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The two-scale convergence method applied to generalized Besicovitch spaces

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The two-scale convergence method has proved to be a very useful tool for dealing with periodic homogenization problems. In the present paper we develop this theory to generalized Besicovitch spaces, which include the almost-periodic functions. The main difficulty comes from the fact that these spaces are not separable. We also show how to apply these results to the homogenization of partial differential problems in this framework.

Keywords: partial differential equations; homogenization; two-scale convergence

1. Introduction

It is usual in homogenization theory to deal with composite periodic materials and structures with very small periods. In order to study their physical behaviour (electrical or thermal conductivity, elastic behaviour, etc.), we need to solve a partial differential equation that, in a model case, can be written in the form

$$-\operatorname{div} A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon = f, \tag{1.1}$$

besides some boundary conditions. Here the matrix A is periodic and ε is a small parameter. From the numerical point of view, it is very difficult to calculate u_ε from this problem. We need to use a discretization of size smaller than ε, and therefore solving a very large system of equations, which requires a lot of computer memory, is time-consuming and involves several stability problems.

Homogenization theory seeks to obtain an approximation of u_ε through the resolution of simpler partial differential equations. The theory of asymptotic expansions (see Bensoussan *et al*. 1978; Sánchez Palencia 1980) provides us with

$$u_\varepsilon(x) \sim u_0(x) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \cdots, \tag{1.2}$$

where the functions u_i are obtained as the solutions of partial differential problems much easier to solve than (1.1). A rigorous way of obtaining this expansion and showing its convergence is the two-scale convergence method of Nguetseng and Allaire (see Allaire 1992; Arbogast *et al.* 1990; Nguetseng 1990). It has proved to be very useful in the homogenization of periodic problems.

We note, however, that although periodic materials are common in engineering applications, they do not usually occur in nature. However, we can see a recurrence in the structures which suggests that a better approximation is to consider these materials as almost-periodic. The aim of the present paper is to extend the two-scale convergence method to the case of almost-periodic (or more general) coefficients, in particular, the sum of periodic functions with different periods, in order to be able to treat more general composite materials than the periodic ones. Similarly to the periodic case, it is necessary to have a characterization of the limit of the expression

$$\int_{\Omega} v_{\varepsilon}(x)\psi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x \tag{1.3}$$

when v_ε is a bounded sequence in a Lebesgue space $L^p(\Omega)$, $p > 1$, and ψ is an almost-periodic smooth function or, in a wider sense, in a generalized Besicovitch space (see Casado Díaz & Gayte 2002; Jikov *et al.* 1994; Zhikov & Krivenko 1983) in its second variable. To prove the corresponding result, the first step is to show the existence of a subsequence of v_ε, still denoted by v_ε, such that there exists the limit of (1.3), for every ψ as above. This is easy, using a diagonal argument, if the space of functions ψ is separable. However, the generalized Besicovitch spaces are not separable in general. This is the main difficulty in obtaining our result. To solve this problem, we propose an abstract theorem generalizing the well-known result about the weak sequential compactness of the unit ball in a reflexive space. In a simpler situation this was carried out in Casado Díaz & Gayte (1996). The case where the almost-periodic functions are in a separable space has been considered by Nguetseng (2000).

As an example of how our results can be used in the study of the asymptotic behaviour of composite materials, we study the nonlinear problem

$$\begin{aligned} -\operatorname{div} a\left(\frac{x}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon\right) &= f \quad \text{in } \Omega, \\ u_\varepsilon &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where a is a Carathéodory function which defines a pseudomonotone operator of order p and belongs to a Besicovitch space in its second variable. In this case we obtain the limit equation and a corrector result related to (1.2). To complete this introduction, we mention that an adaptation of the two-scale convergence to stochastic homogenization problems has been given in Bourgeat *et al.* (1994) (assuming separability). The notion of a stochastic weak derivative given in this article is strongly related to the mean derivative we use in the present paper. To the study of homogenization problems in a stochastic frame, we also refer to Dal Maso & Modica (1986) and Abddaimi *et al.* (1997).

2. A compactness theorem

It is well known that, for a bounded sequence $\{f_n\}$ in the dual space X' of a reflexive space X, there exists a subsequence of $\{f_n\}$ which converges weakly-$*$ to some $f \in X'$, i.e. $\{f_n\}$ pointwise converges to f. The purpose of the present section is to generalize this result, by showing that it is necessary to assume neither f_n continuous nor X complete.

Theorem 2.1. *Let X be a subspace (not necessarily closed) of a reflexive space Y and let $f_n : X \mapsto \mathbb{R}$ be a sequence of linear functionals (not necessarily continuous). Assume there exists a constant $C > 0$ which satisfies*

$$\limsup_n f_n(x) \leqslant C\|x\|, \quad \forall x \in X. \tag{2.1}$$

There then exist a subsequence $\{n_k\}$ of $\{n\}$ and a functional $f \in Y'$ such that

$$\exists \lim_k f_{n_k}(x) = \langle f, x \rangle, \quad \forall x \in X. \tag{2.2}$$

Remark 2.2. If X is complete and f_n is continuous, theorem 2.1 easily follows from the Banach–Steinhauss theorem and the weak-$*$ sequential compactness of the unit ball in a reflexive space. It is also clear that theorem 2.1 holds if we replace the hypothesis X included in a reflexive space by X separable. The aim of theorem 2.1 is precisely the application to spaces that are not separable.

Remark 2.3. Theorem 2.1 has been established in Casado Díaz & Gayte (1996) when Y is a Hilbert space.

In order to prove theorem 2.1, we need to recall some results about smooth norms (see Cioranescu 1990).

Definition 2.4. Let Y be a Banach space. The norm in Y is called smooth if for every $y \in Y$ with $\|y\| = 1$ there exists a unique $f \in Y'$ such that $\|f\| = 1$ and $\langle f, y \rangle = 1$.

The following theorem is due to Asplund and Lindenstrauss (see Cioranescu 1990; Lindenstrauss 1966).

Theorem 2.5. *Every reflexive Banach space has an equivalent smooth norm.*

Proof of theorem 2.1. By theorem 2.5, it is not restrictive to assume that the norm in Y is smooth.

First step. Let us prove that there exist a subsequence $\{n_k\}$ of $\{n\}$, a constant $\bar{C} \geqslant 0$ and a sequence $\{z_j\} \subset X$ such that

$$\|z_j\| = 1, \tag{2.3}$$

$$\limsup_k f_{n_k}(x) \leqslant \bar{C}\|x\|, \quad \forall x \in X, \tag{2.4}$$

$$\exists \lim_k f_{n_k}(z_j) \geqslant \bar{C} - \frac{1}{j}, \quad \forall j \in \mathbb{N}. \tag{2.5}$$

To this end, we define

$$C_1 = \sup\Big\{\limsup_n f_n(x) : x \in X, \ \|x\| = 1\Big\}.$$

This supremum is finite because of (2.1).

By definition of C_1 there exist $z_1 \in X$ with $\|z_1\| = 1$ and a subsequence $\{n_1(k)\}_k$ of $\{n\}$ such that

$$\exists \lim_k f_{n_1(k)}(z_1) \geqslant C_1 - 1.$$

Then we define C_2 by

$$C_2 = \sup\Big\{\limsup_k f_{n_1(k)}(x) : x \in X,\ \|x\| = 1\Big\}.$$

Obviously, $C_2 \leqslant C_1$ and there exist $z_2 \in X$ with $\|z_2\| = 1$ and a subsequence $\{n_2(k)\}_k$ of $\{n_1(k)\}_k$ such that

$$\exists \lim_k f_{n_2(k)}(z_2) \geqslant C_2 - \tfrac{1}{2}.$$

Repeating this reasoning, we deduce that for every $j \in \mathbb{N}$ there exist $C_j \in \mathbb{R}$, $z_j \in X$ and $\{n_j(k)\}_k$ such that, denoting $n_0(k) = k$ for every $k \in \mathbb{N}$, we have

$$\|z_j\| = 1, \tag{2.6}$$

$$C_j = \sup\Big\{\limsup_k f_{n_{j-1}(k)}(x) : x \in X,\ \|x\| = 1\Big\}, \tag{2.7}$$

$$\{n_j(k)\}_k \text{ is a subsequence of } \{n_{j-1}(k)\}_k, \tag{2.8}$$

$$\exists \lim_k f_{n_j(k)}(z_j) \geqslant C_j - \frac{1}{j}, \tag{2.9}$$

$$0 \leqslant C_{j+1} \leqslant C_j. \tag{2.10}$$

Taking the diagonal subsequence $\{n_k(k)\}$, which we denote by $\{n_k\}$, we have

$$\limsup_k f_{n_k}(x) \leqslant C_j\|x\|, \quad \forall x \in X,\ \forall j \in \mathbb{N},$$

$$\exists \lim_k f_{n_k}(z_j) \geqslant C_j - \frac{1}{j}, \quad \forall j \in \mathbb{N}.$$

Therefore, for $\bar{C} = \lim_j C_j$ statements (2.4) and (2.5) hold.

