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NONEXPANSIVE MAPPINGS AND MONOTONE VECTOR FIELDS IN HADAMARD MANIFOLDS

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ABSTRACT.

This paper briefly surveys some recent advances in the investigation of nonexpansive mappings and monotone vector fields, focusing in the extension of basic results of the classical nonlinear functional analysis from Banach spaces to the class of nonpositive sectional curvature Riemannian manifolds called Hadamard manifolds. Within this setting, we first analyze the problem of finding fixed points of nonexpansive mappings. Later on, different classes of monotonicity for set-valued vector fields and the relationship between some of them will be presented, followed by the study of the existence and approximation of singularities for such vector fields. We will discuss about variational inequality and minimization problems in Hadamard manifolds, stressing the fact that these problems can be solved by means of the iterative approaches for monotone vector fields.

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1. INTRODUCTION AND MOTIVATION

Riemannian manifolds constitute a broad and fruitful framework for the development of different fields in mathematic, such as convexity, dynamical systems, optimization or mathematical programming, and other scientific areas, where some of its approaches and methods have successfully been extended from Euclidean spaces. The nonpositive sectional curvature is an important property which is enjoyed by a large class of Riemannian manifolds and it is strong enough to imply tight topological restrictions and rigidity phenomena. Hence Riemannian manifolds with this property have awakened the interest of many researches. Specially, Hadamard manifolds, which are complete simply connected and finite dimensional Riemannian manifolds of nonpositive sectional curvature, have worked out a suitable setting for diverse disciplines, being an example of hyperbolic spaces and geodesic spaces such as Busemann nonpositive curvature (NPC) spaces and CAT(0) spaces, see [24, 20, 53].

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Within the setting of Riemannian manifolds we are particularly concerned about two problems: the approach of fixed points of nonexpansive mappings and the existence and approximation of singularities of monotone vector fields.

In a metric space X, a mapping $T : X \to X$ is nonexpansive if for any two points $x, y \in X$ the following inequality holds:

$$d(T(x), T(y)) \leq d(x, y).$$

The study of the asymptotic behavior of such mappings and its relevant role in the fixed point theory have seen a significant increase in interest on the part of scientists from nonlinear analysis and many other disciplines where it finds application. Two types of algorithms for approximating fixed points of nonexpansive mappings have been successfully considered in the normed linear framework: Halpern's iteration, given by the formula

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \quad n \geq 0, \tag{1.1}$$

where $u \in X$ is an arbitrary point; and Mann's iteration, whose iterative scheme is

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0. \tag{1.2}$$

A large number of works about the convergence and some modifications of these two methods can be found (see [44, 55, 28] and reference therein), but just few results have been obtained out of the framework of linear spaces. With preciseness, some progresses have been made in the setting of geodesic spaces, hyperbolic metric spaces and in the special case of Hadamard manifolds. See [16, 17, 45, 23, 24, 27].

If we denote by X^* the dual space of the Banach space X, recall that a (set-valued) operator $A : X \to 2^{X^*}$ is said to be monotone provided that

$$(x^* - y^*, x - y) \geq 0, \quad \forall x, y \in \mathcal{D}(A) \text{ and } x^* \in A(x),\ y^* \in A(y),$$

where $\mathcal{D}(A)$ denotes the domain of A defined by $\mathcal{D}(A) := \{x \in X : \ A(x) \neq \emptyset\}$. The concept of a monotone operator has turned out to be very powerful in various areas of mathematics such as operator theory, numerical analysis, differentiability of convex functions and partial differential equations, because it is broad enough to cover both linear positive semi-definite operators and subdifferentials of convex functions (see [42, 31, 32]). It is the latter one which has received most of the recent attention in diverse frameworks, due to its increasing importance in optimization theory.

On the other hand, extensions of concepts and techniques for optimization in $\mathbb{R}^n$ to Riemannian manifolds are natural, and this has been done frequently in order to develop theoretical results and get efficient algorithms, see [50, 53, 20]. The study of this optimization methods' extension to solve minimization problems on Riemannian manifolds has been the subject of many works, solving non-convex constrained minimization problems in Euclidean spaces by means of convex problems on Riemannian manifolds, see [7, 8, 13, 15, 14, 39, 40]. A generalization of the convex minimization problem is the variational inequality problem. In the study of this problem in the framework of Riemannian manifolds several classes of monotone vector fields have been introduced (see [34, 35, 10] for single-valued vector field and [9] for point-to-set vector fields) and convergence properties of iterative methods to solve them have been presented (see for instance [11, 13]).

