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Compact Bilinear Commutators: The Weighted Case

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Abstract. Commutators of bilinear Calderón–Zygmund operators and multiplication by functions in a certain subspace of the space of functions of bounded mean oscillation are shown to be compact on appropriate products of weighted Lebesgue spaces.

1. Introduction and Statements of Main Results

The study in harmonic analysis of commutators of singular integrals with pointwise multiplication by functions in *BMO* started with the by now well-known 1976 work of Coifman, Rochberg, and Weiss [6]. A couple of years later, in another classic work in the subject, Uchiyama [16] proved that the L^p-boundedness result in [6] can be refined to a compactness one if the space *BMO* is replaced by the smaller space *CMO*. Recently, Bényi and Torres [2] revisited a notion of compactness in a bilinear setting, which was first introduced by Calderón in his fundamental paper on interpolation [3]. They showed in [2] that commutators of bilinear Calderón–Zygmund operators with multiplication by *CMO* functions are compact bilinear operators from $L^{p_1} \times L^{p_2} \to L^p$ for $1 < p_1, p_2 < \infty$ and $1/p_1 + 1/p_2 = 1/p \leq 1$, thus giving an extension to the bilinear setting of result in [16] for the linear case. In a subsequent joint work with Damián and Moen [1], the scope of the notion of compactness was expanded to include the commutators of a larger family of operators that contains bilinear Calderón–Zygmund ones and several singular bilinear fractional integrals. All these compactness results rely on the Frechét–Kolmogorov–Riesz characterization of precompact sets in unweighted Lebesgues spaces L^p; see Yosida's book [17, p. 275] and the expository note of Hanche-Olsen and Holden [10].

What happens if we change the Lebesgue measure dx with weighted versions $w\,dx$? This article originates in this natural question. Although seemingly simple, the answer to this question turns out to be more delicate than in the unweighted case. As we shall see, the compactness on products of weighted Lebesgue spaces depends rather crucially on the class of weights w considered. We note that in the linear case the compactness of the commutator on weighted spaces was not

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known until the recent work of Clop and Cruz [5]. We will rely on their work for the selection of weights and some computations.

Let then T be a bilinear Calderón–Zygmund operator. For the purposes of this article, this means that T is a bounded map from $L^{p_1} \times L^{p_2}$ to L^p with $1 < p_1, p_2 < \infty$ and

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}, \tag{1}$$

and there exists a kernel $K(x, y, z)$ defined away from the diagonal $x = y = z$ such that

$$|K(x, y, z)| \lesssim \frac{1}{(|x-y| + |x-z|)^{2n}}, \tag{2}$$

$$|\nabla K(x, y, z)| \lesssim \frac{1}{(|x-y| + |x-z|)^{2n+1}}, \tag{3}$$

and such that for $f, g \in L_c^\infty$, we have

$$T(f, g)(x) = \iint_{\mathbb{R}^{2n}} K(x, y, z) f(y) g(z)\, dy\, dz, \quad x \notin \operatorname{supp} f \cap \operatorname{supp} g. \tag{4}$$

See [8] and the references therein for more on this type of operators.

We will consider the commutators of bilinear Calderón–Zygmund operators with functions in an appropriate subspace of BMO. Recall that BMO consists of all locally integrable functions b with $\|b\|_{BMO} < \infty$, where

$$\|b\|_{BMO} = \sup_Q ⨍_Q \left| b(x) - ⨍_Q b \right| dx,$$

the supremum is taken over all cubes $Q \in \mathbb{R}^n$, and, as usual, $⨍_Q b = b_Q$ denotes the average of b over Q:

$$⨍_Q b = \frac{1}{|Q|} \int_Q b(x)\, dx.$$

The relevant subspace of BMO of multiplicative symbols of our focus is CMO, which is defined to be the closure of $C_c^\infty(\mathbb{R}^n)$ in the BMO norm.

Given a bilinear Calderón–Zygmund operator T and a function b in BMO, we consider the following commutators with b:

$$[b, T]_1(f, g) = T(bf, g) - bT(f, g)$$

and

$$[b, T]_2(f, g) = T(f, bg) - bT(f, g).$$

Furthermore, given $\mathbf{b} = (b_1, b_2)$ in $BMO \times BMO$, we consider the iterated commutator

$$[\mathbf{b}, T] = [b_2, [b_1, T]_1]_2 = [b_1, [b_2, T]_2]_1.$$

In fact, for bilinear Calderón–Zygmund operators T and $\mathbf{b} = (b_1, b_2)$, we can define $[\mathbf{b}, T]_\alpha$ for any multiindex $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$, formally as

$$[\mathbf{b}, T]_\alpha(f, g)(x) = \iint (b_1(y) - b_1(x))^{\alpha_1} (b_2(z) - b_2(x))^{\alpha_2} K(x, y, z) f(y) g(z)\, dy\, dz.$$

Recall that a bilinear operator is said to be (jointly) compact if the image of the bi-unit ball

$$\{(f, g) : \ \|f\|_{L^{p_1}} \le 1, \|g\|_{L^{p_2}} \le 1\}$$

under its action is a precompact set in L^p. (The only notion of compactness in the bilinear setting used here is referred to as *joint compactness* in the related previous works, to differentiate it from the weaker notion of *separate compactness*. The latter being the compactness of the linear operators obtained when one of the entries in the bilinear one is kept fixed.) When $1 < p_1, p_2 < \infty$, $p = p_1 p_2/(p_1 + p_2) \ge 1$, $\alpha_1, \alpha_2 = 0$ or 1, and $\mathbf{b}$ in $CMO \times CMO$, we have that

$$[\mathbf{b}, T]_\alpha : L^{p_1} \times L^{p_2} \to L^p$$

is a compact bilinear operator; see [2]. In this note we will consider what happens on weighted Lebesgue spaces.

