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BEST PROXIMITY PAIR THEOREMS FOR NONCYCLIC MAPPINGS IN BANACH AND METRIC SPACES

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Abstract. Let A and B be two nonempty subsets of a metric space X. A mapping $T : A\cup B \to A\cup B$ is said to be noncyclic if $T(A) \subseteq A$ and $T(B) \subseteq B$. For such a mapping, a pair $(x, y) \in A \times B$ such that $Tx = x$, $Ty = y$ and $d(x, y) = \operatorname{dist}(A, B)$ is called a best proximity pair. In this paper we give some best proximity pair results for noncyclic mappings under certain contractive conditions.
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1. Introduction

Given a metric space (X, d), a self-mapping T on X is called a *contraction* provided that there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \le \alpha d(x, y),$$

for all $x, y \in X$. The well-known Banach contraction principle states that if (X, d) is a complete metric space and T is a contraction map, then T has a unique fixed point. The Banach contraction principle is a very important tool in nonlinear analysis. Moreover, many extensions of this principle have arisen in the literature so far (see [20] for more information). In 1968, R. Kannan [18] introduced the so-called Kannan contraction mappings in order to investigate the existence and uniqueness of fixed point for such mappings. A mapping $T : X \to X$ is said to be a *Kannan contraction* if there exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \le \alpha[d(x, Tx) + d(y, Ty)],$$

for each $x, y \in X$. It is known that if (X, d) is a complete metric space then every Kannan contraction mapping has a unique fixed point [18]. It is interesting to note that the notions of contraction and Kannan contraction are independent, that is, there exist contraction mappings which are not Kannan contractions and Kannan contraction mappings which are not contractions. Consequently, both conditions cannot be compared directly. Moreover, contractions are always continuous while Kannan contractions may not be. On the other hand, in 1972, S. K. Chatterjea [8] introduced the so-called Chatterjea contractions with a similar purpose. A mapping $T : X \to X$ is a *Chatterjea contraction* if there exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx)],$$

for each $x, y \in X$. In that occasion, existence and uniqueness of fixed point was also obtained for these mappings in the setting of complete metric spaces.

In the last years, specifically from 2003 with [23] and 2005 with [10], certain extensions of the previous fixed point problems have arisen. In general terms, these extensions are based on considering a smaller set of points where the metric assumption on the mapping T is fulfilled while certain condition of cyclic nature is assumed on the mapping under discussion. In particular, in this paper, we mainly work with the so-called noncyclic mappings. Given (A, B) a nonempty pair of subsets of a metric space (X, d), a mapping $T : A \cup B \to A \cup B$ is called *noncyclic* provided that $T(A) \subseteq A$ and $T(B) \subseteq B$. When dealing with these mappings, it is interesting to ask whether it is possible to find $(x^*, y^*) \in A \times B$ such that

$$Tx^* = x^*, \ Ty^* = y^* \text{ and } d(x^*, y^*) = \operatorname{dist}(A, B), \tag{1.1}$$

where $\operatorname{dist}(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$. Notice that such a pair of points (x^*, y^*) is actually a solution of the following minimization problem: Find $(x, y) \in A \times B$ such that

$$\min_{x \in A} d(x, Tx), \quad \min_{y \in B} d(y, Ty) \quad \text{and} \quad \min_{(x,y) \in A \times B} d(x, y). \tag{1.2}$$

Thus, a point $(x^*, y^*) \in A \times B$ is said to be a *best proximity pair* for the noncyclic mapping $T : A \cup B \to A \cup B$ if it is an absolute optimal solution of the minimization problem (1.2), i.e., if it is a point satisfying (1.1). Notice that in [1] the authors studied sufficient conditions to ensure the existence of solutions of the nonlinear programming problem (1.2).

Another condition of cyclic nature that has been considered in the literature is the one of cyclic mapping. A mapping $T : A \cup B \to A \cup B$ is called *cyclic* if $T(A) \subseteq B$ and $T(B) \subseteq A$. In this case, the fixed point equation $Tx = x$ may make no sense. Consequently, the interest of the theory focuses then on the study of the existence of best proximity points, that is, points $x^* \in A \cup B$ such that $d(x^*, Tx^*) = \operatorname{dist}(A, B)$.

In the current article, we first give some best proximity pair theorems in Banach spaces for various classes of noncyclic mappings such as noncyclic contractions, noncyclic contractions in the sense of Kannan, noncyclic contractions in the sense of Chatterjea and strongly noncyclic relatively nonexpansive mappings in the sense of Chatterjea. Finally, we devote Section 5 to show that the results proved in the previous sections also hold in metric settings that are more general.

2. Preliminaries

Let (A, B) be a nonempty pair of subsets in a metric space (X, d). We say that the pair (A, B) satisfies a property if both A and B satisfy that property. For example, (A, B) is closed if and only if both A and B are closed. Likewise, $(A, B) \subseteq (C, D)$

if and only if $A \subseteq C$ and $B \subseteq D$. Throughout this paper we shall use the following notations and definitions:

$$\delta_x(A) := \sup\{d(x,y) : y \in A\} \text{ for all } x \in X,$$

$$\delta(A,B) := \sup\{\delta_x(B) : x \in A\}, \quad \operatorname{diam}(A) := \delta(A,A).$$

From now on, $B(a;r)$ denotes the closed ball in the space X centered at $a \in X$ with radius $r > 0$.

The *proximal pair* of a given pair (A,B) of nonempty subsets of X is the pair (A_0, B_0) given by

$$A_0 = \{x \in A\colon d(x,y') = \operatorname{dist}(A,B) \text{ for some } y' \in B\},$$

$$B_0 = \{y \in B\colon d(x',y) = \operatorname{dist}(A,B) \text{ for some } x' \in A\}.$$

Proximal pairs may be empty but, in particular, if A and B are nonempty, weakly compact and convex in a Banach space X, then (A_0, B_0) is a nonempty weakly compact convex pair in X. A pair of sets (A,B) is said to be proximal if $A = A_0$ and $B = B_0$. In fact, given $x \in A$, we say that $y \in B$ is a *proximal point* of x if $d(x,y) = \operatorname{dist}(A,B)$. Similarly, we can define the proximal point of $y \in B$. The *closed convex hull* of a set A in a Banach space X will be denoted by $\overline{\operatorname{con}}(A)$.

From now on in the paper, unless otherwise stated, we will use the notation d in a linear setting to refer to the induced metric by the norm in a Banach space X.

Definition 2.1. *A Banach space X is said to be strictly convex if the following implication holds for every $x, y, p \in X$ and $R > 0$:*

$$\begin{cases} d(x,p) \le R, \\ d(y,p) \le R, \\ x \neq y \end{cases} \Rightarrow d\left(\frac{x+y}{2}, p\right) < R.$$

In the last section, we will mainly work in the setting of geodesic metric spaces. A metric space (X,d) is said to be a *(uniquely) geodesic space* if every two points x and y of X are joined by a *(unique) geodesic*, i.e, a map $c : [0,l] \subseteq \mathbb{R} \to X$ such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0,l]$. A subset A of a geodesic space X is said to be *convex* if the image of any geodesic that joins each pair of points x and y of A (geodesic segment $[x,y]$) is contained in A. A point z in X belongs to a geodesic segment $[x,y]$ if and only if there exists $t \in [0,1]$ such that $d(x,z) = td(x,y)$ and $d(y,z) = (1-t)d(x,y)$ and we write $z = (1-t)x + ty$ for simplicity. Notice that this point may not be unique. When $t = \frac{1}{2}$, we often use the notation $\frac{x+y}{2}$ to denote $\frac{1}{2}x + \frac{1}{2}y$. Any Banach space is for instance a geodesic space with usual segments as geodesic segments.

A geodesic metric space X is said to be *reflexive* if for every decreasing chain $\{C_\alpha\} \subset X$ with $\alpha \in I$ such that C_α is closed convex bounded and nonempty for all $\alpha \in I$ we have that $\bigcap_{\alpha \in I} C_\alpha \neq \emptyset$. It is immediate to see that reflexive metric spaces extend the notion of reflexivity from Banach to metric spaces. It is well-known that every complete uniformly convex metric space with either a monotone or lower semicontinuous from the right modulus of uniform convexity is reflexive (see for instance [24, 13] for more information about uniform convexity in geodesic spaces). Other well-known examples of this type of spaces are complete CAT(0) [4] spaces or uniformly convex Banach spaces.

Let (X, d) be a uniquely geodesic space. A *metric* $d\colon X \times X \to \mathbb{R}$ is said to be *convex* if for any $x, y, z \in X$ one has

$$d(x, (1-t)y + tz) \leq (1-t)d(x, y) + td(x, z) \text{ for all } t \in [0, 1].$$

A geodesic space (X, d) is *Busemann convex* (introduced in [7]) if given any pair of geodesics $c_1 : [0, l_1] \to X$ and $c_2 : [0, l_2] \to X$ one has

$$d(c_1(tl_1), c_2(tl_2)) \leq (1-t)d(c_1(0), c_2(0)) + td(c_1(l_1), c_2(l_2)) \text{ for all } t \in [0, 1].$$

It is well-known that Busemann convex spaces are strictly convex [14] and with convex metric. A metric space is said to be strictly convex (see [3] for more on this property) if X is a geodesic space and for every $r > 0$, a, x and $y \in X$ with $d(x, a) \leq r$, $d(y, a) \leq r$ and $x \neq y$, it is the case that $d(a, p) < r$, where p is any point between x and y such that $p \neq x$ and $p \neq y$, i.e., p is any point in the interior of a geodesic segment that joins x and y. It is immediate that every strictly convex metric space is uniquely geodesic. Also notice that a reflexive and Busemann convex geodesic space is complete (see [15, Lemma 4.1]).

