

**A q -EXTENSION OF THE GENERALIZED
HERMITE POLYNOMIALS WITH THE CONTINUOUS
ORTHOGONALITY PROPERTY ON \mathbb{R}**

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Abstract: In this paper we study in detail a q -extension of the generalized Hermite polynomials of Szegő. A continuous orthogonality property on \mathbb{R} with respect to the positive weight function is established, a q -difference equation and a three-term recurrence relation are derived for this family of q -polynomials.

AMS Subject Classification: 26C05, 33D45, 39A13

Key Words: generalized Hermite polynomials, continuous orthogonality, q -difference equation, three-term recurrence relation

Received: November 12, 2003

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1. Introduction

The generalized Hermite polynomials were introduced by Szegő [12] as

$$\begin{aligned}
 H_{2n}^{(\mu)}(x) &:= (-1)^n 2^{2n} n! L_n^{(\mu-1/2)}(x^2), \\
 H_{2n+1}^{(\mu)}(x) &:= (-1)^n 2^{2n+1} n! x L_n^{(\mu+1/2)}(x^2),
 \end{aligned}
 \tag{1.1}$$

where $\mu > -1/2$, $L_n^{(\alpha)}(x)$ are the Laguerre polynomials,

$$\begin{aligned}
 L_n^{(\alpha)}(z) &:= \frac{(\alpha + 1)_n}{n!} {}_1F_1\left(\begin{matrix} -n \\ \alpha + 1 \end{matrix} \middle| z\right) \\
 &= \frac{(\alpha + 1)_n}{n!} \sum_{k=0}^n \frac{(-n)_k}{(\alpha + 1)_k} \frac{z^k}{k!},
 \end{aligned}
 \tag{1.2}$$

and $(a)_n = \Gamma(a + n)/\Gamma(a)$, $n = 0, 1, 2, \dots$, is the shifted factorial. Observe that the zero value of the parameter μ in (1.1) corresponds to the ordinary Hermite polynomials $H_n(x)$, i.e., $H_n^{(0)}(x) = H_n(x)$.

The generalized Hermite polynomials (1.1) are orthogonal with respect to the weight function $|x|^{2\mu} e^{-x^2}$, $x \in \mathbb{R}$, i.e.,

$$\begin{aligned}
 \int_{-\infty}^{\infty} H_n^{(\mu)}(x) H_m^{(\mu)}(x) |x|^{2\mu} e^{-x^2} dx \\
 = 2^{2n} \left[\frac{n}{2} \right]! \Gamma\left(\left[\frac{n+1}{2} \right] + \mu + \frac{1}{2}\right) \delta_{nm},
 \end{aligned}
 \tag{1.3}$$

where $[x]$ denotes the greatest integer not exceeding x . They satisfy a three-term recurrence relation

$$2x H_n^{(\mu)}(x) = H_{n+1}^{(\mu)}(x) + 2(n + 2\mu \theta_n) H_{n-1}^{(\mu)}(x), \quad n \geq 0, \tag{1.4}$$

and a second-order differential equation

$$\left[x \frac{d^2}{dx^2} + 2(\mu - x^2) \frac{d}{dx} + 2nx - 2\mu \theta_n x^{-1} \right] H_n^{(\mu)}(x) = 0, \quad n \geq 0, \tag{1.5}$$

with $\theta_n := n - 2[n/2]$ (see Szegő [12], Chihara [4]). A detailed discussion of other properties of $H_n^{(\mu)}(x)$ can be found in Market [9], Rosenblum [11].

The reason for interest in studying the generalized Hermite polynomials (1.1) is twofold. Pure mathematically they are of interest as an explicit example of the complete orthonormal set in $L^2_\mu(\mathbb{R})$, the Hilbert space of Lebesgue measurable functions $f(x)$, $x \in \mathbb{R}$, with

$$\|f\|_\mu := \left(\int_{-\infty}^\infty |f|^2 |x|^{2\mu} dx \right)^{1/2} < \infty. \tag{1.6}$$

Hence one can build the Bose-like oscillator calculus in terms of these polynomials, which generalizes the well-known calculus, based on the quantum-mechanical harmonic oscillator in physics (see, for example, Rosenblum [11]). So we try to make one step further by considering a generalization of the classical Hermite polynomials $H_n(x)$ with two additional parameters, μ and q .