Given $\mathbf{p} = (p_1, p_2) \in (1, \infty) \times (1, \infty)$ and a vector weight $\mathbf{w} = (w_1, w_2)$, let

$$\nu_{\mathbf{w}} = \nu_{\mathbf{w},\mathbf{p}} = w_1^{p/p_1} w_2^{p/p_2}.$$

The vector weight $\mathbf{w}$ belongs to the class $\mathbf{A_p}$, provided that

$$[\mathbf{w}]_{\mathbf{A_p}} = \sup_Q \left(\fint_Q \nu_{\mathbf{w}} \right) \left(\fint_Q w_1^{1-p_1'} \right)^{p/p_1'} \left(\fint_Q w_2^{1-p_2'} \right)^{p/p_2'} < \infty.$$

In [12], Lerner et al. proved that

$$\mathbf{w} \in \mathbf{A_p} \quad \Leftrightarrow \quad \begin{cases} \nu_{\mathbf{w}} \in A_{2p}, \\ \sigma_1 = w_1^{1-p_1'} \in A_{2p_1'}, \\ \sigma_2 = w_2^{1-p_2'} \in A_{2p_2'}. \end{cases} \tag{5}$$

Recall that the classical Muckenhoupt class A_p consists of nonnegative weights w that are locally integrable and such that

$$[w]_{A_p} = \sup_Q \left(\fint_Q w \right) \left(\fint_Q w^{1-p'} \right)^{p/p'} < \infty. \tag{6}$$

The weights in the class $\mathbf{A_p}$ characterize the boundedness of the maximal function

$$\mathcal{M} : L^{p_1}(w_1) \times L^{p_1}(w_1) \to L^p(\nu_{\mathbf{w}}),$$

where

$$\mathcal{M}(f, g)(x) = \sup_{Q \ni x} \left(\fint_Q |f(y)|\, dy \right) \left(\fint_Q |g(z)|\, dz \right).$$

From (5) we can see that when $p \ge 1$, we have

$$A_p \times A_p \subsetneq A_{\min(p_1,p_2)} \times A_{\min(p_1,p_2)} \subsetneq A_{p_1} \times A_{p_2} \subsetneq \mathbf{A_p}. \tag{7}$$

The first two containments follow from well-known properties of the (scalar) A_p classes, and the last containment is proved in [12] (see Section 3 for a new example of the strictness of this containment). Moreover, we also note that

$$\mathbf{w} \in A_p \times A_p \quad \Longrightarrow \quad \nu_{\mathbf{w}} \in A_p. \tag{8}$$

Indeed, by Hölder's inequality

$$\left(\fint_Q \nu_{\mathbf{w},\mathbf{p}}\right)\left(\fint_Q \nu_{\mathbf{w},\mathbf{p}}^{1-p'}\right)^{p-1} \leq [w_1]_{A_p}^{p/p_1}[w_2]_{A_p}^{p/p_2}.$$

It was shown in [12] that if $w \in \mathbf{A_p}$ and T is a bilinear Calderón–Zygmund operator, then T is bounded from $L^{p_1}(w_1) \times L^{p_2}(w_2)$ into $L^p(\nu_{\mathbf{w}})$, and the same result holds for the first-order commutator. The boundedness of the iterated commutator on weighted Lebesgue spaces in the case of $\mathbf{A_p}$ weights was obtained by Pérez et al. [13]. The case of product of classical weights was considered also by Tang [15].

The goal of this paper is to show that the improving effect of the bilinear commutators caries over to the weighted setting when we consider the "appropriate" class of weights. We have the following theorem.

THEOREM 1.1. *Suppose that* $\mathbf{p} \in (1,\infty) \times (1,\infty)$, $p = p_1p_2/(p_1+p_2) > 1$, $b \in$ *CMO, and* $\mathbf{w} \in A_p \times A_p$. *Then* $[b,T]_1$ *and* $[b,T]_2$ *are compact operators from* $L^{p_1}(w_1) \times L^{p_2}(w_2)$ *to* $L^p(\nu_{\mathbf{w}})$.

A similar result holds also for the iterated commutator.

THEOREM 1.2. *Suppose that* $\mathbf{p} \in (1,\infty) \times (1,\infty)$, $p = p_1p_2/(p_1+p_2) > 1$, $\mathbf{b} \in CMO \times CMO$, *and* $\mathbf{w} \in A_p \times A_p$. *Then* $[\mathbf{b},T]$ *is a compact operator from* $L^{p_1}(w_1) \times L^{p_2}(w_2)$ *to* $L^p(\nu_{\mathbf{w}})$.

