**UNIVERSIDAD DE SEVILLA**

Facultad de Matemáticas

Dpto. de Ecuaciones Diferenciales y Análisis Numérico

# DISEÑO ÓPTIMO DINÁMICO EN CONDUCTIVIDAD CON COSTES DEPENDIENDO EN DERIVADAS

Tesis Doctoral

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Tesis presentada por
Faustino Maestre Caballero
para optar al grado de
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Fdo.: Pablo Pedregal Tercero
Director de Tesis

Fdo.: Faustino Maestre Caballero

Adquiere sabiduría, adquiere inteligencia;
No te olvides ni te apartes de las razones de mi boca;
No la abandones, y ella cuidará de ti;
Ámala, y ella te protegerá.
Sabiduría ante todo; adquiere sabiduría;
Y sobre todas tus posesiones adquiere inteligencia.
Honralá, y ella te engrandecerá;
Ella te honrará, cuando tu la hayas abrazado.
Adorno de gracia dará a tu cabeza;
Corona de hermosura te entregará.

Prov. 4,5-9.

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# Índice general

# Capítulo 1

# Diseño óptimo dinámico

## 1.1 Motivación y planteamiento del problema

A lo largo del tiempo la humanidad se ha planteado muchos problemas de optimización de diferente forma y naturaleza. Desde encontrar la distancia mas corta entro dos puntos hasta encontrar el mejor diseño aerodinámico de una nave aeroespacial. Los problemas de diseño óptimo están dirigidos a encontrar minimizadores de ciertas propiedades sujetas a algunas restricciones o características que tienen que verificar los minimizadores. Estas características pueden ser de diferente naturaleza y son las que hacen interesante a los problemas de diseño óptimo. Algunas de estas propiedades están relacionadas con la rigidez, grosor, forma, tamaño, y por último pero no menos, el aspecto estético. En este sentido, dependiendo de la propiedad que tenemos que minimizar podemos clasificar los problemas de diseño óptimo, de forma amplia, como:

1. *Problemas de optimización de forma*, que consisten en encontrar la forma óptima de una región la cual, por ejemplo, maximiza la conductividad o rigidez del cuerpo. Estos problemas pueden ser "interpretados" como problemas de optimización de dos fases, en el sentido que los hoyos en el dominio pueden ser rellenados por un material muy débil, que tiene propiedad nula o casi nula.

2. *Problemas de optimización estructural*, que consisten en diseñar los mejores materiales o estructuras de todos los posibles diseños verificando ciertas propiedades. En este tipo de problemas de diseño óptimo son usuales los materiales compuestos con microestructuras.

Atendiendo a la clasificación anterior, podemos distinguir dos grandes campos de trabajo de los problemas de diseño ópitmo, de forma que el primer campo es a nivel macroscópico, considerando propiedades macroscópicas del cuerpo como el peso o la geometría de la frontera; y desde el otro punto de vista los problemas son analizados desde un nivel microscópico, pues el objetivo es deducir propiedades macroscópicas o efectivas de los materiales compuestos relacionados con la microestructura.

En la literatura las restricciones que tienen que verificar los minimizadores suelen ser independientes de la variable temporal. En este sentido, una novedad en este trabajo es que nosotros analizamos problemas que dependen del tiempo, en el cual las propiedades a minimizar y las restricciones dependen del tiempo además de la posición, y por lo tanto buscamos soluciones no estacionarias, soluciones que dependen explícitamente de la variable temporal.

En este trabajo analizamos problemas de diseño óptimo de dos fases, los cuales de forma general consisten en, dado la cantidad de dos materiales, encontrar la mejor distribución (en espacio y tiempo) de estos materiales que tienen diferentes propiedades, de manera que minimice (o maximice) un objetivo propuesto. En ausencia de regularidad o hipótesis topológicas es usual la falta de soluciones clásicas (soluciones donde los dos materiales son distribuidos de forma separada) en problemas de optimización estructural, pues la distribución óptima de los materiales tiene lugar con finas inclusiones de un material en otro, de manera que en el proceso de optimización estas inclusiones son cada vez mas finas, hasta el punto que se pierde la presencia de los dos materiales por separado. Por tanto, es natural la presencia de materiales compuestos en este tipo de problemas. Entendemos por un material compuesto, como un material estructural formado por dos o mas constituyentes combinados a nivel macroscópico y que no son solubles el uno en el otro.

El material compuesto más útil que utilizaremos son los materiales laminados. Formalmente, un laminado es un material formado por un grupo de capas unas junta a otras, el cual está determinado por la dirección de laminación y la fracción de volumen de cada material. En este sentido, cuando no existen soluciones clásicas, el siguiente paso consiste en introducir diseños generalizados, i.e., agrandamos apropiadamente el conjunto de diseños admisibles para probar la existencia de solución óptima en el nuevo conjunto. Este proceso es conocido como *relajación*. Durante las últimas décadas, una herramienta importante en relajación ha sido el método de Homogeneización, donde se han introducido nuevos materiales compuestos gracias a los conceptos de $H-$convergencia y $G-$convergencia. Este método es especialmente útil cuando el coste a minimizar no depende de gradientes o cuando esta presencia es asociada al concepto de $G-$convergencia. Otro importante método, el cual será la principal herramienta en esta tesis está basado en el uso de las medidas de Young como diseños generalizados. Esta herramienta nos permitirá analizar problemas de diseño óptimo más generales.