A more general problem is the search of zeros of monotone operators, i.e. solutions of the inclusion $0 \in A(x)$. In this direction, the concept of maximal monotone operators and its relationship with the notion of upper semicontinuity have been very useful, see [41, 2]. In [26] both definitions for set-valued vector fields were extended to Hadamard manifolds and proved to be equivalent. Many diverse approaches for approximating zeros in the setting of Banach spaces have been investigated by many authors (see [5, 6, 21, 29, 47]). It is worth mentioning the widely studied proximal point algorithm, inspired by Moreau and Martinet [29, 30, 33] and defined by Rockafellar [47] by means of the following iterative scheme

$$0 \in A(x_{n+1}) + \lambda_n(x_{n+1} - x_n),$$

where $\{\lambda_n\}$ is a sequence of real positive numbers and x_0 is an initial point. Another successful iterative method is the extragradient algorithm proposed first by Korpelevich [25] and studied by many authors (see, for instance, [18, 51]). As regards the counterpart of these methods in Hadamard manifolds, convergence of the proximal point algorithm was proved for subdifferential of convex functions in minimization problems (see [15]) and for maximal monotone vector fields in general (see [26]). An extragradient-type method for continuous monotone vector fields was developed by Ferreira et al [13].

Most of this extended methods requires the Riemannian manifold to have nonpositive sectional curvature, specially, to be a Hadamard manifold. This is due to the fact that, in general, the exponential map cannot be defined in the whole tangent bundle and it is not invertible (see reference in section 2 for details). Then we will focus in the case of Hadamard manifolds, remarking the statements which remain true in Riemannian manifolds in general.

The purpose of this paper is to describe the role of nonexpansive mappings and monotone vector fields in Hadamard manifolds, as well as the problems mentioned above which involve such operators and make its closeness quite clear. In section 2 we introduce some notations and we provide references about basic concepts on Riemannian manifolds. The fixed point problem for nonexpansive mappings will be discussed in section 3. Section 4 is devoted to introduce different classes of monotonicity, establishing some relationship between them, and analyze the existence and approximation of singularities of monotone vector fields. We deal with variational inequality and minimization problems in Hadamard manifolds in sections 5, stressing the fact that these problems can be solved by means of the iterative approaches for monotone vector fields.

2. NOTATION

In this section we fix the notations used throughout this paper. We assume that the reader is familiar with the basic concepts and properties on Riemannian manifold. As general references in Riemannian geometry, we recommend [12, 43, 48, 53, 54]. Moreover, basic knowledge of fixed point theory [24], monotone operators [41, 56], variational inequalities [22] and optimization theory [1, 49] could ease the reading of these notes.

Let M be a Hadamard manifold, that is a complete simply connected m-dimensional Riemannian manifold of nonpositive sectional curvature. Given $p \in M$ we denote the tangent space of M at p by T_pM and the tangent bundle of M by $TM = \bigcup_{p\in M} T_pM$. Let $\mathcal{X}(M)$ denote the set of all set-valued *vector fields* $A : M \to 2^{TM}$ such that $A(x) \subseteq T_xM$ for each $x \in M$. We assume that $\mathcal{D}(A) = \{x \in M : A(x) \neq \emptyset\}$, the domain of A, is closed and convex. Recall that a subset $K \subseteq M$ is convex if for any two points p and q in K, the geodesic joining p to q is contained in K.

For any $x \in M$ and $K \subset M$ closed convex set, there exists a unique $x^* \in K$ such that $d(x, x^*) \leq d(x, y)$ for all $y \in K$. That unique point is called the projection of x onto the convex set K and is denoted by $P_K(x)$.

We use $P_{x,y}$ to denote the parallel transport on the tangent bundle TM along the geodesic γ starting at x and ending at y. This is an isometry from T_xM to T_yM.