3. Noncyclic contractive type mappings

In [23, 25], some fixed point results were given for certain cyclic mappings $T : A \cup B \to A \cup B$ under contractive conditions. Notice that a mapping $T : A \cup B \to A \cup B$ is cyclic if $T(A) \subseteq B$ and $T(B) \subseteq A$. Many generalizations of those conditions have been considered to study the problem of finding best proximity points in absence of fixed points. For instance, the so-called cyclic contractions [12], weak cyclic Kannan contractions [26] or cyclic Meir-Keeler contractions [9], among others, have been studied. However noncyclic mappings have been much less studied in the literature. A mapping $T : A \cup B \to A \cup B$ is called noncyclic if $T(A) \subseteq A$ and $T(B) \subseteq B$. In the sequel we give some comments and results regarding these noncyclic natural extensions. First we point out that, for this noncyclic approach, we mainly focus on finding best proximity pairs, i.e., pairs $(x, y) \in A \times B$ such that $Tx = x$, $Ty = y$ and $d(x, y) = \operatorname{dist}(A, B)$.

The first case that is natural to study in the noncyclic context is the following.

Proposition 3.1. *Let A and B be two nonempty closed subsets of a complete metric space (X, d). Assume that $T : A \cup B \to A \cup B$ is a noncyclic mapping such that for some $\alpha \in (0, 1)$*

$$d(Tx, Ty) \leq \alpha d(x, y), \text{ for all } (x, y) \in A \times B.$$

Then dist$(A, B) = 0$. Moreover, the mapping T has a fixed point in $A \cup B$ if and only if $A \cap B \neq \emptyset$.

Proof. Let $\{x_n\}, \{y_n\}$ be sequences in A and B respectively such that $d(x_n, y_n) \to \operatorname{dist}(A, B)$. Then $\operatorname{dist}(A, B) \leq d(Tx_n, Ty_n) \leq \alpha d(x_n, y_n)$. Taking limit when n tends to infinity, we see that necessarily $\operatorname{dist}(A, B) = 0$.

Suppose first that $A \cap B \neq \emptyset$. If we apply the Banach contraction principle in $A \cap B$, we get that there exists a fixed point of the mapping T that in fact is unique in $A \cap B$.

On the other hand, suppose that T has a fixed point y^* in $A \cup B$. Without loss of generality, suppose that $y^* \in B$. Then, given a point $x_0 \in A$, if we denote $x_n = T^n x_0$, we have that

$$d(x_n, y^*) \leq \alpha d(x_{n-1}, y^*) \leq \alpha^n d(x_0, y^*).$$

Consequently, we get that $\{x_n\}$ converges to y^*. Since A is closed, $y^* \in A \cap B$ and the result follows. $\square$

In 2005, the more general case of $\alpha = 1$ in the previous proposition was taken under consideration [10]. In this regarding, the notion of proximal normal structure was introduced.

Definition 3.2. *A convex pair (K_1, K_2) in a Banach space X is said to have proximal normal structure if for any bounded closed convex and proximal pair $(H_1, H_2) \subseteq (K_1, K_2)$ for which dist$(H_1, H_2) =$ dist(K_1, K_2) and $\delta(H_1, H_2) >$ dist(H_1, H_2), there exits $(x_1, x_2) \in H_1 \times H_2$ such that*

$$\delta_{x_1}(H_2) < \delta(H_1, H_2), \quad \delta_{x_2}(H_1) < \delta(H_1, H_2).$$

Note that if in the above definition $K_1 = K_2$, then we get the notion of normal structure introduced by Brodski and Milman [5].

Also in [10], it was announced that every nonempty bounded closed convex pair of subsets of a uniformly convex Banach space X has proximal normal structure. The next theorem, established by Eldred, Kirk and Veeramani in [10], is the main result in the cited paper concerning the so-called noncyclic relatively nonexpansive mappings. A mapping $T : A \cup B \to A \cup B$ is a noncyclic relatively nonexpansive mapping if T is noncyclic on $A \cup B$ and $d(Tx, Ty) \leq d(x, y)$ for all $(x, y) \in A \times B$.

Theorem 3.3. *(Theorem 2.2 in* [10]*) Let (A, B) be a nonempty, weakly compact convex pair in a strictly convex Banach space X and $T : A \cup B \to A \cup B$ a noncyclic relatively nonexpansive mapping. Assume that the pair (A, B) has proximal normal structure. Then T has a best proximity pair.*

One year later [12], a weaker condition of contractive type was considered, but only for cyclic mappings. The resultant cyclic mappings were called cyclic contractions and in particular satisfy

$$d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha)\mathrm{dist}(A, B),$$

for some $\alpha \in (0, 1)$ and for every $x \in A$ and $y \in B$. In [2], a best proximity pair result has been given for the counterpart noncyclic mappings. This result states that if A and B are two nonempty and weakly compact convex sets in a strictly convex Banach space, then any noncyclic contraction, i.e., any noncyclic mapping $T : A \cup B \to A \cup B$ such that $d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha)\mathrm{dist}(A, B)$ for every $x \in A$ and $y \in B$, has a best proximity pair. In that occasion, the result was proved by using Zorn's Lemma. Next we give a proof of this result without using this lemma. It is remarkable that the hypothesis on compactness for the sets A and B are removed.

First, we need the following result on the boundedness of the iterates of the mapping T.

Proposition 3.4. *Let A and B be nonempty subsets of a metric space X. Suppose that $T : A \cup B \to A \cup B$ is a noncyclic contraction. Let $x_0 \in A$, $y_0 \in B$, $x_n = T^n x_0$ and $y_n = T^n y_0$ for every $n \in \mathbb{N}$. Then $\lim_n d(x_n, y_n) = dist(A, B)$. Moreover, $\{x_n\}$ and $\{y_n\}$ are bounded.*

Proof. The proof follows similar patterns to those considered in [12] in Proposition 3.3. The fact that $\lim_n d(x_n, y_n) = \operatorname{dist}(A, B)$ is immediate. Then, it suffices to see that $x_n = T^n x_0$ is bounded. Suppose that $\{x_n\}$ is not bounded. Then let M be a natural number such that

$$M \geq \max\Big\{\big\{\frac{\alpha}{(1-\alpha)} d(y_0, Ty_0) + \operatorname{dist}(A, B)\big\}, d(Ty_0, x_0)\Big\},$$

where α is the contraction constant. For this natural number, there exists $N_0 \in \mathbb{N}$ such that

$$d(Ty_0, T^{N_0} x_0) > M \text{ and } d(Ty_0, T^{N_0 - 1} x_0) \leq M.$$

Thus,

$$\begin{aligned}\frac{M - \operatorname{dist}(A, B)}{\alpha} + \operatorname{dist}(A, B) &< d(y_0, T^{N_0-1} x_0) \\ &\leq d(y_0, Ty_0) + d(Ty_0, T^{N_0-1} x_0) \leq d(y_0, Ty_0) + M,\end{aligned}$$

which is a contradiction and the result follows. □

Next we give the following best proximity pair result for noncyclic contractions.

Theorem 3.5. *Let A and B be two closed convex subsets of a strictly convex and reflexive Banach space. Suppose that $T : A \cup B \to A \cup B$ is a noncyclic contraction. Then T has a best proximity pair.*

Proof. Let $d^*(x, y) := d(x, y) - \operatorname{dist}(A, B)$. Then $d^*(Tx, Ty) \leq \alpha d^*(x, y)$ for every $x \in A$ and $y \in B$. Let x_0 and y_0 be two arbitrary but fixed points in A and B respectively.

Fix $m_0 \in N$ and let $m = m_0 + k$ with $k \in \mathbb{N}$. Then

$$\begin{gathered}d^*(T^m x_0, T^{m_0} y_0) \leq \alpha^{m_0} d^*(T^k x_0, y_0) \\ \leq \alpha^{m_0} \sup\{d(T^k x_0, y_0) : \ k \in \mathbb{N}\} = \alpha^{m_0} M(x_0, y_0).\end{gathered}$$

Since the orbits of a noncyclic contraction map are bounded at every point $z \in A \cup B$, we have that, for every $\varepsilon > 0$, there exists $m_0(\varepsilon) \in \mathbb{N}$ such that $T^m x_0 \in B(T^{m_0(\varepsilon)} y_0; \operatorname{dist}(A, B) + \varepsilon)$ for every $m \geq m_0(\varepsilon)$.

