the Cauchy property of (x_n):

$$\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists N \forall i, j \in [N, N + g(N)]\ \left(\|x_i - x_j\| < 2^{-k}\right). \qquad (1)$$

This is known in logic as Kreisel's [20, 21] no-counterexample interpretation of the Cauchy property and it was popularized in the last years under the name of *metastability* by Tao [32, 33]. In [33], Tao generalized the mean ergodic theorem for multiple commuting measure-preserving transformations, by deducing it from a finitary norm convergence result, expressed in terms of metastability. Recently, Walsh [34] used again metastability to show the L^2-convergence of multiple polynomial ergodic averages arising from nilpotent groups of measure-preserving transformations.

Logical metatheorems developed by Kohlenbach [13] show that, from wide classes of mathematical proofs one can extract effective bounds on $\exists N$ in (1). Thus, taking $\varepsilon > 0$ instead of 2^{-k}, we define a *rate of metastability* as a functional $\Phi : (0,\infty) \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ satisfying

$$\forall \varepsilon > 0 \, \forall g : \mathbb{N} \to \mathbb{N} \, \exists N \leq \Phi(\varepsilon, g) \, \forall i, j \in [N, N + g(N)] \; (\|x_i - x_j\| < \varepsilon). \quad (2)$$

Thus, a natural direction of research is to obtain finitary, quantitative versions of convergence statements for sequences (x_n) by providing effective rates of metastability. A qualitative feature of these quantitative versions is that the rates of metastability are highly uniform and independent or have only a weak dependence on the input data. Furthermore, these quantitative versions can be thereafter generalized to new structures, obtaining as an immediate consequence the generalization of the initial (non-quantitative) Cauchy statement to these structures. The main quantitative result of this paper is obtained in this way. We refer to [15] for another example in the context of the asymptotic behaviour of nonlinear iterations.

Avigad, Gerhardy and Towsner [1] computed for the first time explicit and uniform rates of metastability for the ergodic averages, by a logical analysis of Riesz' proof of the mean ergodic theorem. Their result was generalized, with better bounds, to uniformly convex Banach spaces by Kohlenbach and the first author [14], applying proof mining methods, but this time to a proof of Garrett Birkhoff [3]. In fact, Avigad and Rute [2] realized that the computations in [16] allow one to obtain an effective bound on the number of ε-fluctuations (i.e. pairs (i, j) with $i > j$ and $\|x_i - x_j\| > \varepsilon$). A very nice discussion on the different types of quantitative information (metastability, effective learnability, bounds on the number of oscillations) that can be extracted from convergence proofs is done in a recent paper by Kohlenbach and Safarik [19].

In the important paper [35], Wittmann obtained the following nonlinear generalization of the mean ergodic theorem.

Theorem 1.2. *[35] Let C be a bounded closed convex subset of a Hilbert space X, $T : C \to C$ a nonexpansive mapping and $(\lambda_n)_{n\geq 1}$ a sequence in $[0, 1]$. For any $u \in C$, define*

$$x_0 = u, \quad x_{n+1} = \lambda_{n+1} u + (1 - \lambda_{n+1}) T x_n. \quad (3)$$

Assume that (λ_n) satisfies

$$\lim_{n\to\infty} \lambda_n = 0, \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n = \infty \quad (4)$$

Then for any $u \in C$, (x_n) converges to the projection $P_{Fix(T)} u$ of u onto the (nonempty) set of fixed points $Fix(T)$.

One can easily see that (x_n) coincides with the Cesàro mean when T is linear and $\lambda_n = \dfrac{1}{n+1}$. The iteration (x_n) is known as the Halpern iteration, as it was introduced by Halpern [9] for the special case $u = 0$. We refer to [17, Section 3] for a discussion on results in the literature on Halpern iterations, obtained by considering different conditions on (λ_n) or more general spaces.

Kohlenbach's logical metatheorem for Hilbert spaces [13] guarantees also in the case of Wittmann's theorem that from its proof one can extract a rate of metastability Φ of (x_n), uniform in the following sense: it depends only on ε and g, an upper bound on the diameter of C and moduli on (λ_n), given by the quantitative version of (4). Thus, Φ is independent with respect to the starting point u of the iteration, the nonexpansive mapping T, the Hilbert space X and depends on C only via its diameter. Kohlenbach [12] computed such a uniform rate of metastability by a logical analysis of Wittmann's proof.

Furthermore, Kohlenbach and the first author [16, 17, 18] extracted rates of metastability from the proofs of two generalizations of Wittmann's theorem given by Shioji and Takahashi [31] for a class of Banach spaces with a uniformly Gâteaux differentiable norm and by Saejung [29] for CAT(0) spaces. Both Saejung's and Shioji-Takahashi's proofs use Banach limits (whose existence requires the axiom of choice), inspired by Lorentz' seminal paper [25], introducing almost convergence. Our quantitative results were obtained by developing in [17] a method to eliminate the use of Banach limits from these proofs and get, in this way, elementary proofs to which general logical metatheorems for CAT(0) spaces [13] and for uniformly smooth Banach spaces [16] can be applied to guarantee the extractability of effective bounds. We point out that the use of Lorentz' almost convergence (and hence, Banach limits) in nonlinear ergodic theory was introduced by Reich [27], while Bruck and Reich [7] applied Banach limits for the first time to the study of Halpern iterations (see also [8, Sections 12, 14]).

Geodesic spaces provide a suitable setting for extending the notion of sectional curvature from Riemannian manifolds. An important class of geodesic spaces of bounded curvature are CAT(κ) spaces, where geodesic triangles are in some sense "thin". Such spaces enjoy nice properties inherited from the comparison with the model spaces and proved to be relevant in various problems and aspects in geometry (see [4]).

Recently, Piątek [26] extended Wittmann's result to the context of CAT(κ) spaces with $\kappa > 0$. *In this paper we extract an effective and uniform rate of metastability for this generalization of Wittmann's theorem.*

Our main quantitative result (Theorem 3.4) is obtained by generalizing to CAT(κ) spaces the quantitative proof for CAT(0) spaces from [17]. Thus, we apply again the general method developed in [17], together with the remark that, in fact, our logical analysis of Saejung's proof for CAT(0) spaces results in the elimination of any contribution of Banach limits, hence even the finitary lemmas proved in [17, Section 8] are no longer needed (see [18]). Despite this simplification, the proofs we give in this paper are much more involved, since we work in the setting of CAT(κ) spaces. However, we still get a rate of metastability having a form similar to the one described in [19].

As the first step in the convergence proof is to obtain the asymptotic regularity, our first important result (Proposition 3.2) consists in the computation of a uniform rate of asymptotic regularity.

For the rest of the paper $\mathbb{N} = \{0, 1, 2, \ldots\}$ *and* $\mathbb{Z}_+ = \{1, 2, \ldots\}$. *Furthermore, we consider CAT*(κ) *spaces with* $\kappa > 0$.

2 CAT(κ) spaces

Let (X, d) be a metric space. A *geodesic path* from x to y is a mapping $c : [0, l] \subseteq \mathbb{R} \to X$ such that $c(0) = x, c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for every $t, t' \in [0, l]$. The image $c([0, l])$ of c forms a *geodesic segment* which joins x and y. Note that a geodesic segment from x to y is not necessarily unique. If no confusion arises, we use $[x, y]$ to denote a geodesic segment joining x and y. (X, d) is called a *(uniquely) geodesic space* if every two points $x, y \in X$ can be joined by a (unique) geodesic path. A point $z \in X$ belongs to the geodesic segment $[x, y]$ if and only if there exists $t \in [0, 1]$ such that $d(z, x) = td(x, y)$ and $d(z, y) = (1 - t)d(x, y)$, and we write $z = (1 - t)x + ty$ for simplicity. This, too, may not be unique. A subset C of X is *convex* if C contains any geodesic segment that joins every two points in C. A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ consists of three points x_1, x_2 and x_3 in X (its *vertices*) and three geodesic segments corresponding to each pair of points (its *edges*).

CAT(κ) spaces are defined in terms of comparisons with the model spaces M^n_κ. We focus here on CAT(κ) spaces with $\kappa > 0$. We give below the precise definition and briefly describe some of their properties that play an essential role in this work. For a detailed discussion on geodesic metric spaces and, in particular, on CAT(κ) spaces, one may check, for example, [4].

The *n-dimensional sphere* $\mathbb{S}^n$ is the set $\{x \in \mathbb{R}^{n+1} : (x \mid x) = 1\}$, where $(\cdot \mid \cdot)$ stands for the Euclidean scalar product. Consider the mapping $d : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{R}$ by assigning to each $(x, y) \in \mathbb{S}^n \times \mathbb{S}^n$ the unique number $d(x, y) \in [0, \pi]$ such that $\cos d(x, y) = (x \mid y)$. Then, $(\mathbb{S}^n, d)$ is a metric space called the spherical space. This space is also geodesic and, if $d(x, y) < \pi$, then there exists a unique geodesic segment joining x and y. Moreover, open (resp. closed) balls of radius $\leq \pi/2$ (resp. $< \pi/2$) are convex. The *spherical law of cosines* states that in a spherical triangle with vertices $x, y, z \in \mathbb{S}^n$ and γ the spherical angle between the geodesic segments $[x, y]$ and $[x, z]$ we have

$$\cos d(y, z) = \cos d(x, y) \cos d(x, z) + \sin d(x, y) \sin d(x, z) \cos \gamma.$$

Let $\kappa > 0$ and $n \in \mathbb{N}$. The classical *model spaces* M^n_κ are obtained from the spherical space $\mathbb{S}^n$ by multiplying the spherical distance with $1/\sqrt{\kappa}$. These spaces inherit the geometrical properties from the spherical space. Thus, there is a unique geodesic path joining $x, y \in M^n_\kappa$ if and only if $d(x, y) < \pi/\sqrt{\kappa}$. Furthermore, closed balls of radius $< \pi/(2\sqrt{\kappa})$ are convex and we have a counterpart of the spherical law of cosines. We denote the *diameter of* M^n_κ by $D_\kappa = \pi/\sqrt{\kappa}$.