Una gran clase de problemas de optimización se puede encajar bajo la formulación estándar del Cálculo de Variaciones. El problema consiste en minimizar un funcional integral del tipo

$$\int_{\Omega} f(x, u(x), \nabla u(x)) dx$$

sujeto a algunas condiciones frontera (o iniciales) junto a otras restricciones sobre $u(x)$ o $\nabla u(x)$. Algunas de estas restricciones están escritas en forma integral, de forma que tenemos que asegurar que cierta integral es igual o es acotada por un valor conocido (restricción de volumen). Otras restricciones usuales que aparecen in estos problemas están relacionadas con leyes descritas por ecuaciones diferenciales ordinarias o ecuaciones en derivadas parciales, las cuales introducen un caracter no local al problema. En particular, estamos interesados en minimizar funcionales sujetos a una ecuación de estado de la forma

$$-\mathrm{div}(A\nabla u(x)) = g$$

donde $A$ es un operador lineal. En este marco de trabajo, podemos hablar de problemas de control óptimo gobernados por ecuaciones diferenciales. En estos problemas, la ecuación de estado es la ley que gobierna el sistema, y el control es el mecanismo que cambia el comportamiento natural del mismo. Todas las posibilidades que podemos usar para actuar sobre el sistema, forman el conjunto de controles admisibles. Otra clase más general de problemas de diseño es la clase de problemas en la que la función coste depende del control y de los estado(s) asociado(s) al control también.

En este trabajo, estamos interesados en problemas donde el control actúa en el termino principal de la ecuación en derivadas parciales, i.e., en los coeficientes del término de mayor orden. Este tipo de problemas de control son conocidos como problemas de diseño óptimo. El objetivo de esta tesis es analizar el siguiente problema de diseño óptimo:

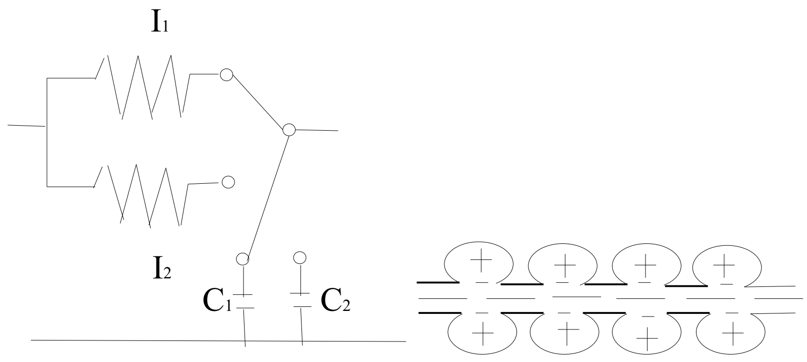



*Figura 1.1: Aplicaciones.*

Consideramos $\Omega \subset \mathbb{R}^2$, $T > 0$ y $0 < \alpha < \beta$, $V_\alpha \in (0,1)$ el problema consiste en minimizar

$$(P): \qquad I(\chi) = \int_0^T \int_\Omega \left[ u_t^2(t,x) + a(t,x,\chi) |\nabla_x u(t,x)|^2 \right] \, dx \, dt \quad (1.1)$$

donde $u$ es la única solución de

$$\begin{aligned} u_{tt} - div_x([\alpha\chi + \beta(1-\chi)]\nabla_x u) = 0 \quad &\text{en} \quad (0,T) \times \Omega, \\ u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x) \quad &\text{en} \quad \Omega, && (1.2) \\ u(t,0) = f(t), \;\; u(t,1) = g(t) \quad &\text{en} \quad [0,T], && (1.3) \end{aligned}$$

$$\int_\Omega \chi(t,x) dx \leq V_\alpha |\Omega|, \quad \forall t \in [0,T].$$

y las funciones $a, u_0, u_1, f$ y $g$ son conocidas. La función $\chi \in L^\infty([0,T] \times \Omega; \{0,1\})$ es la variable de diseño.

Nos gustaría enfatizar el carácter hiperbólico de la ecuación de estado. En la literatura, los problemas de diseño óptimo en conductividad son bastante conocidos en el caso elíptico. Sin embargo son mucho menos conocidos en el caso dinámico o hiperbólico. En la forma más general, los problemas elípticos consisten en determinar para cada punto del espacio, qué material debe ser puesto en ese lugar. En el caso hiperbólico esta decisión debe ser tomada tanto en la variable espacial como temporal, por lo que hablamos de compuestos dependientes del tiempo, los cuales son llamados también, materiales dinámicos.

Son muchas las posibles aplicaciones de este problema. Nos gustaría resaltar unas pocas.

*Figura 1.2: Puente de Tacoma.*

1. En ingeniería eléctrica, es usual el uso de sistemas distributivos, que generan y trasportan ondas a traves de una excitación paramétrica. Estos mecanismos son construidos usando materiales ferromagnéticos que permiten una activación externa de su inductancia y capacitancia (ver Figura 1.1).

2. En acústica, se construyen mecanismos en propagación de ondas a través de conductos con sección transversal variable. Estos conductos son rellenados con vapor de agua y, usando la concentración se cambia la velocidad de la onda (ver Figura 1.1).

3. En dinámica estructural, donde son bien conocidos los fenómenos de resonancia. Una adecuada distribución espacio-temporal de los materiales podría ser un buen mecanismo para evitar estos fenómenos catastróficos. Un famoso desastre debido a este fenómeno tuvo lugar en el puente de Tacoma (EE.UU.) en 1940 donde el soplo transversal del viento hizo vibrar el puente hasta destruirlo(Figure 1.2).