The aim of this paper is to investigate in detail a q -extension of the generalized Hermite polynomials (1.1) with the continuous orthogonality property on \mathbb{R} (the case of discrete orthogonality requires a different technique, see, for example, Berg et al [3]). In Section 2 we introduce this family $\{\mathcal{H}_n^{(\mu)}(x; q)\}$ in terms of the q -Laguerre polynomials and find a relevant q -difference equation for it. In Section 3 the continuous orthogonality property for $\{\mathcal{H}_n^{(\mu)}(x; q)\}$ with respect to the positive weight function on \mathbb{R} is explicitly formulated. Section 4 is devoted to the derivation of a three-term recurrence relation for this family of q -polynomials.

2. Generalized Hermite Polynomials

It is known from Hahn [7], Exton [5], and Moak [10] that the q -Laguerre polynomials $L_n^{(\alpha)}(x; q)$ are explicitly given as

$$\begin{aligned} L_n^{(\alpha)}(x; q) &:= \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} {}_1\phi_1 \left(\begin{matrix} q^{-n} \\ q^{\alpha+1} \end{matrix} \middle| q, -q^{n+\alpha+1} x \right) \\ &= \frac{1}{(q; q)_n} {}_2\phi_1 \left(\begin{matrix} q^{-n}, -x \\ 0 \end{matrix} \middle| q, q^{n+\alpha+1} \right), \end{aligned} \tag{2.1}$$

where $(a; q)_0 = 1$ and $(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j)$, $n = 1, 2, \dots$, is the q -shifted factorial, and

$$\begin{aligned}
 & {}_r\phi_p \left(\begin{matrix} q^{-n}, a_2, \dots, a_r \\ b_1, b_2, \dots, b_p \end{matrix} \middle| q, z \right) \\
 &= \sum_{k=0}^n \frac{(q^{-n}; q)_k (a_2; q)_k \cdots (a_r; q)_k}{(b_1; q)_k (b_2; q)_k \cdots (b_p; q)_k} \frac{z^k}{(q; q)_k} \left[(-1)^k q^{k(k-1)/2} \right]^{p-r+1}
 \end{aligned} \tag{2.2}$$

is the basic hypergeometric polynomial of degree n in the variable z (throughout this paper, we will employ the standard notations of the q -special functions theory, see Gasper et al [6] or Andrews et al [2]). The q -Laguerre polynomials (2.1) satisfy two kinds of orthogonality relations, an absolutely continuous one and a discrete one. The former orthogonality relation, in which we are interested in the present paper, is given by

$$\int_0^\infty \frac{x^\alpha}{E_q(x)} L_m^{(\alpha)}(x; q) L_n^{(\alpha)}(x; q) dx = d_n^{-1}(\alpha) \delta_{mn}, \quad \alpha > -1, \tag{2.3}$$

where $E_q(x)$ is the Jackson q -exponential function,

$$E_q(z) := \sum_{n=0}^\infty \frac{q^{n(n-1)/2}}{(q; q)_n} z^n = (-z; q)_\infty, \tag{2.4}$$

and the normalization constant $d_n(\alpha)$ is equal to

$$d_n(\alpha) = \frac{1}{\pi} \sin \pi(\alpha + 1) \frac{q^n (q; q)_n}{(q^{\alpha+1}; q)_n} \frac{(q; q)_\infty}{(q^{-\alpha}; q)_\infty}. \tag{2.5}$$

The q -Laguerre polynomials (2.1) are defined in such a way that in the limit as $q \rightarrow 1$ they reduce to the ordinary Laguerre polynomials $L_n^{(\alpha)}(x)$, i.e.,

$$\lim_{q \rightarrow 1} L_n^{(\alpha)}((1 - q)x; q) = L_n^{(\alpha)}(x). \tag{2.6}$$