For a geodesic triangle $\Delta = \Delta(x_1, x_2, x_3)$, a *$\kappa$-comparison triangle* is a triangle $\bar{\Delta} = \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in M^2_κ such that $d(x_i, x_j) = d_{M^2_\kappa}(\bar{x}_i, \bar{x}_j)$ for $i, j \in \{1, 2, 3\}$. For κ fixed, κ-comparison triangles of geodesic triangles (having perimeter less than $2D_\kappa$) always exist and are unique up to isometry.

A geodesic triangle Δ of perimeter less than $2D_\kappa$ satisfies the *CAT(κ) inequality* if for every κ-comparison triangle $\bar{\Delta}$ of Δ and for every $x, y \in \Delta$ we have

$$d(x,y) \le d_{M_\kappa^2}(\bar{x},\bar{y}),$$

where $\bar{x}, \bar{y} \in \bar{\Delta}$ are the comparison points of x and y, i.e., if $x = (1-t)x_i + tx_j$ then $\bar{x} = (1-t)\bar{x}_i + t\bar{x}_j$ for $i, j \in \{1,2,3\}$.

A metric space is called a *CAT(κ) space* if every two points at distance less than D_κ can be joined by a geodesic segment and every geodesic triangle having perimeter less than $2D_\kappa$ satisfies the CAT(κ) inequality. CAT(0) spaces are defined in a similar way considering the model space M_0^2 to be the Euclidean plane of infinite diameter.

3 Main results

Let (X,d) be a geodesic space, $C \subseteq X$ a convex subset, $T : C \to C$ a nonexpansive mapping and (λ_n) a sequence in $[0,1]$. The Halpern iteration starting at $u \in C$ can be defined by

$$x_0 = u, \quad x_{n+1} = \lambda_{n+1}u + (1-\lambda_{n+1})Tx_n. \tag{5}$$

The main purpose of our work is to prove a quantitative version of the following generalization of Wittmann's theorem to CAT(κ) spaces, obtained recently by Piątek [26].

Theorem 3.1. *Let X be a complete CAT(κ) space, $C \subseteq X$ a bounded closed convex subset with diameter $d_C < \dfrac{D_\kappa}{2}$ and $T : C \to C$ a nonexpansive mapping. Assume that (λ_n) satisfies (4). Then for any $u \in C$, the iteration (x_n) starting from u converges to the fixed point of T which is nearest to u.*

A first important result of this paper is the extraction of an effective rate of asymptotic regularity for the Halpern iteration, that is, a rate of the convergence of $(d(x_n, Tx_n))$ towards 0. In order to state this result, we need to make the hypotheses (4) on (λ_n) quantitative.

For brevity, we say that the sequence (λ_n) and the functions $\alpha : (0,\infty) \to \mathbb{Z}_+$, $\gamma : (0,\infty) \to \mathbb{Z}_+$ and $\theta : \mathbb{Z}_+ \to \mathbb{Z}_+$ satisfy (*) if the following conditions hold:

(i) $\lim\limits_{n\to\infty} \lambda_{n+1} = 0$ with rate of convergence α, i.e.,

$$\lambda_{n+1} \le \varepsilon, \quad \text{for all } \varepsilon > 0 \text{ and all } n \ge \alpha(\varepsilon);$$

(ii) $\sum\limits_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n|$ converges with Cauchy modulus γ, i.e.,

$$\sum_{i=\gamma(\varepsilon)+1}^{\gamma(\varepsilon)+n} |\lambda_{i+1} - \lambda_i| \le \varepsilon, \quad \text{for all } \varepsilon > 0 \text{ and all } n \in \mathbb{Z}_+;$$

(iii) $\sum_{n=1}^{\infty} \lambda_{n+1} = \infty$ with rate of divergence θ, i.e.,

$$\sum_{k=1}^{\theta(n)} \lambda_{k+1} \geq n, \quad \text{for all } n \in \mathbb{Z}_+.$$

Proposition 3.2. *Let X be a CAT(κ) space, $C \subseteq X$ a bounded convex subset, $T : C \to C$ nonexpansive and $M < \frac{D_\kappa}{2}$ an upper bound on the finite diameter d_C of C. Assume furthermore that $(\lambda_n), \alpha, \gamma, \theta$ satisfy (*).*

Then $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ with rate of convergence $\tilde{\Phi}$ given by

$$\tilde{\Phi}(\varepsilon, \kappa, M, \gamma, \theta) = \theta\left(\left\lceil \frac{1}{\cos(M\sqrt{\kappa})} \right\rceil \left(\gamma\left(\frac{\varepsilon}{2M}\right) + \max\left\{\left\lceil \ln\left(\frac{2M}{\varepsilon}\right)\right\rceil, 1\right\}\right)\right) \tag{6}$$

and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ with rate of convergence Φ given by

$$\Phi(\varepsilon, \kappa, M, \gamma, \theta, \alpha) = \max\left\{\tilde{\Phi}\left(\frac{\varepsilon}{2}, \kappa, M, \gamma, \theta\right), \alpha\left(\frac{\varepsilon}{2M}\right)\right\}. \tag{7}$$

Proof. See Section 5. □

If $\lambda_n = \dfrac{1}{n+1}$ one can easily obtain rates α, γ, θ:

$$\alpha(\varepsilon) = \gamma(\varepsilon) = \left\lceil \frac{1}{\varepsilon} \right\rceil, \qquad \theta(n) = \exp\left((n+1)\ln 4\right). \tag{8}$$

As an immediate consequence we get the following:

Corollary 3.3. *Assume that $\lambda_n = \dfrac{1}{n+1}$, $n \geq 1$. Then*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_n, Tx_n) = 0$$

with a common rate of convergence

$$\Psi(\varepsilon, \kappa, M) = \exp\left(\left\lceil \frac{1}{\cos(M\sqrt{\kappa})} \right\rceil \left\lceil \frac{8M}{\varepsilon} + 2 \right\rceil \ln 4\right), \tag{9}$$

which is exponential in $\dfrac{1}{\varepsilon}$.

We point out that exponential rates of asymptotic regularity for the Halpern iteration were obtained by the first author for Banach spaces in [23] and for the so-called W-hyperbolic spaces in [24]. Kohlenbach [12] remarked that the proof in [23] can be simplified and, as a consequence, one gets quadratic rates in Banach spaces. For CAT(0) spaces, Kohlenbach and the first author provide in [17] a quantitative asymptotic regularity result for general (λ_n) by considering instead of $\sum_{n=1}^{\infty} \lambda_{n+1} = \infty$ the equivalent condition $\prod_{n=1}^{\infty}(1-\lambda_{n+1}) = 0$. As a corollary, one obtains again quadratic rates of asymptotic regularity. However, the method used in [17] for CAT(0) spaces does not hold for CAT(κ) spaces.

The main result of the paper is the following quantitative version of Theorem 3.1, which provides an explicit uniform rate of metastability for the Halpern iteration in CAT(κ) spaces. To get such a result we apply again the general method developed by Kohlenbach and the first author in [17] for the Halpern iteration in CAT(0) spaces and applied again in [16] for uniformly smooth Banach spaces as well as in [30] for a modified Halpern iteration in CAT(0) spaces. As noticed in [18], in the end we do not need the finitary Lemmas 8.3 and 8.4 from [17], since, as a consequence of the proof mining methods applied to Saejung's proofs, one gets a proof where no contributions of Banach limits can be traced.

Theorem 3.4. *Let X be a complete CAT(κ) space, $C \subseteq X$ a bounded closed convex subset, $T : C \to C$ nonexpansive and $M < \frac{D_\kappa}{2}$ an upper bound on the finite diameter d_C of C. Assume furthermore that $(\lambda_n), \alpha, \gamma, \theta$ satisfy (*). Then for all $\varepsilon \in (0,2)$ and $g : \mathbb{N} \to \mathbb{N}$,*

$$\exists N \leq \Sigma(\varepsilon, g, \kappa, M, \theta, \alpha, \gamma)\ \forall m, n \in [N, N+g(N)]\ (d(x_n, x_m) \leq \varepsilon),$$

with $N = \Theta_{K_0}\left(\sin^2 \frac{\varepsilon\sqrt{\kappa}}{4}\right)$ *for some* $\left\lceil \frac{1}{\varepsilon_0} \right\rceil \leq K_0 \leq \widetilde{f^*}^{B_{\varepsilon,\kappa,M}}(0) + \left\lceil \frac{1}{\varepsilon_0} \right\rceil$ *and*

$$\Sigma(\varepsilon, g, \kappa, M, \theta, \alpha, \gamma) = A_{\varepsilon,\kappa,M,\theta,\alpha,\gamma}\left(\widetilde{f^*}^{B_{\varepsilon,\kappa,M}}(0) + \left\lceil \frac{1}{\varepsilon_0} \right\rceil\right),$$

where the above constants and functionals are specified in Table 1.

Proof. We refer to Section 7 for the proof. We point here only the main steps:

(i) extract a rate of asymptotic regularity (this is done in Proposition 3.2);

(ii) obtain a quantitative Browder theorem (see Proposition 6.2);

(iii) define in an appropriate way an approximate fixed point sequence γ_n^t (see (25));

(iv) apply Lemma 7.3, a quantitative lemma on sequences of real numbers.

□

Hence, we compute a rate of metastability which is uniform in the starting point x_0 of the iteration and the nonexpansive mapping T. Moreover, it depends on the space X and the set C only via κ and the diameter d_C of C. The dependence on (λ_n) is through the rates α, γ, θ, which can be computed very easily for the natural choice $\lambda_n = \dfrac{1}{n+1}$.