4. En materiales no isótropos, si consideramos la variable temporal $t$ como otra variable espacial, la ecuación de estado puede ser interpretada como una ley para materiales no isótropos

$$-\mathrm{div}(\chi A_\alpha + (1-\chi)A_\beta) = 0$$

   donde las matrices no isótropas $A_\gamma$ son de la forma

$$A_\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\gamma & 0 \\ 0 & 0 & -\gamma \end{pmatrix}, \gamma = \alpha, \beta.$$

## 1.2 Dificultades y enfoque variacional

Los problemas de diseño óptimo han sido estudiados de diferente forma a lo largo del tiempo. La teoría clásica de problemas variacionales sujetos a restricciones diferenciales ha analizado este tipo de problemas usando métodos de compacidad-compensada, ([16], [45], [46], [60], [61]). Esta teoría trata de encontrar condiciones necesarias y suficientes de semicontinuidad inferior de funcionales sin ningún tipo de restricción de convexidad. Un resultado típico de esta teoría es el conocido *Div-Curl Lemma*, el cual básicamente dice que *dadas dos sucesiones débilmente convergentes tal que los operadores div y curl son compactos respectivamente, entonces el producto de ambas sucesiones es débilmente convergente.*

Durante las dos o tres últimas décadas, otra teoría para estudiar problemas de diseño óptimo es introducida por la homogeneización a través de los conceptos de $H-$convergencia y $G-$convergencia ([1],[47],[48],[49]). Más recientemente se introdujo el concepto de $\Gamma-$convergencia ([18], [11]). Gracias a la teoría de la homogeneización se han obtenido resultados importantes, especialmente cuando el funcional coste no depende de las derivadas del estado. Es importante resaltar la dependencia explícita de gradientes del estado en nuestra función coste. Este motivo es la razón principal por la que usamos otra vía alternativa, estudiando a una formulación variacional y usando medidas parametrizadas (o de Young) como diseños generalizados ([52], [3]).

Nos gustaría hacer hincapié en las dificultades mas importantes que encontramos en este tipo de problemas de diseño óptimo.

- Los problemas de diseño óptimo bajo leyes elípticas son bastante conocidos, sin embargo lo son mucho menos en el caso hiperbólico. Con este trabajo nos gustaría entender el comportamiento de este tipo de problemas y las diferencias eventuales con respecto al caso elíptico.

- Este tipo de problemas tienen carácter no local, en el sentido que las variables de la función coste, el par $(u, \chi)$, están conectadas por una restricción no local como es la ecuación de estado (un ecuación en derivadas parciales). Esto significa que para todo punto $x \in \Omega$ el valor de la función estado $u(x)$ depende de todos los valores de $\chi(x')$ con $x' \in \Omega$.

- Por otra parte, cuando aplicamos el Método Directo del Cálculo de Variaciones para resolver estos problemas de diseño óptimo, se observa que las sucesiones minimizantes son altamente oscilantes entre las dos fases. Este fenómeno es debido a la falta de convexidad del problema, y está asociado con la falta de solución clásica y la presencia de microestructuras.

- Otros aspecto importante es la presencia explícita de gradientes del estado en la función coste. Este hecho es la razón principal por la que, salvo casos particulares, el método de la homogeneización no es válido para resolver estos problemas, pues la convergencia débil no se transmite por derivadas (ver [25],[12]).

### 1.2.1 Formulación variacional

Como comentamos anteriormente, la falta de solución clásica de estos problemas es bien conocida ([44]). En los párrafos anteriores hablamos de un camino para buscar soluciones generalizadas: la teoría de la homogeneización y los $H-$límites como diseños generalizados. La presencia de gradientes del estado en la función coste sugiere utilizar otra herramienta para analizar nuestro problema de diseño óptimo. La otra dificultad principal asociada con nuestro problema, la cual tenemos que superar, es el carácter no local de la ley de estado. La estrategia general para superar estas dificultades es introducir "potenciales" de manera que podamos evitar el carácter no local de la ecuación de estado, y transformar el problema de diseño óptimo dentro del formato del Cálculo de Variaciones. Con esta reformulación, evitamos el carácter no local, pero en cambio el nuevo problema reformulado es un problema variacional vectorial, el cual debe ser tratado apropiadamente.

Mostraremos esta reformulación con algún detalle. La idea básica recae sobre una identificación adecuada de los campos con divergencia nula. Esta identificación tiene una naturaleza diferente dependiendo de la dimensión. Este hecho hace que el problema tenga que ser tratado de diferente forma dependiendo de la dimensión. Básicamente usamos el hecho que, bajo alguna hipótesis de regularidad sobre $Q \subset \mathbb{R}^{n+1}$ con $n = 1, 2$, todos los campos $F : Q \to \mathbb{R}^{n+1}$ en $L^2(Q)$ tal que

$$\operatorname{div}(F) = 0 \text{ en } Q,$$

entonces existen $v_1, v_n : Q \to \mathbb{R}$ en $H^1(Q)$, tal que

$$F(z) = E_n(v_1(z), v_n(z))$$

donde la función $E_n$ está definida como

$$E_n(v_1(z), v_n(z)) = \begin{cases} R\nabla v_1(z), & \text{si } n = 1 \\ \nabla v_1(z) \times \nabla v_2(z), & \text{si } n = 2. \end{cases}$$

y $R$ es la rotación en el plano de ángulo $\pi/2$ con el sentido contrario de las agujas del reloj, $\times$ representa el producto vectorial de dos vectores, y las funciones $v_1, v_2$ son conocidas como potenciales de Clebsh ([31], [51], [65]).