In a similar way,

$$\begin{gathered}d^*(T^m x_0, T^{m_0+1} y_0) \leq \alpha^{m_0} d^*(T^k x_0, Ty_0) \\ \leq \alpha^{m_0} \sup\{d(T^k x_0, Ty_0) : \ k \in \mathbb{N}\} = \alpha^{m_0} M'(x_0, y_0).\end{gathered}$$

Let $\varepsilon_n = 1/n$. Then there is $m_0(\varepsilon_n) \in \mathbb{N}$ such that $T^m x_0 \in B(T^{m_0(\varepsilon_n)} y_0; \operatorname{dist}(A, B) + \varepsilon_n)$ and $T^m x_0 \in B(T^{m_0(\varepsilon_n)+1} y_0; \operatorname{dist}(A, B) + \varepsilon_n)$ for every $m \geq m_0(\varepsilon_n)$. Denote $B_n = B(T^{m_0(\varepsilon_n)} y_0; \operatorname{dist}(A, B) + \varepsilon_n)$ and $B'_n = B(T^{m_0(\varepsilon_n)+1} y_0; \operatorname{dist}(A, B) + \varepsilon_n)$ and consider the sequence of subsets $\{C_n\}$ of X such that $C_1 = A \cap B_1 \cap B_1'$ and $C_n = C_{n-1} \cap B_n \cap B_n'$ for every $n \geq 2$.

By definition, it is immediate to see that, for every $n \in \mathbb{N}$, C_n is closed, convex, bounded and nonempty. Then, by means of the reflexivity of the space, we conclude that $\bigcap_{n\in\mathbb{N}} C_n \neq \emptyset$. Let $p \in \bigcap_{n\in\mathbb{N}} C_n \subset A$. Taking $m_0(\varepsilon_n)$ as an increasing sequence depending on n, we have that the sequences of terms $z_n = T^{m_0(\varepsilon_n)}y_0$ and $z_n' = T^{m_0(\varepsilon_n)+1}y_0$ for every $n \in \mathbb{N}$ are subsequences of $\{T^n y_0\}$. Consequently, since $p \in C_n$ for every $n \in \mathbb{N}$, $d(z_n, p) \to \operatorname{dist}(A, B)$ and $d(z_n', p) \to \operatorname{dist}(A, B)$. Besides, we have that $d(z_n', Tp) = d(Tz_n, Tp) \to \operatorname{dist}(A, B)$. Since $\{z_n'\}$ is a bounded sequence, it has a weakly convergent subsequence to a point $q \in B$. Then, by using the lower semicontinuity of the norm, we have that $d(q, p) = d(q, Tp) = \operatorname{dist}(A, B)$, which implies in a strictly convex space that $p = Tp$. Moreover, since $d(Tp, Tq) = d(p, q) = d(Tp, q) = \operatorname{dist}(A, B)$, we also get that $q = Tq$ and the result follows. □

Remark 3.6. *The assumption of reflexivity for the space X can be dropped if A and B are supposed to be weakly compact and nonempty.*

The next example shows that the hypothesis on convexity and completeness for the sets A and B are necessary conditions in the previous theorem.

Example 3.7. *Let $k \in (0, 1)$ and let A and B the subsets of the space ℓ_p, $1 \leq p \leq \infty$, defined as $A = \{((1 + k^{2n})e_{2n}) : n \in \mathbb{N}\}$ and $B = \{((1 + k^{2m+1})e_{2m+1}) : m \in \mathbb{N}\}$. It is easy to see that dist$(A, B) = 2^{1/p}$ in ℓ_p. Consider the mapping $T : A \cup B \to A \cup B$ defined by*

$$Tx = \begin{cases} (1 + k^{2m+2})e_{2m+2} & \textit{if} \quad x = (1 + k^{2m})e_{2m} \textit{ for some } m \geq 1 \\ (1 + k^{2n+1})e_{2n+1} & \textit{if} \quad x = (1 + k^{2n-1})e_{2n-1} \textit{ for some } n \geq 1. \end{cases}$$

Then T is noncyclic on $A \cup B$. Moreover, T is a noncyclic contraction mapping. We omit the proof of this fact since it follows similar patterns to those considered in Example 3.6 in [12]. *As it was pointed out in that paper, $A_0 = B_0 = \emptyset$ and therefore T does not have any best proximity pair. In fact, notice that T does not even have fixed points.*

In 2011, certain existence results on best proximity points were given for the so-called weak cyclic Kannan contractions [26]. It seems natural to analyse what that weak Kannan condition implies in the noncyclic case.

To establish our results, we introduce the following new class of noncyclic maps.

Definition 3.8. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is said to be a noncyclic contraction in the sense of Kannan if T is noncyclic on $A \cup B$ and there exists $\alpha \in [0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)] + (1 - 2\alpha)\,dist(A, B),$$

for all $(x, y) \in A \times B$.

It is clear that if in the above definition $A = B$, then we get the notion of Kannan contraction [18]. Next we show that, unlike the cyclic case, this contractive condition seems to be very restrictive in the noncyclic case.

Proposition 3.9. *Let (A, B) be a nonempty pair of subsets of a complete metric space (X, d). Let $T : A \cup B \to A \cup B$ be a noncyclic contraction in the sense of Kannan. Then T has a best proximity pair $(x, y) \in A \times B$ if and only if $A \cap B \neq \emptyset$.*

Proof. Suppose that $(x, y) \in A \times B$ is a best proximity pair of T, that is, $d(x, y) = \operatorname{dist}(A, B)$ and $x = Tx$, $y = Ty$. Since T is a noncyclic contraction in the sense of Kannan, we have that

$$\operatorname{dist}(A, B) = d(x, y) = d(Tx, Ty) \leq$$
$$\leq \alpha[d(x, Tx) + d(Ty, y)] + (1 - 2\alpha)\operatorname{dist}(A, B) = (1 - 2\alpha)\operatorname{dist}(A, B),$$

where α is the Kannan contraction constant of the mapping T. This implies that $d(A, B) = 0$. Moreover, since $d(x, y) = \operatorname{dist}(A, B)$, the result follows. The other implication holds directly by [18]. $\square$

Next we give an example of a noncyclic contraction in the sense of Kannan that is neither noncyclic relatively nonexpansive nor a noncyclic contraction.

Example 3.10. *Let X be the Banach space $\mathbb{R}$ endowed with the Euclidean norm. Let $A := [-1, 1]$ and $B := [0, 2]$. Define the noncyclic mapping $T : A \cup B \to A \cup B$ as follows:*

$$Tx = \begin{cases} \frac{1}{5} & \text{if } x \neq 1, \\ 0 & \text{if } x = 1. \end{cases}$$

Then T is noncyclic on $A \cup B$. We now claim that T is a noncyclic contraction in the sense of Kannan. Let $x \in A$ and $y \in B$. If $x = 1$, then

$$\mid Tx - Ty \mid = \tfrac{1}{5} \text{ and } \mid x - Tx \mid + \mid y - Ty \mid = 1 + \mid y - \tfrac{1}{5} \mid .$$

Thus,

$$\mid Tx - Ty \mid \leq \alpha[\mid x - Tx \mid + \mid y - Ty \mid],$$

for each $\alpha \in [\frac{1}{6}, \frac{1}{2})$.
Moreover, let $y_0 = \frac{9}{10} \in B$ and $x_0 = 1 \in A$. Then $d(Tx_0, Ty_0) = \frac{1}{5} > d(x_0, y_0) = \frac{1}{9}$. Thus T is not relatively nonexpansive and therefore neither a noncyclic contraction. Also notice that $y^* = \frac{1}{5}$ is a fixed point of T.

As a consequence of Proposition 3.9, we see that the problem of finding a best proximity pair for these mappings reduces to find a fixed point in the intersection of A and B.

Next we consider a metric condition on A and B that guarantees the existence of best proximity pairs for T, or, in other words, that guarantees the nonempty intersection of A and B when there is a noncyclic contraction in the sense of Kannan defined on them.

Definition 3.11. *A pair (A, B) of subsets of a metric space X satisfies property (H) provided that for every nonempty closed convex bounded pair $(K_1, K_2) \subseteq (A, B)$ we have*

$$\max\{diam(K_1), diam(K_2)\} \leq \delta(K_1, K_2).$$

Remark 3.12. *Notice that if a pair of nonempty and convex sets (A, B) satisfies property (H), then the intersection of A and B contains as most one point.*

Theorem 3.13. *Let (A,B) be a nonempty weakly compact convex pair in a strictly convex Banach space X. Suppose that $T : A\cup B \to A\cup B$ is a noncyclic contraction in the sense of Kannan. If the pair (A,B) has property (H), then T has a unique fixed point in $A\cap B$.*