Recall that the *exponential map* at $p \in M$, $\exp_p : T_pM \to M$, is defined by $\exp_p v = \gamma_v(1, x)$, where $\gamma_v(., p)$ is the geodesic starting at p with velocity v. Then, for any value of t, $\exp_p tv = \gamma_v(t, p)$. Since M is Hadamard, $\exp_p$ is a diffeomorphism from M to $\mathbb{R}^m$. Then, M has the same topology and differential structure as $\mathbb{R}^m$.

Let $f : M \to (-\infty, +\infty]$ be a proper extended real-valued function with domain $\mathcal{D}(f) := \{x \in M : \ f(x) \neq +\infty\}$. The function f is said to be convex if for any geodesic γ in M, the composition function $f \circ \gamma : \mathbb{R} \to (-\infty, +\infty]$ is convex.

3. FIXED POINTS OF NONEXPANSIVE MAPPINGS

Let $F := \mathrm{Fix}(T)$ denote the set of all fixed points of a nonexpansive mapping $T : K \to K$, where K is a closed convex subset of M, and assume that $F \neq \emptyset$. We recall that from Brouwer's theorem the existence of fixed points is ensured provided that K is bounded. In order to solve the problem of approximating a fixed point of T, Kirk provided in [24] an implicit algorithm in general metric spaces called geodesic spaces which contains the class of Hadamard manifolds. Applying the general result due to Kirk to Hadamard manifolds, and writing the algorithm in terms of the exponential map, one obtain the following theorem.

Theorem 3.1. [24] *Let $u \in K$, and for each $t \in [0,1)$ let x_t be the unique point such that*

$$x_t = \exp_u t \exp_u^{-1} T(x_t).$$

Then $\lim_{t\to 1^-} x_t = \overline{x}$, the unique nearest point to x in F.

In an Euclidean space $\mathbb{R}^n$, this iterative scheme turns into $x_t = (1-t)x + tT(x_t)$, which coincides with the implicit Halpern-type algorithm called Browder's iteration, see [4]. An analogue of Halpern's algorithm (1.1) was developed for nonexpansive mappings on Hadamard manifolds, whose iterative scheme is

$$x_{n+1} = \exp_u(1-\alpha_n) \exp_u^{-1} T(x_n), \quad \forall n \geq 0, \tag{3.1}$$

where $x_0,\ u \in M$ and $\{\alpha_n\} \subset (0,1)$ and which coincides with the Halpern's one in the particular case of an Euclidean space.

Theorem 3.2. [27] *Let $u, x_0 \in M$. Suppose that $\{\alpha_n\} \in (0,1)$ satisfies*

(H1) $\lim_{n\to\infty} \alpha_n = 0$;
(H2) $\sum_{n\geq 0} \alpha_n = \infty$;
(H3) *either* $\sum_{n\geq 0} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty}(\alpha_n - \alpha_{n-1})/\alpha_n = 0$.

Then the sequence $\{x_n\}$ generated by the algorithm (3.1) converges to $P_F(u)$.

As regards Mann's iteration (1.2), an extension of this algorithm and its convergence results to the framework of metric spaces is due to Goebel-Kirk [23, 16] and Reich-Shafrir [45]. In spaces of *hyperbolic type*, which includes Hadamard manifolds, they provided an iterative scheme in terms of geodesic segments, coinciding with Mann's algorithm in Euclidean spaces. In particular, under the assumption that $\{\alpha_n\}$ is bounded away from 0 and 1, Reich and Shafrir proved the convergence of such iteration to a fixed point of T defined in the Hilbert ball. Goebel and Reich [17] studied the behavior of the sequence of the iterates $x_{n+1} = T(x_n)$ in hyperbolic metric spaces for a class of nonexpansive mappings. Recently, the authors of [27] proved the convergence of the algorithm

$$x_{n+1} = \exp_{x_n}(1-\alpha_n) \exp_{x_n}^{-1} T(x_n), \quad \forall n \geq 0, \tag{3.2}$$

in the setting of Hadamard manifolds.

Theorem 3.3. [27] *Suppose that $\{\alpha_n\} \subset (0,1)$ satisfy the condition*

$$\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty.$$

Let $x_0 \in M$ and let $\{x_n\}$ be the sequence generated by the algorithm (3.2). Then $\{x_n\}$ converges to a fixed point of T.