Second step. Let us now prove that there exists $f \in Y'$ such that $\{n_k\}$ and f satisfy (2.2). Note that we can suppose $\bar{C} > 0$ because if not, by (2.4) we immediately get (2.2) with $f = 0$. Since Y is reflexive and $\{z_j\}$ is bounded, there exist a subsequence, still denoted by $\{z_j\}$, and $z_0 \in Y$ such that

$$z_j \rightharpoonup z_0 \quad \text{in } Y. \tag{2.11}$$

By (2.6), z_0 satisfies

$$\|z_0\| \leqslant 1. \tag{2.12}$$

Let $x \in X$ be arbitrary. Since $\{f_{n_k}(x)\}_k$ is bounded, there exists a subsequence $\{n_{k(j)}\}_j$ of $\{n_k\}_k$, depending on x, such that

$$\exists \lim_j f_{n_{k(j)}}(x). \tag{2.13}$$

Denoting $S = \operatorname{span}(\{x\} \bigcup \{z_n : n \in \mathbb{N}\})$, statements (2.13), (2.5) and $f_{n_{k(j)}}$ linear imply

$$\exists \lim_j f_{n_{k(j)}}(s), \quad \forall s \in S.$$

We define $f : S \mapsto \mathbb{R}$ by

$$f(s) = \lim_j f_{n_{k(j)}}(s), \quad \forall s \in S.$$

By (2.4) and because $n_{k(j)}$ is a subsequence of n_k for every $s \in S \subset X$, we have

$$f(s) = \lim_j f_{n_{k(j)}}(s) \leqslant \limsup_k f_{n_k}(s) \leqslant \bar{C}\|s\|_X,$$

and so f belongs to S' and satisfies $\|f\|_{S'} \leqslant \bar{C}$. On the other hand, by (2.5) and (2.6), for every $j \in \mathbb{N}$ we have

$$f(z_j) \geqslant \bar{C} - \frac{1}{j} = \left(\bar{C} - \frac{1}{j}\right)\|z_j\|. \tag{2.14}$$

Thus $\|f\| \geqslant \bar{C} - 1/j$ for every $j \in \mathbb{N}$ and so

$$\|f\|_{S'} = \bar{C}. \tag{2.15}$$

By the Hahn–Banach theorem, we can extend f to a functional of Y', still denoted by f, which satisfies $\|f\|_{Y'} = \bar{C}$. By (2.14), (2.15), (2.11) and (2.6), we deduce

$$\bar{C} = \lim_j f(z_j) = \langle f, z_0\rangle. \tag{2.16}$$

Using (2.12) and (2.15), we then have

$$\langle f, z_0\rangle = \bar{C}, \quad \|f\| = \bar{C}, \quad \|z_0\| = 1. \tag{2.17}$$

Since Y is smooth there exists a unique element $f \in Y'$ satisfying (2.17), so in (2.13) it is not necessary to take a subsequence, and the whole of the sequence $\{f_{n_k}\}$ pointwise converges to f in X. ∎

Following the idea of theorem 2.1, we can also prove the following theorem, which generalizes theorem 2.1 and contains the case where X is separable. The proof can be found in Gayte Delgado (1998).

Theorem 2.6. *Let X be a normed space (not necessarily complete) such that the unit sphere of X' endowed with the weak-$*$ topology is first countable. Let $f_n : X \mapsto \mathbb{R}$ be a sequence of linear functionals (not necessarily continuous) satisfying (2.1). There then exist a subsequence $\{n_k\}$ of $\{n\}$ and a functional $f \in X'$ such that (2.2) holds.*

Remark 2.7. It can be proved that the unit sphere of X' endowed with the weak-$*$ topology is first countable if and only if for every $f \in X'$, $\|f\| = 1$, there exists a sequence $\{z_n\} \subset X$ such that if $g \in X'$ satisfies $\|g\| = 1$ and $\langle g, z_n\rangle = \langle f, z_n\rangle$, for every $n \in \mathbb{N}$. Then $g = f$.

3. Preliminaries on generalized Besicovitch spaces

In this section we recall some results on generalized Besicovitch spaces we will need later. They have been proved in Casado Díaz & Gayte (2002) (see also Gayte Delgado (1998); Jikov *et al.* (1994)). We recall the definition of the mean value.

Definition 3.1. We say that a function $f : \mathbb{R}^N \to \mathbb{R}$ has a mean value if there exists a real number $M\{f\}$ such that for every bounded measurable set $K \subset \mathbb{R}^N$ with $|K| > 0$ we have

$$M\{f\} = \lim_{T\to+\infty} \frac{1}{|TK|} \int_{TK} f(y)\,\mathrm{d}y. \tag{3.1}$$

In this case, we say that $M\{f\}$ is the mean value of f.

Following Jikov *et al.* (1994, 7.5, p. 242), we now give the definition of an algebra with mean value.

Definition 3.2. A linear space, X, of real-valued functions defined in $\mathbb{R}^N$ is a Banach algebra with mean value if the following conditions are satisfied.

(i) The elements of X are bounded, uniformly continuous and possess a mean value (see (3.1)).

(ii) The constant functions belong to X.

(iii) X is an algebra.

(iv) X endowed with the uniform convergence topology is complete.

(v) For every $f \in X$ and $s \in \mathbb{R}$, the function $f(\cdot + s)$ belongs to X.

Remark 3.3. As examples of X, we have the space of continuous $(0,1)^N$-periodic functions and the space of uniformly almost-periodic functions.

The above definition allows us to define the generalized Besicovitch spaces in the following way.

Definition 3.4. We define the generalized Besicovitch space of order p (relative to X), with $1 \leqslant p < +\infty$, and we denote it by B^p, as the closure of X for the seminorm

$$[f]_p = \left(\limsup_{T \to +\infty} \frac{1}{|B_T|} \int_{B_T} |f(x)|^p \, \mathrm{d}x \right)^{1/p}, \tag{3.2}$$

i.e.

$$B^p = \{ f : \mathbb{R}^N \to \mathbb{R} \text{ measurable: } \forall \varepsilon > 0, \ \exists \varphi \in X \text{ with } [f - \varphi]_p < \varepsilon \}.$$

The generalized Besicovitch space of order ∞ (relative to X), B^∞, is defined by

$$B^\infty = \Big\{ f \in B^1 \text{ such that } [f]_\infty = \sup_{p \geqslant 1} [f]_p < +\infty \Big\}.$$

The spaces B^p are seminormed spaces. The quotient of B^p with the kernel of $[\cdot]_p$ is denoted by $\mathcal{B}^p$ and it is a normed space.

Remark 3.5. When X is the space of continuous $(0,1)^N$-periodic functions, B^p is the space of functions in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ which are $(0,1)^N$-periodic. The space of almost-periodic functions in the sense of Besicovitch (see, for example, Besicovitch 1954; Bohr 1951) is obtained by taking X as the space of uniformly almost-periodic functions.

The following theorem shows that the spaces B^p are analogous to the spaces L^p for a probability measure (see Casado Díaz & Gayte (2002) and Gayte Delgado (1998) for the proof).

Theorem 3.6. *The spaces B^p satisfy the following properties.*

(i) *For $1 \leqslant p \leqslant +\infty$, B^p and then $\mathcal{B}^p$ are complete.*

(ii) *For every $f \in B^p$, $1 \leqslant p < +\infty$, there exist $M\{f\}$ and $M\{|f|^p\}$. Besides, $[f]_p = M\{|f|^p\}^{1/p}$.*

(iii) *For $f \in B^\infty$ and $\alpha = [f]_\infty$, the function $T_\alpha(f) \in L^\infty(\mathbb{R}^N)$ satisfies*

$$\|T_\alpha\|_{L^\infty(\mathbb{R}^N)} = [f]_\infty, \quad [T_\alpha(f) - f]_\infty = 0,$$

where T_α is defined by

$$T_\alpha(s) = \begin{cases} \alpha & \text{if } s > \alpha, \\ s & \text{if } |s| \leqslant \alpha, \\ -\alpha & \text{if } s < -\alpha. \end{cases}$$

(iv) *If $p < q$, then $B^q \subset B^p$ and $[\cdot]_p \leqslant [\cdot]_q$. Moreover, if $f, g \in B^q$ satisfy $[f-g]_p = 0$, then $[f-g]_q = 0$, and we can also then see that $\mathcal{B}^q$ is a subspace of $\mathcal{B}^p$.*

(v) *The dual space of $\mathcal{B}^p$, for $1 \leqslant p < +\infty$, can be identified with $\mathcal{B}^{p'}$ through the following isometric isomorphism:*

$$\mathcal{F} : \mathcal{B}^{p'} \to (\mathcal{B}^p)'$$

$$\langle \mathcal{F}(f), g \rangle = M\{\tilde{f}\tilde{g}\} \quad \forall \tilde{f} \in f \in \mathcal{B}^{p'},\ \forall \tilde{g} \in g \in \mathcal{B}^p,\ \text{if } 1 < p < +\infty,$$

and

$$\langle \mathcal{F}(f), g \rangle = M\{T_{[f]_\infty}(\tilde{f})\tilde{g}\} \quad \forall \tilde{f} \in f \in \mathcal{B}^\infty,\ \forall \tilde{g} \in g \in \mathcal{B}^1, \text{if } p = 1.$$

To finish this section, we recall some results related to the derivation theory for generalized Besicovitch spaces (see Casado Díaz & Gayte 2002).

We start by introducing the space D^∞, which plays in the spaces B^p the same role as the spaces $C_0^\infty(\mathbb{R}^N)$ in the distributional theory.

Definition 3.7. We define D^∞ as

$$D^\infty = \{\varphi \in C^\infty(\mathbb{R}^N) : D^\alpha\varphi \in B^1 \cap L^\infty(\mathbb{R}^N)\ \forall \alpha \in (\mathbb{N} \cup \{0\})^N\}.$$

Reasoning by convolution (see Casado Díaz & Gayte 2002), we can show the following proposition.

Proposition 3.8. *The space D^∞ is dense in B^p for $1 \leqslant p < +\infty$.*

For every $f \in B^\infty$, there exists a sequence $\{f_n\}$ in D^∞ which converges to f in B^1 and is bounded in B^∞.

Analogously to distributional theory, we use the spaces D^∞ to give a definition of the derivative in B^p.

Definition 3.9. For $f \in B^1$, we define the mean partial i derivative of f, $1 \leqslant i \leqslant N$, and we denote it by $\partial_{i,m} f$, as the linear application of D^∞ in $\mathbb{R}$ given by

$$\partial_{i,m} f(\varphi) = -M\left\{ f \frac{\partial \varphi}{\partial x_i} \right\}, \quad \forall \varphi \in D^\infty. \tag{3.3}$$

We also define the mean gradient of $f \in B^1$, $\nabla_m f$, as $\nabla_m f = (\partial_{1,m} f, \dots, \partial_{N,m} f)$ and the mean divergence of $F \in (B^1)^N$, $\operatorname{div}_m F$, as $\operatorname{div}_m F = \sum_{i=1}^N \partial_{i,m} F_i$. Clearly, these definitions can also be extended to $\mathcal{B}^1$.