Proof. Let Σ denote the collection of all nonempty weakly compact convex pairs (E,F) which are subsets of (A,B) and such that T is noncyclic on $E\cup F$. Since $(A,B)\in\Sigma$, then Σ is nonempty. Note that Σ is partially ordered by the reverse inclusion, that is, $(A,B)\le(C,D)\Leftrightarrow(C,D)\subseteq(A,B)$. By the fact that (A,B) is a weakly compact and convex pair in X, every increasing chain in Σ is bounded above. Hence, by using Zorn's lemma we can get a maximal element, say $(K_1,K_2)\in\Sigma$, that is minimal respect to the set inclusion. We have

$$(\overline{con}(T(K_1)),\overline{con}(T(K_2)))\subseteq(K_1,K_2).$$

Moreover,

$$T(\overline{con}(T(K_1)))\subseteq T(K_1)\subseteq\overline{con}(T(K_1)),$$

and also

$$T(\overline{con}(T(K_2)))\subseteq\overline{con}(T(K_2)).$$

That is, T is noncyclic on $\overline{con}(T(K_1))\cup\overline{con}(T(K_2))$. Now, by the minimality of (K_1,K_2), we have $\overline{con}(T(K_1))=K_1$, $\overline{con}(T(K_2))=K_2$. Let $a\in K_1$. Then $K_2\subseteq B(a;\delta_a(K_2))$. Now, if $y\in K_2$, then by the fact that (A,B) has property (H) we obtain

$$\begin{aligned}d(Ta,Ty)&\le\alpha\{d(a,Ta)+d(Ty,y)\}+(1-2\alpha)\mathrm{dist}(A,B)\\&\le 2\alpha\max\{\mathrm{diam}(K_1),\mathrm{diam}(K_2)\}+(1-2\alpha)\mathrm{dist}(A,B)\\&\le 2\alpha\delta(K_1,K_2)+(1-2\alpha)\mathrm{dist}(A,B).\end{aligned}$$

So, for each $y\in K_2$, we have

$$T(K_2)\subseteq B(Ta;2\alpha\delta(K_1,K_2)+(1-2\alpha)\mathrm{dist}(A,B)).$$

Hence,

$$K_2=\overline{con}(T(K_2))\subseteq B(Ta;2\alpha\delta(K_1,K_2)+(1-2\alpha)\mathrm{dist}(A,B)).$$

This implies that

$$d(y,Ta)\le 2\alpha\delta(K_1,K_2)+(1-2\alpha)\mathrm{dist}(A,B)\quad\text{for all}\quad y\in K_2,$$

and then

$$\delta_{Ta}(K_2)\le 2\alpha\delta(K_1,K_2)+(1-2\alpha)\mathrm{dist}(A,B). \tag{3.1}$$

Similarly, if $b\in K_2$, we see that

$$\delta_{Tb}(K_1)\le 2\alpha\delta(K_1,K_2)+(1-2\alpha)\mathrm{dist}(A,B). \tag{3.2}$$

Set

$$\begin{aligned}E_1&:=\{x\in K_1:\delta_x(K_2)\le 2\alpha\delta(K_1,K_2)+(1-2\alpha)\mathrm{dist}(A,B)\},\\E_2&:=\{y\in K_2:\delta_y(K_1)\le 2\alpha\delta(K_1,K_2)+(1-2\alpha)\mathrm{dist}(A,B)\}.\end{aligned}$$

Note that (E_1, E_2) is a nonempty closed convex pair. Also, it is easy to see that

$$E_1 = \bigcap_{y \in K_2} B(y; 2\alpha\delta(K_1, K_2) + (1 - 2\alpha)\mathrm{dist}(A, B)) \cap K_1,$$

$$E_2 = \bigcap_{x \in K_1} B(x; 2\alpha\delta(K_1, K_2) + (1 - 2\alpha)\mathrm{dist}(A, B)) \cap K_2.$$

Besides, T is noncyclic on $E_1 \cup E_2$ by the relations (3.1) and (3.2). Minimality of (K_1, K_2) implies that $E_1 = K_1$ and $E_2 = K_2$. Thus, we obtain

$$\delta_x(K_2) \leq 2\alpha\delta(K_1, K_2) + (1 - 2\alpha)\mathrm{dist}(A, B), \quad \text{for all } \ x \in K_1,$$

which implies that

$$\delta(K_1, K_2) = \sup_{x \in K_1} \delta_x(K_2) \leq 2\alpha\delta(K_1, K_2) + (1 - 2\alpha)\mathrm{dist}(A, B).$$

Hence,

$$\delta(K_1, K_2) = \mathrm{dist}(A, B).$$

Therefore, for each $(x^*, y^*) \in K_1 \times K_2$ we have

$$d(x^*, y^*) = d(Tx^*, Ty^*) = \mathrm{dist}(A, B).$$

Now, if $(x^*, y^*) \neq (Tx^*, Ty^*)$, by the strict convexity of X we must have

$$\begin{aligned}\mathrm{dist}(A, B) &\leq \left\| \frac{x^* + Tx^*}{2} - \frac{y^* + Ty^*}{2} \right\| = \left\| \frac{x^* - y^*}{2} + \frac{Tx^* - Ty^*}{2} \right\| \\ &< \frac{1}{2}(\|x^* - y^*\| + \|Tx^* - Ty^*\|) = \mathrm{dist}(A, B),\end{aligned}$$

which is a contradiction. Then, the unique pair of points $(x, y) \in K_1 \times K_2$ is a best proximity pair of T. Consequently, from Proposition 3.9, $A \cap B \neq \emptyset$. The uniqueness comes from the fact that the fixed point in the intersection is unique. $\square$

Next we show an example where the previous theorem applies.

Example 3.14. *Let X be the Banach space $\mathbb{R}$ endowed with the Euclidean norm. Let $A := [-1, 0]$ and $B := [0, 1]$. It is not difficult to see that (A, B) is a closed convex bounded pair that satisfies property (H). Define the noncyclic mapping $T : A \cup B \to A \cup B$ as follows:*

$$Tx = \begin{cases} 0 & \text{if } \ x \neq -1, \\ \frac{-1}{10} & \text{if } \ x = -1. \end{cases}$$

Then T is noncyclic on $A \cup B$. We now claim that T is a noncyclic contraction in the sense of Kannan. Let $x \in A$ and $y \in B$. If $x = -1$, then

$$\mid Tx - Ty \mid = \tfrac{1}{10} \ \text{and} \mid x - Tx \mid + \mid y - Ty \mid = \tfrac{9}{10} + y.$$

Thus,

$$\mid Tx - Ty \mid \leq \alpha[\mid x - Tx \mid + \mid y - Ty \mid],$$

for each $\alpha \in [\frac{1}{9}, \frac{1}{2})$. Therefore, T is a noncyclic contraction in the sense of Kannan. Now we can apply Theorem 3.13 to conclude that $A \cap B$ is nonempty and that T has a unique fixed point in $A \cap B$, that is $x^* = 0$.

The next example shows that the strict convexity of the underlying space is just a sufficient condition in Theorem 3.13.

Example 3.15. *Let X be the Banach space $\mathbb{R}^3$ endowed with the supremum norm and let $\{e_1, e_2, e_3\}$ be the canonical basis of $\mathbb{R}^3$. Suppose that e_0 is the zero of $\mathbb{R}^3$. Let*

$$A := \overline{con}\{te_1, te_3 : 0 \leq t \leq 1\} \quad \text{and} \quad B := \{te_2 : 0 \leq t \leq 1\}.$$

It is clear that (A, B) is a closed convex bounded pair that satisfies property (H). Define the noncyclic mapping $T : A \cup B \to A \cup B$ as follows:

$$Tx = \begin{cases} e_0 & \text{if } x \neq e_1, \\ \frac{1}{10} e_3 & \text{if } x = e_1. \end{cases}$$

Then T is noncyclic on $A \cup B$. We now claim that T is a noncyclic contraction in the sense of Kannan. Let $x \in A$ and $y := ye_2 \in B$, with $y \in [0, 1]$. If $x = e_1$, then

$$\|Tx - Ty\|_\infty = \|\tfrac{1}{10}e_3 - e_0\|_\infty = \tfrac{1}{10} \text{ and } \|x - Tx\|_\infty + \|y - Ty\|_\infty = 1 + y.$$

Thus,

$$\|Tx - Ty\|_\infty \leq \alpha[\|x - Tx\|_\infty + \|y - Ty\|_\infty],$$

for each $\alpha \in [\frac{1}{10}, \frac{1}{2})$. Notice that $A \cap B$ is a nonempty set and that T has a unique fixed point in $A \cap B$ which is $x^* = e_0$.

The next corollary trivially follows from Theorem 3.13.

Corollary 3.16. *Let (A, B) be a nonempty bounded closed convex pair in a strictly convex and reflexive Banach space X. Assume that $T : A \cup B \to A \cup B$ is a noncyclic contraction in the sense of Kannan. If the pair (A, B) has property (H), then T has a unique fixed point in $A \cap B$.*

Remark 3.17. *Notice that every noncyclic mapping T satisfying that $d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)]$ for some $\alpha \in [0, \frac{1}{2})$ and for all $(x, y) \in A \times B$ is also a noncyclic contraction in the sense of Kannan. Therefore, all the results given above for noncyclic contractions in the sense of Kannan also hold for this smaller class of noncyclic mappings.*

We finish this section by studying a new family of contractive type mappings. Since the Kannan contraction mappings have been generalized consider the problem of finding best proximity points or best proximity pairs, it seems natural to wonder whether it is possible to weaken the classical Chatterjea contractive condition [8] to proceed in a similar way. In this regarding, several fixed point results have been given in the literature (see for instance [25, 19]). In the sequel, we consider the following Chatterjea contractive type condition.

Definition 3.18. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is said to be a noncyclic contraction in the sense of Chatterjea if T is noncyclic and there exists $\alpha \in [0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx)] + (1 - 2\alpha)\, dist(A, B),$$

for all $(x, y) \in A \times B$.

