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Asymptotic centers and fixed points for multivalued nonexpansive mappings

Dedicated to W.A. Kirk on the occasion of
His Honorary Doctorate of
Maria Curie-Skłodowska University

Abstract. Let X be a nearly uniformly convex Banach space, C a convex closed bounded subset of X and $T : C \to 2^C$ a multivalued nonexpansive mapping with convex compact values. We prove that T has a fixed point. This result improves former results in [4] and solves an open problem appearing in [17].

1. Introduction. In 1969 Nadler [15] extended the Banach Contraction Principle to multivalued contractive mappings in complete metric spaces. Namely, he proved: Let X be a complete metric space and $T : X \to 2^X$ a contraction with closed bounded values. Then T has a fixed point. Since then, many authors have studied the possibility of extending classical fixed point theorems for single-valued nonexpansive mappings to the setting of multivalued nonexpansive mappings. Even though several authors have

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obtained fixed point results for Banach spaces satisfying some strong geometric restriction (see [14], [2], [12] and the survey [17]) many problems remain open in this theory. For instance the following very general problem is still open [16]: Let X be a Banach space satisfying the fixed point property (FPP), i.e. every nonexpansive single valued mapping defined from a convex bounded closed subset of X into itself has a fixed point. Does X satisfy the same property for multivalued nonexpansive mappings with closed bounded values? The answer to this question could be strongly connected with the problem of obtaining a nonexpansive selection for any nonexpansive multivalued mapping. In spite of the well known Michael selection theorem which gives a continuous selection for multivalued upper semicontinuous mappings, almost nothing is known about obtaining a nonexpansive selection. A positive result in this direction was obtained by W.A. Kirk, M.A. Khamsi and C. Martínez Yañez [6] for a class of nonexpansive multivalued mappings in hyperconvex metric spaces. However these problems are too general and we cannot expect a positive answer for them. Thus, it seems to be more convenient to study particular problems. For instance, the celebrated Kirk's theorem [7] which states the FPP for reflexive Banach spaces with normal structure yields to a very natural question: Do reflexive Banach spaces with normal structure have the FPP for multivalued nonexpansive mappings ? The answer is unknown, either. Since normal structure is implied by different geometrical properties of Banach spaces, it is natural to consider the following problem: Do these properties imply the FPP for multivalued mappings? Let us list some of the properties implying reflexivity and normal structure:

(1) X is uniformly convex (UC),

(2) X is nearly uniformly convex (NUC),

(3) $\varepsilon_0(X) < 1$ where $\varepsilon_0(X)$ is the characteristic of convexity,

(4) $\varepsilon_\alpha(X) < 1$ where $\varepsilon_\alpha(X)$ is the characteristic of noncompact convexity for the Kuratowski measure of noncompactness,

(5) $\varepsilon_\beta(X) < 1$ where $\varepsilon_\beta(X)$ is the characteristic of noncompact convexity for the separation measure of noncompactness.

Furthermore, we have the following relationships between these notions:

$$\begin{array}{ccc} \text{UC} & \Rightarrow & \text{NUC} \\ \Downarrow & & \Downarrow \\ \varepsilon_0(X) < 1 & \Rightarrow \varepsilon_\alpha(X) < 1 \Rightarrow & \varepsilon_\beta(X) < 1 \end{array}$$

Hence the following question arises: Does any of the above properties imply the FPP for multivalued nonexpansive mappings? Of course, these questions are scaled. A positive answer for the case $\varepsilon_\beta(X) < 1$ solves all cases.

T.C. Lim [13] obtained a fixed point theorem for a multivalued nonexpansive self-mapping in a uniformly convex Banach space. W.A. Kirk [8] gave an extension of Lim's theorem proving the existence of a fixed point in a Banach space for which the asymptotic center of a bounded sequence in a closed bounded convex subset is nonempty and compact (note that the asymptotic center is a singleton in UC spaces). First, he proved the following result.