Furthermore, as in [16, 17] as well as in other case studies in proof mining, the rate of metastability has the form described by Kohlenbach and Safarik [19].

$$B_{\varepsilon,\kappa,M} = \left\lceil \frac{M\sqrt{\kappa}\tan(M\sqrt{\kappa})}{1-\cos(\varepsilon_0)} \right\rceil, \quad \varepsilon_0 = \frac{\cos(M\sqrt{\kappa})}{36}\sin^2\frac{\varepsilon\sqrt{\kappa}}{4},$$

$$A_{\varepsilon,\kappa,M,\theta,\alpha,\gamma}(n) = \theta^+\left(\left\lceil \frac{1}{\cos(M\sqrt{\kappa})}\right\rceil (\Gamma(n) - 1 + \max\{S,1\})\right) + 1,$$

$$\Gamma(n) = \max\left\{\chi_i^*\left(\frac{1}{3}\sin^2\frac{\varepsilon\sqrt{\kappa}}{4}\right) : \left\lceil\frac{1}{\varepsilon_0}\right\rceil \le i \le n\right\}, \quad \theta^+(n) = \max_{1\le i\le n}\theta(i),$$

$$S = \left\lceil \ln\left(\frac{3\sin^2\frac{M\sqrt{\kappa}}{4}}{\sin^2\frac{\varepsilon\sqrt{\kappa}}{4}}\right)\right\rceil, \quad \chi_i^*(\varepsilon) = \chi_i\left(\frac{\varepsilon}{2}\cos(M\sqrt{\kappa})\right), \quad L_i = \frac{\cos(M\sqrt{\kappa})\varepsilon}{4M\sqrt{\kappa}(i+1)},$$

$$\chi_i(\varepsilon) = \max\left\{\theta\left(\left\lceil\frac{1}{\cos(M\sqrt{\kappa})}\right\rceil\left(\gamma(L_i) + \max\left\{\left\lceil\ln\left(\frac{1}{L_i}\right)\right\rceil, 1\right\}\right)\right), \alpha(2L_i)\right\},$$

$$\Theta_i(\varepsilon) = \theta\left(\left\lceil\frac{1}{\cos(M\sqrt{\kappa})}\right\rceil\left(\chi_i^*\left(\frac{\varepsilon}{3}\right) - 1 + \max\{T,1\}\right)\right) + 1,$$

$$T = \left\lceil \ln\left(\frac{3}{\varepsilon}\sin^2\frac{M\sqrt{\kappa}}{2}\right)\right\rceil, \quad \Delta_i^*(\varepsilon, g) = \frac{\varepsilon}{3\Theta_i(\varepsilon) - 3\chi_i^*(\frac{\varepsilon}{3}) + 3g\left(\Theta_i(\varepsilon)\right)},$$

$$f(i) = \max\left\{\left\lceil\frac{M\sqrt{\kappa}}{\Delta_i^*(\sin^2\frac{\varepsilon\sqrt{\kappa}}{4}, g)}\right\rceil, i\right\} - i, \quad f^*(i) = f\left(i + \left\lceil\frac{1}{\varepsilon_0}\right\rceil\right) + \left\lceil\frac{1}{\varepsilon_0}\right\rceil$$

and $\widetilde{f^*}(i) = i + f^*(i)$.

Table 1

Thus, g does not appear at all in the definition of the mappings $A_{\varepsilon,\kappa,M,\theta,\alpha,\gamma}$ and $B_{\varepsilon,\kappa,M}$, and $\widetilde{f^*}(i)$ only uses g on one argument, $\Theta_i(\sin^2(\varepsilon\sqrt{\kappa}/4))$, which itself does not depend on g. We refer to [19] for a logical explanation of this phenomenon in terms of effective learnability and bounds on the number of mind changes.

If $\lambda_n = \dfrac{1}{n+1}$, with α, γ, θ given by (8), one can easily see that $(\chi_i^*)_i$ is nondecreasing. As a consequence we obtain the following:

Corollary 3.5. *Assume that* $\lambda_n = \dfrac{1}{n+1}$ *for all* $n \ge 1$*. Then for all* $\varepsilon \in (0,2)$ *and* $g : \mathbb{N} \to \mathbb{N}$,

$$\exists N \le \Sigma(\varepsilon, g, \kappa, M)\ \forall m,n \in [N, N+g(N)]\ (d(x_n, x_m) \le \varepsilon),$$

where

$$\Sigma(\varepsilon, g, \kappa, M) = A_{\varepsilon,\kappa,M}\left(\widetilde{f^*}^{B_{\varepsilon,\kappa,M}}(0) + \left\lceil\frac{1}{\varepsilon_0}\right\rceil\right),$$

with

$$A_{\varepsilon,\kappa,M}(n) = \exp\left(\left(\left\lceil\frac{1}{\cos(M\sqrt{\kappa})}\right\rceil(\Gamma(n) - 1 + \max\{S,1\}) + 1\right)\ln 4\right) + 1,$$

$$\Gamma(n) = \chi_n^*\left(\frac{1}{3}\sin^2\frac{\varepsilon\sqrt{\kappa}}{4}\right),$$

$$\chi_i(\varepsilon) = \exp\left(\left(\left\lceil\frac{1}{\cos(M\sqrt{\kappa})}\right\rceil\left(\left\lceil\frac{1}{L_i}\right\rceil + \max\left\{\left\lceil\ln\left(\frac{1}{L_i}\right)\right\rceil, 1\right\}\right) + 1\right)\ln 4\right),$$

$$\Theta_i(\varepsilon) = \exp\left(\left(\left\lceil\frac{1}{\cos(M\sqrt{\kappa})}\right\rceil\left(\chi_i^*\left(\frac{\varepsilon}{3}\right) - 1 + \max\{T,1\}\right) + 1\right)\ln 4\right) + 1$$

and the other constants and functionals are defined as in Theorem 3.4.

4 Some technical lemmas

Throughout the paper, we shall use the following well-known facts:

(i) $x \geq \sin x$ for all $x \geq 0$.

(ii) $\sin(tx) \geq t \sin x$ for all $x \in [0,\pi]$ and all $t \in [0,1]$.

(iii) The function $f : (0,\pi) \to (0,1)$, $f(x) = \dfrac{\sin x}{x}$ is decreasing.

(iv) Given $t \in [0,1]$, the mapping $f : (0,\pi) \to (0,\infty), f(x) = \dfrac{\sin(tx)}{\sin x}$ is increasing.

The following very useful result is proved in [26] for $\kappa = 1$. The proof for general $\kappa > 0$ is an immediate rescaling.

Lemma 4.1. *Let $\Delta(x,y,z)$ be a triangle in X and $M \leq \frac{D_\kappa}{2}$ be an upper bound on the lengths of the sides of $\Delta(x,y,z)$. Then for all $t \in (0,1)$,*

$$d((1-t)x + tz, (1-t)y + tz) \leq \frac{\sin\left((1-t)M\sqrt{\kappa}\right)}{\sin\left(M\sqrt{\kappa}\right)} d(x,y) \leq d(x,y).$$

Let X be a CAT(κ) space. The next results gather some useful properties which will be needed in the subsequent sections.

Lemma 4.2. *Let $\Delta(x,y,z)$ be a triangle in X with perimeter $< 2D_\kappa$. Let w be a point on the segment joining x and z. Suppose that $\cos(d(y,z)\sqrt{\kappa}) \geq \cos(d(y,w)\sqrt{\kappa})\cos(d(w,z)\sqrt{\kappa})$. Then $d(x,w) \leq d(x,y)$. Moreover, if $\Delta(\bar{x},\bar{y},\bar{z})$ is a κ-comparison triangle for $\Delta(x,y,z)$, then $\angle_{\bar{w}}(\bar{y},\bar{x}) \geq \frac{\pi}{2}$.*

Proof. Let $\Delta(\bar{x}, \bar{y}, \bar{z})$ be a κ-comparison triangle for $\Delta(x, y, z)$ and $\alpha = \angle_{\bar{w}}(\bar{y}, \bar{z})$. Suppose that $\alpha > \dfrac{\pi}{2}$. Then

$$\begin{aligned}
\cos(d(y,z)\sqrt{\kappa}) &= \cos(d(\bar{y},\bar{w})\sqrt{\kappa})\cos(d(\bar{w},\bar{z})\sqrt{\kappa}) \\
&\quad + \sin(d(\bar{y},\bar{w})\sqrt{\kappa})\sin(d(\bar{w},\bar{z})\sqrt{\kappa})\cos\alpha \\
&< \cos(d(\bar{y},\bar{w})\sqrt{\kappa})\cos(d(\bar{w},\bar{z})\sqrt{\kappa}) \\
&\leq \cos(d(y,w)\sqrt{\kappa})\cos(d(w,z)\sqrt{\kappa}),
\end{aligned}$$

which contradicts the hypothesis. Thus, $\alpha \leq \frac{\pi}{2}$ and $\beta = \angle_{\bar{w}}(\bar{y}, \bar{x}) \geq \dfrac{\pi}{2}$. It follows that

$$\begin{aligned}
\cos(d(\bar{x},\bar{y})\sqrt{\kappa}) &= \cos(d(\bar{x},\bar{w})\sqrt{\kappa})\cos(d(\bar{w},\bar{y})\sqrt{\kappa}) \\
&\quad + \sin(d(\bar{x},\bar{w})\sqrt{\kappa})\sin(d(\bar{w},\bar{y})\sqrt{\kappa})\cos\beta \\
&\leq \cos(d(\bar{x},\bar{w})\sqrt{\kappa}),
\end{aligned}$$

hence $d(\bar{x}, \bar{y}) \geq d(\bar{x}, \bar{w})$. Thus, $d(x, w) \leq d(x, y)$. □

Assume $C \subseteq X$ is bounded with $M < \frac{D_\kappa}{2}$ an upper bound on its diameter. In the sequel x, y, z are pairwise distinct points of C and $w \in [x, y]$, $v \in [x, z]$.