Note that the theorem above generalized the convergence result of the Mann's algorithm in the case of an Euclidean space. In order to illustrate the behavior of both Halpern's and Mann's iteration in Hadamard manifolds, a numerical example in a finite-dimensional hyperbolic space is presented in [27].

4. SINGULARITIES OF MONOTONE VECTOR FIELDS

The concepts of monotonicity and strict monotonicity of vector fields defined on a Riemannian manifold were introduced by Németh in [35]. In [10] the strong monotonicity was defined, and they established the relationship between different classes of vector fields' monotonicity and their differential operators' definiteness. The authors of [13] provided a class of monotone vector fields, those which are gradients of convex functions, and the complementary vector field of a nonexpansive mappings is proved to be monotone in [34]. For more examples and relations between different kinds of generalized monotone vector fields in Riemannian manifolds see [35, 36, 37].

The first appearance of the concept of monotone set-valued vector field is found in [9] where it is shown that the subdifferential operator of a Riemannian convex function is a monotone set-valued vector field. The notion of maximal monotonicity for set-valued vector fields was given in [26] for Hadamard manifolds. The definitions below can be rewritten in general Riemannian manifold in term of geodesics.

Definition 4.1. *Let $A \in \mathcal{X}(M)$. A is called*

(a) monotone *if for any $x, y \in \mathcal{D}(A)$ the following condition holds:*

$$\langle u, \exp_x^{-1} y\rangle \leq \langle v, -\exp_y^{-1} x\rangle, \quad \forall u \in A(x) \text{ and } \forall v \in A(y); \tag{4.1}$$

(b) strictly monotone *if* (4.1) *is satisfied with strict inequality for any $x, y \in \mathcal{D}(A)$;*

(c) strongly monotone *if there exists $\rho > 0$ such that, for any $x, y \in \mathcal{D}(A)$,*

$$\langle u, \exp_x^{-1} y\rangle - \langle v, -\exp_y^{-1} x\rangle \leq -\rho d^2(x, y), \quad \forall u \in A(x) \text{ and } \forall v \in A(y); \tag{4.2}$$

(d) maximal monotone *if it is monotone and, for any $x \in \mathcal{D}(A)$ and $u \in T_x M$, the condition that*

$$\langle u, \exp_x^{-1} y\rangle \leq \langle v, -\exp_y^{-1} x\rangle, \quad \forall y \in \mathcal{D}(A) \text{ and } v \in A(y) \tag{4.3}$$

implies that $u \in A(x)$.

With the aim of characterize the maximal monotone vector fields, the notions of upper semicontinuity and upper Kuratowski semicontinuity (cf. [49, p. 55]) as well as the local boundedness for operators in Banach spaces were extended to the setting of Hadamard manifolds in the following definition (see [26]).

Definition 4.2. *Let $A \in \mathcal{X}(M)$ and $x_0 \in \mathcal{D}(A)$. A is called*

(a) upper semicontinuous *at* x_0 *if for any open set* V *satisfying* $A(x_0) \subseteq V \subseteq T_{x_0}M$, *there exists an open neighborhood* $U(x_0)$ *of* x_0 *such that* $P_{x,x_0}A(x) \subseteq V$ *for any* $x \in U(x_0)$*;*

(b) upper Kuratowski semicontinuous *at* x_0 *if for any sequences* $\{x_k\} \subseteq \mathcal{D}(A)$ *and* $\{u_k\} \subset TM$ *with each* $u_k \in A(x_k)$*, the relations* $\lim_{k\to\infty} x_k = x_0$ *and* $\lim_{k\to\infty} u_k = u_0$ *imply* $u_0 \in A(x_0)$*;*

(c) locally bounded *at* x_0 *if there exists an open neighborhood* $U(x_0)$ *of* x_0 *such that the set* $\bigcup_{x\in U(x_0)} A(x)$ *is bounded.*

(d) *upper semicontinuous (resp. upper Kuratowski semicontinuous, locally bounded) on* M *if it is upper semicontinuous (resp. upper Kuratowski semicontinuous, locally bounded) at each* $x_0 \in \mathcal{D}(A)$*.*

Clearly, the upper semicontinuity implies the upper Kuratowski semicontinuity, and we can deduce that the converse is also true provided that A is locally bounded. The authors of [26] proved that the maximality implies the upper Kuratowski semicontinuity, and if the domain of the vector field is all the manifold then we get the extension of the well-known equivalence between the maximal monotonicity and the upper semicontinuity for a set-valued operator with closed and convex values in a Hilbert space (cf. [41]).