The remainder of the paper is structured as follows. In Section 2 we give the proofs of Theorems 1.1 and 1.2, whereas in Section 3 we provide a discussion regarding the class of weights assumed in our main results.

2. Proofs of Theorems

As pointed out in [5], in the linear setting the idea of considering truncated operators to prove compactness results goes back to Krantz and Li [11]. We will follow this approach too, but we find convenient to introduce a smooth truncation. (This approach could also be used to simplify some of the computations in [5] in the linear case.)

Let $\varphi = \varphi(x,y,z)$ be a nonnegative function in $C_c^\infty(\mathbb{R}^{3n})$ such that

$$\operatorname{supp}\varphi \subset \{(x,y,z) : \max(|x|,|y|,|z|) < 1\}$$

and

$$\int_{\mathbb{R}^{3n}} \varphi(u)\,du = 1.$$

For $\delta > 0$, let $\chi^\delta = \chi^\delta(x, y, z)$ be the characteristic function of the set

$$\left\{(x, y, z):\ \max(|x-y|, |x-z|) \geq \frac{3\delta}{2}\right\},$$

and let

$$\psi^\delta = \varphi_\delta * \chi^\delta,$$

where

$$\varphi_\delta(x, y, z) = (\delta/4)^{-3n}\varphi_\delta(4x/\delta, 4y/\delta, 4z/\delta).$$

Clearly, we have that $\psi^\delta \in C^\infty$,

$$\operatorname{supp}\psi^\delta \subset \{(x, y, z):\ \max(|x-y|, |x-z|) \geq \delta\},$$

$\psi^\delta(x, y, z) = 1$ if $\max(|x-y|, |x-z|) > 2\delta$, and $\|\psi^\delta\|_{L^\infty} \leq 1$. Moreover, $\nabla\psi^\delta$ is not zero only if $\max(|x-y|, |x-z|) \approx \delta$ and $\|\nabla\psi^\delta\|_{L^\infty} \lesssim 1/\delta$. Given a kernel K associated to a Calderón–Zygmund operator T, we define the truncated kernel

$$K^\delta(x, y, z) = \psi^\delta(x, y, z)K(x, y, z).$$

It follows that K^δ satisfies the same size and regularity estimates of K, (2) and (3), with a constant C independent of δ. We let $T^\delta(f, g)$ be the operator defined pointwise by K^δ through (4), now for all $x \in \mathbb{R}^n$. We have the following lemma.

Lemma 2.1. *If* $\mathbf{b} \in C_c^\infty \times C_c^\infty$, *then*

$$|[\mathbf{b}, T^\delta]_\alpha(f, g)(x) - [\mathbf{b}, T]_\alpha(f, g)(x)| \lesssim \|\nabla b_1\|_\infty^{\alpha_1}\|\nabla b_2\|_\infty^{\alpha_2}\delta^{|\alpha|}\mathcal{M}(f, g)(x).$$

Consequently, if $\mathbf{w} \in \mathbf{A_p}$, *then we have*

$$\lim_{\delta\to 0}\|[\mathbf{b}, T^\delta]_\alpha - [\mathbf{b}, T]_\alpha\|_{L^{p_1}(w_1)\times L^{p_2}(w_2)\to L^p(\nu_{\mathbf{w}})} = 0.$$

Proof. We adapt the proof given in [5, Lemma 7] for the linear version of the result. For simplicity, we consider the case $\alpha = (1, 0)$; the other cases are similar. We have:

$$\begin{aligned}
&|[b, T^\delta]_1(f, g)(x) - [b, T]_1(f, g)(x)| \\
&\quad\lesssim \iint_{\max(|x-y|,|x-z|)\leq 2\delta} |b(y)-b(x)||K(x, y, z)f(y)g(z)|\,dy\,dz \\
&\qquad + \iint_{\delta\leq\max(|x-y|,|x-z|)\leq 2\delta} |b(y)-b(x)||K^\delta(x, y, z)f(y)g(z)|\,dy\,dz \\
&\quad\lesssim \|\nabla b\|_{L^\infty}\iint_{\max(|x-y|,|x-z|)\leq 2\delta} \frac{|f(y)||g(z)|}{(|x-y|+|x-z|)^{2n-1}}\,dy\,dz \\
&\quad\lesssim \|\nabla b\|_{L^\infty}\sum_{j\geq 0}\iint_{2^{-j}\delta\leq\max(|x-y|,|x-z|)\leq 2^{-j+1}\delta} \frac{|f(y)||g(z)|}{(|x-y|+|x-z|)^{2n-1}}\,dy\,dz \\
&\quad\lesssim \|\nabla b\|_{L^\infty}\sum_{j\geq 0}\bigg(\int_{|x-y|\leq 2^{-j+1}\delta}|f(y)|\,dy\int_{2^{-j}\delta\leq|x-z|\leq 2^{-j+1}\delta}\frac{|g(z)|}{|x-z|^{2n-1}}\,dz \\
&\qquad + \int_{2^{-j}\delta\leq|x-y|\leq 2^{-j+1}\delta}\frac{|f(y)|}{|x-y|^{2n-1}}\,dy\int_{|x-z|\leq 2^{-j}\delta}|g(z)|\,dz\bigg)
\end{aligned}$$