We shall use the following notation:

$$\begin{aligned}
&S_1 = \sin(d(x,w)\sqrt{\kappa})\sin(d(x,v)\sqrt{\kappa}), \quad S_2 = \sin(d(x,y)\sqrt{\kappa})\sin(d(x,z)\sqrt{\kappa}), \\
&S_3 = \sin(d(x,w)\sqrt{\kappa})\sin(d(x,z)\sqrt{\kappa}), \quad S_4 = \sin(d(y,w)\sqrt{\kappa})\sin(d(x,z)\sqrt{\kappa}), \\
&S_5 = \sin(d(x,w)\sqrt{\kappa})\sin(d(z,v)\sqrt{\kappa}), \\
&C_1 = \cos(d(x,w)\sqrt{\kappa})\cos(d(x,v)\sqrt{\kappa}), \quad C_2 = \cos(d(x,y)\sqrt{\kappa})\cos(d(x,z)\sqrt{\kappa}).
\end{aligned}$$

Lemma 4.3.

$$S_2 - S_3 \leq S_4 \cos(d(x,w)\sqrt{\kappa}), \tag{10}$$

$$S_3 - S_1 \leq S_5 \cos(d(x,v)\sqrt{\kappa}), \tag{11}$$

$$S_2C_1 - S_1C_2 = S_4 \cos(d(x,v)\sqrt{\kappa}) + S_5 \cos(d(x,y)\sqrt{\kappa}), \tag{12}$$

$$S_2 - S_3 - S_4\cos(d(x,v)\sqrt{\kappa}) \leq 2S_4 \left(\sin^2 \frac{d(x,v)\sqrt{\kappa}}{2} - \sin^2 \frac{d(x,w)\sqrt{\kappa}}{2}\right), \tag{13}$$

$$S_3 - S_1 - S_5\cos(d(x,y)\sqrt{\kappa}) \leq 2S_5 \left(\sin^2 \frac{d(x,y)\sqrt{\kappa}}{2} - \sin^2 \frac{d(x,v)\sqrt{\kappa}}{2}\right). \tag{14}$$

Proof.

$$\begin{aligned}
S_2 - S_3 &= \big(\sin(d(x,y)\sqrt{\kappa}) - \sin(d(x,w)\sqrt{\kappa})\big)\sin(d(x,z)\sqrt{\kappa}) \\
&= 2\sin\frac{(d(x,y)-d(x,w))\sqrt{\kappa}}{2}\cos\frac{(d(x,y)+d(x,w))\sqrt{\kappa}}{2}\sin(d(x,z)\sqrt{\kappa}) \\
&= 2\sin\frac{d(y,w)\sqrt{\kappa}}{2}\cos\left(\left(d(x,w)+\frac{d(y,w)}{2}\right)\sqrt{\kappa}\right)\sin(d(x,z)\sqrt{\kappa}) \\
&\leq 2\sin\frac{d(w,y)\sqrt{\kappa}}{2}\cos(d(x,w)\sqrt{\kappa})\cos\frac{d(w,y)\sqrt{\kappa}}{2}\sin(d(x,z)\sqrt{\kappa}) \\
&= \sin(d(w,y)\sqrt{\kappa})\cos(d(x,w)\sqrt{\kappa})\sin(d(x,z)\sqrt{\kappa}) = S_4\cos(d(x,w)\sqrt{\kappa}).
\end{aligned}$$

Similarly, one gets that $S_3 - S_1 \leq S_5\cos(d(x,v)\sqrt{\kappa})$.

$$\begin{aligned}
S_2C_1 - S_1C_2 &= \sin(d(x,y)\sqrt{\kappa})\sin(d(x,z)\sqrt{\kappa})\cos(d(x,w)\sqrt{\kappa})\cos(d(x,v)\sqrt{\kappa}) \\
&\quad - \sin(d(x,w)\sqrt{\kappa})\sin(d(x,v)\sqrt{\kappa})\cos(d(x,y)\sqrt{\kappa})\cos(d(x,z)\sqrt{\kappa}) \\
&= \sin(d(x,z)\sqrt{\kappa})\cos(d(x,v)\sqrt{\kappa})\sin\big((d(x,y)-d(x,w))\sqrt{\kappa}\big) \\
&\quad + \sin(d(x,w)\sqrt{\kappa})\cos(d(x,y)\sqrt{\kappa})\sin\big((d(x,z)-d(x,v))\sqrt{\kappa}\big) \\
&= \sin(d(x,z)\sqrt{\kappa})\cos(d(x,v)\sqrt{\kappa})\sin(d(y,w)\sqrt{\kappa}) \\
&\quad + \sin(d(x,w)\sqrt{\kappa})\cos(d(x,y)\sqrt{\kappa})\sin(d(z,v)\sqrt{\kappa}) \\
&= S_4\cos(d(x,v)\sqrt{\kappa}) + S_5\cos(d(x,y)\sqrt{\kappa}).
\end{aligned}$$

Items (13) and (14) follow easily from (10) and (11), respectively. ☐

Proposition 4.4.

$$\sin^2\frac{d(w,v)\sqrt{\kappa}}{2} \leq \frac{S_1}{S_2}\sin^2\frac{d(y,z)\sqrt{\kappa}}{2} + \frac{1}{2}(1-C_1) - \frac{S_1}{2S_2}(1-C_2). \tag{15}$$

Proof. Let $\Delta(\bar{x},\bar{y},\bar{z})$ be a κ-comparison triangle for $\Delta(x,y,z)$. Denote $\alpha = \angle_{\bar{x}}(\bar{y},\bar{z}) = \angle_{\bar{x}}(\bar{w},\bar{v})$. Using the cosine law we have

$$\begin{aligned}
\cos(d(\bar{w},\bar{v})\sqrt{\kappa}) &= \cos(d(\bar{x},\bar{w})\sqrt{\kappa})\cos(d(\bar{x},\bar{v})\sqrt{\kappa}) \\
&\quad + \sin(d(\bar{x},\bar{w})\sqrt{\kappa})\sin(d(\bar{x},\bar{v})\sqrt{\kappa})\cos\alpha
\end{aligned}$$

and

$$\begin{aligned}
\cos(d(\bar{y},\bar{z})\sqrt{\kappa}) &= \cos(d(\bar{x},\bar{y})\sqrt{\kappa})\cos(d(\bar{x},\bar{z})\sqrt{\kappa}) \\
&\quad + \sin(d(\bar{x},\bar{y})\sqrt{\kappa})\sin(d(\bar{x},\bar{z})\sqrt{\kappa})\cos\alpha.
\end{aligned}$$

Thus,

$$\begin{aligned}
\cos(d(\bar{w},\bar{v})\sqrt{\kappa}) &= \cos(d(\bar{x},\bar{w})\sqrt{\kappa})\cos(d(\bar{x},\bar{v})\sqrt{\kappa}) \\
&\quad + \frac{\sin(d(\bar{x},\bar{w})\sqrt{\kappa})\sin(d(\bar{x},\bar{v})\sqrt{\kappa})}{\sin(d(\bar{x},\bar{y})\sqrt{\kappa})\sin(d(\bar{x},\bar{z})\sqrt{\kappa})}\Big(\cos(d(\bar{y},\bar{z})\sqrt{\kappa}) \\
&\quad - \cos(d(\bar{x},\bar{y})\sqrt{\kappa})\cos(d(\bar{x},\bar{z})\sqrt{\kappa})\Big) \\
&= \frac{S_1}{S_2}\cos(d(y,z)\sqrt{\kappa}) + C_1 - \frac{S_1}{S_2}C_2.
\end{aligned}$$

It follows that

$$\frac{1-\cos(d(w,v)\sqrt{\kappa})}{2} \quad \leq \quad \frac{1}{2}+\frac{S_1}{S_2}\left(\frac{1-\cos(d(y,z)\sqrt{\kappa})}{2}-\frac{1}{2}\right)-\frac{1}{2}C_1+\frac{S_1}{2S_2}C_2.$$

Hence,

$$\sin^2\frac{d(w,v)\sqrt{\kappa}}{2} \quad \leq \quad \frac{S_1}{S_2}\sin^2\frac{d(y,z)\sqrt{\kappa}}{2}+\frac{1}{2}(1-C_1)-\frac{S_1}{2S_2}(1-C_2).$$

□

Proposition 4.5. *(i)*

$$\begin{aligned}\sin^2\frac{d(w,v)\sqrt{\kappa}}{2} \leq \frac{\sin(d(x,w)\sqrt{\kappa})}{\sin(d(x,y)\sqrt{\kappa})}\sin^2\frac{d(y,z)\sqrt{\kappa}}{2} \\ + \frac{\sin(d(y,w)\sqrt{\kappa})}{\sin(d(x,y)\sqrt{\kappa})}\left(\sin^2\frac{d(x,v)\sqrt{\kappa}}{2}-\sin^2\frac{d(x,w)\sqrt{\kappa}}{2}\right) \\ + \frac{\sin(d(z,v)\sqrt{\kappa})}{\sin(d(x,z)\sqrt{\kappa})}\sin^2\frac{d(x,y)\sqrt{\kappa}}{2}.\end{aligned}$$

(ii) Assume that $v = sx+(1-s)z$, $s\in[0,1]$ *and* $w = rx+(1-r)y$, $r\in[0,1]$. *Then,*

$$\begin{aligned}\sin^2\frac{d(w,v)\sqrt{\kappa}}{2} \leq \frac{\sin((1-r)M\sqrt{\kappa})}{\sin(M\sqrt{\kappa})}\sin^2\frac{d(y,z)\sqrt{\kappa}}{2} \\ + \frac{\sin(rM\sqrt{\kappa})}{\sin(M\sqrt{\kappa})}\max\left\{\sin^2\frac{d(x,v)\sqrt{\kappa}}{2}-\sin^2\frac{d(x,w)\sqrt{\kappa}}{2},0\right\} \\ + \frac{\sin(sM\sqrt{\kappa})}{\sin(M\sqrt{\kappa})}\sin^2\frac{M\sqrt{\kappa}}{2}.\end{aligned}$$

Proof. (i) We apply Proposition 4.4 to get that

$$\begin{aligned}\sin^2\frac{d(w,v)\sqrt{\kappa}}{2} &\leq \frac{S_1}{S_2}\sin^2\frac{d(y,z)\sqrt{\kappa}}{2}+\frac{1}{2}(1-C_1)-\frac{S_1}{2S_2}(1-C_2) \\ &= \frac{S_1}{S_2}\sin^2\frac{d(y,z)\sqrt{\kappa}}{2} \\ &\quad + \frac{S_2-S_1-S_4\cos(d(x,v)\sqrt{\kappa})-S_5\cos(d(x,y)\sqrt{\kappa})}{2S_2} \\ &\quad \text{by (12)} \\ &\leq \frac{S_1}{S_2}\sin^2\frac{d(y,z)\sqrt{\kappa}}{2}+\frac{S_4}{S_2}\left(\sin^2\frac{d(x,v)\sqrt{\kappa}}{2}-\sin^2\frac{d(x,w)\sqrt{\kappa}}{2}\right) \\ &\quad + \frac{S_5}{S_2}\sin^2\frac{d(x,y)\sqrt{\kappa}}{2} \\ &\quad \text{by (13) and (14),}\end{aligned}$$

which yields the desired inequality.