Tenemos un problema de minimización donde la ecuación de estado está escrita en forma de divergencia

$$(ODP) \qquad \min I(\chi) = \int_Q g(\chi(x), \nabla u(x))dx$$

sujeto a,

$$-\operatorname{div}\,(A(\chi)\nabla u) = 0 \quad \text{en} \quad \Omega,$$

y algunas condiciones de frontera e iniciales adicionales, bajo una restricción de volumen del tipo

$$\int_Q \chi(z)dz \le V_\alpha |Q|$$

donde $A(\cdot)$ es una función lineal en $\chi$.

De esta forma, utilizando la identificación anterior, la ecuación de estado es equivalente a

$$A(\chi)\nabla u = E_n(v_1, v_n) \text{ e.c.t. } x \in Q. \tag{1.4}$$

Por lo tanto el problema reformulado consiste en remplazar la ecuación de estado no local introduciendo los nuevos potenciales y forzando la restricción puntual 1.4.

Ponemos $U = (u, v_1, v_n) \in \mathbb{R}^{n+1}$, con $\nabla U = (\nabla u, \nabla v_1, \nabla v_n) \in \mathbb{R}^{(n+1)\times(n+1)}$. Consideramos $\Lambda$ el conjunto de matrices donde se "verifica" la ecuación de estado

$$\Lambda = \left\{ \nabla U \in M^{(n+1)\times(n+1)} : A(\chi)\nabla u = E_n(v_1, v_n) \right\}$$

Como $\chi$ es una variable binaria que toma los valores $\{0, 1\}$, podemos descomponer $\Lambda$ como

$$\Lambda = \Lambda_\alpha \cup \Lambda_\beta$$

donde $\Lambda_\alpha$ son las matrices que pertenecen a $\Lambda$ con $\chi = 1$, y $\Lambda_\beta$ el complementario (cuando $\chi = 0$).

No es difícil comprobar que podemos identificar los pares $(u, \chi)$ con las ternas $(u, v_1, v_2)$, y viceversa. Por lo tanto consideraremos $U = (u, v_1, v_2)$ como nuestra nueva variable de diseño.

Por tanto de todo esto deducimos que el problema de diseño óptimo (ODP) puede ser escrito dentro del marco clásico del Cálculo de Variaciones. El problema reformulado consiste en minimizar

$$(VP) \quad \underset{U}{min} \quad \hat{I}\ (U) = \int_Q W(\nabla U(z))dz$$

sujeto a

$$U = (U^{(1)}, U^{(2)}, U^{(3)}) \in H^1(Q)^{n+1},$$
$$\int_Q V(\nabla U(z))dz \leq V_\alpha |Q|,$$

donde $U^{(1)}$ satisface las condiciones iniciales y/o de frontera como en (ODP). Los dos integrandos involucrados son

$$W(A) = \begin{cases} g(1, A^{(1)}), & \text{si } A \in \Lambda_\alpha, \\ g(0, A^{(1)}), & \text{si } A \in \Lambda_\beta \setminus \Lambda_\alpha, \\ +\infty, & \text{e.c.c.,} \end{cases}$$

$$V(A) = \begin{cases} 1, & \text{si } A \in \Lambda_\alpha, \\ 0, & \text{si } A \in \Lambda_\beta \setminus \Lambda_\alpha, \\ +\infty, & \text{e.c.c.} \end{cases}$$

Tenemos una reformulación equivalente del problema de diseño óptimo original como un problema variacional vectorial con una restricción integral.

Al ser equivalente al problema de diseño óptimo original, también carece de solución óptima. Si analizamos la reformulación variacional, aunque hemos evitado el carácter no local, hemos introducido dos nuevas dificultades. La primera, es que tenemos un problema variacional vectorial; y la segunda, que las funciones $W$ y $V$ no son funciones de Carathéodory (pues toman el valor $+\infty$ en gran parte del dominio).

La reformulación variacional no está por lo tanto dentro del marco clásico del Cálculo de Variaciones pues la función $W$ no es de Carathéodory al tomar el valor $+\infty$ abruptamente. Gracias al uso de las medidas de Young , se pueden obtener resultados similares como para integrandos de Carathéodory ([52]). El carácter vectorial da una naturaleza diferente al problema reformulado. Básicamente son dos los caminos para tratar tales problemas variacionales, analizando los condiciones de optimalidad (método indirecto) y caracterizando los minimizadores; o abordando directamente el problema, examinando directamente el comportamiento de las sucesiones minimizantes, el Método Directo. Descartamos el método indirecto debido al carácter vectorial de nuestro problema, pues las condiciones de optimalidad de primer orden, la ecuación Euler-Lagrange, son en general un sistema de ecuaciones en derivadas parciales, que resulta difícil de resolver. Es claro por que el Método Directo ha sido el camino usual para abordar este tipo de problemas y analizar la existencia de minimizadores. Nosotros proponemos usar el Método Directo para analizar el problema reformulado. Estudiaremos condiciones necesarias para tener la existencia de soluciones. En este camino, tenemos un resultado general y muy conocido, el teorema de Weierstrass (ver Apéndice).