$$\lesssim \|\nabla b\|_{L^\infty} \sum_{j\geq 0} (2^{-j}\delta)^{1-2n}$$
$$\times \left(\int_{|x-y|\lesssim 2^{-j+1}\delta} |f(y)|\, dy \int_{|z-y|\lesssim 2^{-j+1}\delta} |g(z)|\, dz \right)$$
$$\lesssim \|\nabla b\|_{L^\infty} \delta \sum_{j\geq 0} 2^{-j} \mathcal{M}(f,g)(x),$$

and the rest of the result follows from the boundedness properties of the maximal function $\mathcal{M}$. □

Lemma 2.1 shows that $[b, T^\delta]_\alpha$ converges in operator norm to $[b, T^\delta]_\alpha$, provided that the functions b_1 and b_2 are smooth enough. Therefore, in order to prove that any of the commutators $[\mathbf{b}, T]_\alpha$ is compact, it suffices (as in the linear case, the limit in the operator norm of compact operators is compact) to work with $[\mathbf{b}, T^\delta]_\alpha$ for a fixed δ, and our estimates may depend on δ. Moreover, since the bounds of the commutators with *BMO* functions are of the form

$$\|[\mathbf{b}, T]_\alpha(f,g)\|_{L^p(\nu_{\mathbf{w}})} \lesssim \|b_1\|_{BMO}^{\alpha_1} \|b_2\|_{BMO}^{\alpha_2} \|f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)},$$

to show compactness when working with symbols in *CMO*, we may also assume that $\mathbf{b} \in C_c^\infty \times C_c^\infty$ and the estimates may depend on $\mathbf{b}$ too.

A relevant observation made in [5, Theorem 5] is that there exists a sufficient condition for precompactness in $L^r(w)$ when the weight is assumed, crucially for the argument to work, in A_r. By adapting the arguments in [10] and, in particular, circumventing the nontranslation invariance of $L^r(w)$, the authors in [5] obtained the following weighted variant of the Frechét–Kolmogorov–Riesz result:

Let $1 < r < \infty$ and $w \in A_r$, and let $\mathcal{K} \subset L^r(w)$. If

(i) *$\mathcal{K}$ is bounded in $L^r(w)$,*
(ii) $\lim_{A\to\infty} \int_{|x|>A} |f(x)|^r w(x)\, dx = 0$ *uniformly for $f \in \mathcal{K}$, and*
(iii) $\lim_{t\to 0} \|f(\cdot + t) - f\|_{L^r(w)} = 0$ *uniformly for $f \in \mathcal{K}$,*

then $\mathcal{K}$ is precompact in $L^r(w)$.

Let us immediately note now that our choice for the class of vector weights in Theorems 1.1 and 1.2 is dictated by the previous compactness criterion. In both our results we will need the weight $\nu_{\mathbf{w},\mathbf{p}} \in A_p$ to apply the above version of the Frechét–Kolmogorov–Riesz theorem. In general, if $\mathbf{w} \in A_{p_1} \times A_{p_2}$ or $\mathbf{w} \in \mathbf{A_p}$, then the best class that $\nu_{\mathbf{w},\mathbf{p}}$ belongs to is A_{2p}. However, as we noticed in (8), if $\mathbf{w} \in A_p \times A_p$, then $\nu_{\mathbf{w},\mathbf{p}}$ is actually in A_p. We also point out that there exist examples with $\mathbf{w} \in \mathbf{A_p}$ and $\nu_{\mathbf{w}} \in A_p$, but $\mathbf{w} \notin A_p \times A_p$ (see Section 3).

Proof of Theorem 1.1. We will work with the commutator in the first variable; by symmetry, the proof for the other commutator is identical. As already pointed out, we may fix $\delta > 0$, assume that $b \in C_c^\infty$, and study $[b, T^\delta]_1$. Suppose that f, g belong to

$$B_1(L^{p_1}(w_1)) \times B_1(L^{p_2}(w_2)) = \{(f,g) :\ \|f\|_{L^{p_1}(w_1)}, \|g\|_{L^{p_2}(w_2)} \leq 1\}$$

with w_1 and w_2 in A_p. We need to show that the following conditions hold:

(a) $[b, T^\delta]_1(B_1(L^{p_1}(w_1)) \times B_1(L^{p_2}(w_2)))$ is bounded in $L^p(\nu_{\mathbf{w}})$;
(b) $\lim_{R\to\infty} \int_{|x|>R} [b, T^\delta]_1(f, g)(x)^p \nu_{\mathbf{w}}\, dx = 0$;
(c) $\lim_{t\to 0} \|[b, T^\delta]_1(f, g)(\cdot + t) - [b, T^\delta]_1(f, g)\|_{L^p(\nu_{\mathbf{w}})} = 0$.