(ii) We have that

$$\sin^2\frac{d(w,v)\sqrt{\kappa}}{2} \le \frac{\sin((1-r)d(x,y)\sqrt{\kappa})}{\sin(d(x,y)\sqrt{\kappa})}\sin^2\frac{d(y,z)\sqrt{\kappa}}{2}$$
$$+\frac{\sin(rd(x,y)\sqrt{\kappa})}{\sin(d(x,y)\sqrt{\kappa})}\max\left\{\sin^2\frac{d(x,v)\sqrt{\kappa}}{2}-\sin^2\frac{d(x,w)\sqrt{\kappa}}{2},0\right\}$$
$$+\frac{\sin(sd(x,z)\sqrt{\kappa})}{\sin(d(x,z)\sqrt{\kappa})}\sin^2\frac{d(x,y)\sqrt{\kappa}}{2}$$
$$\le \frac{\sin((1-r)M\sqrt{\kappa})}{\sin(M\sqrt{\kappa})}\sin^2\frac{d(y,z)\sqrt{\kappa}}{2}$$
$$+\frac{\sin(rM\sqrt{\kappa})}{\sin(M\sqrt{\kappa})}\max\left\{\sin^2\frac{d(x,v)\sqrt{\kappa}}{2}-\sin^2\frac{d(x,w)\sqrt{\kappa}}{2},0\right\}$$
$$+\frac{\sin(sM\sqrt{\kappa})}{\sin(M\sqrt{\kappa})}\sin^2\frac{M\sqrt{\kappa}}{2}.$$

□

For the rest of the section, we assume that $v = sx + (1-s)z$, $s \in (0,1)$. We use the additional notation

$$L_1 = \frac{S_1}{S_3} = \frac{\sin(d(x,v)\sqrt{\kappa})}{\sin(d(x,z)\sqrt{\kappa})}, \quad L_2 = \frac{S_5}{S_3} = \frac{\sin(d(v,z)\sqrt{\kappa})}{\sin(d(x,z)\sqrt{\kappa})}.$$

Lemma 4.6.

$$0 < 1 - L_1 \le L_2\cos(d(x,v)\sqrt{\kappa}), \tag{16}$$
$$\frac{L_1}{1-L_1} \le \frac{1}{s\cos(M\sqrt{\kappa})}. \tag{17}$$

Proof.

$$(1-L_1)\sin(d(x,z)\sqrt{\kappa}) = \sin(d(x,z)\sqrt{\kappa}) - \sin(d(x,v)\sqrt{\kappa})$$
$$= 2\sin\frac{(d(x,z)-d(x,v))\sqrt{\kappa}}{2}\cos\frac{(d(x,z)+d(x,v))\sqrt{\kappa}}{2}$$
$$\le 2\sin\frac{d(z,v)\sqrt{\kappa}}{2}\cos\frac{d(z,v)\sqrt{\kappa}}{2}\cos(d(x,v)\sqrt{\kappa})$$
$$= \sin(d(z,v)\sqrt{\kappa})\cos(d(x,v)\sqrt{\kappa}).$$

Thus, $1 - L_1 \le L_2\cos(d(x,v)\sqrt{\kappa})$.

$$\frac{L_1}{1-L_1} = \frac{\sin(d(x,v)\sqrt{\kappa})}{\sin(d(x,z)\sqrt{\kappa}) - \sin(d(x,v)\sqrt{\kappa})} \le \frac{\sin(d(x,z)\sqrt{\kappa})}{\sin(d(x,z)\sqrt{\kappa}) - \sin(d(x,v)\sqrt{\kappa})}$$
$$= \frac{\sin(d(x,z)\sqrt{\kappa})}{2\sin\frac{d(z,v)\sqrt{\kappa}}{2}\cos\left(\left(d(x,z)-\frac{d(z,v)}{2}\right)\sqrt{\kappa}\right)}$$
$$\le \frac{\sin(d(x,z)\sqrt{\kappa})}{2\sin\frac{sd(x,z)\sqrt{\kappa}}{2}\cos(d(x,z)\sqrt{\kappa})} \le \frac{1}{s\cos(M\sqrt{\kappa})}.$$

□

Proposition 4.7. *(i)*

$$\sin^2\frac{d(y,v)\sqrt{\kappa}}{2}\le L_1\sin^2\frac{d(y,z)\sqrt{\kappa}}{2}+\frac{1-L_1}{2}-\frac{1}{2}\cos(d(x,y)\sqrt{\kappa})L_2$$
$$=L_2\sin^2\frac{d(x,y)\sqrt{\kappa}}{2}+\frac{1}{2}(1-L_1-L_2)+L_1\sin^2\frac{d(y,z)\sqrt{\kappa}}{2}.$$

(ii) Let $q\in C$ be such that $d(q,z)\le d(y,v)$. Assume that

$$\sin^2\frac{d(x,y)\sqrt{\kappa}}{2}-\sin^2\frac{d(x,v)\sqrt{\kappa}}{2}\le 0. \tag{18}$$

Then,

$$\sin^2\frac{d(y,v)\sqrt{\kappa}}{2} \le \sin^2\frac{d(x,y)\sqrt{\kappa}}{2}-\sin^2\frac{d(x,v)\sqrt{\kappa}}{2} +\frac{1}{s\cos(M\sqrt{\kappa})}\left(\sin^2\frac{d(y,q)\sqrt{\kappa}}{2}+\sin\frac{d(y,q)\sqrt{\kappa}}{2}\right).$$

Proof. (i) We apply Proposition 4.4 with $w=y$ to get that

$$\begin{aligned}\sin^2\frac{d(y,v)\sqrt{\kappa}}{2}&\le L_1\sin^2\frac{d(y,z)\sqrt{\kappa}}{2}+\frac{1}{2}(1-\cos(d(x,y)\sqrt{\kappa})\cos(d(x,v)\sqrt{\kappa}))\\&\quad-\frac{1}{2}L_1(1-\cos(d(x,y)\sqrt{\kappa})\cos(d(x,z)\sqrt{\kappa}))\\&=L_1\sin^2\frac{d(y,z)\sqrt{\kappa}}{2}+\frac{1-L_1}{2}\\&\quad-\frac{\cos(d(x,y)\sqrt{\kappa})}{2\sin(d(x,z)\sqrt{\kappa})}\big(\cos(d(x,v)\sqrt{\kappa})\sin(d(x,z)\sqrt{\kappa})-\\&\quad-\sin(d(x,v)\sqrt{\kappa})\cos(d(x,z)\sqrt{\kappa})\big)\\&=L_1\sin^2\frac{d(y,z)\sqrt{\kappa}}{2}+\frac{1-L_1}{2}-\frac{\cos(d(x,y)\sqrt{\kappa})}{2\sin(d(x,z)\sqrt{\kappa})}\sin(d(v,z)\sqrt{\kappa})\\&=L_1\sin^2\frac{d(y,z)\sqrt{\kappa}}{2}+\frac{1-L_1}{2}-\frac{1}{2}\cos(d(x,y)\sqrt{\kappa})L_2.\end{aligned}$$

(ii)

$$\begin{aligned}\sin^2\frac{d(y,v)\sqrt{\kappa}}{2}&\le L_2\sin^2\frac{d(x,y)\sqrt{\kappa}}{2}+\frac{1}{2}(1-L_1-L_2)+L_1\sin^2\frac{d(y,z)\sqrt{\kappa}}{2}\\&\le L_2\sin^2\frac{d(x,y)\sqrt{\kappa}}{2}+\frac{1}{2}(1-L_1-L_2)+L_1\sin^2\frac{(d(y,q)+d(q,z))\sqrt{\kappa}}{2}\\&\le L_2\sin^2\frac{d(x,y)\sqrt{\kappa}}{2}+\frac{1}{2}(1-L_1-L_2)+L_1\sin^2\frac{(d(y,q)+d(y,v))\sqrt{\kappa}}{2}\\&\le L_2\sin^2\frac{d(x,y)\sqrt{\kappa}}{2}+\frac{1}{2}(1-L_1-L_2)\\&\quad+L_1\left(\sin^2\frac{d(y,q)\sqrt{\kappa}}{2}+\sin^2\frac{d(y,v)\sqrt{\kappa}}{2}+\frac{1}{2}\sin(d(y,q)\sqrt{\kappa})\right)\end{aligned}$$

since $\sin^2\frac{a+b}{2}\le\sin^2\frac{a}{2}+\sin^2\frac{b}{2}+\frac{1}{2}\sin a$ for $a,b\in[0,\pi]$.

It follows that

$$\begin{aligned}\sin^2\frac{d(y,v)\sqrt{\kappa}}{2}(1-L_1) &\leq L_2\sin^2\frac{d(x,y)\sqrt{\kappa}}{2}+\frac{1}{2}(1-L_1-L_2)\\ &\quad + L_1\left(\sin^2\frac{d(y,q)\sqrt{\kappa}}{2}+\frac{1}{2}\sin(d(y,q)\sqrt{\kappa})\right)\\ &\leq L_2\sin^2\frac{d(x,y)\sqrt{\kappa}}{2}+\frac{1}{2}(1-L_1-L_2)\\ &\quad + L_1\left(\sin^2\frac{d(y,q)\sqrt{\kappa}}{2}+\sin\frac{d(y,q)\sqrt{\kappa}}{2}\right)\\ &\leq L_2\sin^2\frac{d(x,y)\sqrt{\kappa}}{2}-\frac{1}{2}L_2(1-\cos(d(x,v)\sqrt{\kappa}))\\ &\quad + L_1\left(\sin^2\frac{d(y,q)\sqrt{\kappa}}{2}+\sin\frac{d(y,q)\sqrt{\kappa}}{2}\right)\\ &\quad \text{by (16).}\end{aligned}$$

Thus,

$$\begin{aligned}\sin^2\frac{d(y,v)\sqrt{\kappa}}{2} \quad &\leq \quad \frac{L_2}{1-L_1}\left(\sin^2\frac{d(x,y)\sqrt{\kappa}}{2}-\sin^2\frac{d(x,v)\sqrt{\kappa}}{2}\right)\\ &\quad +\frac{L_1}{1-L_1}\left(\sin^2\frac{d(y,q)\sqrt{\kappa}}{2}+\sin\frac{d(y,q)\sqrt{\kappa}}{2}\right).\end{aligned}$$

By assumption, we have that $\sin^2\frac{d(x,y)\sqrt{\kappa}}{2}-\sin^2\frac{d(x,v)\sqrt{\kappa}}{2}\leq 0$. Using the fact that $\frac{L_2}{1-L_1}\geq 1$ and (17), it follows that

$$\begin{aligned}\sin^2\frac{d(y,v)\sqrt{\kappa}}{2} \quad &\leq \quad \sin^2\frac{d(x,y)\sqrt{\kappa}}{2}-\sin^2\frac{d(x,v)\sqrt{\kappa}}{2}\\ &\quad +\frac{1}{s\cos(M\sqrt{\kappa})}\left(\sin^2\frac{d(y,q)\sqrt{\kappa}}{2}+\sin\frac{d(y,q)\sqrt{\kappa}}{2}\right).\end{aligned}$$

□

5 Effective rates of asymptotic regularity

We assume the hypothesis of Proposition 3.2. As in [23, 16, 17], the main tool in obtaining rates of asymptotic regularity is the following quantitative lemma, which is a slight reformulation of [17, Lemma 1].