Cuando analizamos las hipótesis del teorema de Weierstrass, comprobamos que el estamento más difícil de verificar es el concerniente a la semicontinuidad inferior. Esta es una propiedad difícil de verificar y es la principal causa de la no existencia de soluciones. Cuando analizamos con profundidad esta condición, y buscamos propiedades necesarias y suficientes que nos digan cuando un funcional $I$ es semicontinuo inferior débil, en el caso escalar esta propiedad es la convexidad de la densidad coste, sin embargo para el caso vectorial, por fortuna, necesitamos una noción de convexidad mas general: esta es la cuasiconvexidad (para más detalles sobre cuasiconvexidad, ver Apéndice).

La cuasiconvexidad de la densidad en el coste en la variable gradiente juega un papel importante en la existencia de minimizadores para el problema variacional reformulado. Ante la falta de (cuasi)convexidad tenemos que proceder vía relajación. Un resultado importante en este camino fue establecido por B. Dacorogna, el cual probó que si remplazamos la densidad en el coste otra función cuasiconvexa determinada, el ínfimo de ambos problemas coincide, y más aún, si asumimos ciertas condiciones de coercividad el ínfimo del problema relajado es realmente un mínimo. La envoltura cuasiconvexa de una función $g$ se define (clásicamente) como

$$Qg = \sup\{h \leq g : h \text{ quasiconvex}\}$$

Cuando aparecen en el problema otras restricciones de tipo integral, necesitamos introducir una nueva noción de convexidad, la cual es llamada *cuasiconvexidad restringida* ([53]).

Con la ayuda del método variacional, no sólo calculamos el valor óptimo de la función coste, sino que podemos encontrar la microestructura óptima asociada con este valor óptimo. Nuevamente, hacemos uso de las medidas de Young como diseños generalizados ([52]). Es más, incluso podemos escribir la envoltura anterior en término de las medidas de Young. Por lo tanto esta herramienta juega un papel importante y básico en nuestro análisis. Particularmente, estamos especialmente interesados en las medidas de Young gradientes, pues el problema variacional reformulado está escrito en términos de gradientes.

De igual manera que en la teoría de la homogeneización, donde los laminados juegan un papel importante en la identificación de la G-clausura, será también una herramienta fundamental en este método o enfoque variacional. Las medidas de Young gradientes pueden ser examinadas por un proceso de "dualidad" con la desigualdad de Jensen (este es un interesante trabajo llevado a cabo por D. Kinderleher y P. Pedregal [28], [29]). La propiedad principal que caracteriza a las medidas de Young gradientes es que, deben verificar la desigualdad de Jensen para toda función cuasiconvexa. Los laminados son una subclase de medidas de Young gradientes muy útil. Como en el método de homogeneización, las microestructuras laminadas pueden ser interpretadas en nuestro caso como medidas de Young, las cuales introducen un nuevo concepto de convexidad, *convexidad de rango uno.* Esta es una noción de convexidad a lo largo de direcciones de rango uno. Estas medidas están caracterizadas porque verifican la desigualdad de Jensen para todas las funciones convexas de rango uno (para mas detalles sobre medidas de Young y convexidad ver Apéndice).

Con el método variacional calculamos la relajación en forma de "cuasiconvexidad restringida" usando las medidas de Young, y luego comprobando que la medida de Young óptima es en realidad un microestructura, i.e., que la medida óptima es un laminado. Una de las ventajas de este método variacional es que no necesitamos calcular previamente la $G-$clausura del problema (lo cual es generalmente un problema difícil), pues nosotros obtenemos directamente la microestructura óptima.

Finalmente, nos gustaría decir algo sobre la relación entre estos dos distintos métodos, homogeneización y formulación variacional. Esta conexión fue estudiada por J.C. Bellido y P. Pedregal ([6], [5], [54]). Los autores mostraron algunas equivalencias entre ambos métodos en el sentido que $G-$convergencia de sucesiones de matrices en homogeneización puede ser interpretada como convergencia débil de la variable $\nabla U(z)$ en el problema reformulado. Ver también [12],[25].

## 1.3 Resultados obtenidos en esta tesis doctoral

En esta sección describimos y establecemos las contribuciones originales que hemos obtenido en el estudio del problema de diseño óptimo hiperbólico (P) dado en el principio. Se puede comprobar la falta de soluciones clásicas de este tipo de problemas. Ver algunos ejemplos en [44]. En este sentido, nuestra estrategia consiste en estudiar una relajación de estos problemas. Analizaremos la reformulación variacional del problema de diseño óptimo dada en la sección previa. Como la dimensión juega un papel importante en la caracterización de los campos de divergencia nula, es por tanto que nuestro análisis está dividido en tres secciones correspondientes a n=1,2,3.

### 1.3.1 El caso dinámico/hiperbólico unidimensional

En este caso, $\Omega \subset \mathbb{R}$ es un intervalo $\Omega = (0,1)$ y $T > 0$ fijo. El problema de diseño óptimo consiste en minimizar

$$I(\chi) = \int_0^T \int_\Omega \left[u_t^2(t,x) + \big(a_\alpha(t,x)\chi + (1-\chi)a_\beta(t,x)\big)u_x^2(t,x)\right] \, dx\, dt$$

donde $u$ es la única solución de

$$\begin{aligned} u_{tt} - ([\alpha\chi + \beta(1-\chi)]u_x)_x = 0 \quad & \text{en} \quad (0,T)\times(0,1), \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) \quad & \text{en} \quad \Omega, \\ u(t,0) = f(t),\ u(t,1) = g(t) \quad & \text{en} \quad [0,T], \end{aligned}$$

$$\int_\Omega \chi(t,x)dx \leq V_\alpha|\Omega|, \quad \forall t \in [0,T].$$

y las funciones $a$, $u_0$, $u_1$, $f$ y $g$ son conocidas y la función $a$ satisface ciertas cotas.