It is clear that the first condition (a) holds since

$$[b, T^\delta]_1 : L^{p_1}(w_1) \times L^{p_2}(w_2) \to L^p(\nu_{\mathbf{w}})$$

is bounded when $\mathbf{w} \in A_p \times A_p \subset \mathbf{A_p}$.

We now show that the second condition (b) holds. It is worth pointing out that for our calculations to work, we need the restrictive assumption $\nu_{\mathbf{w}} \in A_p$, which holds since $\mathbf{w} \in A_p \times A_p$. Let A be large enough so that $\operatorname{supp} b \subset B_A(0)$, and let $R \geq \max(2A, 1)$. Then, for $|x| > R$, we have

$$\begin{aligned}
|[b, T^\delta]_1(f, g)(x)| &\leq \|b\|_\infty \int_{\operatorname{supp} b} \int_{\mathbb{R}^n} \frac{|f(y)||g(z)|}{(|x-y| + |x-z|)^{2n}}\, dy\, dz \\
&\lesssim \|b\|_\infty \int_{\operatorname{supp} b} |f(y)| \int_{\mathbb{R}^n} \frac{|g(z)|}{(|x| + |x-z|)^{2n}}\, dy\, dz \\
&\leq \|b\|_\infty \|f\|_{L^{p_1}(w_1)} \sigma_1(B_A(0))^{1/p_1'} \int_{\mathbb{R}^n} \frac{|g(z)|}{(|x| + |x-z|)^{2n}}\, dz \\
&\leq \frac{\|b\|_\infty}{|x|^n} \|f\|_{L^{p_1}(w_1)} \sigma_1(B_A(0))^{1/p_1'} \int \frac{|g(z)|}{(|x| + |x-z|)^{n}}\, dz.
\end{aligned}$$

Now, since $|x| > 1$, it follows that $|z| + 1 \lesssim |z - x| + |x|$ and

$$\int_{\mathbb{R}^n} \frac{|g(z)|}{(|x| + |x-z|)^{n}}\, dz \lesssim \|g\|_{L^{p_2}(w_2)} \left(\int_{\mathbb{R}^n} \frac{\sigma_2(z)}{(1+|z|)^{np_2'}}\, dz \right)^{1/p_2'}.$$

Since $w_2 \in A_p \subset A_{p_2}$, we have $\sigma_2 \in A_{p_2'}$, and hence

$$\int_{\mathbb{R}^n} \frac{\sigma_2(z)}{(1+|z|)^{np_2'}}\, dz < \infty;$$

see, for example, [7, p. 412] or [14, p. 209]. It follows that for $|x| > R$,

$$|[b, T^\delta]_1(f, g)(x)| \lesssim \frac{1}{|x|^n}.$$

Raising both sides of the last inequality to the power p and integrating over $|x| > R$, we have

$$\int_{|x|>R} |[b, T^\delta]_1(f, g)(x)|^p \nu_{\mathbf{w}}\, dx \lesssim_{b,\mathbf{p},\mathbf{w}} \int_{|x|>R} \frac{\nu_{\mathbf{w}}(x)}{|x|^{np}}\, dx \to 0, \qquad R \to \infty,$$

where we used again the fact that for $v \in A_r$, $r > 1$,

$$\int_{\mathbb{R}^n} \frac{v(x)}{(1+|x|)^{nr}}\, dx < \infty.$$

We now show the uniform equicontinuity estimate (c). Note that

$$\begin{aligned}
&[b, T^\delta]_1(f, g)(x+t) - [b, T^\delta]_1(f, g)(x) \\
&\quad = \iint_{\mathbb{R}^{2n}} (b(y) - b(x+t)) K^\delta(x+t, y, z) f(y) g(z)\, dy\, dz
\end{aligned}$$

$$\begin{aligned}&- \iint_{\mathbb{R}^{2n}} (b(y) - b(x)) K^\delta(x, y, z) f(y) g(z)\, dy\, dz \\ &= (b(x) - b(x+t)) \iint_{\mathbb{R}^{2n}} K^\delta(x, y, z) f(y) g(z)\, dy\, dz \\ &\quad + \iint_{\mathbb{R}^{2n}} (b(y) - b(x+t))(K^\delta(x+t, y, z) - K^\delta(x, y, z)) f(y) g(z)\, dy\, dz \\ &= I_1(t, x) + I_2(t, x).\end{aligned}$$

To estimate I_1, we first observe that

$$|I_1(t, x)| \le |t| \|\nabla b\|_\infty \tilde{T}_*(f, g)(x),$$

where $\tilde{T}_*(f, g)$ denotes the maximal truncated bilinear singular integral operator

$$\tilde{T}_*(f, g)(x) = \sup_{\delta>0} |T^\delta(f, g)(x)| = \sup_{\delta>0} \left| \iint_{\mathbb{R}^{2n}} K^\delta(x, y, z) f(y) g(z)\, dy\, dz \right|.$$

Note that with arguments similar to those used in the proof of Lemma 2.1,

$$\begin{aligned}&\left| T^\delta(f, g)(x) - \iint_{\max(|x-y|, |x-z|) \ge \delta} K(x, y, z) f(y) g(z)\, dy\, dz \right| \\ &\qquad \lesssim \left| \iint_{\delta < \max(|x-y|, |x-z|) \le 2\delta} \frac{|f(y) g(z)|}{(|x-y| + |x-z|)^{2n}}\, dy\, dz \right| \lesssim \mathcal{M}(f, g)(x).\end{aligned}$$