Lemma 5.1. *Let $(\alpha_n)_{n\geq 1}$ be a sequence in $[0,1]$ and $(a_n)_{n\geq 1},(b_n)_{n\geq 1}$ be sequences in $\mathbb{R}_+$ such that*

$$a_{n+1}\leq (1-\alpha_{n+1})a_n+b_n \quad \textit{for all } n\in\mathbb{Z}_+. \tag{19}$$

Assume that $\sum_{n=1}^{\infty} b_n$ *is convergent with Cauchy modulus* γ *and* $\sum_{n=1}^{\infty} \alpha_{n+1}$ *diverges with rate of divergence* θ.

Then, $\lim_{n\to\infty} a_n = 0$ *with rate of convergence* Σ *given by*

$$\Sigma(\varepsilon, P, \gamma, \theta) = \theta\left(\gamma\left(\frac{\varepsilon}{2}\right) + \max\left\{\left\lceil \ln\left(\frac{2P}{\varepsilon}\right)\right\rceil, 1\right\}\right) + 1, \tag{20}$$

where $P > 0$ *is an upper bound on* (a_n).

A second useful result, which is also needed in the metastability proof, is the following:

Lemma 5.2. *For all* $n \geq 1$, *let*

$$\mu_n = 1 - \frac{\sin\left((1-\lambda_n)M\sqrt{\kappa}\right)}{\sin(M\sqrt{\kappa})} \in (0, 1). \tag{21}$$

Then

(i) $\mu_n \geq \lambda_n \cos(M\sqrt{\kappa})$ *for all* $n \geq 1$.

(ii) $\sum_{n=1}^{\infty} \lambda_{n+1} = \infty$ *with rate of divergence* θ *yields* $\sum_{n=1}^{\infty} \mu_{n+1} = \infty$ *with rate of divergence* $\tilde{\theta}(n) = \theta\left(\left\lceil \frac{1}{\cos(M\sqrt{\kappa})}\right\rceil n\right)$.

Proof. (i) One has

$$\begin{aligned} \mu_n &= \frac{2\sin\frac{\lambda_n M\sqrt{\kappa}}{2}\cos\frac{(2-\lambda_n)M\sqrt{\kappa}}{2}}{\sin(M\sqrt{\kappa})} \geq \frac{2\sin\frac{\lambda_n M\sqrt{\kappa}}{2}\cos(M\sqrt{\kappa})}{\sin(M\sqrt{\kappa})} \\ &\geq \lambda_n \cos\left(M\sqrt{\kappa}\right). \end{aligned}$$

(ii) Follows immediately from (i).

∎

Lemma 5.3. *For all* $n \in \mathbb{Z}_+$

$$d(x_n, x_{n+1}) \leq (1 - \mu_{n+1})d(x_{n-1}, x_n) + M|\lambda_{n+1} - \lambda_n|. \tag{22}$$

Proof. Let us denote for simplicity $u_n = \lambda_{n+1}u + (1-\lambda_{n+1})Tx_{n-1}$. Then,

$$\begin{aligned} d(x_n, u_n) &= |\lambda_{n+1} - \lambda_n| d(u, Tx_{n-1}) \leq M|\lambda_{n+1} - \lambda_n| \quad \text{and} \\ d(u_n, x_{n+1}) &\leq \frac{\sin\left((1-\lambda_{n+1})M\sqrt{\kappa}\right)}{\sin\left(M\sqrt{\kappa}\right)} d(x_{n-1}, x_n) \quad \text{by Lemma 4.1.} \end{aligned}$$

∎

5.1 Proof of Proposition 3.2

Let $\tilde{\Phi}$, Φ be given by (6) and (7). Apply Lemma 5.1 with

$$a_n = d(x_n, x_{n-1}), \quad b_n = M|\lambda_{n+1} - \lambda_n| \quad \text{and} \quad \alpha_n = \mu_n,$$

and use Lemma 5.2.(ii) and the fact that $\sum_{n=1}^{\infty} b_n$ is convergent with Cauchy modulus $\tilde{\gamma}(\varepsilon) = \gamma\left(\frac{\varepsilon}{M}\right)$ to conclude that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ with rate of convergence $\tilde{\Phi}$.

Since $d(x_n, Tx_n) \le d(x_n, x_{n+1}) + M\lambda_{n+1}$ for all $n \ge 1$, it follows easily that Φ is a rate of asymptotic regularity. □

6 A quantitative Browder theorem

Let X be a complete CAT(κ) space, $C \subseteq X$ a bounded closed convex subset with diameter $d_C < \frac{D_\kappa}{2}$ and $T : C \to C$ be nonexpansive.

A very important step in the convergence proof for Halpern iterations is the construction of a sequence of approximants converging strongly to a fixed point of T. Given $t \in (0,1)$ and $u \in C$, Lemma 4.1 yields that the mapping

$$T_t^u : C \to C, \quad T_t^u(y) = tu + (1-t)Ty \tag{23}$$

is a contraction, hence it has a unique fixed point $z_t^u \in C$. Thus,

$$z_t^u = tu + (1-t)Tz_t^u. \tag{24}$$

Piątek [26] obtained the following generalization to CAT(κ) spaces of an essential result due to Browder [5, 6].

Theorem 6.1. *[26] In the above hypothesis, $\lim_{t\to 0^+} z_t^u$ exists and is a fixed point of T.*

In the setting of Hilbert spaces, Browder proved the result using weak sequential compactness and a projection argument (to the set of fixed points of T). A new and elementary proof of Browder's result was given by Halpern [9] when C is the closed unit ball and the starting point is $u = 0$. Generalizations of Browder's theorem were obtained by Reich [28] for uniformly smooth Banach spaces, Goebel and Reich [8] for the Hilbert ball and Kirk [10] for CAT(0) spaces.

Kohlenbach [12] applied proof mining methods to both Browder's original proof and the extension of Halpern's proof to bounded closed convex C and arbitrary $u \in C$, obtaining in this way quantitative versions of Browder's theorem with uniform effective rates of metastability. As pointed out in [12, Remark 1.4], one cannot expect in general to get effective rates of convergence. Since Kirk's proof of the generalization of Browder's theorem to CAT(0) spaces is obtained by a slight change of Halpern's argument, Kohlenbach's quantitative result goes through basically unchanged to CAT(0) spaces (see [17, Proposition 9.3]).

In this section we obtain a quantitative version of Theorem 6.1. As a consequence of Halpern's proof, for any nonincreasing sequence (t_n) in $(0,1)$, one gets that $(z^u_{t_n})$ converges strongly to some point $z \in C$, which is a fixed point of T if $\lim\limits_{n\to\infty} t_n = 0$. Our quantitative result gives rates of metastability for such sequences $(z^u_{t_n})$ and this suffices for the proof of our main Theorem 3.4.

Proposition 6.2. *Let X be a complete CAT(κ) space, $C \subseteq X$ bounded closed convex with diameter $d_C < \dfrac{D_\kappa}{2}$ and $T : C \to C$ be nonexpansive. Assume that $(t_n) \subseteq (0,1)$ is nonincreasing. Then for every $\varepsilon \in (0,1)$ and $g : \mathbb{N} \to \mathbb{N}$,*

$$\exists K_0 \le K(\varepsilon, g, M)\ \forall i,j \in [K_0, K_0 + g(K_0)]\ \Big(d(z^u_{t_i}, z^u_{t_j}) \le \frac{\varepsilon}{\sqrt{\kappa}}\Big),$$

where

$$K(\varepsilon, g, M) = \widetilde{g}^{\left(\left\lceil \frac{M\sqrt{\kappa}\tan(M\sqrt{\kappa})}{1-\cos\varepsilon}\right\rceil\right)}(0),$$

with $d_C \le M < \dfrac{D_\kappa}{2}$ and $\widetilde{g}(n) = n + g(n)$.

Proof. Let $\varepsilon \in (0,1)$ and $g : \mathbb{N} \to \mathbb{N}$. We assume without loss of generality that $i < j$, hence $t_j \le t_i$. Denote $u_{i,j} = t_j u + (1-t_j)Tz^u_{t_i}$. Then,

$$d(u, z^u_{t_i}) = (1-t_i)d(u, Tz^u_{t_i}) \le (1-t_j)d(u, Tz^u_{t_i}) = d(u, u_{i,j}),$$

so $z^u_{t_i} \in [u, u_{i,j}]$. It follows by Lemma 4.1 that $d(z^u_{t_j}, u_{i,j}) \le d(Tz^u_{t_j}, Tz^u_{t_i}) \le d(z^u_{t_j}, z^u_{t_i})$.