Theorem 4.3. [26] *Suppose that* $A \in \mathcal{X}(M)$ *is monotone and that* $\mathcal{D}(A) = M$*. Then the following statements are equivalent:*

(i) *A is maximal monotone.*

(ii) *A is upper semicontinuous on M and $A(x)$ is closed and convex for each $x \in M$.*

Recall that $x \in \mathcal{D}(A)$ is a *singularity* of A if $0 \in A(x)$. We denote the set of all singularities of A by $A^{-1}(0) := \{x \in \mathcal{D}(A) : 0 \in A(x)\}$.

Regarding the existence of singularities, it is a direct consequence of the definition that a strictly monotone vector field has at most one singularity. In [10, 13] it was proved that strongly monotone single-valued vector fields on Hadamard manifolds with $\mathcal{D}(A) = M$ have at least one singularity, that is, since the strong monotonicity implies the strictly monotonicity, existence and uniqueness are ensured. This result was extended to the set-valued case for maximal strong monotonicity, using the equivalence established in Theorem 4.3 and the coercivity condition of finite-dimensional Banach spaces.

When the domain of a monotone vector field A is open and convex, it is known that in Riemannian manifolds the set of singularities is convex, see [13]. Some topological and metric consequences of the existence of strictly monotone vector fields have been studied in both single-valued and set-valued case, involving the sectional curvature, dimension or volume of the Riemannian manifold, see [9, 10].

The study of approximating zeros of monotone operators within the framework of a Banach space has awakened the interest of many researchers. Motivated by the proximal point algorithm on $\mathbb{R}^m$ introduced and studied by Moreau [33], Martinet [29] and Rockafellar [47], and with the aim of generalize the iterative method given in [15] for approximating solutions of convex minimization problems, the following proximal point algorithm for set-valued vector fields on Hadamard manifolds was presented. Let $x_0 \in \mathcal{D}(A)$ and $\{\lambda_n\} \subset (0,1)$. Having x_n, define x_{n+1} such that

$$0 \in A(x_{n+1}) - \lambda_n \exp^{-1}_{x_{n+1}} x_n. \tag{4.4}$$

Note that this algorithm is implicit. But defining the vector field $B_n(x) := A(x) - \lambda_n \exp^{-1}_x x_n$ for each $x \in \mathcal{D}(A)$ and $n \geq 0$, if $\mathcal{D}(A) = M$ and A is maximal monotone, B_n is maximal strongly monotone and then the algorithm (4.4) is well-defined.

The convergence of the method was established in the following theorem.

Theorem 4.4. [26] *Let $A \in \mathcal{X}(M)$ be such that $A^{-1}(0) \neq \emptyset$. Suppose that A is upper Kuratowski semicontinuous and monotone. Let $\{\lambda_n\} \subset (0,1)$ satisfy*

$$\sup\{\lambda_n : n \geq 0\} < \infty. \tag{4.5}$$

Let $x_0 \in \mathcal{D}(A)$ and suppose that the sequence $\{x_n\}$ generated by the algorithm (4.4) *is well-defined. Then $\{x_n\}$ converges to a singularity of A.*

In the assumptions of the previous theorem, if we assume that $\mathcal{D}(A) = M$ and A is maximal monotone, we obtain as consequence that x_n generated by (4.4) is well-defined and converges to a singularity of A, which is the equivalence result for the proximal point algorithm in a Hilbert space (see [47]).