The following result, which relates the distributional derivative with the mean derivative, is shown in Casado Díaz & Gayte (2002).

Proposition 3.10. *If $f \in B^1$ is such that there exists $i \in \{1,\dots,N\}$ with $\partial f/\partial x_i \in B^1$, then $\partial_{i,m} f = \partial f/\partial x_i$ in the following sense:*

$$\langle \partial_{i,m} f, \varphi\rangle = M\left\{\frac{\partial f}{\partial x_i}\varphi\right\}, \quad \forall \varphi \in D^\infty.$$

The following space plays a very important role in applications.

Definition 3.11. For $1 \leqslant p < +\infty$ we define

$$W^p = \{f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N) : \nabla f \in (B^p)^N,\ M\{\nabla f\} = 0\}$$

and

$$\nabla W^p = \{\nabla f : f \in W^p\}.$$

Identifying an element of ∇W^p with its class in $(\mathcal{B}^p)^N$, ∇W^p will be considered as a subspace of $(\mathcal{B}^p)^N$. Moreover, we identify $w_1, w_2 \in W^p$ if $[\nabla(w_1 - w_2)]_p = 0$, and then we can consider W^p as a normed space for the norm $\|w\| = [\nabla w]_p$.

The following theorem gives some interesting properties of W^p (see Casado Díaz & Gayte (2002) and Gayte Delgado (1998) for the proof).

Theorem 3.12. *The subspace ∇W^p is closed in $(B^p)^N$ (and then W^p is Banach). If $f \in B^p$, $1 \leqslant p < +\infty$, is such that $\nabla_m f$ belongs to $(B^p)^N$ then there exists $g \in W^p$ such that $\nabla_m f = \nabla g$ in $(B^p)^N$.*

To obtain further properties of W^p the algebra must satisfy another property.

Definition 3.13. An algebra X is called ergodic if for every $f \in B^1$ such that $[f - f(\cdot + s)]_1 = 0$ for every $s \in \mathbb{R}^N$ (equivalently $\nabla_m f = 0$), we have $[f - M\{f\}]_1 = 0$.

Proposition 3.14. *An algebra is ergodic if and only if*

$$\lim_{R\to+\infty}\left[\frac{1}{|B_R|}\int_{B_R} f(x+y)\,\mathrm{d}y - M\{f\}\right]_p = 0, \quad \forall f \in B^p,\ 1 \leqslant p < +\infty. \tag{3.4}$$

In the ergodic algebras we have the following density result.

Theorem 3.15. *If the algebra is ergodic, then ∇D^∞ is dense in ∇W^p.*

4. The two-scale convergence method

In this section, we present the extension of the two-scale convergence theory (see Allaire 1992; Nguetseng 1990) to the generalized Besicovitch spaces B^p relative to an algebra with mean value X. We start by giving the definition of two-scale convergence.

Definition 4.1. Let $\Omega \subset \mathbb{R}^N$ be open. We say that a sequence $\{u_\varepsilon\} \subset L^1_{\mathrm{loc}}(\Omega)$ two-scale converges to $u \in L^1_{\mathrm{loc}}(\Omega; \mathcal{B}^1)$ if for every $g \in B^1 \cap L^\infty(\mathbb{R}^N)$ and every

$E \subset\subset \Omega$ bounded, measurable, we have

$$\exists \lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \int_\Omega M_y\{u(x,y)\psi(x,y)\}\,\mathrm{d}x, \tag{4.1}$$

where $\psi(x, y) = g(y)X_E(x)$ for $x \in \Omega$, $y \in R^N$. We will denote $u_\varepsilon \overset{2e}{\rightharpoonup} u$.

Remark 4.2. The left-hand side of (4.1) makes no sense if g is only in $\mathcal{B}^1$ since two representatives of g may differ in every point of $\mathbb{R}^N$. The right-hand side does not depend on the representative of u chosen.

Remark 4.3. Since $\mathcal{B}^\infty$ can be identified with the dual of $\mathcal{B}^1$, it is easy to deduce that the two-scale limit, if that exists, is unique.

Remark 4.4. Our definition of two-scale convergence can seem different to the usual one for the periodic case, which establishes that (4.1) holds for every ψ periodic in the second variable and smooth enough (in general, in the space of admissible functions (see Allaire 1992)). As established in proposition 4.6, this is equivalent to our definition, because if (4.1) holds for ψ as in definition 4.1, then it holds for ψ in all of the spaces which appear with the usual definition. We have chosen the definition given above because it makes it easier to check if a sequence two-scale converges, and when u_ε is bounded in $L^p(\Omega)$ for some $p \in (1, +\infty)$ (usual situation), it does not depend on p.

Although we have defined the two-scale convergence merely for a sequence $\{u_\varepsilon\}$ in $L^1_{\mathrm{loc}}(\Omega)$, in the applications we will usually have a bounded sequence in $L^p(\Omega)$ for some $p \in [1, +\infty]$. In this case, we have the following result.

Proposition 4.5. *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^p(\Omega)$ for some $p \in [1, +\infty]$, which two-scale converges to a function $u \in L^1_{\mathrm{loc}}(\Omega; \mathcal{B}^1)$. Then u belongs to $L^p(\Omega; \mathcal{B}^p)$, the sequence $\{u_\varepsilon\}$ converges weakly in $L^p(\Omega)$ (weakly-$*$ if $p = +\infty$) to $u_0 = M_y\{u(\cdot, y)\}$ and we have*

$$\liminf_{\varepsilon\to 0} \|u_\varepsilon\|_{L^p(\Omega)} \geqslant \|u\|_{L^p(\Omega;\mathcal{B}^p)} \geqslant \|u_0\|_{L^p(\Omega)}. \tag{4.2}$$

Proof. We denote by $St_c(\Omega; B^{p'})$ the set of simple functions which have the support strictly included in Ω. For $\psi \in St_c(\Omega; B^{p'})$, $|\psi| \leqslant \|\psi\|_{L^\infty(\Omega;\mathcal{B}^\infty)}$ a.e. in $\Omega \times \mathbb{R}^N$ if $p = 1$, we have

$$\left|\int_\Omega u_\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x\right| \leqslant \|u_\varepsilon\|_{L^p(\Omega)} \left\|\psi\left(x, \frac{x}{\varepsilon}\right)\right\|_{L^{p'}(\Omega)},$$

where passing to the limit when ε tends to zero, we deduce

$$\left|\int_\Omega M_y\{u(x,y)\psi(x,y)\}\,\mathrm{d}x\right| \leqslant \liminf_{\varepsilon\to 0} \|u_\varepsilon\|_{L^p(\Omega)} \|\psi\|_{L^{p'}(\Omega;B^{p'})} \quad \forall \psi \in St_c(\Omega; B^\infty). \tag{4.3}$$

From this inequality and theorem 3.6 we easily deduce that u belongs to $L^p(\Omega; \mathcal{B}^p)$ and that (4.2) holds. ∎

The following result, which is easy to prove, extends definition 4.1 to a wide class of admissible function ψ.

Proposition 4.6. *Assuming $\{u_\varepsilon\}$ bounded in $L^p(\Omega)$, $p \in [1, +\infty)$, the equality (4.1) is verified when ψ belongs, for example, to the following spaces:*

$$\Big\{\sum_{i\in I} f_i(x)g_i(y) : I \subset \mathbb{N} \text{ finite},\ f_i \in L^{p'}(\Omega),\ g_i \in B^{p'}\ \forall i \in I\Big\},$$

$$\Big\{\sum_{i\in I} f_i(x,y)g_i(y) : I \subset \mathbb{N} \text{ finite},\ f_i \in L^{p'}(\Omega; X),\ g_i \in B^1 \cap L^\infty(\mathbb{R}^N)\ \forall i \in I\Big\},$$

$$\Big\{\sum_{i\in I} f_i(x,y)g_i(y) : I \subset \mathbb{N} \text{ finite},\ f_i \in C^0(\bar{\Omega}; B^1) \cap L^\infty(\Omega \times \mathbb{R}^N),\ g_i \in B^{p'}\ \forall i \in I\Big\}.$$

Moreover, if ψ belongs to some of these spaces, the sequence $\psi(\cdot, \cdot/\varepsilon)$ two-scale converges to the class of ψ in $\mathcal{B}^{p'}$, and we have

$$\lim_{\varepsilon\to 0}\left\|\psi\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\|_{L^{p'}(\Omega)} = \|\psi\|_{L^{p'}(\Omega;\mathcal{B}^{p'})}.$$

The following result shows how, in several cases, the two-scale limit u of a bounded sequence u_ε in $L^p(\Omega)$ provides a good approach of u_ε in $L^p(\Omega)$ (i.e. a corrector result).

Proposition 4.7. *Let $\{u_\varepsilon\} \subset L^p(\Omega)$, $1 < p < +\infty$, which two-scale converges to a function $u \in L^p(\Omega; \mathcal{B}^p)$. If*

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{L^p(\Omega)} = \|u\|_{L^p(\Omega;\mathcal{B}^p)} \tag{4.4}$$

and u has a representative in some of the spaces given in proposition 4.6, then, choosing this representative for u, we have

$$\lim_{\varepsilon\to 0}\left\|u_\varepsilon(\cdot) - u\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\|_{L^p(\Omega)} = 0. \tag{4.5}$$

Proof. We assume that $\|u\|_{L^p(\Omega;\mathcal{B}^p)}$ is non-zero, and, if not, (4.5) is clearly satisfied.