We can apply now Lemma 4.2 with $x = u, y = z^u_{t_j}, z = u_{i,j}, w = z^u_{t_i}$ to get that $d(u, z^u_{t_i}) \le d(u, z^u_{t_j})$ and for $\Delta(\bar{u}, \bar{z}^u_{t_j}, \bar{u}_{i,j})$ a κ-comparison triangle of $\Delta(u, z^u_{t_j}, u_{i,j})$, one has $\angle_{\bar{z}^u_{t_i}}(\bar{u}, \bar{z}^u_{t_j}) \ge \pi/2$. Since $(d(u, z^u_{t_n}))_n$ is a nondecreasing sequence in $[0, M]$, by an application of [12, Lemma 4.1], there exists $K_0 \le K(\varepsilon, g, M)$ such that

$$\forall i,j \in [K_0, K_0 + g(K_0)] \left(|d(u, z^u_{t_j}) - d(u, z^u_{t_i})| \le \frac{1-\cos\varepsilon}{\sqrt{\kappa}\tan(M\sqrt{\kappa})} \right).$$

Let now $i < j \in [K_0, K_0 + g(K_0)]$. Then,

$$\begin{aligned} \cos(d(u, z^u_{t_i})\sqrt{\kappa}) - \cos(d(u, z^u_{t_j})\sqrt{\kappa}) &\le \sin(M\sqrt{\kappa})\frac{1-\cos\varepsilon}{\tan(M\sqrt{\kappa})} \\ &= (1-\cos\varepsilon)\cos(M\sqrt{\kappa}). \end{aligned}$$

Furthermore, by the cosine law and the fact that $\angle_{\bar{z}^u_{t_i}}(\bar{u}, \bar{z}^u_{t_j}) \ge \frac{\pi}{2}$, we have that

$$\cos(d(\bar{u}, \bar{z}^u_{t_j})\sqrt{\kappa}) \le \cos(d(\bar{u}, \bar{z}^u_{t_i})\sqrt{\kappa})\cos(d(\bar{z}^u_{t_i}, \bar{z}^u_{t_j})\sqrt{\kappa}).$$

It follows that

$$\begin{aligned}
&\cos(d(u, z^u_{t_i})\sqrt{\kappa}) - (1-\cos\varepsilon)\cos(M\sqrt{\kappa}) \\
&\quad \le \cos(d(u, z^u_{t_j})\sqrt{\kappa}) = \cos(d(\bar{u}, \bar{z}^u_{t_j})\sqrt{\kappa}) \\
&\quad \le \cos(d(\bar{u}, \bar{z}^u_{t_i})\sqrt{\kappa})\cos(d(\bar{z}^u_{t_j}, \bar{z}^u_{t_i})\sqrt{\kappa}) \\
&\quad \le \cos(d(u, z^u_{t_i})\sqrt{\kappa})\cos(d(z^u_{t_j}, z^u_{t_i})\sqrt{\kappa}).
\end{aligned}$$

Hence

$$\cos(d(z^u_{t_j}, z^u_{t_i})\sqrt{\kappa}) \ge 1 - (1-\cos\varepsilon)\frac{\cos(M\sqrt{\kappa})}{\cos(d(u, z^u_{t_i})\sqrt{\kappa})} \ge \cos\varepsilon.$$

Thus, $d(z^u_{t_j}, z^u_{t_i})\sqrt{\kappa} \le \varepsilon$ and the proof is complete. □

7 Effective rates of metastability

In this section we shall prove the main result of our paper, Theorem 3.4, hence we assume that its hypotheses are satisfied. We give first some technical results that will be needed in the proof.

7.1 Some useful lemmas

As in [16, 17], one of the main ingredients of our proof is a sequence obtained by combining the Halpern iteration (x_n) and the points z^u_t. However, in the setting of CAT(κ) spaces, its definition and the proofs of the necessary properties are based on the much more involved technical lemmas from Section 4.

If (a_n) is a real sequence, we say that $\limsup_{n\to\infty} a_n \le 0$ with effective rate $\Psi : (0,\infty) \to \mathbb{Z}_+$ if

$$\forall \varepsilon > 0\, \forall n \ge \Psi(\varepsilon)\ (a_n \le \varepsilon).$$

Let us define

$$\gamma^t_n = \sin^2\frac{d(u, z^u_t)\sqrt{\kappa}}{2} - \sin^2\frac{d(u, x_{n+1})\sqrt{\kappa}}{2}. \tag{25}$$

Proposition 7.1. *(i) For $n \ge 1$, if $\gamma^t_n \ge 0$, then*

$$\gamma^t_n \le \frac{a_n}{t} - \sin^2\frac{d(x_{n+1}, z^u_t)\sqrt{\kappa}}{2}, \tag{26}$$

where

$$a_n = \frac{1}{\cos(M\sqrt{\kappa})}\left(\sin^2\frac{d(x_{n+1}, Tx_{n+1})\sqrt{\kappa}}{2} + \sin\frac{d(x_{n+1}, Tx_{n+1})\sqrt{\kappa}}{2}\right). \tag{27}$$

(ii) $\gamma^t_n \le \dfrac{a_n}{t}$ for all $n \ge 1$.

(iii) $\limsup_{n\to\infty} \gamma_n^t \le 0$ *with effective rate* $\Psi(\varepsilon, \kappa, M, t, \gamma, \theta, \alpha)$ *given by*

$$\Psi = \max\left\{\theta\left(\left\lceil \frac{1}{\cos(M\sqrt{\kappa})}\right\rceil \left(\gamma(L) + \max\left\{\left\lceil \ln\left(\frac{1}{L}\right)\right\rceil, 1\right\}\right)\right), \alpha(2L)\right\}, \quad (28)$$

where $L = \dfrac{\cos(M\sqrt{\kappa})t\varepsilon}{4M\sqrt{\kappa}}$.

(iv) *For* $n \ge 1$,

$$\begin{aligned}\sin^2 \frac{d(x_{n+1}, z_t^u)\sqrt{\kappa}}{2} &\le \frac{\sin\left((1-\lambda_{n+1})M\sqrt{\kappa}\right)}{\sin(M\sqrt{\kappa})} \sin^2 \frac{d(x_n, z_t^u)\sqrt{\kappa}}{2} \\ &\quad + \frac{\sin\left(\lambda_{n+1}M\sqrt{\kappa}\right)}{\sin(M\sqrt{\kappa})} \max\{\gamma_n^t, 0\} \\ &\quad + \frac{\sin\left(tM\sqrt{\kappa}\right)}{\sin(M\sqrt{\kappa})} \sin^2 \frac{M\sqrt{\kappa}}{2}.\end{aligned}$$

Proof. (i) Apply Proposition 4.7.(ii) with $x = u, y = x_{n+1}, z = Tz_t^u, v = z_t^u, s = t, q = Tx_{n+1}$ and note that

$$\sin^2 \frac{d(u, x_{n+1})\sqrt{\kappa}}{2} - \sin^2 \frac{d(u, z_t^u)\sqrt{\kappa}}{2} = -\gamma_n^t \le 0.$$

It follows that $\sin^2 \dfrac{d(x_{n+1}, z_t^u)\sqrt{\kappa}}{2} \le -\gamma_n^t + \dfrac{a_n}{t}$, hence (i).

(ii) Obviously, since $\dfrac{a_n}{t} \ge 0$.

(iii) Since $a_n \le \dfrac{1}{\cos(M\sqrt{\kappa})} d(x_{n+1}, Tx_{n+1})\sqrt{\kappa}$ and, by Proposition 3.2, the sequence $(d(x_n, Tx_n))$ converges to 0 with rate of convergence Φ given by (7), we get that $\limsup_{n\to\infty} \gamma_n^t \le 0$ with effective rate

$$\Psi(\varepsilon, \kappa, M, t, \gamma, \theta, \alpha) = \Phi\left(\frac{\cos(M\sqrt{\kappa})t\varepsilon}{\sqrt{\kappa}}, \kappa, M, \gamma, \theta, \alpha\right).$$

(iv) By Proposition 4.5.(ii) with $x = u, y = Tx_n, z = Tz_t^u, w = x_{n+1}, v = z_t^u, r = \lambda_{n+1}$ and $s = t$.

□

In fact, it suffices for the proof of the main theorem to consider the case $t_i = \dfrac{1}{i+1}, i \ge 0$. Then (t_i) converges towards 0 with rate $\left\lceil \dfrac{1}{\varepsilon} \right\rceil$.

We shall denote $\gamma_n^{t_i}$ with γ_n^i. Furthermore, $z_{t_i}^u$ will be simply denoted by z_i^u. Thus,

$$\gamma_n^i = \sin^2 \frac{d(u, z_i^u)\sqrt{\kappa}}{2} - \sin^2 \frac{d(u, x_{n+1})\sqrt{\kappa}}{2}. \quad (29)$$

Lemma 7.2. *Assume that $i, j \geq 0$ and $\delta \in (0,1)$ are such that $d(u, Tz_i^u) - d(u, Tz_j^u) \leq \frac{\delta}{\sqrt{\kappa}}$. Then,*

$$\gamma_n^i \leq \gamma_n^j + \sin^2 \frac{M\sqrt{\kappa}}{2(j+1)} + 2\sin \frac{M\sqrt{\kappa}}{2(j+1)} + \sin^2 \frac{\delta}{2} + 2\sin\frac{\delta}{2} \sin \frac{M\sqrt{\kappa}}{2}.$$

Proof. We have that

$$\begin{aligned}
\gamma_n^i &= \sin^2 \frac{\frac{i}{i+1} d(u, Tz_i^u)\sqrt{\kappa}}{2} - \sin^2 \frac{d(u, x_{n+1})\sqrt{\kappa}}{2} \\
&\leq \sin^2 \frac{d(u, Tz_i^u)\sqrt{\kappa}}{2} - \sin^2 \frac{d(u, x_{n+1})\sqrt{\kappa}}{2} \\
&\leq \left(\sin \frac{d(u, Tz_j^u)\sqrt{\kappa}}{2} + \sin \frac{\delta}{2} \right)^2 - \sin^2 \frac{d(u, x_{n+1})\sqrt{\kappa}}{2} \\
&\leq \sin^2 \frac{d(u, Tz_j^u)\sqrt{\kappa}}{2} - \sin^2 \frac{d(u, x_{n+1})\sqrt{\kappa}}{2} + \sin^2\frac{\delta}{2} + 2\sin\frac{\delta}{2}\sin\frac{M\sqrt{\kappa}}{2}.
\end{aligned}$$

Note that

$$\begin{aligned}
\sin^2 \frac{d(u, Tz_j^u)\sqrt{\kappa}}{2} &= \sin^2 \frac{\frac{j}{j+1} d(u, Tz_j^u)\sqrt{\kappa} + \frac{1}{j+1} d(u, Tz_j^u)\sqrt{\kappa}}{2} \\
&\leq \sin^2 \frac{\frac{j}{j+1} d(u, Tz_j^u)\sqrt{\kappa}}{2} + \sin^2 \frac{M\sqrt{\kappa}}{2(j+1)} + 2\sin\frac{M\sqrt{\kappa}}{2(j+1)}.
\end{aligned}$$

□

Finally, let us recall the following slight reformulation of [17, Lemma 5.2].