It follows then that

$$\tilde{T}_*(f, g)(x) \lesssim T_*(f, g)(x) + \mathcal{M}(f, g)(x), \tag{9}$$

where now

$$T_*(f, g)(x) = \sup_{\delta>0} \left| \iint_{\max(|x-y|, |x-z|) \ge \delta} K(x, y, z) f(y) g(z)\, dy\, dz \right|.$$

By the pointwise estimate [9, (2.1)], for all $\eta > 0$,

$$T_*(f, g)(x) \lesssim_\eta (M(|T(f, g)|^\eta)(x))^{1/\eta} + Mf(x) Mg(x), \tag{10}$$

where M is the Hardy–Littlewood maximal function. From (9) and (10) (with $\eta = 1$ in our current situation) it easily follows that

$$\tilde{T}_* : L^{p_1}(w_1) \times L^{p_2}(w_2) \to L^p(\nu_{\mathbf{w}})$$

for $\mathbf{w} \in A_p \times A_p$. We obtain then

$$\|I_1(t, x)\|_{L^p(\nu_{\mathbf{w}})} \lesssim |t|.$$

To estimate I_2, we observe that, if $t < \delta/4$, then

$$K^\delta(x+t, y, z) - K^\delta(x, y, z) = 0$$

when $\max(|x-y|, |x-z|) \le \delta/2$. Therefore, with what are by now familiar arguments,

$$\begin{aligned}&\left| \iint (b(y) - b(x+t))(K^\delta(x+t, y, z) - K^\delta(x, y, z)) f(y) g(z)\, dy\, dz \right| \\ &\qquad \lesssim \|b\|_\infty |t| \iint_{\max\{|x-y|, |x-z|\} > \delta/2} \frac{|f(y)||g(z)|}{(|x-y| + |x-z|)^{2n+1}}\, dy\, dz\end{aligned}$$

$$\lesssim \|b\|_\infty |t| \sum_{j\geq 0} \iint_{2^{j-1}\delta<\max\{|x-y|,|x-z|\}\leq 2^j\delta} \frac{|f(y)||g(z)|}{(|x-y|+|x-z|)^{2n+1}}\, dy\, dz$$
$$\lesssim \frac{\|b\|_\infty |t|}{\delta} \mathcal{M}(f,g)(x).$$

From the boundedness properties of $\mathcal{M}$ we obtain the desired estimate

$$\|I_2(t,x)\|_{L^p(\nu_{\mathbf{w}})} \lesssim |t|. \qquad \square$$

We concentrate now on the compactness of the iterated commutator. We will show that $[\mathbf{b}, T^\delta]$ satisfies the corresponding conditions (a), (b), and (c) listed at the beginning of the proof of Theorem 1.1. The proof is similar to that of Theorem 1.1, but it is worth pointing out that for the iterated commutator, these conditions hold under the weakest assumption on the class of weights, that is, $\mathbf{w} \in \mathbf{A_p}$. We indicate the needed modifications in the proof below.

Proof of Theorem 1.2. As before, we may assume that $\mathbf{b} \in C_c^\infty \times C_c^\infty$, fix $\delta > 0$, and study $[\mathbf{b}, T^\delta]$. Once again, condition (a) holds since $[\mathbf{b}, T^\delta]$ is bounded from $L^{p_1}(w_1) \times L^{p_2}(w_2)$ to $L^p(\nu_{\mathbf{w}})$ when $\mathbf{w} \in \mathbf{A_p}$. Next, we show that condition (b) holds. Let A be large enough so that $\operatorname{supp} b_1 \cup \operatorname{supp} b_2 \subset B_A(0)$, and let $R \geq 2A$. Then, for $|x| \geq R$, we have

$$\begin{aligned}
&|[\mathbf{b}, T^\delta](f,g)(x)| \\
&\quad \lesssim \|b_1\|_\infty \|b_2\|_\infty \int_{\operatorname{supp} b_1} \int_{\operatorname{supp} b_2} \frac{|f(y)||g(z)|}{(|x-y|+|x-z|)^{2n}}\, dy\, dz \\
&\quad \lesssim \frac{1}{|x|^{2n}} \|b_1\|_\infty \|b_2\|_\infty \int_{\operatorname{supp} b_1} |f(y)|\, dy \int_{\operatorname{supp} b_2} |g(z)|\, dz \\
&\quad \lesssim \frac{1}{|x|^{2n}} \|b_1\|_\infty \|b_2\|_\infty \|f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)} \\
&\qquad \times \sigma_1(\operatorname{supp} b_1)^{1/p_1'} \sigma_2(\operatorname{supp} b_2)^{1/p_2'}.
\end{aligned}$$

We can raise the previous pointwise estimate to the power p and integrate over $|x| > R$ to get

$$\begin{aligned}
&\int_{|x|>R} |[\mathbf{b}, T^\delta](f,g)(x)|^p \nu_{\mathbf{w}}(x)\, dx \\
&\quad \leq (\|f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)} \sigma_1(\operatorname{supp} b_1)^{1/p_1'} \sigma_2(\operatorname{supp} b_2)^{1/p_2'})^p \\
&\qquad \times \int_{|x|>R} \frac{\nu_{\mathbf{w}}(x)}{|x|^{2np}}\, dx,
\end{aligned}$$

which tends to zero as $R \to \infty$ even if $\nu_{\mathbf{w}} \in A_{2p}$ and gives (b).