Since

$$\lim_{\varepsilon\to 0}\left\|u\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\|_{L^p(\Omega)} = \lim_{\varepsilon\to 0}\|u_\varepsilon\|_{L^p(\Omega)} = \|u\|_{L^p(\Omega;\mathcal{B}^p)} \neq 0, \tag{4.6}$$

the sequences

$$v_\varepsilon = \|u_\varepsilon\|^{-1}_{L^p(\Omega)} u_\varepsilon, \qquad \tilde{v}_\varepsilon = \left\|u\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right\|^{-1}_{L^p(\Omega)} u\left(\cdot, \frac{\cdot}{\varepsilon}\right)$$

are well defined for $\varepsilon > 0$ small enough and satisfy

$$\|v_\varepsilon\|_{L^p(\Omega)} = \|\tilde{v}_\varepsilon\|_{L^p(\Omega)} = 1, \qquad v_\varepsilon \overset{2e}{\rightharpoonup} \|u\|^{-1}_{L^p(\Omega;\mathcal{B}^p)} u, \qquad \tilde{v}_\varepsilon \overset{2e}{\rightharpoonup} \|u\|^{-1}_{L^p(\Omega;\mathcal{B}^p)} u.$$

Thus

$$\tfrac{1}{2}(v_\varepsilon + \tilde{v}_\varepsilon) \overset{2e}{\rightharpoonup} \|u\|^{-1}_{L^p(\Omega;\mathcal{B}^p)} u,$$

which by (4.2) implies

$$\exists \lim_{\varepsilon\to 0} \|\tfrac{1}{2}(v_\varepsilon + \tilde{v}_\varepsilon)\|_{L^p(\Omega)} = 1.$$

The uniform convexity of $L^p(\Omega)$ and (4.6) then implies

$$\lim_{\varepsilon\to 0}\left\|u_\varepsilon(\cdot)-u\left(\cdot,\frac{\cdot}{\varepsilon}\right)\right\|_{L^p(\Omega)}=\frac{1}{\|u\|_{L^p(\Omega;\mathcal{B}^p)}}\lim_{\varepsilon\to 0}\|v_\varepsilon-\tilde v_\varepsilon\|_{L^p(\Omega)}=0.$$

■

The most important result about two-scale convergence is the following compactness theorem which generalizes the corresponding one for the periodic case due to Nguetseng (see Allaire 1992; Nguetseng 1990).

Theorem 4.8. *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^p(\Omega)$, $1<p\leqslant+\infty$. Then there exist a subsequence, still denoted by $\{u_\varepsilon\}$, and a function $u\in L^p(\Omega;\mathcal{B}^p)$ such that $\{u_\varepsilon\}$ two-scale converges to u.*

Let us use the following lemma.

Lemma 4.9. *Let V be a vectorial space and $M\subset V$ a subspace, then there exists a linear application $f:V/M\to V$ such that $f(v)\in v$ for every $v\in V/M$.*

Proof. By the axiom of choice, there exists a basis $B=\{e_i:i\in I\}\subset V/M$, i.e. B is such that for every $v\in V/M\setminus\{0\}$ there exist $e_{i_1},\dots,e_{i_n}\in B$, $\alpha_{i_1},\dots,\alpha_{i_n}\in\mathbb{R}\setminus\{0\}$, unique, such that

$$v=\sum_{j=1}^n\alpha_{i_j}e_{i_j}.$$

Choose, then, for every $e_i\in B$, $\tilde e_i\in V$ such that $\tilde e_i\in e_i$ and define f as the unique linear application from V/M to V such that $f(e_i)=\tilde e_i$ for every $i\in I$. ■

Proof of theorem 4.8. Using the lemma, we can consider $f:\mathcal{B}^p\to B^p$ linear such that $f(v)\in v$ for every $v\in\mathcal{B}^p$. Assume $1<p<+\infty$ and let $F_\varepsilon:L^{p'}(\Omega)\otimes\mathcal{B}^{p'}\subset L^{p'}(\Omega;\mathcal{B}^{p'})\to\mathbb{R}$ be defined by

$$F_\varepsilon(\psi)=\int_\Omega u_\varepsilon(x)(f\circ\psi)\left(x,\frac{x}{\varepsilon}\right)\mathrm{d}x,\quad\forall\psi\in L^{p'}(\Omega)\otimes\mathcal{B}^{p'}.\tag{4.7}$$

Using that $\{u_\varepsilon\}$ is bounded in $L^p(\Omega)$, we have

$$\limsup_{\varepsilon\to 0}|F_\varepsilon(\psi)|\leqslant C\lim_{\varepsilon\to 0}\left\|(f\circ\psi)\left(x,\frac{x}{\varepsilon}\right)\right\|_{L^{p'}(\Omega)}=C\|\psi\|_{L^{p'}(\Omega;\mathcal{B}^{p'})}.$$

Since $\mathcal{B}^{p'}$ is reflexive, we can apply theorems 2.1 and 3.6 to deduce the existence of a subsequence of $\{F_\varepsilon\}$, still denoted by $\{F_\varepsilon\}$, and $u\in L^p(\Omega;\mathcal{B}^p)$, such that

$$\exists\lim_{\varepsilon\to 0}F_\varepsilon(\psi)=\int_\Omega M_y\{u(x,y)\psi(x,y)\}\,\mathrm{d}x\quad\forall\psi\in L^{p'}(\Omega)\otimes B^{p'},$$

i.e. u_ε two-scale converges to u. When $p=+\infty$ the result easily holds from $\{u_\varepsilon\}$ bounded in $L^2_{\mathrm{loc}}(\Omega')$ for every $\Omega'\subset\subset\Omega$ open and proposition 4.5. ■

For $p=1$ we have the following result.

Theorem 4.10. *Let $\{u_\varepsilon\}$ be a bounded and locally equi-integrable sequence in $L^1(\Omega)$. Then there exist a subsequence, still denoted by $\{u_\varepsilon\}$, and a function $u \in L^1(\Omega;\mathcal{B}^1)$ such that $\{u_\varepsilon\}$ two-scale converges to u.*

Proof. Clearly, we can assume Ω bounded and $\{u_\varepsilon\}$ equi-integrable. For every $k \in \mathbb{N}$, the sequence $\{T_k(u_\varepsilon)\}_\varepsilon$ is bounded in $L^\infty(\Omega)$. So, by theorem 4.8, there exist a subsequence ε_k of ε and $\bar{u}_k \in L^\infty(\Omega;\mathcal{B}^\infty)$ such that $T_k(u_{\varepsilon_k})$ two-scale converges to $\bar{u}_k$. By a diagonal procedure, we can assume that the subsequence is the same for every k.

On the other hand, for $k', k \in \mathbb{N}$, $k' \leqslant k$, we have

$$\int_\Omega |T_k(u_\varepsilon) - T_{k'}(u_\varepsilon)|\,\mathrm{d}x \leqslant \int_{\Omega\cap\{|u_\varepsilon|\geqslant k'\}} |u_\varepsilon|\,\mathrm{d}x,$$

which by proposition 4.5 implies

$$\|\bar{u}_k - \bar{u}_{k'}\|_{L^1(\Omega;\mathcal{B}^1)} \leqslant \liminf_{\varepsilon\to 0} \int_{\Omega\cap\{|u_\varepsilon|\geqslant k'\}} |u_\varepsilon|\,\mathrm{d}x. \tag{4.8}$$

Since the integrability of u_ε implies that the limit in k' on the right-hand side of (4.8) tends to zero, we deduce that $\{\bar{u}_k\}$ is a Cauchy sequence in $L^1(\Omega;\mathcal{B}^1)$ and then there exists $u \in L^1(\Omega;\mathcal{B}^1)$ such that

$$\bar{u}_k \to u \quad \text{in } L^1(\Omega;\mathcal{B}^1). \tag{4.9}$$

Let us prove that $\{u_\varepsilon\}$ two-scale converges to u. For $\psi(x,y) = X_E(x)g(y)$, with $E \subset\subset \Omega$ bounded, measurable and $g \in B^1 \cap L^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}
\left|\int_E u_\varepsilon(x) g\left(\frac{x}{\varepsilon}\right)\mathrm{d}x - \int_E M_y\{u(x,y)g(y)\}\,\mathrm{d}x\right| \hspace{-6em}&\\
&\leqslant \left|\int_E u_\varepsilon(x) g\left(\frac{x}{\varepsilon}\right)\mathrm{d}x - \int_E T_k(u_\varepsilon(x)) g\left(\frac{x}{\varepsilon}\right)\mathrm{d}x\right| \\
&\quad + \left|\int_E T_k(u_\varepsilon(x)) g\left(\frac{x}{\varepsilon}\right)\mathrm{d}x - \int_E M_y\{\bar{u}_k(x,y)g(y)\}\,\mathrm{d}x\right| \\
&\quad + \left|\int_E M_y\{(\bar{u}_k(x,y) - u(x,y))g(y)\}\,\mathrm{d}x\right| \\
&\leqslant \|g\|_{L^\infty(\mathbb{R}^N)} \int_{E\cap\{|u_\varepsilon|>k\}} |u_\varepsilon|\,\mathrm{d}x \\
&\quad + \left|\int_E T_k(u_\varepsilon(x)) g\left(\frac{x}{\varepsilon}\right)\mathrm{d}x - \int_E M_y\{\bar{u}_k(x,y)g(y)\}\,\mathrm{d}x\right| \\
&\quad + \left|\int_E M_y\{(\bar{u}_k(x,y) - u(x,y))g(y)\}\,\mathrm{d}x\right|.
\end{aligned} \tag{4.10}$$

Taking the limit in this expression, first in ε and then in k, and taking into account the equi-integrability of $\{u_\varepsilon\}$, we deduce the result. ■

The following proposition proves that the smoothness of u in theorems 4.8 and 4.10 cannot be improved.

Proposition 4.11. *For every $u \in L^p(\Omega; \mathcal{B}^p)$, $1 \leqslant p \leqslant +\infty$, there exists a bounded sequence in $L^p(\Omega)$ which two-scale converges to u.*

Proof. Let us denote $\tilde{p} = p$ if $p < +\infty$ and $\tilde{p} = 1$ if $p = +\infty$.