Another approach for finding zeros of monotone operators is the well-known extragradient algorithm proposed first by Korpelevich [25] and studied by many authors specially for solving variational inequalities (see, for instance, [18, 51]). An extragradient-type method for approximating singularities of continuous monotone vector fields defined on a constant curvature Hadamard manifold was presented and proved to converge by Ferreira et al [13]. They also provided an example of computation of such algorithm in a finite-dimensional hyperbolic space,

We remark that in the definition and convergence of both previous algorithms we need to impose restrictions on the manifold, namely nonpositive sectional curvature. It would be desirable to extend these methods to an arbitrary Riemannian manifold, but for the moment, we are unable to solve this problem because, in general, the exponential map cannot be defined in the whole tangent bundle and it is not invertible.

It is worth mentioning that the concept of monotonicity has been introduced in the setting of infinite-dimensional manifolds, together with other notions such as resolvent, Yosida approximation and accretive operators, and some convergence results for approximating the solutions of the equation

$$0 \in \frac{d}{dt}u(t) + Au(t), \quad t > 0,$$

in this context, have been proved (see [19] and references therein).

5. VARIATIONAL INEQUALITY AND MINIMIZATION PROBLEMS

In this section we consider a convex subset K of M and $V : K \to TM$ a single-valued vector field, that is, $V(x) \in T_xM$ for each $x \in K$. The problem of finding $x \in K$ such that

$$\langle V(x), \exp_x^{-1} y\rangle \geq 0, \quad \forall y \in K, \tag{5.1}$$

is called a *variational inequality* on K. The first step in analyzing this problem on Hadamard manifolds was taken by Németh [38, 13], establishing some existence and uniqueness theorems and studying the properties of the solution set of a variational inequality. He proved the existence of solutions of the variational inequality (5.1) for continuous vector fields defined on a compact convex subset of M, by using an extension of the well-known Brouwer fixed point Theorem to Hadamard manifolds (see [38]), whose unclear proof was completed in [26]. This fact could be deduced from more general results involving some abstract concepts as the Lefschetz number on acyclic spaces (see [3]), and the NPC spaces (see [52]).

Lemma 5.1. [38] *Let K be a compact convex subset of M. Let $F : K \to K$ be a continuous map. Then F has a fixed point in K.*

The following iterative approach to solve variational inequalities, deduced from the proximal point algorithm (4.4), was developed in [26]. Clearly, a point $x \in K$ is a solution of the variational inequality (5.1) if and only if x satisfies that

$$0 \in V(x) + N_K(x),$$

that is, x is a singularity of the set-valued vector field $A := V + N_K$, where $N_K(x)$ denote the normal cone of the set K at $x \in K$:

$$N_K(x) := \{u \in T_xM : \langle u, \exp_x^{-1} y\rangle \leq 0, \quad \forall y \in K\}.$$

Applying the algorithm (4.4) to A, we get the following proximal point algorithm with initial point x_0 for finding solutions of the variational inequality (5.1):

$$0 \in V(x_{n+1}) + N_K(x_{n+1}) - \lambda_n \exp_{x_{n+1}}^{-1} x_n, \quad \forall n \geq 0. \tag{5.2}$$

By the characterization of maximal monotone vector fields (Theorem 4.3), the convergence of the proximal point algorithm (Theorem 4.4) and Lemma 5.1 we obtain the well behavior of the algorithm (5.2).

Theorem 5.2. [26] *Let K be a closed convex subset of M and $V : K \to TM$ a single-valued continuous monotone vector field. Let $x_0 \in K$ and $\{\lambda_n\} \subset (0,1)$ satisfy* (4.5)*. Then, the sequence $\{x_n\}$ generated by the algorithm* (5.2) *is well-defined and converges to a solution of the variational inequality* (5.1)*, whenever it exists.*

As far as we know, it is still an open problem the formulation of an extragradient-type algorithm to solve variational inequalities as defined in (5.1).

Let $f : M \to (-\infty, +\infty]$ be a convex function. The minimization problem

$$\min_{x \in M} f(x) \tag{5.3}$$

has been the subject of various works from different approaches. Iterative and numerical methods were developed by Udriste in [53], where a general descent algorithm and the steepest descent method (or so-called gradient method) were extended to the framework of Riemannian manifolds. The latter one, involving the gradient function of the objective function f, was improved by introduction of a proximal regularization, see [8], allowing to solve constrained non-convex problem in $\mathbb{R}^n$ when the constrain set is a Hadamard manifold. Jost [20] presented an iterative procedure using the Moreau-Yosida approximation for finding minimizers in a certain class of Riemannian manifolds, and he gathered some specific minimization problems such as the center of mass. The classical subgradient algorithm for nondifferentiable minimization problems was extended to Hadamard manifolds by Ferreira and Oliveira, see [14]. We also find the use of the subdifferential vector field of the function f,