En este problema usaremos la caracterización de campos bidimensionales (para (x,t)) con divergencia nula. El problema variacional equivalente es

$$\underset{U}{min} \quad \hat{I}\ (U) = \int_0^T \int_\Omega W(\nabla U(t,x))dxdt$$

sujeto a

$$
\begin{aligned}
U &= (U^{(1)}, U^{(2)}) \in H^1([0,T]\times\Omega)^2,\\
U^{(1)}(0,x) &= u_0(x),\ U_t^{(1)}(0,x) = u_1(x) \quad \text{en} \quad \Omega,\\
U^{(1)}(t,0) &= f(t),\ U^{(1)}(t,1) = g(t) \quad \text{en} \quad [0,T],\\
\int_\Omega & V(\nabla U(t,x))dx \le V_\alpha|\Omega| \quad \forall t\in[0,T].
\end{aligned}
$$

donde las funciones $W$ y $V$ están definidas en la sección previa.

**El resultado principal en esta sección es calcular explícitamente el problema cuasiconvexificado (Capítulo 2)**. Consideremos el siguiente problema variacional

$$\min_{U,s} \int_0^T \int_\Omega \varphi(\nabla U(t,x), s(t,x))dxdt$$

sujeto a

$$
\begin{aligned}
0 \le s(t,x) \le 1, \int_\Omega s(t,x)\,dx &\le V_\alpha|\Omega| \quad \forall t\in[0,T],\\
U \in H^1([0,T]\times\Omega)^2,\ \ & tr(\nabla U(t,x)) = 0,\\
U^{(1)}(0,x) = u_0(x),\ U_t^{(1)}(0,x) &= u_1(x) \quad \text{en} \quad \Omega,\\
U^{(1)}(t,1) = f(t),\ U^{(1)}(t,0) &= g(t) \quad \text{en} \quad [0,T],
\end{aligned}
$$

donde notamos por $tr$ la traza de una matriz, and $\varphi(F,s)$ es dada explícitamente por

$$
\begin{cases}
\dfrac{h}{s\beta(\beta-\alpha)^2}(\beta^2|F_{12}|^2 + |F_{21}|^2 + 2\beta F_{12}F_{21}) + |F_{11}|^2 - \dfrac{a_\beta}{\beta}F_{12}F_{21} \\
\hfill \text{si } h(x,t)\ge 0, \psi(F,s)\le 0,\\
\dfrac{-h}{(1-s)\alpha(\beta-\alpha)^2}(\alpha^2|F_{12}|^2 + |F_{21}|^2 + 2\alpha F_{12}F_{21}) + |F_{11}|^2 - \dfrac{a_\alpha}{\alpha}F_{12}F_{21}, \\
\hfill \text{si } h(x,t)\le 0, \psi(F,s)\le 0,\\
-detF + \dfrac{1}{s(1-s)(\beta-\alpha)^2}\Big(\big((1-s)\beta^2(\alpha+a_\alpha) + s\alpha^2(\beta+a_\beta)\big)|F_{12}|^2 \\
\quad +\big((1-s)(\alpha+a_\alpha) + s(\beta+a_\beta)\big)|F_{21}|^2 + 2\big((\alpha+a_\alpha)\beta - sh\big)F_{12}F_{21}\Big) \\
\hfill \text{si } \psi(F,s)\ge 0.
\end{cases}
$$

y

$$\psi(F,s) = F_{12}F_{21} + \frac{\alpha}{s(\beta-\alpha)^2}(\beta F_{12}+F_{21})^2 + \frac{\beta}{(1-s)(\beta-\alpha)^2}(\alpha F_{12}+F_{21})^2.$$

con $F_{ij}$ las componentes de la matriz $F$, y

$$h(t,x) = \beta a_\alpha(t,x) - \alpha a_\beta(t,x).$$

Los siguientes teoremas nos proporciona el resultado principal de esta sección

**Teorema 1** *El problema variacional precedente es una relajación del problema de optimización inicial en el sentido que*

*a) el ínfimo de ambos problemas coincide;*

*b) existen soluciones óptimas para el problema relajado;*

*c) estas soluciones codifican (en el sentido de las medidas de Young) las microestructuras óptimas del problema de diseño original.*

En adición, podemos proporcionar de forma explícita las microestructuras óptimas.

**Teorema 2** *Las medidas de Young óptimas asociadas al problema relajado son siempre laminados, los cuales podemos concer de forma explícita (ver Capítulo 2)*

Una vez que tenemos una relajación completa del problema de diseño original, nos gustaría realizar algunas simulaciones numéricas. El hecho que en el problema relajado la densidad cuasiconvexificada depende de gradientes, verifica ciertas restricciones puntuales, y toma el valor $+\infty$ en ciertos dominios; todo ello hace difícil el análisis numérico del problema relajado.