We can now define, in complete analogy with the relationship (1.1), a q -extension of the generalized Hermite polynomials $H_n^{(\mu)}(x)$ of the form

$$\begin{aligned}
 \mathcal{H}_{2n}^{(\mu)}(x; q) &:= (-1)^n (q; q)_n L_n^{(\mu-1/2)}(x^2; q), \\
 \mathcal{H}_{2n+1}^{(\mu)}(x; q) &:= (-1)^n (q; q)_n x L_n^{(\mu+1/2)}(x^2; q),
 \end{aligned} \tag{2.7}$$

which are orthogonal on the real line \mathbb{R} . Indeed, since

$$\lim_{q \rightarrow 1} \frac{(q^a; q)_n}{(1 - q)^n} = (a)_n, \tag{2.8}$$

with the aid of (2.6) one readily verifies that

$$\lim_{q \rightarrow 1} (1 - q)^{-n/2} \mathcal{H}_n^{(\mu)}(\sqrt{1 - q} x; q) = 2^{-n} H_n^{(\mu)}(x). \tag{2.9}$$

Observe also that the zero value of the parameter μ in (2.7) corresponds to polynomials $\mathcal{H}_n(x; q) \equiv \mathcal{H}_n^{(0)}(x; q)$. The sequence $\{\mathcal{H}_n(x; q)\}$ can be expressed either in terms of the q -Laguerre polynomials $L_n^{(\alpha)}(x; q)$, $\alpha = \pm 1/2$ (as it obvious from definition (2.7) itself), or through the discrete q -Hermite polynomials $\tilde{h}_n(x; q)$ of type II:

$$\mathcal{H}_n(x; q^2) = q^{n(n-1)/2} \tilde{h}_n(x; q). \tag{2.10}$$

A detailed discussion of the properties of the polynomials $\mathcal{H}_n(x; q)$ can be found in our previous paper Álvarez-Nodarse et al [1] on this subject.

A q -difference equation for the introduced polynomials $\mathcal{H}_n^{(\mu)}(x; q)$ is, in fact, an easy consequence of the known q -difference equation

$$\begin{aligned} q^\alpha (1 + x) L_n^{(\alpha)}(q x; q) + L_n^{(\alpha)}(q^{-1} x; q) \\ = [1 + q^\alpha (1 + q^n x)] L_n^{(\alpha)}(x; q) \end{aligned} \tag{2.11}$$

for the q -Laguerre polynomials (see, for example, formula (3.21.6) in Koekoek et al [8]). Indeed, from this q -difference equation and definition (2.7) it follows immediately that

$$\begin{aligned} q^{\mu-1/2} (1 + x^2) \mathcal{H}_n^{(\mu)}(q^{1/2} x; q) + \mathcal{H}_n^{(\mu)}(q^{-1/2} x; q) \\ = \left[q^{-\theta_n/2} + q^{\mu+(\theta_n-1)/2} (1 + q^{[n/2]} x^2) \right] \mathcal{H}_n^{(\mu)}(x; q), \end{aligned} \tag{2.12}$$

where, as before, $\theta_n = n - 2[n/2]$. Taking into account that the dilations $x \rightarrow q^{\pm 1} x$ are represented by the operators $q^{\pm x} \frac{d}{dx}$, that is, $q^{\pm x} \frac{d}{dx} f(x) = f(q^{\pm 1} x)$, one now readily verifies that the q -difference equation (2.12) coincides with the second-order differential equation (1.5) in the limit as $q \rightarrow 1$.