For $u \in L^p(\Omega; B^p)$, we consider $\{u_n\} \subset St(\Omega; B^p)$ such that

$$\lim_n \int_\Omega M_y\{|u_n - u|^{\tilde{p}}\} = 0 \tag{4.11}$$

and $\|u_n\|_{L^\infty(\Omega\times\mathbb{R}^N)}$ bounded if $p = +\infty$.

Since

$$\lim_{\varepsilon\to 0}\left(\int_\Omega \left|u_n\left(x, \frac{x}{\varepsilon}\right) - u_m\left(x, \frac{x}{\varepsilon}\right)\right|^{\tilde{p}} \mathrm{d}x\right)^{1/\tilde{p}} = \left(\int_\Omega M_y\{|u_n(x,y) - u_m(x,y)|^{\tilde{p}}\}\,\mathrm{d}x\right)^{1/\tilde{p}}, \quad \forall n, m \in \mathbb{N},$$

and (4.11), there exists a subsequence of n, still denoted by n, such that

$$\lim_{\varepsilon\to 0}\left(\int_\Omega \left|u_n\left(x, \frac{x}{\varepsilon}\right) - u_m\left(x, \frac{x}{\varepsilon}\right)\right|^{\tilde{p}} \mathrm{d}x\right)^{1/\tilde{p}} < \frac{1}{2^m}, \quad \forall n \geqslant m.$$

Then, for every $n \in \mathbb{N}$, we can choose ε_n, decreasing in n, such that

$$\left(\int_\Omega \left|u_n\left(x, \frac{x}{\varepsilon_n}\right) - u_m\left(x, \frac{x}{\varepsilon_n}\right)\right|^{\tilde{p}} \mathrm{d}x\right)^{1/\tilde{p}} < \frac{1}{2^m}, \quad 1 \leqslant m \leqslant n. \tag{4.12}$$

By (4.12),

$$\begin{aligned}\left\|u_n\left(x, \frac{x}{\varepsilon_n}\right)\right\|_{L^{\tilde{p}}(\Omega)} &\leqslant \left\|u_n\left(x, \frac{x}{\varepsilon_n}\right) - u_1\left(x, \frac{x}{\varepsilon_n}\right)\right\|_{L^{\tilde{p}}(\Omega)} + \left\|u_1\left(x, \frac{x}{\varepsilon_n}\right)\right\|_{L^{\tilde{p}}(\Omega)} \\ &\leqslant \frac{1}{2} + \left\|u_1\left(x, \frac{x}{\varepsilon_n}\right)\right\|_{L^{\tilde{p}}(\Omega)}.\end{aligned}$$

Then $\{\|u_n(x, x/\varepsilon_n)\|_{L^p(\Omega)}\}$ is bounded if $1 \leqslant p < +\infty$. Clearly, this is also true for $p = +\infty$.

We now consider $g \in B^1 \cap L^\infty(\mathbb{R}^N)$ and $E \subset\subset \Omega$ measurable and bounded. Then for every $j \in \mathbb{N}$ we have

$$\begin{aligned}\limsup_{n\to\infty}&\left|\int_E u_n\left(x, \frac{x}{\varepsilon_n}\right) g\left(\frac{x}{\varepsilon_n}\right) \mathrm{d}x - \int_E M_y\{u(x,y)g(y)\}\,\mathrm{d}x\right| \\ &\leqslant \limsup_{n\to\infty}\left|\int_E \left(u_n\left(x, \frac{x}{\varepsilon_n}\right) - u_j\left(x, \frac{x}{\varepsilon_n}\right)\right) g\left(\frac{x}{\varepsilon_n}\right) \mathrm{d}x\right| \\ &\quad + \limsup_{n\to\infty}\left|\int_E u_j\left(x, \frac{x}{\varepsilon_n}\right) g\left(\frac{x}{\varepsilon_n}\right) \mathrm{d}x - \int_E M_y\{u_j(x,y)g(y)\}\,\mathrm{d}x\right| \\ &\quad + \left|\int_E M_y\{(u_j(x,y) - u(x,y))g(y)\}\,\mathrm{d}x\right| \\ &\leqslant \|g\|^{1/\tilde{p}'}_{L^\infty(\mathbb{R}^N)} \frac{|E|^{1/\tilde{p}'}}{2^j} + \left|\int_E M_y\{(u_j(x,y) - u(x,y))g(y)\}\,\mathrm{d}x\right|.\end{aligned}$$

Passing to the limit in j in this inequality and taking into account the arbitrariness of g and E, we deduce that $\{u_n(\cdot, \cdot/\varepsilon_n)\}$ two-scale converges to u. ∎

It is usual in homogenization to have a bounded sequence in a Sobolev space and not only in an L^p space. Theorem 4.13 characterizes the two-scale limit of the gradients of such sequence.

Let us use the following lemma.

Lemma 4.12. *Let $\{u_\varepsilon\}$ be a locally equi-integrable sequence in $L^1_{\mathrm{loc}}(\Omega)$ which two-scale converges to $u \in L^1(\Omega; \mathcal{B}^1)$. Then*

(a) *the sequence $\{u_\varepsilon(\cdot - \varepsilon z)\}$ two-scale converges to $u(\cdot, \cdot - z)$ for every $z \in \mathbb{R}^N$,*

(b) *for every $R > 0$, the sequence*

$$\left\{ \frac{1}{|B_R|} \int_{B_R} u_\varepsilon(\cdot + \varepsilon\rho)\, \mathrm{d}\rho \right\}$$

two-scale converges to

$$\frac{1}{|B_R|} \int_{B_R} u(\cdot, \cdot + \rho)\, \mathrm{d}\rho.$$

Proof. For $g \in B^1 \cap L^\infty(\mathbb{R}^N)$ and $E \subset\subset \Omega$ bounded, measurable we have

$$\begin{aligned} \int_E u_\varepsilon(x - \varepsilon z) g\left(\frac{x}{\varepsilon}\right) \mathrm{d}x &= \int_{E-\varepsilon z} u_\varepsilon(x) g\left(\frac{x}{\varepsilon} + z\right) \mathrm{d}x \\ &= \int_E u_\varepsilon(x) g\left(\frac{x}{\varepsilon} + z\right) \mathrm{d}x - \int_{E\setminus(E-\varepsilon z)} u_\varepsilon(x) g\left(\frac{x}{\varepsilon} + z\right) \mathrm{d}x \\ &\quad + \int_{(E-\varepsilon z)\setminus E} u_\varepsilon(x) g\left(\frac{x}{\varepsilon} + z\right) \mathrm{d}x, \end{aligned} \tag{4.13}$$

where the locally equi-integrability of u_ε and the inequality

$$\int_{(E-\varepsilon z)\Delta E} |u_\varepsilon(x)| \left| g\left(\frac{x}{\varepsilon} + z\right) \right| \mathrm{d}x \leqslant \|g\|_{L^\infty(\Omega)} \int_{(E-\varepsilon z)\Delta E} |u_\varepsilon|\, \mathrm{d}x$$

imply that the last two terms of the right-hand side tend to zero. Using then that $\{u_\varepsilon\}$ two-scale converges to u and $g(\cdot + z)$ belongs to $B^1 \cap L^\infty(\mathbb{R}^N)$, we deduce that

$$\lim_{\varepsilon\to 0} \int_E u_\varepsilon(x - \varepsilon z) g\left(\frac{x}{\varepsilon}\right) \mathrm{d}x = \int_E M_y\{u(x, y) g(y + z)\}\, \mathrm{d}x = \int_E M_y\{u(x, y - z) g(y)\}\, \mathrm{d}x,$$

for every $g \in B^1 \cap L^\infty(\mathbb{R}^N)$ and $E \subset\subset \Omega$ bounded, measurable. This proves (a). Let us now prove (b). For $E \subset\subset \Omega$ bounded, measurable and $g \in B^1 \cap L^\infty(\mathbb{R}^N)$, the Fubini's theorem gives

$$\int_E \left(\frac{1}{|B_R|} \int_{B_R} u_\varepsilon(x + \varepsilon\rho)\, \mathrm{d}\rho \right) g\left(\frac{x}{\varepsilon}\right) \mathrm{d}x = \frac{1}{|B_R|} \int_{B_R} \int_E u_\varepsilon(x + \varepsilon\rho) g\left(\frac{x}{\varepsilon}\right) \mathrm{d}x\, \mathrm{d}\rho.$$

Statement (a) implies

$$\lim_{\varepsilon\to 0} \int_E u_\varepsilon(x + \varepsilon\rho) g\left(\frac{x}{\varepsilon}\right) \mathrm{d}x = \int_E M_y\{u(x, y + \rho) g(y)\}\, \mathrm{d}x, \quad \forall \rho \in B_R.$$

Using then that the sequence

$$\left\{ \int_E u_\varepsilon(x+\varepsilon\rho) g\left(\frac{x}{\varepsilon}\right) \mathrm{d}x \right\}_\varepsilon$$

is bounded in $L^\infty(B_R)$, we can apply the Lebesgue-dominated convergence theorem to deduce

$$\lim_{\varepsilon\to 0} \int_E \left(\frac{1}{|B_R|} \int_{B_R} u_\varepsilon(x+\varepsilon\rho)\,\mathrm{d}\rho \right) g\left(\frac{x}{\varepsilon}\right) \mathrm{d}x = \frac{1}{|B_R|} \int_{B_R} \int_E M_y\{u(x,y+\rho)g(y)\}\,\mathrm{d}x\,\mathrm{d}\rho,$$

which by the arbitrariness of g and E finishes the proof of (b). ■

Theorem 4.13. *We assume that X is an ergodic algebra. Then, for every bounded sequence $\{u_\varepsilon\}$ in $W^{1,p}(\Omega)$ with $1 < p \leqslant +\infty$, there exist a subsequence, still denoted by $\{u_\varepsilon\}$, a function $u \in W^{1,p}(\Omega)$ and a function $u_1 \in L^p(\Omega; W^p)$ such that*

$$u_\varepsilon \rightharpoonup u \quad \text{in } W^{1,p}(\Omega) \ (*\text{-weak if } p = +\infty), \tag{4.14}$$

$$\nabla u_\varepsilon \overset{2e}{\rightharpoonup} \nabla_x u + \nabla_y u_1. \tag{4.15}$$

If $p = 1$, the corresponding result is also true assuming that $\{u_\varepsilon\}$ weakly converges in $W^{1,1}(\Omega)$.