Una posibilidad para hacer simulaciones numéricas con el problema relajado, es calcular las condiciones de optimalidad de primer orden, la ecuación de Euler-Lagrange. Esta condición es un sistema de ecuaciones en derivadas parciales además de las otras muchas restricciones propias del problema. De esta forma, proponemos un método alternativo para evitar esta complejidad. Si examinamos el problema relajado con cierto detalle, se puede comprobar fácilmente la conexión entre el problema de diseño óptimo original, el problema cuasiconvexificado y el problema establecido abajo en la Conjetura 1.

Conjeturamos que en algunos casos el problema relajado admite una formulación alternativa y mas fácil ([37]).

**Conjetura 1** *Supongamos los coeficientes* $a_\alpha = 1$, $a_\beta = 1$. *Entonces, el problema de optimización* $(\widetilde{RP})$

$$\min_s \tilde{I}(s) = \int_0^T \int_\Omega u_t^2(t,x) + u_x^2(t,x)\, dx\, dt$$

*donde* $u$ *es la única solución de*

$$\begin{aligned} u_{tt} - div\Big([\alpha s + \beta(1-s)]u_x\Big) = 0 \quad & en \quad (0,T)\times(0,1), \qquad (1.5)\\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) \quad & en \quad \Omega,\\ u(t,0) = f(t),\ u(t,1) = g(t) \quad & en \quad [0,T],\\ \int_\Omega s(t,x)dx \le V_\alpha|\Omega|, \quad & \qquad \forall t\in[0,T],\\ 0 \le s(t,x) \le 1. & \end{aligned}$$

*es equivalente al problema de diseño óptimo original (P), en el sentido que*

*a) el ínfimo de ambos problemas coincide, i.e., inf($\widetilde{RP}$)=inf(P);*

*b) el problema de diseño óptimo anterior ($\widetilde{RP}$) admite soluciones óptimas;*

*c) estas soluciones (en el sentido de las medidas de Young) muestran que las microestructuras óptimas son siempre laminados de primer orden con dirección normal $n = (0,1)$ y fracción de volumen s.*

Este resultado está en concordancia con un trabajo previo [63], donde el autor caracteriza las sucesiones minimizantes para problemas de diseño óptimo elípticos y asegura que este tipo de relajación es una relajación buena cuando el campo eléctrico deseado pertenece a un conjunto $G_\delta$ denso, pero desconocido. Con nuestra conjetura y en vista con los experimentos numéricos obtenidos, deducimos que el campo cero pertenece a este conjunto $G_\delta$.

Ahora brevemente describiremos el algoritmo numérico empleado para resoler ($\widetilde{RP}$) (ver el Capítulo 2 para más detalles). Usamos un método de descenso tipo gradiente. Dado un diseño admisible $s$, calculamos una perturbación $s^\eta = s + \nu s_1$ tal que $\tilde{I}(s^\eta) \leq \tilde{I}(s)$, con $\eta \in \mathbb{R}^+$ suficientemente pequeño.

Tenemos el siguiente resultado.

**Teorema 3** *Si $(u_0, u_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, entonces la derivada de $\widetilde{I}$ respecto a s en cualquier dirección $s_1$ existe y es de la forma*

$$\frac{\partial \widetilde{I}(s)}{\partial s} \cdot s_1 = \int_0^T \int_\Omega s_1 \Big( (a_\alpha - a_\beta) u_x^2 + (\alpha - \beta) u_x p_x \Big) dxdt$$

*donde u es la solución de (1.5) y p es la solución en $C^1([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ del problema adjunto*

$$\begin{cases} div(p_t, -[s\alpha + (1-s)\beta]p_x) = div(u_t, a(t,x,s)u_x) & en \quad (0,T) \times \Omega, \\ p = 0 & en \quad (0,T) \times \partial\Omega, \\ p(T,x) = 0, \quad p_t(T,x) = u_t(T,x) & en \quad \Omega. \end{cases}$$

Por otro lado, para tener en cuenta la restricción de volumen introducimos el típico multiplicador de Lagrange $\gamma \in L^\infty((0,T); \mathbb{R})$, que es dado por

$$\gamma(t) = \frac{(\int_\Omega s(t,x)dx - V_\alpha|\Omega|) - \int_\Omega \eta(t,x) \Big( (a_\alpha - a_\beta) u_x^2 + (\alpha - \beta) u_x p_x \Big) dx}{\int_\Omega \eta(t,x) dx}.$$

Finalmente elegimos $\eta$ de manera que el nuevo diseño (control) $s^\eta = s + \nu s_1$ pertenezca a $[0,1]$ para todo $x \in \Omega$ y $t \in (0,T)$.

Es importante enfatizar algunos aspectos de este algoritmo. El primer punto interesante es que tenemos que resolver dos ecuaciones de ondas con coeficientes dependiendo de la variable espacial y temporal, lo cual es bastante desconocido desde un punto de vista numérico. Otro aspecto importante en nuestro algoritmo es la presencia de derivadas de la función $u$ y $p$ en la dirección de descenso, lo cual introduce algunos irregularidades al esquema. En este sentido añadimos un término dispersivo y de viscosidad nula del tipo $\epsilon^2 \mathrm{div}([s\alpha + (1-s)\beta] u_{xtt})$ con $\epsilon$ del orden de $h$ (el parametro de discretización en espacio) y el cual introduce un efecto regulador de la dirección de descenso. De esta forma conseguimos un algoritmo convergente.