3. Orthogonality Relation

We begin this section with the following theorem:

Theorem 1. *The sequence of the q -polynomials $\{\mathcal{H}_n^{(\mu)}(x; q)\}$, which are defined by the relations (2.7), satisfies the orthogonality relation*

$$\begin{aligned} & \int_{-\infty}^{\infty} \mathcal{H}_m^{(\mu)}(x; q) \mathcal{H}_n^{(\mu)}(x; q) \frac{|x|^{2\mu} dx}{E_q(x^2)} \\ &= \frac{\pi}{\cos \pi \mu} \frac{(q^{1/2-\mu}; q)_{\infty}}{(q; q)_{\infty}} q^{-\frac{n}{2}-\mu \theta_n} (q; q)_{[\frac{n}{2}]} (q^{\mu+1/2}; q)_{[\frac{n+1}{2}]} \delta_{mn}, \end{aligned} \tag{3.1}$$

on the whole real line \mathbb{R} with respect to the continuous positive weight function $w(x) = 1/E_q(x^2)$.

Proof. Since the weight function in (3.1) is an even function of the independent variable x and $\mathcal{H}_n^{(\mu)}(-x; q) = (-1)^n \mathcal{H}_n^{(\mu)}(x; q)$ by the definition (2.7), the q -polynomials of an even degree $\mathcal{H}_{2m}^{(\mu)}(x; q)$ and of an odd degree $\mathcal{H}_{2n+1}^{(\mu)}(x; q)$, $m, n = 0, 1, 2, \dots$, are evidently orthogonal to each other. Consequently, it suffices to prove only those cases in (3.1), when degrees of polynomials m and n are either simultaneously even or odd. Let us consider first the former case. From (2.7) and (2.3) it follows that

$$\begin{aligned} & \int_{-\infty}^{\infty} \mathcal{H}_{2m}^{(\mu)}(x; q) \mathcal{H}_{2n}^{(\mu)}(x; q) \frac{|x|^{2\mu} dx}{E_q(x^2)} \\ &= (-1)^{m+n} (q; q)_m (q; q)_n \int_{-\infty}^{\infty} L_m^{(\mu-1/2)}(x^2; q) L_n^{(\mu-1/2)}(x^2; q) \frac{|x|^{2\mu} dx}{E_q(x^2)} \\ &= 2(-1)^{m+n} (q; q)_m (q; q)_n \int_0^{\infty} L_m^{(\mu-1/2)}(x^2; q) L_n^{(\mu-1/2)}(x^2; q) \frac{x^{2\mu} dx}{E_q(x^2)} \\ &= (-1)^{m+n} (q; q)_m (q; q)_n \int_0^{\infty} L_m^{(\mu-1/2)}(y; q) L_n^{(\mu-1/2)}(y; q) \frac{y^{\mu-1/2} dy}{E_q(y)} \\ &= (q; q)_n^2 d_n^{-1}(\mu - 1/2) \delta_{mn}, \end{aligned}$$

where the normalization constant $d_n(\alpha)$ is defined in (2.5). Thus

$$\int_{-\infty}^{\infty} \mathcal{H}_{2m}^{(\mu)}(x; q) \mathcal{H}_{2n}^{(\mu)}(x; q) \frac{|x|^{2\mu} dx}{E_q(x^2)} = \frac{\pi}{\cos \pi \mu} \frac{(q^{1/2-\mu}; q)_{\infty}}{(q; q)_{\infty}} q^{-n} (q; q)_n (q^{\mu+1/2}; q)_n \delta_{mn}. \quad (3.2)$$

Likewise, one finds that in the latter case

$$\begin{aligned} & \int_{-\infty}^{\infty} \mathcal{H}_{2m+1}^{(\mu)}(x; q) \mathcal{H}_{2n+1}^{(\mu)}(x; q) \frac{|x|^{2\mu} dx}{E_q(x^2)} \\ &= 2(-1)^{m+n} (q; q)_m (q; q)_n \int_0^{\infty} L_m^{(\mu+1/2)}(x^2; q) L_n^{(\mu+1/2)}(x^2; q) \frac{x^{2(\mu+1)} dx}{E_q(x^2)} \\ &= (-1)^{m+n} (q; q)_m (q; q)_n \int_0^{\infty} L_m^{(\mu+1/2)}(y; q) L_n^{(\mu+1/2)}(y; q) \frac{y^{\mu+1/2} dy}{E_q(y)} \\ &= (q; q)_n^2 d_n^{-1}(\mu + 1/2) \delta_{mn}. \end{aligned}$$