Notice that if in the above definition $A = B$, then we have a Chatterjea contraction. Next we give an example of a noncyclic contraction in the sense of Chatterjea. Notice first that the mapping T defined in Example 3.10 is not a noncyclic contraction in the sense of Chatterjea.

Example 3.19. *Let X be the set $\mathbb{R}$ endowed with the Euclidean norm. Let $A := [0, 1]$, $B := [2, 3]$ and $T : A \cup B \to A \cup B$ a noncyclic mappings defined as*

$$Tx = \begin{cases} 1 & \text{if } x \in A - \{0\}, \\ \frac{2}{3} & \text{if } x = 0, \\ 2 & \text{if } x \in B. \end{cases}$$

Then, T is a noncyclic contraction in the sense of Chatterjea for every $\alpha \in [\frac{1}{4}, \frac{1}{2})$. It is not difficult to see that T is not a noncyclic contraction in the sense on Kannan for any $\alpha \in [0, \frac{1}{2})$.

Since it is very well-known in the literature that there exist self-mappings T that are Chatterjea contractions but are neither nonexpansive nor Kannan contractions, we also have that the extensions given in the noncyclic case of these mappings are independent. In fact, it is not very difficult to give an example similar to Example 3.10 that is a noncyclic contraction in the sense of Chatterjea but is neither nonexpansive nor Kannan contraction. Notice that in the previous example the mapping T is noncyclic relatively nonexpansive since every noncyclic contraction in the sense of Chatterjea defined on those sets A and B is so.

From now on in this section, we omit most of the proofs since they are quite similar to those given for the counterpart Kannan case. We first study the existence of best proximity pairs for noncyclic contractions in the sense of Chatterjea.

Theorem 3.20. *Let (A, B) be a nonempty weakly compact convex pair in a strictly convex Banach space X. Suppose that $T : A \cup B \to A \cup B$ is a noncyclic mapping that is a contraction in the sense of Chatterjea. Then T has a best proximity pair $(x, y) \in A \times B$.*

Proof. The proof of this result is similar to the proof of Theorem 3.13. The single difference between them is that in Chatterjea's version we do not need to assume that the pair (A, B) has property (H). $\square$

Remark 3.21. *As far as we know, the existence of best proximity pairs for a noncyclic mapping $T : A \cup B \to A \cup B$ that satisfies $d(Tx, Ty) \le \alpha[d(x, Ty) + d(y, Tx)]$ for all $(x, y) \in A \times B$ has not been studied. Notice that the previous result also applies for this type of Chatterjea contractive condition.*

To be consistent with the existent theory so far, we study the cyclic case for this Chatterjea contractive type condition. First of all we give the following result. We omit the proof since it follows similar patterns to those given for the proof of Lemma 3 in [26].

Proposition 3.22. *Let (A, B) be a nonempty pair in a metric space space X. Suppose that $T : A \cup B \to A \cup B$ is a cyclic mapping that is a contraction in the sense of*

Chatterjea, i.e., for which there exists $\alpha \in [0, \frac{1}{2})$ such that $d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx)] + (1 - 2\alpha)dist(A, B)$ for all $(x, y) \in A \times B$. Then

1. *$d(Tx, T^2x) \leq \alpha d(x, Tx) + (1 - \alpha)dist(A, B)$ for every $x \in A \cup B$.*
2. *$d(T^n x, T^{n+1}x) \leq \alpha^n d(x, Tx) + (1 - \alpha^n)dist(A, B)$ for every $x \in A \cup B$.*
3. *If X is a normed space that is strictly convex, then T has a best proximity point if and only if T^2 has a fixed point.*

Proposition 3.23. *Let (A, B) be a nonempty pair of subsets of a strictly convex Banach space (X, d). Let $T : A \cup B \to A \cup B$ be a cyclic contraction in the sense of Chatterjea. Then T has a best proximity point in $A \cup B$ if and only if $A \cap B \neq \emptyset$.*

Proof. Suppose $x \in A \cup B$ is such that $d(x, Tx) = \text{dist}(A, B)$. Then, by the strict convexity of the space we have that $T^2x = x$. Since T is a cyclic contraction in the sense of Chatterjea, we have that

$$\begin{gathered}\text{dist}(A, B) = d(x, Tx) = d(T^2x, Tx) \leq \\ \leq \alpha d(T^2x, x) + \alpha d(Tx, Tx) + (1 - 2\alpha)\text{dist}(A, B),\end{gathered}$$

which implies that $\text{dist}(A, B) = 0$ and the result follows.

The other implication holds directly by [8]. □

Lemma 3.24. *Let (A, B) be a nonempty pair of subsets of a complete metric space (X, d). Let $T : A \cup B \to A \cup B$ be a cyclic contraction in the sense of Chatterjea. If $x \in A \cup B$ and $\{T^{2n}x\}$ has a subsequence which converges to a point $z \in A \cup B$, then z is a best proximity point T.*

Proof. Let $\{T^{2n_k}x\}$ the convergent subsequence to $z \in A$. Then

$$\begin{aligned} d(z, Tz) &= \lim_{k\to\infty} d(T^{2n_k}x, Tz) \\ &\leq \limsup_{k\to\infty}(\alpha d(T^{2n_k}x, z) + \alpha d(T^{2n_k-1}x, Tz)) + (1 - 2\alpha)\text{dist}(A, B) \\ &\leq \limsup_{k\to\infty}(\alpha d(T^{2n_k}x, z) + \alpha d(T^{2n_k-1}x, T^{2n_k}x) + \alpha d(T^{2n_k}x, Tz)) \\ &\quad +(1 - 2\alpha)\text{dist}(A, B) \\ &= (1 - \alpha)\text{dist}(A, B) + \alpha d(z, Tz), \end{aligned}$$

which implies that $d(z, Tz) = \text{dist}(A, B)$ and the result follows. □

We omit the proof of the following result since it follows similar patterns to those given to prove Theorem 3.13.

Theorem 3.25. *Let (A, B) be a nonempty weakly compact convex pair in a Banach space X. Suppose that $T : A \cup B \to A \cup B$ is a cyclic contraction in the sense of Chatterjea. If (A, B) satisfies property (H), then T has a best proximity point in $A \cup B$.*

Note that in the previous result the Banach space may not be strictly convex. This is the reason why it may happen that we do not get a fixed point of T in this case.

Remark 3.26. *Notice that both Theorems 3.20 and 3.25 hold if (A, B) is a nonempty pair of closed, convex and bounded subsets of a reflexive Banach space.*

4. Strongly noncyclic relatively nonexpansive mappings in the sense of Chatterjea

After studying in the previous section the existence of best proximity pairs for mappings satisfying contractive type conditions, it is natural to wonder whether it is possible to obtain similar results when we slightly weaken the contractive conditions of the mappings under consideration. In this direction, we first introduce two new classes of noncyclic mappings.

Definition 4.1. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is said to be noncyclic relatively nonexpansive in the sense of Chatterjea provided that T is noncyclic on $A \cup B$ and satisfies*

$$d(Tx, Ty) \leq \frac{1}{2}[d(x, Ty) + d(y, Tx)], \tag{4.1}$$

for all $(x, y) \in A \times B$.

Definition 4.2. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is said to be strongly noncyclic relatively nonexpansive in the sense of Chatterjea provided that T is noncyclic on $A \cup B$ and satisfies*

$$d(Tx, Ty) \leq \min\{d(x, Ty), d(y, Tx)\}, \tag{4.2}$$

for all $(x, y) \in A \times B$.

Note that every strongly noncyclic relatively nonexpansive mapping in the sense of Chatterjea is noncyclic relatively nonexpansive in the sense of Chatterjea. We also point out here that, as far as we know, conditions (4.1) and (4.2) have not been considered in the literature even in the classical case where the conditions are satisfied for every $x, y \in X$.

Next we give two easy examples of strongly noncyclic relatively nonexpansive mappings in the sense of Chatterjea.

Example 4.3. *Let X be the set $\mathbb{R}$ endowed with the Euclidean norm. Let $A := [0, 1]$, $B := [2, 3]$, and T_1 and T_2 noncyclic mappings defined as*

$$T_1 x = \begin{cases} x & \text{if} \quad x \in A \\ 2 & \text{if} \quad x \in B \end{cases} \qquad T_2 x = \begin{cases} 1 & \text{if} \quad A - \{0\}, \\ 0 & \text{if} \quad x = 0, \\ 2 & \text{if} \quad x \in B. \end{cases}$$

Then both T_1 and T_2 are strongly noncyclic relatively nonexpansive in the sense of Chatterjea. However, notice that this mappings are not noncyclic contractions in the sense of Chatterjea for any $[0, \frac{1}{2})$.

Remark 4.4. *Let (X, d) be a metric space. We may observe that if $T : X \to X$ satisfies the metric condition (4.2) $d(Tx, Ty) \leq \min\{d(x, Ty), d(y, Tx)\}$ for every $x, y \in X$, then every point $z \in T(X)$ is a fixed point of T. The same happens if T is supposed to be cyclic instead of noncyclic. This is the reason why we will only give an existence result for this type of metric condition in the noncyclic case.*

The following example shows that in general the family of strongly noncyclic relatively nonexpansive mappings in the sense of Chatterjea is different from the one of noncyclic relatively nonexpansive mappings.