Proof. Assume that $\{u_\varepsilon\}$ is a bounded sequence in $W^{1,p}(\Omega)$, $1 < p \leqslant +\infty$. It is well known that there exist a subsequence of $\{u_\varepsilon\}$, still denoted by $\{u_\varepsilon\}$, and a function $u \in W^{1,p}(\Omega)$ such that

$$u_\varepsilon \rightharpoonup u \quad \text{in} \quad W^{1,p}(\Omega) \quad (*\text{-weakly if } p = +\infty).$$

By theorem 4.8, there also exists $\xi \in L^p(\Omega; \mathcal{B}^p)^N$ such that

$$\nabla u_\varepsilon \overset{2e}{\rightharpoonup} \xi. \tag{4.16}$$

In order to finish the proof of theorem 4.13, it remains to show that there exists $u_1 \in L^p(\Omega; W^p)$ such that

$$\xi(x,y) = \nabla_x u(x) + \nabla_y u_1(x,y) \quad \text{in } L^p(\Omega; B^p)^N. \tag{4.17}$$

Let $r > 0$. Since u_ε is bounded in $W^{1,p}(\Omega)$, it is not difficult to show

$$\left\{ \frac{1}{\varepsilon}\left(u_\varepsilon - \frac{1}{|B_{\varepsilon r}|} \int_{B_{\varepsilon r}} u_\varepsilon(\cdot+\rho)\,\mathrm{d}\rho \right) \right\}_\varepsilon \tag{4.18}$$

is bounded in $L^p_{\mathrm{loc}}(\Omega)$. So, by theorem 4.8, there exists a function $v_r \in L^p_{\mathrm{loc}}(\Omega; B^p)$ such that (for a subsequence), we have

$$\frac{1}{\varepsilon}\left(u_\varepsilon(x) - \frac{1}{|B_{\varepsilon r}|} \int_{B_{\varepsilon r}} u_\varepsilon(x+\rho)\,\mathrm{d}\rho \right) \overset{2e}{\rightharpoonup} v_r. \tag{4.19}$$

For $\varphi \in C_0^\infty(\Omega)$ and $\psi \in D^\infty$, an integration by parts gives

$$\begin{aligned} \int_\Omega \frac{1}{\varepsilon}\left(u_\varepsilon(x) - \frac{1}{|B_{\varepsilon r}|} \int_{B_{\varepsilon r}} u_\varepsilon(x+\rho)\,\mathrm{d}\rho \right) \nabla\psi\left(\frac{x}{\varepsilon}\right) \varphi(x)\,\mathrm{d}x \\ = -\int_\Omega \left(\nabla u_\varepsilon(x) - \frac{1}{|B_{\varepsilon r}|} \int_{B_{\varepsilon r}} \nabla u_\varepsilon(x+\rho)\,\mathrm{d}\rho \right) \psi\left(\frac{x}{\varepsilon}\right) \varphi(x)\,\mathrm{d}x \\ - \int_\Omega \left(u_\varepsilon(x) - \frac{1}{|B_{\varepsilon r}|} \int_{B_{\varepsilon r}} u_\varepsilon(x+\rho)\,\mathrm{d}\rho \right) \psi\left(\frac{x}{\varepsilon}\right) \nabla\varphi(x)\,\mathrm{d}x. \end{aligned} \tag{4.20}$$

By (4.18), we have

$$u_\varepsilon - \frac{1}{|B_{\varepsilon r}|}\int_{B_{\varepsilon r}} u_\varepsilon(\cdot+\rho)\,\mathrm{d}\rho \to 0 \quad \text{in } L^p_{\mathrm{loc}}(\Omega) \text{ strongly.}$$

So, taking the limit in (4.20), and using (4.19) and lemma 4.12(b), we get

$$\int_\Omega M_y\{v_r(x,y)\nabla\psi(y)\}\varphi(x)\,\mathrm{d}x = -\int_\Omega M_y\biggl\{\biggl(\xi(x,y) - \frac{1}{|B_r|}\int_{B_r}\xi(x,y+\rho)\,\mathrm{d}\rho\biggr)\psi(y)\biggr\}\varphi(x)\,\mathrm{d}x,$$

for every $\varphi \in C_0^\infty(\Omega)$, $\psi \in D^\infty$. This implies that for a.e. $x \in \Omega$,

$$\nabla_{y,m}v_r(x,\cdot) = \xi(x,\cdot) - \frac{1}{|B_r|}\int_{B_r}\xi(x,\cdot+\rho)\,\mathrm{d}\rho, \tag{4.21}$$

and then the right-hand side of (4.21) belongs to ∇W^p for a.e. $x \in \Omega$. Since it converges a.e. in Ω to $\xi(x,\cdot) - M_y\{\xi(x,\cdot)\}$ and since ∇W^p is closed, we deduce that there exists $u_1 \in L^p(\Omega; W^p)$ such that

$$\xi - M_y\{\xi\} = \nabla_y u_1.$$

By proposition 4.5, it is also clear that $M_y\{\xi\} = \nabla u$, which together with the above equality gives (4.17). ∎

Analogously to proposition 4.11, we have the following result, which implies that the regularity of u_1 in theorem 4.13 is optimal at least for $1 \leqslant p < +\infty$.

Proposition 4.14. *Let $u \in W^{1,p}(\Omega)$ and $u_1 \in L^p(\Omega; W^p)$, with $1 \leqslant p < +\infty$. Then there exists a bounded sequence $\{u_{\varepsilon_n}\}$ in $W^{1,p}(\Omega)$, such that*

$$\left.\begin{aligned} u_{\varepsilon_n} &\rightharpoonup u \quad \text{in } W^{1,p}(\Omega)\text{-weak,}\\ \nabla u_{\varepsilon_n} &\stackrel{2\text{-}e}{\rightharpoonup} \nabla u + \nabla_y u_1. \end{aligned}\right\} \tag{4.22}$$

Proof. By theorem 3.15 it is easy to check that there exists a sequence $\{\psi_n\} \subset C_0^1(\Omega; C^1(\mathbb{R}^N))$ such that $\nabla_y\psi_n \in C^0(\Omega; (B^p)^N)$,

$$\nabla_y\psi_n \to \nabla_y u_1 \quad \text{in } L^p(\Omega; (B^p)^N) \tag{4.23}$$

and

$$\lim_{\varepsilon\to 0}\biggl(\int_\Omega\biggl|\nabla_y\psi_n\Bigl(x,\frac{x}{\varepsilon}\Bigr) - \nabla_y\psi_m\Bigl(x,\frac{x}{\varepsilon}\Bigr)\biggr|^p\,\mathrm{d}x\biggr)^{1/p} = \biggl(\int_\Omega[\nabla_y(\psi_n(x,y) - \psi_m(x,y))]_p^p\,\mathrm{d}x\biggr)^{1/p} < \frac{1}{2^m} \quad \forall n \geqslant m. \tag{4.24}$$

Thus, for every $n \in \mathbb{N}$ there exists a decreasing sequence $\{\varepsilon_n\} \subset (0,+\infty)$ such that

$$\biggl(\int_\Omega\biggl|\nabla_y\psi_n\Bigl(x,\frac{x}{\varepsilon_n}\Bigr) - \nabla_y\psi_m\Bigl(x,\frac{x}{\varepsilon_n}\Bigr)\biggr|^p\,\mathrm{d}x\biggr)^{1/p} < \frac{1}{2^m} \quad \forall m,n \in \mathbb{N}, \quad 1 \leqslant m \leqslant n, \tag{4.25}$$

and

$$\left(\int_{\Omega}\left|\varepsilon_n\psi_n\left(x,\frac{x}{\varepsilon_n}\right)\right|^p \mathrm{d}x\right)^{1/p} < \frac{1}{2^n}, \qquad \left(\int_{\Omega}\left|\varepsilon_n\nabla_x\psi_n\left(x,\frac{x}{\varepsilon_n}\right)\right|^p \mathrm{d}x\right)^{1/p} < \frac{1}{2^n}. \tag{4.26}$$

Defining

$$u_{\varepsilon_n}(x) = u + \varepsilon_n\psi_n\left(x,\frac{x}{\varepsilon_n}\right)$$

we deduce (4.14). ∎

5. Applications of homogenization problems

Similarly to the 'classical' two-scale convergence for periodic functions (see Allaire 1992; Nguetseng 1990), the main application of the results obtained in the present paper is the homogenization of partial differential problems with coefficients in the spaces B^p, generated by an ergodic algebra. As an example, for $1 < p < +\infty$, let us consider the nonlinear problem

$$\left.\begin{aligned} -\operatorname{div}(a(x/\varepsilon, u_\varepsilon, \nabla u_\varepsilon)) = f \quad \text{in} \quad W^{-1,p'}(\Omega),\\ u_\varepsilon \in W_0^{1,p}(\Omega), \end{aligned}\right\} \tag{5.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, f belongs to $W^{-1,p'}(\Omega)$ and $a : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}^N$ satisfies

(1) For every $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$, $a(\cdot,s,\xi)$ belongs to $(B^{p'})^N$. For a.e. $x \in \mathbb{R}^N$ and every $s \in \mathbb{R}$, $a(x,s,\cdot)$ is continuous.