Consequently,

$$\int_{-\infty}^{\infty} \mathcal{H}_{2m+1}^{(\mu)}(x; q) \mathcal{H}_{2n+1}^{(\mu)}(x; q) \frac{|x|^{2\mu} dx}{E_q(x^2)} = \frac{\pi}{\cos \pi \mu} \frac{(q^{1/2-\mu}; q)_{\infty}}{(q; q)_{\infty}} q^{-n-\mu-1/2} (q; q)_n (q^{\mu+1/2}; q)_{n+1} \delta_{mn}. \quad (3.3)$$

Putting (3.2) and (3.3) together results in the orthogonality relation (3.1).

The positivity of Jackson q -exponential function $E_q(x^2)$ for $x \in \mathbb{R}$ and $q \in (0, 1)$ is obvious from its definition (2.4): for it is represented as an infinite sum of positive terms (or an infinite product of positive factors). This completes the proof. \square

To conclude this section, we note the obvious fact that in the limit as $q \rightarrow 1$ the (3.1) reduces to the orthogonality relation (1.3) for the generalized Hermite polynomials (1.1). This follows immediately from the limit relations (2.8) and (2.9), upon using the fact that

$$\lim_{q \rightarrow 1} E_q((1 - q)z) = e^z. \quad (3.4)$$

Also, in the event the parameter μ is zero, then the (3.1) coincides with the orthogonality relation for the polynomials (2.10), derived in Álvarez-Nodarse et al [1].

4. Recurrence Relation

In this section we derive a three-term recurrence relation for the q -extension of the generalized Hermite polynomials (2.7). Since an arbitrary family of orthogonal polynomials $p_n(x)$ satisfies a recurrence relation of the form (see Chihara [4, p.19])

$$(a_n x + b_n)p_n(x) = p_{n+1}(x) + c_n p_{n-1}(x), \quad n \geq 0, \quad (4.1)$$

one needs to find coefficients a_n , b_n , and c_n , which correspond to the case under discussion.

Before starting this derivation we note that in what follows it proves convenient to use the following form

$$L_n^{(\alpha)}(x; q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \sum_{k=0}^n \frac{q^{k(k+\alpha)}}{(q^{\alpha+1}; q)_k} \begin{bmatrix} n \\ k \end{bmatrix}_q (-x)^k \quad (4.2)$$

of the q -Laguerre polynomials $L_n^{(\alpha)}(x; q)$, which comes from the first line in definition (2.1), upon using the relation

$$\frac{(q^{-n}; q)_k}{(q; q)_k} = (-1)^k q^{k(k-1)/2 - nk} \begin{bmatrix} n \\ k \end{bmatrix}_q. \quad (4.3)$$

Let us first consider the case when n in (4.1) is even. Then from (2.7) and (4.2) we find that

$$\begin{aligned} & \mathcal{H}_{2n+1}^{(\mu)}(x; q) + c_{2n}(q) \mathcal{H}_{2n-1}^{(\mu)}(x; q) \\ &= (-1)^n x (q^{\mu+3/2}; q)_n \sum_{k=0}^n \frac{q^{k(k+\mu+1/2)}}{(q^{\mu+3/2}; q)_k} \begin{bmatrix} n \\ k \end{bmatrix}_q (-x^2)^k \\ &+ (-1)^{n-1} c_{2n}(q) x (q^{\mu+3/2}; q)_{n-1} \sum_{k=0}^{n-1} \frac{q^{k(k+\mu+1/2)}}{(q^{\mu+3/2}; q)_k} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q (-x^2)^k. \end{aligned} \quad (4.4)$$