Theorem 1.1. *Let C be a nonempty weakly compact and separable subset of a Banach space X, $T : C \to 2^C$ a nonexpansive mapping with compact values and $\{x_n\}$ a sequence in C such that $\lim_n d(x_n, Tx_n) = 0$. Then, there exists a subsequence $\{z_n\}$ of $\{x_n\}$ such that*

$$Tx \cap A \neq \emptyset, \quad \forall x \in A := A(C, \{z_n\}).$$

Noting that we can reduce to a separable setting and using a fixed point theorem for compact operators he proved that Banach spaces such that asymptotic centers of bounded sequences are compact enjoy the FPP for multivalued nonexpansive mappings with convex bounded closed values. However, the asymptotic center of bounded sequences in NUC spaces can be noncompact [11]. Thus, Kirk's result does not solve the other questions arisen above. Using some inequalities relating characteristic of noncompact convexity, Chebyshev centers and asymptotic centers, the following partial extension of Kirk's result is obtained in [4].

Theorem 1.2. *Let X be a Banach space such that $\varepsilon_\beta(X) < 1$ and X satisfies the nonstrict Opial condition, C a convex bounded subset of X and $T : C \to 2^C$ a nonexpansive mapping with convex compact values. Then T has a fixed point.*

An open question in [4] is the possibility of removing the nonstrict Opial property from the assumptions. We will prove in this paper that the above result holds without any assumption on Opial conditions.

2. Notation. Let us fix the notation which will be used. Let C be a nonempty bounded closed subset of a Banach space X and $\{x_n\}$ a bounded sequence in X, we use $r(C, \{x_n\})$ and $A(C, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in C, respectively, i.e.

$$r(C, \{x_n\}) = \inf \left\{ \limsup_n \|x_n - x\| : x \in C \right\},$$

$$A(C, \{x_n\}) = \left\{ x \in C : \limsup_n \|x_n - x\| = r(C, \{x_n\}) \right\}.$$

It is known that $A(C, \{x_n\})$ is a nonempty weakly compact convex set as C is.

If D is a bounded subset of X, the Chebyshev radius of D relative to C is defined by

$$r_C(D) := \inf\left\{\sup\{\|x-y\| : y \in D\} : x \in C\right\}.$$

Let X be a Banach space. We denote by $CB(X)$ the family of all nonempty closed bounded subsets of X and by $K(X)$ (resp. $KC(X)$) the family of all nonempty compact (resp. compact convex) subsets of X. On $CB(X)$ we have the Hausdorff metric H given by

$$H(A,B) := \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}, \quad A,B \in CB(X)$$

where for $x \in X$ and $E \subset X$, $d(x,E) := \inf\{\|x-y\| : y \in E\}$ is the distance from the point x to the subset E.

If C is a closed convex subset of X and $k \in [0,1)$, then a multivalued mapping $T : C \to CB(X)$ is said to be k-contractive if

$$H(Tx,Ty) \le k\|x-y\|, \quad x,y \in C,$$

and T is said to be nonexpansive if

$$H(Tx,Ty) \le \|x-y\|, \quad x,y \in C.$$

A multivalued mapping $T : C \to 2^X$ is called k-ϕ-contractive where ϕ is a measure of noncompactness if, for each bounded subset B of C, we have

$$\phi(T(B)) \le k\phi(B).$$

Let us recall the definition of a nearly uniformly convex space.

Definition 2.1. X is said to be nearly uniformly convex (NUC) if it is reflexive and its norm is uniformly Kadec-Klee, that is, for any positive number ϵ there exists a corresponding number $\delta = \delta(\epsilon) > 0$ such that for any sequence $\{x_n\}$

$$\left.\begin{array}{c} \|x_n\| \le 1 \quad n = 1,2,\dots \\ w\text{-}\lim\limits_n x_n = x \\ \operatorname{sep}(\{x_n\}) := \inf\{\|x_n - x_m\| : n \neq m\} \ge \epsilon \end{array}\right\} \Longrightarrow \|x\| \le 1-\delta.$$