(2) There exists $\alpha > 0$ such that

$$a(x,s,\xi)\cdot\xi \geqslant \alpha|\xi|^p, \quad \forall (s,\xi)\in\mathbb{R}\times\mathbb{R}^N, \quad \text{a.e. } x\in\mathbb{R}^N. \tag{5.2}$$

(3) There exist $h \in B^{p'}$ and $\beta > 0$ such that

$$|a(x,s,\xi)| \leqslant h(x) + \beta(|s|+|\xi|)^{p-1} \quad \forall (s,\xi)\in\mathbb{R}\times\mathbb{R}^N, \quad \text{a.e. } x\in\mathbb{R}^N. \tag{5.3}$$

(4) We have

$$(a(x,s,\xi_1)-a(x,s,\xi_2))\cdot(\xi_1-\xi_2) \geqslant 0 \quad \forall s\in\mathbb{R},\ \forall \xi_1,\xi_2\in\mathbb{R}^N, \quad \text{a.e. } x\in\mathbb{R}^N. \tag{5.4}$$

(5) There exist $\gamma > 0$, $0 < \sigma \leqslant \min\{p-1,1\}$ and $k \in B^{p'}$ such that

$$|a(x,s_1,\xi)-a(x,s_2,\xi)| \leqslant k(x)+\gamma(|\xi|+|s_1|+|s_2|)^{p-1}\min\{|s_1-s_2|,1\}^\sigma, \tag{5.5}$$

for every $s_1, s_2 \in \mathbb{R}$, every $\xi \in \mathbb{R}^N$ and a.e. $x \in \mathbb{R}^N$.

The existence of a solution u_ε of (5.1) can be found in Lions (1969). The next theorem gives the asymptotic behaviour of u_ε.

Theorem 5.1. *Assume the above hypothesis and let* u_ε *be a solution of (5.1) for every* $\varepsilon > 0$*. There then exist a subsequence, still denoted by* $\{\varepsilon\}$*, a function* $u \in W_0^{1,p}(\Omega)$ *and a function* $u_1 \in L^p(\Omega; W^p)$ *such that*

$$u_\varepsilon \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega)\text{-weak},$$

$$\nabla u_\varepsilon \overset{2\text{e}}{\rightharpoonup} \nabla u + \nabla_y u_1,$$

where $(u, u_1) \in W_0^{1,p}(\Omega) \times L^p(\Omega; W^p)$ *is a solution of the two-scale homogenized system*

$$\left.\begin{aligned} -\operatorname{div}_x M_y\{a(y, u, \nabla u + \nabla_y u_1)\} &= f \quad \text{in } W^{-1,p'}(\Omega), \\ -\operatorname{div}_{m,y}\{a(y, u, \nabla u + \nabla_y u_1)\} &= 0 \quad \text{a.e. } x \in \Omega. \end{aligned}\right\} \tag{5.6}$$

Remark 5.2. In general, the problem (5.6) does not have a unique solution, so the convergence of $\{u_\varepsilon\}$ is only for a subsequence. Assuming a further hypothesis, for example, that $a(x, s, \xi)$ does not depend on s and that in (5.4) the inequality is strict for $\xi_1 \neq \xi_2$, we have the uniqueness of solution of the limit problem which assures that the whole of the sequence $\{u_\varepsilon\}$ converges.

Remark 5.3. Defining $b : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ by

$$b(s, \xi) = M_y\{a(y, s, \xi + \nabla_y v_{s,\xi})\} \quad \forall s, \xi \in \mathbb{R} \times \mathbb{R}^N,$$

with $v_{s,\xi}$ the solution of

$$\left.\begin{aligned} -\operatorname{div}_m(a(y, s, \xi + \nabla_y v_{s,\xi})) = 0 \quad \text{in } (W^p)', \\ v_{s,\xi} \in W^p, \end{aligned}\right\}$$

the function u in the statement of theorem 5.1 satisfies

$$\left.\begin{aligned} -\operatorname{div} b(u, \nabla u) = f \quad \text{in } W^{-1,p'}(\Omega), \\ u \in W_0^{1,p}(\Omega). \end{aligned}\right\}$$

Remark 5.4. In the particular case of a linear equation, $a(x/\varepsilon, u_\varepsilon, \nabla u_\varepsilon) = A(x/\varepsilon)\nabla u_\varepsilon$ and A, a matrix whose coefficients are almost-periodic functions, the problem has been studied in Oleinik & Zhikov (1982).

In the case when the equation is monotone, with $a(x/\varepsilon, \nabla u_\varepsilon)$ almost-periodic in the first variable, the homogenization of (5.1) has been done in Braides *et al.* (1992), using approximation results in smoother almost-periodic spaces. Corrector results for these operators are proved in Braides (1991), exploiting the geometric properties of a.

Proof. By (5.2) the sequence $\{u_\varepsilon\}$ is bounded in $W_0^{1,p}(\Omega)$, and then by (5.3), the sequence $\{g_\varepsilon\}$ defined by

$$g_\varepsilon = a\left(\frac{x}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon\right)$$

is bounded in $L^{p'}(\Omega)^N$. Theorems 4.8 and 4.13 then imply that there exists a subsequence, still denoted by $\{\varepsilon\}$, a function $u \in W_0^{1,p}(\Omega)$, a function $u_1 \in L^p(\Omega; W^p)$ and a function $g_0 \in L^{p'}(\Omega; B^{p'})^N$ such that

$$u_\varepsilon \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega), \tag{5.7}$$

$$\nabla u_\varepsilon \overset{2\text{e}}{\rightharpoonup} \nabla u + \nabla_y u_1, \tag{5.8}$$

$$g_\varepsilon \overset{2\text{e}}{\rightharpoonup} g_0. \tag{5.9}$$

Let $\varphi \in C_0^\infty(\Omega)$, $\psi \in C_0^\infty(\Omega)$, $v \in D^\infty$, and consider $\varphi(x) + \varepsilon\psi(x)v(x/\varepsilon)$ as a test function in (5.1). This gives

$$\int_\Omega g_\varepsilon(x)\left(\nabla\varphi(x) + \varepsilon v\left(\frac{x}{\varepsilon}\right)\nabla\psi(x) + \psi(x)\nabla_y v\left(\frac{x}{\varepsilon}\right)\right) \mathrm{d}x = \left\langle f, \varphi(x) + \varepsilon\psi(x)v\left(\frac{x}{\varepsilon}\right)\right\rangle,$$

and then, taking the limit when ε tends to zero, we deduce

$$\int_\Omega M_y\{g_0(x,y)(\nabla\varphi(x) + \psi(x)\nabla_y v(y))\} \,\mathrm{d}x = \langle f, \varphi\rangle,$$

for every φ, ψ, v as above. Reasoning by linearity and density, g_0 satisfies

$$\left.\begin{gathered}\int_\Omega M_y\{g_0(x,y)(\nabla v(x) + \nabla_y v_1(x,y)\} \,\mathrm{d}x = \langle f, v\rangle, \\ \forall v \in W_0^{1,p}(\Omega),\ \forall v_1 \in L^p(\Omega; W^p),\end{gathered}\right\} \tag{5.10}$$

i.e.

$$\left.\begin{aligned} -\operatorname{div}_x M_y\{g_0(x,y)\} &= f \qquad \text{in } W^{-1,p'}(\Omega), \\ -\operatorname{div}_{m,y}\{g_0(x,y)\} &= 0, \qquad \text{a.e. } x \in \Omega.\end{aligned}\right\} \tag{5.11}$$

Let us use the Minty rule to characterize g_0. For $\Psi, \Phi \in St_c(\Omega; (B^p)^N)$, such that $\Psi(x,\cdot)$, $\Phi(x,\cdot)$ belong to D^∞ for a.e. $x \in \Omega$, and $t \in (0,1)$, we define

$$\mu_\varepsilon = \nabla u(x) + \Psi\left(x, \frac{x}{\varepsilon}\right) + t\Phi\left(x, \frac{x}{\varepsilon}\right),$$

which two-scale converges to μ_0 defined by

$$\mu_0(x,y) = \nabla u(x) + \Psi(x,y) + t\Phi(x,y). \tag{5.12}$$

By (5.4) we have

$$\int_\Omega \left(g_\varepsilon - a\left(\frac{x}{\varepsilon}, u_\varepsilon, \mu_\varepsilon\right)\right) \cdot (\nabla u_\varepsilon - \mu_\varepsilon) \,\mathrm{d}x \geqslant 0. \tag{5.13}$$

Let us pass to the limit in the different terms of this inequality. Taking u_ε as a test function in (5.1) and u as a test function in the first equation of (5.11), we get

$$\lim_{\varepsilon\to 0} \int_\Omega g_\varepsilon \nabla u_\varepsilon \,\mathrm{d}x = \lim_{\varepsilon\to 0}\langle f, u_\varepsilon\rangle = \langle f, u\rangle = \int_\Omega M_y\{g_0 \nabla u\} \,\mathrm{d}x. \tag{5.14}$$

By (5.9), we have

$$\lim_{\varepsilon\to 0} \int_\Omega g_\varepsilon \mu_\varepsilon \,\mathrm{d}x = \int_\Omega M_y\{g_0\mu_0\} \,\mathrm{d}x. \tag{5.15}$$

The hypothesis (5.5) of a implies

$$\left|a\left(\frac{x}{\varepsilon}, u_\varepsilon, \mu_\varepsilon\right) - a\left(\frac{x}{\varepsilon}, u, \mu_\varepsilon\right)\right| \leqslant k\left(\frac{x}{\varepsilon}\right) + \gamma(|\mu_\varepsilon| + |u_\varepsilon| + |u|)^{p-1} \min\{|u_\varepsilon - u|, 1\}^\sigma.$$

Taking the power p' and integrating in Ω, we deduce

$$\int_\Omega \left| a\left(\frac{x}{\varepsilon}, u_\varepsilon, \mu_\varepsilon\right) - a\left(\frac{x}{\varepsilon}, u, \mu_\varepsilon\right)\right|^{p'} \mathrm{d}x \leqslant 2^{1/(p-1)} \int_\Omega k\left(\frac{x}{\varepsilon}\right)^{p'} + \gamma^{p'} 3^{p-1}(|\mu_\varepsilon|^p + |u_\varepsilon|^p + |u|^p) \min\{|u_\varepsilon - u|, 1\}^{\sigma p'} \,\mathrm{d}x. \tag{5.16}$$