Lemma 7.3. *Let $\varepsilon \in (0,2)$, $g : \mathbb{N} \to \mathbb{N}$, $L > 0$, $\theta : \mathbb{Z}_+ \to \mathbb{Z}_+$ and $\psi : (0, \infty) \to \mathbb{Z}_+$. Define*

$$\begin{aligned}
\Theta = \Theta(\varepsilon, L, \theta, \psi) &= \theta\left(\psi\left(\frac{\varepsilon}{3}\right) - 1 + \max\left\{ \left\lceil \ln\left(\frac{3L}{\varepsilon}\right) \right\rceil, 1 \right\} \right) + 1, \\
\Delta = \Delta(\varepsilon, g, L, \theta, \psi) &= \frac{\varepsilon}{3 g_\varepsilon(\Theta - \psi(\varepsilon/3))},
\end{aligned}$$

where $g_\varepsilon(n) = n + g(n + \psi(\varepsilon/3))$.
Assume that

(i) (α_n) is a sequence in $[0,1]$ such that the series $\sum_{n=1}^{\infty} \alpha_n$ diverges with rate of divergence θ;

(ii) (t_n) is a sequence of real numbers such that $t_n \leq \frac{\varepsilon}{3}$ for all $n \geq \psi\left(\frac{\varepsilon}{3}\right)$.

Let (s_n) be a bounded sequence of real numbers with upper bound L satisfying

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n t_n + \Delta \qquad \text{for all } n \geq 1. \tag{30}$$

Then $s_n \leq \varepsilon$ for all $n \in [\Theta, \Theta + g(\Theta)]$.

7.2 Proof of Theorem 3.4

Let $\varepsilon \in (0,2)$ and $g : \mathbb{N} \to \mathbb{N}$ be fixed. For simplicity, we omit parameters $\kappa, M, \Phi, \theta, \alpha, \beta$ for all functionals in this proof. Let us define $h : (0,1) \to \mathbb{R}_+$ by

$$h(\delta) = \sin\frac{\delta}{2}\left(\sin\frac{\delta}{2} + 2\sin\frac{M\sqrt{\kappa}}{2}\right) + \sin\frac{\delta M\sqrt{\kappa}}{2}\left(\sin\frac{\delta M\sqrt{\kappa}}{2} + 2\right) \le 6\delta. \quad (31)$$

Take $\varepsilon_0 = \dfrac{\cos(M\sqrt{\kappa})}{36}\sin^2\dfrac{\varepsilon\sqrt{\kappa}}{4}$. Then, $h(\varepsilon_0) \le \dfrac{\cos(M\sqrt{\kappa})}{6}\sin^2\dfrac{\varepsilon\sqrt{\kappa}}{4}$.

Applying Proposition 6.2 for $t_i = \dfrac{1}{i+1}, \varepsilon_0$ and f^*, we get the existence of

$$K_1 \le K(\varepsilon_0, f^*) = \widetilde{f^*}\left(\left\lceil \frac{M\sqrt{\kappa}\tan(M\sqrt{\kappa})}{1-\cos\varepsilon_0}\right\rceil\right)(0)$$

such that $d(z_i^u, z_j^u) \le \dfrac{\varepsilon_0}{\sqrt{\kappa}}$ for all $i, j \in [K_1, \widetilde{f^*}(K_1)]$.

Let $K_0 = K_1 + \left\lceil \dfrac{1}{\varepsilon_0}\right\rceil$ and $J = K_0 + f(K_0) = \widetilde{f^*}(K_1)$. It follows that $d(z_J^u, z_{K_0}^u) \le \dfrac{\varepsilon_0}{\sqrt{\kappa}}$, hence

$$\begin{aligned} d(u, Tz_J^u) &\le d(u, Tz_{K_0}^u) + d(Tz_{K_0}^u, Tz_J^u) \le d(u, Tz_{K_0}^u) + d(z_{K_0}^u, z_J^u) \\ &\le d(u, Tz_{K_0}^u) + \frac{\varepsilon_0}{\sqrt{\kappa}}. \end{aligned}$$

An application of Lemma 7.2 with $i = J$, $j = K_0$ and $\delta = \varepsilon_0$ gives us

$$\begin{aligned} \gamma_n^J &\le \gamma_n^{K_0} + \sin^2\frac{M\sqrt{\kappa}}{2(K_0+1)} + 2\sin\frac{M\sqrt{\kappa}}{2(K_0+1)} + \sin^2\frac{\varepsilon_0}{2} + 2\sin\frac{\varepsilon_0}{2}\sin\frac{M\sqrt{\kappa}}{2} \\ &\le \gamma_n^{K_0} + \sin^2\frac{\varepsilon_0}{2} + 2\sin\frac{\varepsilon_0}{2}\sin\frac{M\sqrt{\kappa}}{2} + \sin^2\frac{M\varepsilon_0\sqrt{\kappa}}{2} + 2\sin\frac{M\varepsilon_0\sqrt{\kappa}}{2} \\ &= \gamma_n^{K_0} + h(\varepsilon_0) \le \gamma_n^{K_0} + \frac{\cos(M\sqrt{\kappa})}{6}\sin^2\frac{\varepsilon\sqrt{\kappa}}{4}. \end{aligned}$$

Applying now Proposition 7.1.(iv) with $t = \frac{1}{J+1}$ and recalling the definition (21) of (μ_n), it follows that for all $n \ge 1$,

$$\begin{aligned} \sin^2\frac{d(x_{n+1}, z_J^u)\sqrt{\kappa}}{2} &\le (1-\mu_{n+1})\sin^2\frac{d(x_n, z_J^u)\sqrt{\kappa}}{2} \\ &\quad + \frac{\sin(\lambda_{n+1}M\sqrt{\kappa})}{\sin(M\sqrt{\kappa})}\max\{\gamma_n^J, 0\} \\ &\quad + \frac{\sin\left(\frac{1}{J+1}M\sqrt{\kappa}\right)}{\sin(M\sqrt{\kappa})}\sin^2\frac{M\sqrt{\kappa}}{2}. \end{aligned}$$

Since $J = K_0 + f(K_0) \ge \left\lceil \dfrac{M\sqrt{\kappa}}{\Delta_{K_0}^*(\sin^2\frac{\varepsilon\sqrt{\kappa}}{4}, g)}\right\rceil$ and

$$\cos(M\sqrt{\kappa})\sin(\lambda_{n+1}M\sqrt{\kappa}) \le \sin(M\sqrt{\kappa}) - \sin\left((1-\lambda_{n+1})M\sqrt{\kappa}\right),$$

it follows that

$$\sin^2 \frac{d(x_{n+1}, z_J^u)\sqrt{\kappa}}{2} \quad \leq \quad (1-\mu_{n+1}) \sin^2 \frac{d(x_n, z_J^u)\sqrt{\kappa}}{2} + \mu_{n+1} \max\left\{\frac{\gamma_n^J}{\cos(M\sqrt{\kappa})}, 0\right\} + \Delta_{K_0}^*\left(\sin^2 \frac{\varepsilon\sqrt{\kappa}}{4}, g\right).$$

Letting $t = \dfrac{1}{K_0+1}$ in Proposition 7.1.(iii), we get that

$$\gamma_n^{K_0} \leq \frac{\cos(M\sqrt{\kappa})}{6} \sin^2 \frac{\varepsilon\sqrt{\kappa}}{4},$$

for all $n \geq \chi_{K_0}^*\left(\dfrac{1}{3}\sin^2 \dfrac{\varepsilon\sqrt{\kappa}}{4}\right) = \chi_{K_0}\left(\dfrac{\cos(M\sqrt{\kappa})}{6}\sin^2 \dfrac{\varepsilon\sqrt{\kappa}}{4}\right)$. Thus,

$$\gamma_n^J \quad \leq \quad \gamma_n^{K_0} + \frac{\cos(M\sqrt{\kappa})}{6} \sin^2 \frac{\varepsilon\sqrt{\kappa}}{4} \leq \frac{\cos(M\sqrt{\kappa})}{3} \sin^2 \frac{\varepsilon\sqrt{\kappa}}{4},$$

and so, $\max\left\{\dfrac{\gamma_n^J}{\cos(M\sqrt{\kappa})}, 0\right\} \leq \dfrac{1}{3}\sin^2 \dfrac{\varepsilon\sqrt{\kappa}}{4}$ for all $n \geq \chi_{K_0}^*\left(\dfrac{1}{3}\sin^2 \dfrac{\varepsilon\sqrt{\kappa}}{4}\right)$.

Furthermore, by Lemma 5.2, we have that $\sum\limits_{n=1}^{\infty} \mu_{n+1} = \infty$ with rate of divergence $\tilde{\theta}(n) = \theta\left(\left\lceil \dfrac{1}{\cos(M\sqrt{\kappa})}\right\rceil n\right)$. Hence, we can apply Lemma 7.3 with

$$s_n = \sin^2 \frac{d(x_n, z_J^u)\sqrt{\kappa}}{2}, \quad t_n = \max\left\{\frac{\gamma_n^J}{\cos(M\sqrt{\kappa})}, 0\right\}, \quad \alpha_n = \mu_{n+1},$$
$$\varepsilon = \sin^2 \frac{\varepsilon\sqrt{\kappa}}{4}, \quad \Delta = \Delta_{K_0}^*\left(\sin^2 \frac{\varepsilon\sqrt{\kappa}}{4}, g\right), \quad L = \sin^2 \frac{M\sqrt{\kappa}}{2}.$$

By letting $N = \Theta_{K_0}\left(\sin^2 \dfrac{\varepsilon\sqrt{\kappa}}{4}\right)$, it follows that for all $n \in [N, N+g(N)]$,

$$\sin^2 \frac{d(x_n, z_J^u)\sqrt{\kappa}}{2} \leq \sin^2 \frac{\varepsilon\sqrt{\kappa}}{4}, \quad \text{and so} \quad d(x_n, z_J^u) \leq \frac{\varepsilon}{2}.$$

Obviously, $d(x_n, x_m) \leq \varepsilon$ for all $m, n \in [N, N+g(N)]$. One can easily see that $N \leq \Sigma(\varepsilon, g)$. □

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