Assume that C is a subset of a Banach space X. Looking at C as a metric space we can consider the Hausdorff measure of noncompactness χ_C defined for any bounded subset A of C by $\chi_C(A) = \inf\{\epsilon > 0 : A$ can be covered by finitely many balls centered at points in C with radii less than $\epsilon\}$. It must be noted that this measure depends on C and it is, in general, different from $\chi := \chi_X$. Furthermore, if C is a convex closed set, it is easy to check that the usual arguments to prove $\chi_X(A) = \chi_X(\overline{\mathrm{co}}(A))$ (see, for instance [1, Theorem 2.4]) equally well apply to prove $\chi_C(A) = \chi_C(\overline{\mathrm{co}}(A))$ for any bounded subset $A \subset C$. Furthermore, if C is separable, for any bounded subset A of C there exists $B \subset A$ such that $\chi_C(B) = \chi_C(A)$ and B is χ_C-minimal, i.e. $\chi_C(B) = \chi_C(D)$ for any infinite subset D of B (for definition and properties of χ_C-minimal sets, see [10], Chapter 8). Apart from χ and χ_C, we shall consider in this paper the separation measure of noncompactness defined by

$$\beta(B) = \sup\{\epsilon\colon \text{there exists a sequence } \{x_n\} \text{ in } B \text{ such that } \operatorname{sep}(\{x_n\}) \geq \epsilon\}$$

for any bounded subset B of a Banach space X.

Definition 2.2. Let X be a Banach space. The modulus of noncompact convexity associated to β is defined in the following way:

$$\Delta_{X,\beta}(\epsilon) = \inf\{1 - d(0, A) : A \subset B_X \text{ is convex, } \beta(A) \geq \epsilon\}$$

(B_X is the unit ball of X).

The characteristic of noncompact convexity of X associated with the measure of noncompactness β is defined by

$$\epsilon_\beta(X) = \sup\{\epsilon \geq 0 : \Delta_{X,\beta}(\epsilon) = 0\}.$$

When X is a reflexive Banach space we have the following alternative expression for the modulus of noncompact convexity associated with β,

$$\Delta_{X,\beta}(\epsilon) = \inf\{1 - \|x\| : \{x_n\} \subset B_X,\ x = w\text{-}\lim_n x_n,\ \operatorname{sep}(\{x_n\}) \geq \epsilon\}.$$

It is known that X is NUC if and only if $\epsilon_\beta(X) = 0$. The above-mentioned definitions and properties can be found in [1].

3. Fixed point results. In the sequel we are going to use the following result.

Theorem 3.1 ([3, Lemma 11.5])**.** *Let X be a Banach space and $\emptyset \neq D \subset X$ be compact convex. Let $F : D \to 2^X$ be upper semicontinuous with closed convex values. If $Fx \cap \overline{I_D(x)} \neq \emptyset$ on D then F has a fixed point. (Here $I_D(x)$ is called the inward set at x defined by $I_D(x) := \{x + \lambda(y - x) : \lambda \geq 0,\ y \in D\}$.)*

The following lemma is the key of this paper, stating a relationship between k-contractive and k-χ_C-contractive mappings.

Theorem 3.2. *Let C be a weakly compact convex separable subset of a Banach space X. Assume that $T : C \to 2^C$ is a multivalued k-contractive mapping with compact values. Then, T is k-χ_C-condensing.*

Proof. Let A be a bounded subset of C. Since C is separable there exists a χ_C-minimal subset $B \subset T(A)$ such that $\chi_C(B) = \chi_C(T(A))$. We can assume that B is countable, i.e. $B = \{y_n : n \in \mathbb{N}\}$. Since C is separable, taking a subsequence, we can assume that $\lim \|y_n - x\|$ exists for any $x \in C$. Then $\chi_C(\{y_n : n \in N\}) = r(C, \{y_n\})$. Indeed, denote $h = \chi_C(\{y_n : n \in \mathbb{N}\})$. For any $\epsilon > 0$ there exist $a_1, \ldots, a_N \in C$ such that $\{y_n : n \in \mathbb{N}\} \subset \bigcup_{i=1}^N B(a_i, h + \epsilon)$ where $B(a, r)$ denotes the open ball centered at a with radius r. Thus, for a subsequence $\{z_n\}$ of $\{y_n\}$ and some $i \in \{1, \ldots, N\}$ we have $\limsup_n \|z_n - a_i\| = \lim_n \|y_n - a_i\| \leq h + \epsilon$. Hence $r(C, \{y_n\}) \leq h + \epsilon$ and, since ϵ is arbitrary, we have $r(C, \{y_n\}) \leq h$. On the other hand, denote $r = r(C, \{y_n\})$. We know that the asymptotic center $A(C, \{y_n\})$ is nonempty. Take $a \in A(C, \{y_n\})$ and $\epsilon > 0$. There exists n_0 such that $\|y_n - a\| < r + \epsilon$ for $n > n_0$. Thus $\chi_C(\{y_n : n \in \mathbb{N}\}) = \chi_C(\{y_n : n > n_0\}) \leq r + \epsilon$ and we obtain the opposite inequality.