To show that condition (c) also holds, we write

$$
\begin{aligned}
&|[\mathbf{b}, T^\delta](f,g)(x) - [\mathbf{b}, T^\delta](f,g)(x+t)| \\
&\qquad = \bigg| \iint_{\mathbb{R}^{2n}} (b_1(y) - b_1(x))(b_2(z) - b_2(x)) K^\delta(x,y,z) f(y) g(z)\, dy\, dz \\
&\qquad\qquad + \iint_{\mathbb{R}^{2n}} (b_1(y) - b_1(x+t))(b_2(z) - b_2(x+t)) \\
&\qquad\qquad \times K^\delta(x+t,y,z) f(y) g(z)\, dy\, dz \bigg| \\
&\qquad \le |I_1(x,t)| + |I_2(x,t)|,
\end{aligned}
$$

where

$$
I_1(x,t) = (b_1(x+t) - b_1(x)) \iint_{\mathbb{R}^{2n}} (b_2(z) - b_2(x)) K^\delta(x,y,z) f(y) g(z)\, dy\, dz
$$

and

$$
\begin{aligned}
&I_2(x,t) \\
&\quad = \iint_{\mathbb{R}^{2n}} (K^\delta(x,y,z)(b_2(z) - b_2(x)) - K^\delta(x+t,y,z)(b_2(z) - b_2(x+t))) \\
&\quad\quad \times (b_1(y) - b_1(x+t)) f(y) g(z)\, dy\, dz.
\end{aligned}
$$

The pointwise estimate of $I_1(x,t)$ can be obtained as in the proof of Theorem 1.1:

$$
|I_1(x,t)| \le |t| \|\nabla b_1\|_\infty (\tilde{T}^*(f, b_2 g)(x) + \|b_2\|_\infty \tilde{T}^*(f,g)(x)).
$$

To invoke now the boundedness of

$$
\tilde{T}_* : L^{p_1}(w_1) \times L^{p_2}(w_2) \to L^p(\nu_{\mathbf{w}})
$$

for all $\mathbf{w} \in \mathbf{A_p}$ and not just $\mathbf{w} \in A_p \times A_p$, we can use instead of (10) a strengthened version of it. Namely,

$$
T_*(f,g)(x) \lesssim_\eta (M(|T(f,g)|^\eta)(x))^{1/\eta} + \mathcal{M}(f,g)(x), \tag{11}
$$

which is implicit in the arguments in [9] and explicit in the article by Chen [4, (2.1)]. Thus, as $|t| \to 0$,

$$
\|I_1\|_{L^p(\nu_{\mathbf{w}})} \lesssim |t| \|\nabla b_1\|_\infty \|b_2\|_\infty \|f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)} \longrightarrow 0.
$$

Now, we split I_2 in two other terms as follows:

$$
\begin{aligned}
I_2(x,t) &= \iint_{\mathbb{R}^{2n}} (K^\delta(x,y,z) - K^\delta(x+t,y,z))(b_2(z) - b_2(x+t)) \\
&\quad \times (b_1(y) - b_1(x+t)) f(y) g(z)\, dy\, dz \\
&\quad + (b_2(x+t) - b_2(x)) \\
&\quad \times \iint_{\mathbb{R}^{2n}} (b_1(y) - b_1(x+t)) K^\delta(x,y,z) f(y) g(z)\, dy\, dz \\
&= I_{21}(x,t) + I_{22}(x,t).
\end{aligned}
$$

As in Theorem 1.1, the estimate of I_{21} for t sufficiently small reduces to

$$
\begin{aligned}
&|I_{21}(x,t)| \\
&\qquad \lesssim |t|\|b_1\|_\infty\|b_2\|_\infty \iint_{\max\{|x-y|,|x-z|\}>\delta/2} \frac{|f(y)||g(z)|}{(|x-y|+|x-z|)^{2n+1}}\,dy\,dz \\
&\qquad \lesssim \frac{|t|}{\delta}\|b_1\|_\infty\|b_2\|_\infty\mathcal{M}(f,g)(x),
\end{aligned}
$$

which is again an appropriate estimate to obtain (c). Finally,

$$
\begin{aligned}
&|I_{22}(x,t)| \\
&\qquad \le |t|\|\nabla b_2\|_\infty \\
&\qquad\quad \times \left|\iint_{\max(|x-y|,|x-z|)\ge\delta} (b_1(y)-b_1(x+t)) \times K^\delta(x,y,z)f(y)g(z)\,dy\,dz\right| \\
&\qquad \le |t|\|\nabla b_2\|_\infty(\tilde{T}^*(b_1f,g)(x)+\|b_1\|_\infty\tilde{T}^*(f,g)(x)).
\end{aligned}
$$