$$\partial f(x) = \{u \in T_x M : \langle u, \exp_x^{-1} y\rangle \leq f(y) - f(x),\ \forall y \in M\},$$

in another of their works, see [15], where they prove the convergence of the proximal point algorithm,

$$0 \in \partial f(x_{n+1}) - \lambda_n \exp_{x_{n+1}}^{-1} x_n, \quad \forall n \geq 0, \tag{5.4}$$

in the case when f is a real-valued convex function on M.

Theorem 5.3. [15] *Let $\{x_n\}$ be the sequence generated by* (5.4)*. If the sequence $\{\lambda_n\}$ satisfies $\sum_{n\geq 0} 1/\lambda_n = +\infty$, then $\lim_{n\to\infty} f(x_n) = \inf_{x\in M} f(x)$. In addition, if the minimum of f is attained, then $\lim_{n\to\infty} x_n = x$, and x is a minimizer of f.*

This algorithm can be rescued from the general proximal point algorithm (4.4) when f is lower semicontinuous because it is known that $x \in M$ is a solution of (5.3) if and only if x is a singularity of the subdifferential ∂f, and it was proved that the subdifferential of f in that case is maximal monotone.

Theorem 5.4. [26] *Let f be a proper lower semicontinuous convex function on M. Then the subdifferential ∂f is a monotone and upper Kuratowski semicontinuous multivalued vector field. Furthermore, if in addition $\mathcal{D}(f) = M$, then the subdifferential ∂f is maximal monotone.*

Hence the convergence is consequence of Theorem 4.4. Following the same reasoning we can obtain an algorithm for approximating solutions of a constrained minimization problem in a Hadamard manifold.

If we consider the following optimization problem with constrains,

$$\min_{x\in K} f(x), \tag{5.5}$$

with $f : M \to \mathbb{R}$ a convex function and K a closed and convex subset of M, this one can be written as the minimization problem (5.3) with f replaced by $f_K := f + \delta_K$, where δ_K is the indicate function defined by $\delta_K(x) = 0$ if $x \in K$ and $\delta_K(x) = +\infty$ otherwise. Since $\partial f_K(x) = \partial f(x) + N_K(x)$, $\forall x \in K$, (see [26]) the following algorithm,

$$0 \in \partial f(x_{n+1}) + N_K(x_{n+1}) - \lambda_n \exp^{-1}_{x_{n+1}} x_n, \quad \forall n \geq 0, \tag{5.6}$$

converges to a solution of the minimization problem (5.5).

Theorem 5.5. [26] *Let $f : M \to \mathbb{R}$ be a convex function and K be a closed convex set of M such that the solution set of the optimization problem* (5.5) *is nonempty. Let $x_0 \in M$ and $\{\lambda_n\}$ satisfy* (4.5). *Then, the sequence $\{x_n\}$ generated by the algorithm* (5.6) *is well-defined and converges to a solution of the optimization problem* (5.5).

As was proved by Rockafellar ([47, 46]) in the setting of Hilbert spaces, the proximal point algorithm in Hadamard manifolds is also capable of computing a saddle-point $\bar{z} = (\bar{x}, \bar{y}) \in M_1 \times M_2$ of the minimax problem:

$$L(x, \bar{y}) \leq L(\bar{x}, \bar{y}) \leq L(\bar{x}, y), \quad \forall z = (x, y) \in M_1 \times M_2, \tag{5.7}$$

where M_1 and M_2 are Hadamard manifolds, and the function $L : M_1 \times M_2 \to \mathbb{R}$ is a saddle-function, that is, $L(x, .)$ is convex on M_2 for each $x \in M_1$ and $L(., y)$ is concave, (i.e $-L(., y)$ is convex) on M_1 for each $y \in M_2$. Defining an associated maximal monotone vector field whose singularities are the saddle-points of L, the proximal point algorithm yields an iterative method for approximating solutions of the minimax problem (5.7), see [26].

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