Choose $x_n \in A$ such that $y_n \in Tx_n$. Taking again a subsequence we can assume that the set $\{x_n : n \in \mathbb{N}\}$ is χ_C-minimal and $\lim_n \|x_n - x\|$ exists for every $x \in C$. The same argument as above proves that $\chi_C(\{x_n : n \in \mathbb{N}\}) = r(C, \{x_n\})$. Let $u \in A(C, \{x_n\})$, i.e. $\lim_n \|x_n - u\| = \chi_C(\{x_n : n \in \mathbb{N}\})$. Since T is compact valued, we can take $u_n \in Tu$ such that $\|y_n - u_n\| = d(y_n, Tu)$. Using the compactness of Tu and taking again a subsequence we can assume that $\{u_n\}$ converges strongly to a point $v \in Tu$. Hence, we have

$$\begin{aligned}\chi_C(T(A)) = r(C, \{y_n\}) &\leq \lim_n \|y_n - v\| = \limsup_n \|y_n - u_n\| \\ &= \limsup_n d(y_n, Tu) \leq \limsup_n H(Tx_n, Tu) \\ &\leq k \lim_n \|x_n - u\| = k\chi_C(\{x_n : n \in \mathbb{N}\}) \\ &\leq k\chi_C(A).\end{aligned}$$

□

Theorem 3.3. *Let C be a weakly compact convex separable subset of a Banach space X. Assume that $T : C \to 2^C$ is a multivalued k-contractive ($k < 1$) mapping with convex compact values. Assume that A is a convex closed subset of C such that $Tx \cap A \neq \emptyset$ for every $x \in A$. Then, T has a fixed point in A.*

Proof. According to Theorem 3.2, the mapping T is k-χ_C-condensing. We could check that the arguments in the proof of [3, Theorem 11.5] equally well work for the measure χ_C when we assume that $T(C) \subset C$. However, we are going to prove that Theorem 3.1 can be directly applied to obtain the fixed point. To do that, we follow an induction argument. Denote $A_1 = A$ and assume that we have defined a finite decreasing sequence of convex closed sets $A_n \subset A_{n-1} \subset \ldots \subset A_1$ such that $Tx \cap A_k \neq \emptyset$ for every $x \in A_k$ and for all $k = 1, \ldots, n$. Define $A_{n+1} = [\overline{\mathrm{co}}\, T(A_n)] \cap A_n$. Then, A_{n+1} is a closed convex subset of A_n. Furthermore, for every $x \in A_{n+1}$ we have $Tx \cap A_n \neq \emptyset$. Since $Tx \subset T(A_n)$ we obtain that $Tx \cap A_{n+1}$ is nonempty. Furthermore

$$\chi_C(A_{n+1}) = \chi_C([\overline{\text{co}}\, T(A_n)] \cap A_n) \leq \chi_C(T(A_n)) \leq k\chi_C(A_n).$$

Then, $A_\infty := \bigcap_{n=1}^{\infty} A_n$ is a nonempty compact convex subset of A. Let $x \in A_\infty$ and take $a_n \in Tx \cap A_n$ which is nonempty. The sequence $\{a_n\}$, which lies in a weakly compact set, has some cluster points for the weak topology. Assume that a is a weak cluster point. Then $a \in A_\infty$. Since Tx is weakly closed and the sequence $\{a_n\}$ lies in Tx we have that $a \in Tx$ and $a \in A_\infty$ which implies that $Tx \cap A_\infty \neq \emptyset$. Since A_∞ is compact we obtain from Theorem 3.1 that T has a fixed point in $A_\infty \subset A$. □

Next, we present a theorem which gives a connection between the asymptotic center of a sequence and the modulus of noncompact convexity.