Therefore, as $|t|\to 0$,

$$\|I_{22}\|_{L^p(\nu_{\mathbf{w}})} \lesssim |t|\|\nabla b_2\|_\infty\|b_1\|_\infty\|f\|_{L^{p_1}(w_1)}\|g\|_{L^{p_2}(w_2)} \longrightarrow 0. \qquad \square$$

3. Closing Remarks

1. Our results on bilinear commutators highlight one more time the fact that the higher the order of the commutator with *CMO* symbols, the less singular the operators. In this article this is reflected in the less restrictive class of weights needed to achieve estimates (a), (b), and (c). Indeed, in Theorem 1.1, the assumption $A_p \times A_p$ on the weight is needed both to check condition (b) and to guarantee that the target weight falls in the right class. However, to obtain bilinear compactness in Theorem 1.2, we require the $A_p \times A_p$ assumption about the vector weight only because the sufficient condition from [5] on $L^p(\nu_{\mathbf{w}})$ precompactness requires $\nu_{\mathbf{w}} \in A_p$. As already mentioned, this last condition fails if $\mathbf{w}$ is only assumed to belong to $\mathbf{A_p}$. Actually, our techniques can be used to obtain a more general theorem by assuming that $\mathbf{w}\in\mathbf{A_p}$ and $\nu_{\mathbf{w}}\in A_p$ instead of $\mathbf{w}\in A_p\times A_p$.

Theorem 3.1. *Suppose that* $\mathbf{p}\in(1,\infty)\times(1,\infty)$, $p=p_1p_2/(p_1+p_2)>1$, $b\in$ *CMO, and* $\mathbf{w}\in\mathbf{A_p}$ *with* $\nu_{\mathbf{w}}\in A_p$*. Then* $[b,T]_1$ *and* $[b,T]_2$ *are compact operators from* $L^{p_1}(w_1)\times L^{p_2}(w_2)$ *to* $L^p(\nu_{\mathbf{w}})$*.*

Theorem 3.2. *Suppose that* $\mathbf{p}\in(1,\infty)\times(1,\infty)$, $p=p_1p_2/(p_1+p_2)>1$, $\mathbf{b}\in$ *CMO* $\times$ *CMO, and* $\mathbf{w}\in\mathbf{A_p}$ *with* $\nu_{\mathbf{w}}\in A_p$*. Then* $[\mathbf{b},T]$ *is a compact operator from* $L^{p_1}(w_1)\times L^{p_2}(w_2)$ *to* $L^p(\nu_{\mathbf{w}})$*.*

As mentioned in the Introduction,

$$\mathbf{w}\in A_p\times A_p \quad\Rightarrow\quad \mathbf{w}\in\mathbf{A_p} \quad\text{and}\quad \nu_{\mathbf{w}}\in A_p.$$

To see that the assumption $\mathbf{w}\in\mathbf{A_p}$ and $\nu_{\mathbf{w}}\in A_p$ is indeed weaker, consider the example $w_1(x)=|x|^{-\alpha}$ where $1<\alpha<p_1/p=1+p_1/p_2$ and $w_2(x)=1$ on $\mathbb{R}$.

Then $\sigma_1(x) = |x|^{\alpha(p_1'-1)}$ belongs to $A_{2p_1'}$ since

$$\alpha < 1 + \frac{p_1}{p_2} < 1 + p_1 = \frac{2p_1' - 1}{p_1' - 1}.$$

Moreover, $\nu_{\mathbf{w}}(x) = |x|^{-\alpha(p/p_1)}$ belongs to $A_1(\subset A_p)$ since $\alpha(p/p_1) < 1$. However, the weight w_1 does not belong to any A_p class since it is not locally integrable. This vector weight also provides a new example of the properness of the containment $A_{p_1} \times A_{p_2} \subsetneq \mathbf{A_p}$ from [12, Section 7].

2. It is natural to ask whether the sufficient condition about $L^p(w)$ precompactness in [5] may be extended to include weights $w \in A_q$ with $q > p$. A simple modification of the argument in [17, p. 275] gives the following result in this setting:

Let $1 < r < \infty$, $w \in A_\infty$, *and* $\mathcal{K} \subset L^r(w)$. *If*

(I) $\mathcal{K}$ *is bounded in* $L^r(w)$,
(II) $\lim_{A\to\infty} \int_{|x|>A} |f(x)|^r w\, dx = 0$ *uniformly for* $f \in \mathcal{K}$, *and*
(III) $\|f(\cdot + t_1) - f(\cdot + t_2)\|_{L^r(w)} \to 0$ *uniformly for* $f \in \mathcal{K}$ *as* $|t_1 - t_2| \to 0$,

then $\mathcal{K}$ *is precompact.*

This is different from the sufficient condition we employed in the proofs of our main theorems, specifically in the third assumption about equicontinuity. Note that, in general, the nontranslation invariance of the measure deems our last condition strictly stronger than the corresponding one in [5]. Unfortunately, the arguments we used to prove Theorem 1.2 do not seem to hold anymore in this setting.

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