Since the power p' of k and the power p of $|\mu_\varepsilon|$ and $|u_\varepsilon|$ have a mean value, the sequence $\{k(x/\varepsilon)^{p'} + \gamma^{p'} 3^{p-1}(|\mu_\varepsilon|^p + |u_\varepsilon|^p + |u|^p)\}$ converges weakly in $L^1(\Omega)$. Moreover, the sequence $\{\min\{|u_\varepsilon - u|, 1\}^{\sigma p'}\}$ is bounded in $L^\infty(\Omega)$ and converges a.e. in Ω to zero. Thus (use Egorov's theorem), we conclude that the right-hand side of (5.16) converges to zero, and then

$$a\left(\frac{x}{\varepsilon}, u_\varepsilon, \mu_\varepsilon\right) - a\left(\frac{x}{\varepsilon}, u, \mu_\varepsilon\right) \to 0 \quad \text{in } L^{p'}(\Omega). \tag{5.17}$$

So, by (5.8) we deduce

$$\lim_{\varepsilon \to 0} \int_\Omega a\left(\frac{x}{\varepsilon}, u_\varepsilon, \mu_\varepsilon\right)(\nabla u_\varepsilon - \mu_\varepsilon)\,\mathrm{d}x = \lim_{\varepsilon \to 0} \int_\Omega a\left(\frac{x}{\varepsilon}, u, \mu_\varepsilon\right)(\nabla u_\varepsilon - \mu_\varepsilon)\,\mathrm{d}x = \int_\Omega M_y\{a(y, u, \mu_0)((\nabla u + \nabla_y u_1) - \mu_0)\}\,\mathrm{d}x. \tag{5.18}$$

Thus, passing to the limit when ε tends to zero in (5.13) we conclude

$$\int_\Omega M_y\{g_0\}\nabla u\,\mathrm{d}x - \int_\Omega M_y\{g_0 \mu_0\}\,\mathrm{d}x - \int_\Omega M_y\{a(y, u, \mu_0)((\nabla u + \nabla_y u_1) - \mu_0)\}\,\mathrm{d}x \geqslant 0.$$

Replacing μ_0 by (5.12) and using the second equation of (5.11) we have

$$\int_\Omega M_y\{[g_0 - a(y, u, \mu_0)][\Psi + t\Phi - \nabla_y u_1]\}\,\mathrm{d}x \leqslant 0. \tag{5.19}$$

Choosing Ψ converging to $\nabla_y u_1$ in $L^p(\Omega; B^p)^N$ and Φ converging to a function $W \in L^p(\Omega; (B^p)^N)$, we deduce

$$t \int_\Omega M_y\{[g_0 - a(y, u, \nabla u + \nabla_y u_1 + tW)W]\}\,\mathrm{d}x \leqslant 0 \quad \forall W \in L^p(\Omega; (B^p)^N).$$

Dividing by t and then taking the limit when t tends to zero, we get

$$\int_\Omega M_y\{(a(y, u, \nabla u + \nabla_y u_1) - g_0)W\}\,\mathrm{d}x \geqslant 0 \quad \forall W \in L^p(\Omega; (B^p)^N),$$

i.e. $g_0 = a(y, u, \nabla u + \nabla_y u_1)$, which by (5.11) finishes the proof of theorem 5.1. ■

In order to obtain a corrector result, let us now assume that a is uniformly monotone, i.e. there exists $\delta > 0$ such that for every $s \in \mathbb{R}$, $\xi_1, \xi_2 \in \mathbb{R}^N$ and a.e. $x \in \Omega$ we have

$$(a(x, s, \xi_1) - a(x, s, \xi_2)) \cdot (\xi_1 - \xi_2) \geqslant \begin{cases} \delta|\xi_1 - \xi_2|^p & \text{if } p \geqslant 2, \\ \delta \dfrac{|\xi_1 - \xi_2|^2}{(|\xi_1| + |\xi_2|)^{2-p}} & \text{if } 1 < p < 2. \end{cases} \tag{5.20}$$

Theorem 5.5. *Under the hypothesis of theorem 5.1 and (5.20), if u_1 is smooth enough (for example, $u_1 \in C^1(\bar{\Omega} \times \mathbb{R}^N)$ with $\nabla_x u_1 \in L^\infty(\Omega \times \mathbb{R}^N)$, $\nabla_y u_1 \in C(\bar{\Omega}; B^1) \cap L^\infty(\Omega \times \mathbb{R}^N)$, then*

$$u_\varepsilon(\cdot) - u(\cdot) - \varepsilon u_1\left(\cdot, \frac{\cdot}{\varepsilon}\right) \to 0 \quad \text{in } W^{1,p}(\Omega). \tag{5.21}$$

Proof. Let $Z_\varepsilon = \nabla u(x) + \nabla_y u_1(x, x/\varepsilon)$. Reasoning similarly to the proof of (5.17), we can prove

$$a\left(\frac{x}{\varepsilon}, u_\varepsilon, Z_\varepsilon\right) - a\left(\frac{x}{\varepsilon}, u, Z_\varepsilon\right) \to 0 \quad \text{in } L^{p'}(\Omega).$$

So,

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_\Omega \left(a\left(\frac{x}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon\right) - a\left(\frac{x}{\varepsilon}, u_\varepsilon, Z_\varepsilon\right)\right) \cdot (\nabla u_\varepsilon - Z_\varepsilon)\, \mathrm{d}x \\ = \lim_{\varepsilon\to 0} \int_\Omega \left(a\left(\frac{x}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon\right) - a\left(\frac{x}{\varepsilon}, u, Z_\varepsilon\right)\right) \cdot (\nabla u_\varepsilon - Z_\varepsilon)\, \mathrm{d}x. \end{aligned} \tag{5.22}$$

In order to pass to the limit on the right-hand side of this equality, it is enough to use the fact that $a(x/\varepsilon, u, \nabla u_\varepsilon)$ two-scale converges to $a(y, u, \nabla u + \nabla_y u_1)$ (see the proof of theorem 5.1), the fact that ∇u_ε two-scale converges to $\nabla u + \nabla_y u_1$ and (5.14), which imply

$$\lim_{\varepsilon\to 0} \int_\Omega \left(a\left(\frac{x}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon\right) - a\left(\frac{x}{\varepsilon}, u, Z_\varepsilon\right)\right) \cdot (\nabla u_\varepsilon - Z_\varepsilon) = 0. \tag{5.23}$$

From (5.22), (5.23) and (5.20) we conclude that $\nabla u_\varepsilon - Z_\varepsilon$ converges strongly to zero in $L^p(\Omega)$ and then (5.21) holds. ∎

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References

Abddaimi, Y., Michaille, G. & Licht, C. 1997 Stochastic homogenization for an integral function of a quasiconvex function with linear growth. *Asymp. Analysis* **15**, 183–212.

Allaire, G. 1992 Homogenization and two-scale convergence. *SIAM J. Math. Analysis* **23**, 1482–1518.

Arbogast, T., Douglas, J. & Hornung, U. 1990 Derivation of the double porosity model of single phase flow via homogenization theory. *SIAM J. Math. Analysis* **21**, 823–836.

Bensoussan, A., Lions, J. L. & Papanicolaou, G. 1978 *Asymptotic analysis for periodic structures.* Amsterdam: North-Holland.

Besicovitch, A. S. 1954 *Almost periodic functions.* Dover.

Bohr, H. 1951 *Almost periodic functions.* New York: Chelsea.

Bourgeat, A., Mikelic, A. & Wright, S. 1994 Stochastic two-scale convergence in the mean and applications. *J. Reine Angew. Math.* **456**, 19–51.

Braides, A. 1991 Correctors for the homogenization of almost periodic monotone operators. *Asymp. Analysis* **5**, 47–74.

Braides, A., Chiadò Piat, V. & Defranceschi, A. 1992 Homogenization of almost periodic monotone operators. *Annls Inst. H. Poincaré Analyse Non Linéaire* **9**, 399–432.

Casado Díaz, J. & Gayte, I. 1996 A general compactness result and its application to the two-scale convergence of almost periodic functions. *C. R. Acad. Sci. Paris Sér. I* **323**, 329–334.

Casado Díaz, J. & Gayte, I. 2002 A derivation theory for generalized Besicovitch spaces and its application for partial differential equations. *Proc. R. Soc. Edinb.* A **132**, 283–315.

Cioranescu, I. 1990 *Geometry of Banach spaces, duality mappings and nonlinear problems.* Dordrecht: Kluwer.

Dal Maso, G. & Modica, L. 1986 Non linear stochastic homogenization and ergodic theory. *J. Reine Angew. Math.* **368**, 28–42.

Gayte Delgado, I. 1998 Espacios de Besicovitch generalizados y convergencia en dos escalas. PhD thesis, University of Seville.

Jikov, V. V., Kozlov, S. M. & Oleinik, O. A. 1994 *Homogenization of differential operators and integral functionals.* Springer.

Lindenstrauss, J. 1966 On non separable reflexive Banach spaces. *Bull. Am. Math. Soc.* **72**, 967–970.

Lions, J. L. 1969 *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Paris: Dunod.

Nguetseng, G. 1990 A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Analysis* **21**, 608–623.

Nguetseng, G. 2000 Almost periodic homogenization: asymptotic analysis of a second order elliptic equation. Publication Mathématiques du Laboratoire d'Analyse Numérique, Université de Yaounde I (/01).

Oleinik, O. A. & Zhikov, V. V. 1982 On the homogenization of elliptic operators with almost-periodic coefficients. *Rend. Sem. Mat. Fis. Milano* **52**, 149–166.

Sánchez Palencia, E. 1980 *Non-Homogeneous media and vibration theory.* Lectures Notes in Physics 127. Springer.

Zhikov, V. V. & Krivenko, E. V. 1983 Averaging of singularly pertubed elliptic operators. *Mat. Zametki* **33**, 571–582. (English transl.: *Math. Notes* **33**, 294–300.)