Theorem 3.4 ([4, Theorem 3.1]). *Let C be a closed convex separable subset of a reflexive Banach space X and let $\{y_n\}$ be a bounded sequence in C. Then, there exists a subsequence $\{x_n\}$ such that*

$$r_C(A(C, \{x_n\})) \leq (1 - \Delta_{X,\beta}(1^-))r(C, \{x_n\}).$$

We can prove now that all conditions (1)–(5), assuring normal structure, in the introduction, imply the FPP for multivalued nonexpansive mappings with compact convex values. In particular, NUC spaces enjoy this property solving an open problem appearing in [17].

Theorem 3.5. *Let C be a nonempty closed bounded convex subset of a Banach space X such that $\epsilon_\beta(X) < 1$, and $T : C \to KC(C)$ be a nonexpansive mapping. Then T has a fixed point.*

Proof. From [11] we can assume that C is separable. We claim that for any bounded closed subset A of C such that $Tx \cap A \neq \emptyset$ for every $x \in A$, there exists an approximated fixed point sequence of T in A, i.e. there exists $\{x_n\} \subset A$ such that $d(x_n, Tx_n) \to 0$. Indeed, let $x_0 \in A$ be fixed and, for each $n \geq 1$, define

$$T_n x := \frac{1}{n}x_0 + \left(1 - \frac{1}{n}\right)Tx, \quad x \in C.$$

Then, T_n is $(1 - 1/n)$-contractive and from Theorem 3.3 has a fixed point x_n. It is easily seen that $\lim_n d(x_n, Tx_n) = 0$. Using this fact, we can follow the proof as in [4, Theorem 4.1]. To do that we consider $A_1 = C$. Assume that sets $A_1, \dots, A_m$ and approximated fixed point sequences $\{x_n^k\} \subset A_k$ are constructed where $A_k = A(C, \{x_n^{k-1}\})$ and $r_C(A_k) \leq (1-\Delta_{X,\beta}(1^-))^k r_C(A_1)$ for $k = 2, \dots, m$. Defining $A_{m+1} = A(C, \{x_n^m\})$ and choosing a suitable approximated fixed point sequence $\{x_n^{m+1}\}$ in A_{m+1} we obtain $r_C(A_{m+1}) \leq$

$(1-\Delta_{X,\beta}(1^-))r_C(A_m)$ and we can continue the induction process. As in [4, Theorem 4.1] it can be proved that the diagonal sequence $\{x_n^n\}$ converges to a fixed point of T. $\square$

Remark. According to [13], every uniformly convex space has the FPP for multivalued nonexpansive mappings with compact values. In Theorem 3.5, we need to assume, in addition, that T has convex values. We do not know if this assumption can be removed, but our method in the proof does not work without convexity. Indeed, the main tool in our proof is to obtain an approximated fixed point sequence for T in a set A such that $Tx \cap A \neq \emptyset$. To do that, we cannot use fixed point results for contractive mappings, because these results do not hold for mappings which are not self-mappings. The following example illustrates this fact: Assume $X = \mathbb{R}$, $A = [0,1]$ and define $T : A \to 2^X$ by

$$Tx = [-1, 2] \setminus \left(\frac{3x}{4} - \frac{1}{2}, \frac{3x}{4} + \frac{1}{2}\right).$$

Then, T is 3/4-contractive and satisfies $Tx \cap A \neq \emptyset$ for every $x \in A$. However, T is fixed point free. Thus, in the proof of Theorem 3.5 we need to use a fixed point result for compact mappings. On the other hand, we cannot expect such a result without convexity assumptions. Indeed, consider the following easy example. Assume that $X = \mathbb{R}^2$ and D is the closed unit disk. Define $T(0) = \partial D$ and $T(x) = \partial D \setminus B(x/\|x\|, \|x\|)$ for $x \neq 0$. Then T is a continuous and fixed point free mapping.

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