The next step is to employ the relation

$$(1 - q^\alpha)(q^{\alpha+1}; q)_n = (1 - q^{n+\alpha})(q^\alpha; q)_n \quad (4.5)$$

in order to rewrite the quotient $(q^{\mu+3/2}; q)_n / (q^{\mu+3/2}; q)_k$ from the first term in the right side of (4.4) as

$$\frac{(q^{\mu+3/2}; q)_n}{(q^{\mu+3/2}; q)_k} = \frac{1 - q^{n+\mu+1/2}}{1 - q^{k+\mu+1/2}} \frac{(q^{\mu+1/2}; q)_n}{(q^{\mu+1/2}; q)_k}. \tag{4.6}$$

In the second term in the right side of (4.4) one can use the evident relation $(q^{\mu+3/2}; q)_{n-1} = (q^{\mu+1/2}; q)_n / (1 - q^{\mu+1/2})$ and the same formula (4.5) for the factor $(q^{\mu+3/2}; q)_k$. We recall also the property of the q -binomial coefficient

$$\begin{bmatrix} n-1 \\ k \end{bmatrix}_q = \frac{1 - q^{n-k}}{1 - q^n} \begin{bmatrix} n \\ k \end{bmatrix}_q. \tag{4.7}$$

Putting this all together, we obtain

$$\begin{aligned} \mathcal{H}_{2n+1}^{(\mu)}(x; q) + c_{2n}(q) \mathcal{H}_{2n-1}^{(\mu)}(x; q) &= (-1)^n x (q^{\mu+1/2}; q)_n \times \\ &\sum_{k=0}^n \frac{q^{k(k+\mu+1/2)}}{(q^{\mu+1/2}; q)_k} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-x^2)^k}{1 - q^{k+\mu+1/2}} \left\{ 1 - q^{n+\mu+1/2} - c_{2n}(q) \frac{1 - q^{n-k}}{1 - q^n} \right\}. \end{aligned} \tag{4.8}$$

The right-hand side of (4.8) should match with

$$\mathcal{H}_{2n}^{(\mu)}(x; q) = (-1)^n (q^{\mu+1/2}; q)_n \sum_{k=0}^n \frac{q^{k(k+\mu-1/2)}}{(q^{\mu+1/2}; q)_k} \begin{bmatrix} n \\ k \end{bmatrix}_q (-x^2)^k, \tag{4.9}$$

multiplied by $a_{2n}(q)x + b_{2n}(q)$. This means that the coefficient $c_{2n}(q)$ can be found from the equation

$$1 - q^{n+\mu+1/2} - c_{2n}(q) \frac{1 - q^{n-k}}{1 - q^n} = d_n(q) q^{-k} (1 - q^{k+\mu+1/2}), \tag{4.10}$$

where $d_n(q)$ is some k -independent factor. It is not difficult to verify that the only solution of the equation (4.10) is $c_{2n}(q) = 1 - q^n$ and $d_n(q) = q^n$. Thus

$$\mathcal{H}_{2n+1}^{(\mu)}(x; q) + (1 - q^n) \mathcal{H}_{2n-1}^{(\mu)}(x; q) = q^n x \mathcal{H}_{2n}^{(\mu)}(x; q). \tag{4.11}$$

Similarly, in the case of an odd n from (4.8) we have

$$\begin{aligned} \mathcal{H}_{2n+2}^{(\mu)}(x; q) + c_{2n+1}(q) \mathcal{H}_{2n}^{(\mu)}(x; q) &= (-1)^{n+1} (q^{\mu+1/2}; q)_{n+1} \sum_{k=0}^{n+1} \frac{q^{k(k+\mu-1/2)}}{(q^{\mu+1/2}; q)_k} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q (-x^2)^k \\ &+ (-1)^n c_{2n+1}(q) (q^{\mu+1/2}; q)_n \sum_{k=0}^n \frac{q^{k(k+\mu-1/2)}}{(q^{\mu+1/2}; q)_k} \begin{bmatrix} n \\ k \end{bmatrix}_q (-x^2)^k. \end{aligned} \tag{4.12}$$