Example 4.5. *Let $X := [-1, 1]$ and define a metric d on X by*

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ \max\{|x|, |y|\} & \text{if } x \neq y. \end{cases}$$

Let $A := [-1, \frac{-1}{2}]$, $B := [0, 1]$ and $T : A \cup B \to A \cup B$ the noncyclic mapping given by

$$Tx = \begin{cases} -1, & \text{if } x \in A, \\ 1 & \text{if } x \in B. \end{cases}$$

Then, the mapping T is strongly noncyclic relatively nonexpansive in the sense of Chatterjea. Indeed, if $x \in A$ and $y \in B$ then $d(Tx, Ty) = 1$ and $\min\{d(x, Ty), d(Tx, y)\} = \min\{d(x, 1), d(-1, y)\} = 1$. However we may take $x_0 = \frac{-1}{2} \in A$ and $y_0 = \frac{1}{2} \in B$ so that $d(Tx_0, Ty_0) = 1 > d(x_0, y_0) = \frac{1}{2}$. Then T is not noncyclic relatively nonexpansive.

Next we show that even in the case where $T : X \to X$ is a self-mapping satisfying the metric condition (4.2) for every $x, y \in X$, T may not be nonexpansive.

Example 4.6. *Let $X = [-1, 1]$ and define a metric d on X by*

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ \max\{|x|, |y|\} & \text{if } x \neq y. \end{cases}$$

Let $T : X \to X$ be the mapping given by

$$Tx = \begin{cases} 1, & \text{if } x \in \{0, 1\}, \\ -1 & \text{if } x \in X - \{0, 1\}. \end{cases}$$

Then, the mapping T satisfies (4.2) for every $x, y \in X$. Indeed, if $x = 0$ and $y \in X - \{0, 1\}$ then $d(Tx, Ty) = 1$ and $\min\{d(x, Ty), d(Tx, y)\} = \min\{d(0, -1), d(1, y)\} = 1$. Similarly, if $x = 1$ and $y \in X - \{0, 1\}$, we have $d(Tx, Ty) \leq \min\{d(x, Ty), d(Tx, y)\}$. However it is immediate to see that T is not nonexpansive. Now, from Remark 4.4, we see that both $x_1 = 1$ and $x_2 = -1$ are fixed points of T.

The next result establishes the existence of best proximity pairs for strongly noncyclic relatively nonexpansive mappings in the sense of Chatterjea in strictly convex Banach spaces by using the geometric notion of proximal normal structure described in Definition 3.2. Notice first that a Banach spaces is strictly convex if and only if is Busemann convex.

Theorem 4.7. *Let (A, B) be a nonempty, weakly compact convex pair in a strictly convex Banach space X and suppose (A, B) has proximal normal structure. Suppose that $T(A_0) \subseteq A_0$ and $T(B_0) \subseteq B_0$. If T is strongly noncyclic relatively nonexpansive mapping in the sense of Chatterjea, then T has a best proximity pair.*

Proof. First note that (A_0, B_0) is a nonempty weakly compact convex proximal pair and $\operatorname{dist}(A, B) = \operatorname{dist}(A_0, B_0)$ (see [22] for more details). Let Σ denote the collection of all nonempty, weakly compact convex pairs $(C, D) \subseteq (A, B)$ for which $\operatorname{dist}(C, D) = \operatorname{dist}(A, B)$, (C, D) is a *proximal pair* and T is noncyclic on $C \cup D$. Since $(A_0, B_0) \in \Sigma$, Σ is nonempty. Proceeding as in [17], we can apply Zorn's lemma to prove that Σ has a minimal element with respect to the set inclusion, say (K_1, K_2), that is a proximal pair and satisfies $\operatorname{dist}(K_1, K_2) = \operatorname{dist}(A, B)$. We may observe that if $\delta(K_1, K_2) = \operatorname{dist}(K_1, K_2)$, then the strict convexity of X implies that K_1 and K_2 are singleton and the conclusion trivially holds. Now, suppose $\delta(K_1, K_2) > \operatorname{dist}(K_1, K_2)$. It now follows from the fact that (A, B) has proximal normal structure that there exist $(p_1, p_2) \in K_1 \times K_2$ and $r \in (0, 1)$ such that

$$\delta_{p_1}(K_2) \leq r\delta(K_1, K_2), \quad \delta_{p_2}(K_1) \leq r\delta(K_1, K_2).$$

Since (K_1, K_2) is a proximal pair, there exists a pair $(q_1, q_2) \in K_1 \times K_2$ such that $d(p_1, q_2) = d(q_1, p_2) = \operatorname{dist}(K_1, K_2)$. Now, for each $y \in K_2$ we have

$$\begin{aligned}\left\|\frac{p_1 + q_1}{2} - y\right\| &\leq \frac{\|p_1 - y\|}{2} + \frac{\|q_1 - y\|}{2} \\ &\leq r\frac{\delta(K_1, K_2)}{2} + \frac{\delta(K_1, K_2)}{2} = \frac{r+1}{2}\delta(K_1, K_2).\end{aligned}$$

If we set $p := \frac{p_1+q_1}{2} \in K_1$ and $h := \frac{r+1}{2}$, we obtain $\delta_p(K_2) \leq h\delta(K_1, K_2)$ with $h \in (0, 1)$. Similarly, if we set $q := \frac{p_2+q_2}{2} \in K_2$, then $\delta_q(K_1) \leq h\delta(K_1, K_2)$. Moreover, $d(p, q) = \operatorname{dist}(K_1, K_2)$. Let

$$\begin{aligned}H_1 = \{x \in K_1 : {}&\delta(x, K_2) \leq \alpha\delta(K_1, K_2) \text{ and for its proximal point } y \in K_2, \\ &\delta(y, K_1) \leq \alpha\delta(K_1, K_2)\},\end{aligned}$$

$$\begin{aligned}H_2 = \{y \in K_2 : {}&\delta(y, K_1) \leq \alpha\delta(K_1, K_2) \text{ and for its proximal point } x \in K_1, \\ &\delta(x, K_2) \leq \alpha\delta(K_1, K_2)\}.\end{aligned}$$

Since $p \in H_1$ and $q \in H_2$, $H_i \neq \emptyset$ for $i = 1, 2$. Next we show that H_i is closed and convex for $i = 1, 2$. We give the details for H_1. For H_2 the proof follows similar patterns. Let $\{v_n\} \subseteq H_1$ be a sequence that converges to a point $v \in K_1$. Since $d(v_n, z) \leq \alpha\delta(K_1, K_2)$ for every $n \in \mathbb{N}$ and $z \in K_2$, we get $\delta(v, K_2) \leq \alpha\delta(K_1, K_2)$. The fact that $v_n \in H_1$ implies

$$\delta(w_n, K_1) \leq \alpha\delta(K_1, K_2), \tag{4.3}$$

where $w_n \in K_2$ is the proximal point of $v_n \in K_1$. Let $w \in K_2$ such that $d(v, w) = \operatorname{dist}(K_1, K_2)$. Then we get

$$d\left(\frac{v_n + v}{2}, \frac{w_n + w}{2}\right) = \operatorname{dist}(K_1, K_2).$$

Using the notion of parallelism in Banach spaces, we may prove that $[w_n, v_n]$ and $[w, v]$ are parallel segments and therefore we have that $d(w_n, w) = d(v_n, v)$ for $n \in \mathbb{N}$, from where $w_n \to w$ (for more on this property, see Proposition 5.7 that gives the same result for more general metric spaces). Taking limit in (4.3), we may conclude $\delta(w, K_1) \leq \alpha\delta(K_1, K_2)$. Consequently $v \in H_1$ and so H_1 is closed. Now, let $p_1, q_1 \in$

H_1. Next we see that the mid-point between p_1 and q_1, $m_1 = \frac{p_1+q_1}{2}$, is in H_1. Let $p_2, q_2 \in K_2$ be the proximal points of p_1 and q_1, respectively. Consider $m_2 = \frac{p_2+q_2}{2}$. Since (K_1, K_2) is proximal and the space X is Busemann convex, $d(m_1, m_2) = \operatorname{dist}(K_1, K_2)$. Let $z \in K_2$. Then

$$d(z, m_1) \leq \frac{1}{2} d(p_1, z) + \frac{1}{2} d(q_1, z) \leq \alpha\delta(K_1, K_2).$$

Thus, $\delta(m_1, K_2) \leq \alpha\delta(K_1, K_2)$. The fact that $\delta(m_2, K_1) \leq \alpha\delta(K_1, K_2)$ follows similarly since $\delta(p_2, K_1)$ and $\delta(q_2, K_1)$ are both $\leq \alpha\delta(K_1, K_2)$. Then $m_1 \in H_1$ and so H_1 is convex. Notice that (H_1, H_2) is also a proximal pair and $\operatorname{dist}(H_1, H_2) = \operatorname{dist}(K_1, K_2)$.