In this case it is even easier to find the coefficient $c_{2n+1}(q)$. Indeed, one will obtain from (4.12) an expression $[a_{2n+1}(q)x + b_{2n+1}(q)]\mathcal{H}_{2n+1}^{(\mu)}(x; q)$ only if the two constant terms in (4.12) with $k = 0$ cancel each other. This means that the $c_{2n+1}(q)$ should satisfy the equation

$$\begin{aligned} (q^{\mu+1/2}; q)_{n+1} - (q^{\mu+1/2}; q)_n c_{2n+1}(q) \\ \equiv (q^{\mu+1/2}; q)_n [1 - q^{n+\mu+1/2} - c_{2n+1}(q)] = 0. \end{aligned} \tag{4.13}$$

Consequently, $c_{2n+1}(q) = 1 - q^{n+\mu+1/2}$ and, therefore, by (4.12) one obtains

$$\begin{aligned} \mathcal{H}_{2n+2}^{(\mu)}(x; q) + (1 - q^{n+\mu+1/2})\mathcal{H}_{2n}^{(\mu)}(x; q) &= (-1)^{n+1} (q^{\mu+1/2}; q)_{n+1} \\ &\times \sum_{k=0}^n \frac{q^{(k+1)(k+\mu+1/2)}}{(q^{\mu+1/2}; q)_{k+1}} \left\{ \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q - \begin{bmatrix} n \\ k+1 \end{bmatrix}_q \right\} (-x^2)^k \\ &= q^{n+\mu+1/2} x \mathcal{H}_{2n+1}^{(\mu)}(x; q), \end{aligned} \tag{4.14}$$

upon using the q -Pascal identity

$$\begin{bmatrix} n+1 \\ m+1 \end{bmatrix}_q - \begin{bmatrix} n \\ m+1 \end{bmatrix}_q = q^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q, \tag{4.15}$$

for the q -binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$. From (4.11) and (4.14) it thus follows that the q -polynomials $\mathcal{H}_n^{(\mu)}(x; q)$ satisfy a three-term recurrence relation of the form ($\theta_n := n - 2[n/2]$)

$$\mathcal{H}_{n+1}^{(\mu)}(x; q) + (1 - q^{n/2+\mu\theta_n})\mathcal{H}_{n-1}^{(\mu)}(x; q) = q^{n/2+\mu\theta_n} x \mathcal{H}_n^{(\mu)}(x; q). \tag{4.16}$$

With the aid of (2.9) one now readily verifies that the (4.16) coincides with the three-term recurrence relation (1.4) for the generalized Hermite polynomials $H_n^{(\mu)}(x)$ in the limit as $q \rightarrow 1$.

5. Concluding Remarks

We conclude this exposition with the following remark. It is well-known that the Hermite functions $H_n(x)e^{-x^2/2}$ (or the wave functions of the linear harmonic oscillator in quantum mechanics) are eigenfunctions of the Fourier integral transform (with respect to the kernel e^{ixy}) with eigenvalues i^n . One can

introduce the generalized Fourier transform operator

$$\mathcal{F}_\mu f(x) := c_\mu^{-1} \int_{-\infty}^{\infty} e_\mu(-ixt) f(t) |t|^\mu dt, \quad (5.1)$$

with a kernel

$$e_\mu(-ix) := \frac{c_\mu}{2} \frac{J_{\mu-1/2}(x) - i J_{\mu+1/2}(x)}{x^{\mu-1/2}}, \quad (5.2)$$

where the constant $c_\mu := 2^{\mu+1/2} \Gamma(\mu+1/2)$ and $J_\alpha(x)$ is the Bessel function. The generalized Hermite polynomials (1.1) are the eigenfunctions of the generalized Fourier transform operator (5.1), see Rosenblum [11]. It is of interest to find a q -extension of (5.1) and (5.2). This study is under way.

Acknowledgements

This research has been supported in part by the Ministerio de Ciencias y Tecnología of Spain under the grant BFM-2000-0206-C04-02, the Junta de Andalucía under grant FQM-262, and by the Mexican UNAM-DGAPA project IN112300.

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