Next we see that T is noncyclic on $H_1 \cup H_2$. Let $x \in H_1$ and $y \in H_2$ such that $d(x, y) = \operatorname{dist}(H_1, H_2)$. We prove that $Tx \in H_1$. Let $z \in K_2$. Since $d(Tx, Tz) \leq d(x, Tz) \leq \delta_x(T(K_2)) \leq \delta_x(K_2) \leq h\delta(K_1, K_2)$, we get

$$T(K_2) \subseteq B(Tx; h\delta(K_1, K_2)) \cap K_2 := K_2'.$$

Then K_2' is closed, convex and nonempty. Let $K_1' \subseteq K_1$ be the set

$$K_1' = \{x \in K_1 : \text{ there exists } y \in K_2' \text{ with } d(x, y) = \operatorname{dist}(K_1, K_2)\}.$$

Then K_1' is closed, convex and nonempty. Moreover, (K_1', K_2') is proximal and satisfies $\operatorname{dist}(K_1', K_2') = \operatorname{dist}(K_1, K_2)$, $T(K_1') \subseteq K_1'$ and $T(K_2') \subseteq K_2'$. Therefore, $K_1' \cup K_2' \in \Sigma$ and by minimality of K it follows that $K_2 \subseteq B(Tx, h\delta(K_1, K_2))$ and therefore $\delta(Tx, K_2) \leq h\delta(K_1, K_2)$. Proceeding similarly, we may see that $\delta(Ty, K_1) \leq h\delta(K_1, K_2)$. Since $Ty \in K_2$ is the proximal point of $Tx \in K_1$, we conclude that $Tx \in H_1$ and therefore $T(H_1) \subseteq H_1$. Similarly, $T(H_2) \subseteq H_2$. As a consequence, $H_1 \cup H_2 \in \Sigma$. Since, $\delta(H_1, H_2) \leq \alpha\delta(K_1, K_2)$, we get a contradiction with the minimality of K.

Next we see that T is noncyclic on $H_1 \cup H_2$. Let $x \in H_1$ and $y \in H_2$ such that $d(x, y) = \operatorname{dist}(H_1, H_2)$. We prove that $Tx \in H_1$. Let $z \in K_2$. Since $d(Tx, Tz) \leq d(x, Tz) \leq \delta_x(T(K_2)) \leq \delta_x(K_2) \leq h\delta(K_1, K_2)$, we get

$$T(K_2) \subseteq B(Tx; h\delta(K_1, K_2)) \cap K_2 := K_2'.$$

Then K_2' is closed, convex and nonempty. Let $K_1' \subseteq K_1$ be the set

$$K_1' = \{x \in K_1 : \text{ there exists } y \in K_2' \text{ with } d(x, y) = \operatorname{dist}(K_1, K_2)\}.$$

Then K_1' is closed, convex and nonempty. Moreover, (K_1', K_2') is proximal and satisfies $\operatorname{dist}(K_1', K_2') = \operatorname{dist}(K_1, K_2)$, $T(K_1') \subseteq K_1'$ and $T(K_2') \subseteq K_2'$. Therefore, $K_1' \cup K_2' \in \Sigma$ and by minimality of K it follows that $K_2 \subseteq B(Tx, h\delta(K_1, K_2))$ and therefore $\delta(Tx, K_2) \leq h\delta(K_1, K_2)$. Proceeding similarly, we may see that $\delta(Ty, K_1) \leq h\delta(K_1, K_2)$. Since $Ty \in K_2$ is the proximal point of $Tx \in K_1$, we conclude that $Tx \in H_1$ and therefore $T(H_1) \subseteq H_1$. Similarly, $T(H_2) \subseteq H_2$. As a consequence, $H_1 \cup H_2 \in \Sigma$. Since, $\delta(H_1, H_2) \leq \alpha\delta(K_1, K_2)$, we get a contradiction with the minimality of K. □

Remark 4.8. *Aside from the cases where T is relatively nonexpansive, there are some other natural conditions on a noncyclic mapping T that guarantee that $T(A_0) \subseteq A_0$ and $T(B_0) \subseteq B_0$. For instance, we cite the condition of being a "noncylic relatively* *u-continuous" mapping, which derives from the concept of "relatively u-continuous mapping" introduced in* [11] *for cyclic mappings.*

Since every pair of nonempty, bounded, closed and convex subsets of a uniformly convex Banach space has proximal normal structure, we get the following corollary.

Corollary 4.9. *Let (A, B) be a nonempty bounded closed convex pair in a uniformly convex Banach space X. Suppose that $T(A_0) \subseteq A_0$ and $T(B_0) \subseteq B_0$. If T is strongly noncyclic relatively nonexpansive in the sense of Chatterjea, then T has a best proximity pair.*

Now we raise the next problem.

Question 4.10. *It is interesting to ask whether Theorem 4.7 holds whenever T is noncyclic relatively nonexpansive in the sense of Chatterjea.*

We finish this section by making the following reflection. In [2], the Kannan type condition $d(Tx, Ty) \leq \frac{1}{2}[d(x, Tx) + d(y, Ty)]$ was considered in the cyclic setting. At that moment, the notion of proximal quasi-normal structure is defined to get an existence result of best proximity points. Note that the concept of proximal quasi-normal structure is the analog to the notion of quasi-normal structure that is used in the literature to prove a fixed point result for mappings $T : X \to X$ satisfying this Kannan type condition for every $x, y \in X$ [20]. Similar techniques to those applied for the cyclic case in [2] seem not to be useful to solve the noncyclic case and therefore, the solution to such a problem is still an open question.

5. Several consequences in geodesic spaces

When the existence of best proximity pairs and best proximity points is studied in the previous sections for certain mappings, it is natural to wonder whether similar results hold in metric settings that are more general. In this regarding, we thoroughly analyse the proofs given above and show that, in most of the cases, the underlying linear structure is not necessary to get the results. In particular, we consider the setting of reflexive metric spaces that are either Busemann convex or at least strictly convex (see the section of preliminaries for definitions).

First we see that the counterpart of Theorem 3.5 holds in geodesic spaces. For this aim, we recall the following property between sets given in [16]. This property was introduced in the cited paper to give the analog result in the cyclic case.

Definition 5.1. *Let A and B be two nonempty subsets of a metric space X. The pair (A, B) is said to satisfy property HW if for every sequence $\{x_n\} \subset A$ and $\{z_n\} \subset B$ and every two points $q \in A$ and $p \in B$ we have that*

$$\left.\begin{array}{c} d(x_n, p) \to dist(A, B) \\ d(x_n, z_n) \to dist(A, B) \\ d(z_n, q) \to dist(A, B) \end{array}\right\} \Rightarrow d(p, q) = dist(A, B).$$

Theorem 5.2. *Let A and B be two closed convex subsets of a strictly convex and reflexive metric space and $T : A \cup B \to A \cup B$ a noncyclic contraction. If (A, B) has property HW then T has a best proximity pair.*

Proof. This result follows by applying similar patterns as in the proof of Theorem 3.5. However, in absence of weak convergence, several changes must be considered to get the result.

As in the linear case, let $d^*(x,y) := d(x,y) - \mathrm{dist}(A,B)$. Then $d^*(Tx,Ty) \leq \alpha d^*(x,y)$ for every $x \in A$ and $y \in B$. Let x_0 and y_0 be two arbitrary but fixed points in A and B respectively.

Fix $m_0 \in N$ and let $m = m_0 + k$ with $k \in \mathbb{N}$. Then

$$d^*(T^m x_0, T^{m_0} y_0) \leq \alpha^{m_0} d^*(T^k x_0, y_0)$$
$$\leq \alpha^{m_0} \sup\{d(T^k x_0, y_0) : \ k \in \mathbb{N}\} = \alpha^{m_0} M(x_0, y_0).$$

In a similar way, we have that

$$d^*(T^m x_0, T^{m_0+1} y_0) \leq \alpha^{m_0} \sup\{d(T^k x_0, Ty_0) : \ k \in \mathbb{N}\} = \alpha^{m_0} M'(x_0, y_0),$$
$$d^*(T^m y_0, T^{m_0} x_0) \leq \alpha^{m_0} \sup\{d(T^k y_0, x_0) : \ k \in \mathbb{N}\} = \alpha^{m_0} M''(x_0, y_0) \text{ and}$$
$$d^*(T^m y_0, T^{m_0+1} x_0) \leq \alpha^{m_0} \sup\{d(T^k y_0, Tx_0) : \ k \in \mathbb{N}\} = \alpha^{m_0} M'''(x_0, y_0).$$

Since the proof of Proposition 3.4 is purely metric, we have that the orbits of T are bounded at every point $z \in A \cup B$. Consequently, for every $n \in \mathbb{N}$, there exists $m_0(n) \in \mathbb{N}$ such that for every $m \geq m_0(n)$, $T^m x_0 \in B(T^{m_0(n)} y_0; \mathrm{dist}(A,B) + 1/n)$, $T^m x_0 \in B(T^{m_0(n)+1} y_0; \mathrm{dist}(A,B) + 1/n)$, $T^m y_0 \in B(T^{m_0(n)} x_0; \mathrm{dist}(A,B) + 1/n)$ and $T^m y_0 \in B(T^{m_0(n)+1} x_0; \mathrm{dist}(A,B) + 1/n)$.

Denote $B_n = B(T^{m_0(n)} y_0; \mathrm{dist}(A,B) + 1/n)$, $B'_n = B(T^{m_0(n)+1} y_0; \mathrm{dist}(A,B) + 1/n)$, $A_n = B(T^{m_0(n)} x_0; \mathrm{dist}(A,B) + 1/n)$ and $A'_n = B(T^{m_0(n)+1} x_0; \mathrm{dist}(A,B) + 1/n)$. Consider the sequence of subsets $\{C_n\}$ and $\{D_n\}$ of X such that $C_1 = A \cap B_1 \cap B'_1$, $D_1 = B \cap A_1 \cap A'_1$, $C_n = C_{n-1} \cap B_n \cap B'_n$ and $D_n = D_{n-1} \cap A_n \cap A'_n$ for every $n \geq 2$.

By definition, it is immediate to see that C_n and D_n are closed, bounded and nonempty for every $n \in \mathbb{N}$. Since X is strictly convex, the balls in X are convex and so C_n and D_n are convex for every $n \in \mathbb{N}$. Thus, by means of the reflexivity of X, we conclude that $\bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$ and $\bigcap_{n \in \mathbb{N}} D_n \neq \emptyset$. Let $p \in \bigcap_{n \in \mathbb{N}} C_n \subset A$ and $q \in \bigcap_{n \in \mathbb{N}} D_n \subset B$. Taking $m_0(n)$ as an increasing sequence on n, we have that the sequences of terms $z_n = T^{m_0(n)} y_0$ and $z'_n = T^{m_0(n)+1} y_0$ for every $n \in \mathbb{N}$ are subsequences of $\{T^n y_0\}$. Similarly, $w_n = T^{m_0(n)} x_0$ and $w'_n = T^{m_0(n)+1} x_0$ are subsequences of $\{T^n x_0\}$. Consequently, since $p \in C_n$ and $q \in D_n$ for every $n \in \mathbb{N}$, (i) $d(z_n, p) \to \mathrm{dist}(A,B)$, (ii) $d(z'_n, p) = d(Tz_n, p) \to \mathrm{dist}(A,B)$, (iii) $d(w_n, q) \to \mathrm{dist}(A,B)$, (iv) $d(w'_n, q) = d(Tw_n, q) \to \mathrm{dist}(A,B)$, (v) $d(z'_n, Tp) = d(Tz_n, Tp) \to \mathrm{dist}(A,B)$ and (vi) $d(Tw_n, Tq) \to \mathrm{dist}(A,B)$.

Moreover, since T is a noncyclic contraction, $d(T^n x, T^n y) \to \mathrm{dist}(A,B)$ for every $x \in A$ and $y \in B$. Thus, (vii) $d(w_n, z_n) = d(T^{m_0(n)} x_0, T^{m_0(n)} y_0) \to \mathrm{dist}(A,B)$ and $(viii)$ $d(Tw_n, Tz_n) \to \mathrm{dist}(A,B)$.

If we apply now property HW to $(i), (iii)$ and (vii), we get $d(q,p) = \mathrm{dist}(A,B)$. Considering now the same property for $(iv), (v)$ and $(viii)$, we get $d(p, Tp) = \mathrm{dist}(A,B)$. Since X is strictly convex, we get $p = Tp$. In a similar way we get that $q = Tq$ and the result follows. □

Remark 5.3. *Notice that the previous theorem trivially extends Theorem 3.5 since every pair* (A, B) *of closed, convex and nonempty subsets of a strictly convex and reflexive Banach space has property HW (see Proposition 3.9 in* [16]*).*

Next we give the counterpart to Theorem 3.13 in geodesic spaces. Specifically, we give, in terms of reflexivity in metric spaces, the counterpart to Corollary 3.16. First, notice that both the notion of convex hull of a set and Definition 3.11 may be trivially extended to metric spaces with a convexity structure.

Theorem 5.4. *Let* (A, B) *be a nonempty closed convex bounded pair in a strictly convex and reflexive metric space* X*. Suppose that* $T : A \cup B \to A \cup B$ *is a noncyclic contraction in the sense of Kannan. If the pair* (A, B) *has property (H), then* T *has a unique fixed point in* $A \cap B$*.*

Proof. This result follows by applying similar patterns as in the proof of Theorem 3.13. Consequently, we just give a sketch of the proof and comment the steps where we find the main differences. Let Σ denotes the collection of all nonempty closed convex bounded pairs (E, F) which are subsets of (A, B) and such that T is noncyclic on $E \cup F$. Proceeding as in Theorem 3.13, we may use the reflexivity of X to find a minimal element, say $(K_1, K_2) \in \Sigma$. By using the counterpart concept of convex hull in geodesic spaces, we get similarly that T is noncyclic on $\overline{con}(T(K_1)) \cup \overline{con}(T(K_2))$. Now, by the minimality of (K_1, K_2), we have that $\overline{con}(T(K_1)) = K_1$, $\overline{con}(T(K_2)) = K_2$.

Reasoning as in linear case, we define the sets

$$E_1 := \{x \in K_1 : \delta_x(K_2) \leq 2\alpha\delta(K_1, K_2) + (1 - 2\alpha)\mathrm{dist}(A, B)\},$$
$$E_2 := \{y \in K_2 : \delta_y(K_1) \leq 2\alpha\delta(K_1, K_2) + (1 - 2\alpha)\mathrm{dist}(A, B)\}.$$

Notice that E_1 and E_2 are nonempty and closed. Moreover, since

$$E_1 = \bigcap_{y \in K_2} B(y; 2\alpha\delta(K_1, K_2) + (1 - 2\alpha)\mathrm{dist}(A, B)) \cap K_1,$$
$$E_2 = \bigcap_{x \in K_1} B(x; 2\alpha\delta(K_1, K_2) + (1 - 2\alpha)\mathrm{dist}(A, B)) \cap K_2,$$

the convexity of the balls in X implies that (E_1, E_2) is convex. Proceeding now as in the linear case we get that

$$\delta(K_1, K_2) = \mathrm{dist}(A, B).$$

Thus, given $(x, y) \in K_1 \times K_2$ we have

$$d(x, Ty) = d(Ty, Tx) = \mathrm{dist}(A, B).$$

Then, if $\frac{Tx+x}{2}$ is the mid-point between x and Tx in the uniquely geodesic space X, we may use the strict convexity of X to get

$$\mathrm{dist}(A, B) \leq d\bigg(Ty, \frac{Tx + x}{2}\bigg) < \mathrm{dist}(A, B),$$

which is a contradiction. Then, the unique pair of points $(x, y) \in K_1 \times K_2$ is a best proximity pair of T. Consequently, from Proposition 3.9, $A \cap B \neq \emptyset$. The uniqueness comes from the fact that the fixed point in the intersection is unique. $\square$

In the context of noncyclic and cyclic contractions in the sense of Chatterjea, we may proceed in a similar way to get the following results. Notice first that both Propositions 3.22 and 3.23 also hold when the space under consideration is a strictly convex metric space.

Theorem 5.5. *Let (A, B) be a closed convex bounded pair in a strictly convex and reflexive metric space X. Suppose that $T : A \cup B \to A \cup B$ is a noncyclic contraction in the sense of Chatterjea. Then T has a best proximity pair $(x, y) \in A \times B$.*

Theorem 5.6. *Let (A, B) be a closed convex bounded pair in a strictly convex and reflexive metric space X. Suppose that $T : A \cup B \to A \cup B$ is a cyclic contraction in the sense of Chatterjea. If (A, B) has the (H) property, then T has a best proximity point in $A \cap B$.*

Finally we point out that the counterpart to Theorem 4.7 also holds in some geodesic spaces. In this case we have to assume X to be Busemann convex. When dealing with cyclic and noncyclic mappings in the context of Busemann convex spaces, it is very useful the following property on parallel segments by H. Busemann [6]. A proof of this result is given in [17].

Proposition 5.7. *Let x, y, z, w be four points in a Busemann convex geodesic space. Suppose that $d(x, y) = d(m_1, m_2) = d(z, w)$, where m_1 and m_2 are the midpoints of the geodesic segments $[x, z]$ and $[y, w]$. Then $d(x, z) = d(m_3, m_4) = d(y, w)$, where m_3 and m_4 are the midpoints of $[x, y]$ and $[z, w]$.*

The notion of proximal normal structure given in Banach spaces may be equally defined in a metric space with a convexity structure. Considering this concept in geodesic spaces we get the following result.

Theorem 5.8. *Let (A, B) be a nonempty closed convex bounded pair in a reflexive and Busemann convex metric space and suppose that (A, B) has proximal normal structure. Let $T : A \cup B \to A \cup B$ be a strongly noncyclic relatively nonexpansive mapping in the sense of Chatterjea such that $T(A_0) \subseteq A_0$ and $T(B_0) \subseteq B_0$. Then T has a best proximity pair.*

It is not difficult to see that the proof of Theorem 4.7 also works for the previous result. We may observe that in that theorem we have mainly used the convexity structure of the Banach space, the convexity of the metric induced by the norm and the Busemann convexity of the space. In fact, thanks to Proposition 5.7, we may even get similar conclusions on parallel segments when dealing with these objects in Busemann convex metric spaces.

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