Mostraremos muchos e interesantes ejemplos dentro del contexto de materiales dieléctricos. Para todas nuestras simulaciones, el dominio de diseño es e intervalo $\Omega = (0,1)$, el tiempo positivo $T = 2$, y la función $a$ es tal que $(a_\alpha, a_\beta) = (1,1)$, i.e., el caso cuadrático. Por simplicidad consideraremos las condición de frontera $f \equiv 0$ y $g \equiv 0$, y analizaremos los resultados en función de los valores de $\alpha, \beta, V_\alpha$ y las condiciones iniciales. Usamos una aproximación con elementos finitos $C^0$ respecto a la variable espacial y diferencias finitas centradas con respecto a la variable temporal.

En el primer ejemplo, usamos la condición inicial $u_0(x) = \sin(\pi x)$, $u_1(x) = 0$, y tenemos la restricción sobre la cantidad de material $\beta$ que podemos usar (buen dieléctrico) del 70 % del dominio de diseño para todo tiempo $t$, lo que significa $V_\alpha = 0{,}3$. En este ejemplo fijamos el valor de $\alpha$ igual a 1, y usamos dos posibilidades para $\beta$: $\beta = 1{,}1$ and $\beta = 6$ (bajo y alto contraste).

En los dibujos, representamos los iso-valores de la densidad límite $s(t,x)$. Usamos una escala graduada donde las zonas azules corresponden al material $\alpha$ y las rojas al material $\beta$. Los colores intermedios representan las microestructuras óptimas en cada punto. Estas microestructuras son siempre laminados de primer orden con dirección normal constante $n(t,x) = (0,1)$ (i.e., los laminados son siempre ortogonales al eje del espacio). Como se puede observar de la Figura 1.3, la figura de arriba es "casi" una función característica mientras en la de abajo aparecen muchos microestructuras. Pensamos que cuando la diferencia entre $\alpha$ y $\beta$ es suficientemente pequeña, el problema original está bien puesto, pero cuando la diferencia es suficientemente grande, el problema original está realmente mal puesto, y por tanto es necesaria una relajación.

Finalmente, en los casos donde no existe una solución óptima razonable, proponemos un método de penalización para encontrar una función característica para el problema original. Con este fin, descomponemos el dominio de diseño completo (en espacio y tiempo, $Q = \Omega \times (0,T))$), en subdominios más pequeños y aproximamos la densidad óptima por su media en cada intervalo. De esta forma, si notamos por $\tilde{N}$, el tamaño de la descomposición, esperamos que la densidad aproximada converja a la densidad óptima, cuando $\tilde{N}$ tiende a infinito.

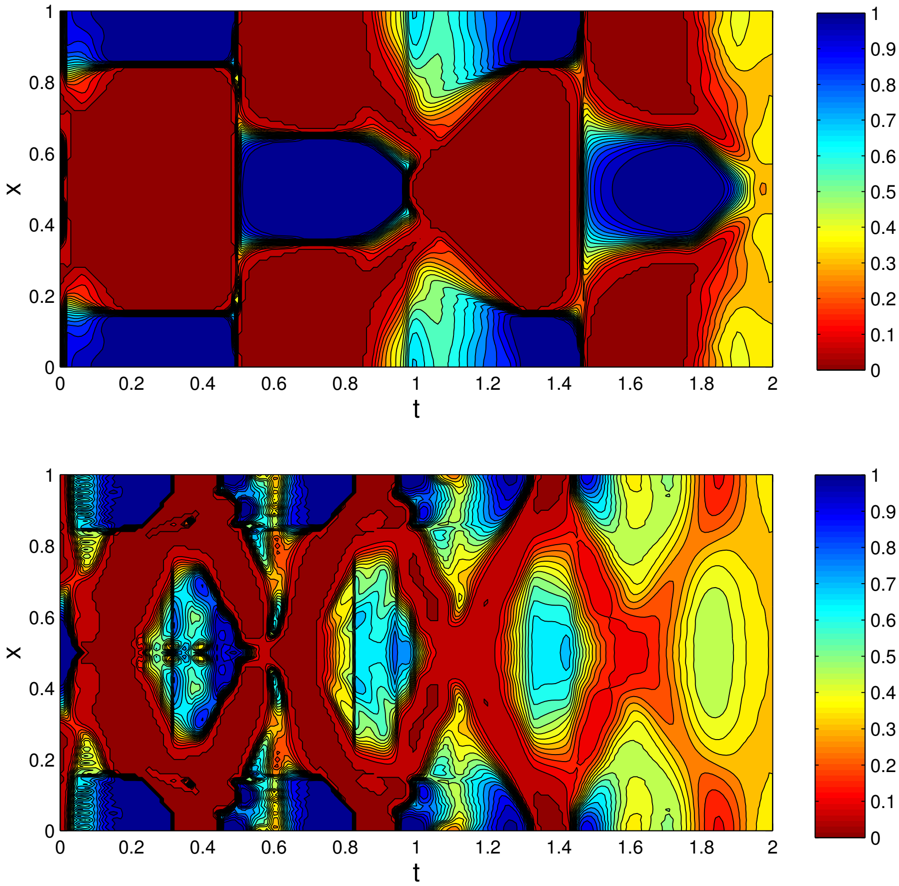



*Figura 1.3: Ejemplo 1- Arriba:* $(\alpha, \beta) = (1{,}1{,}1)$ *- Abajo:* $(\alpha, \beta) = (1, 6)$

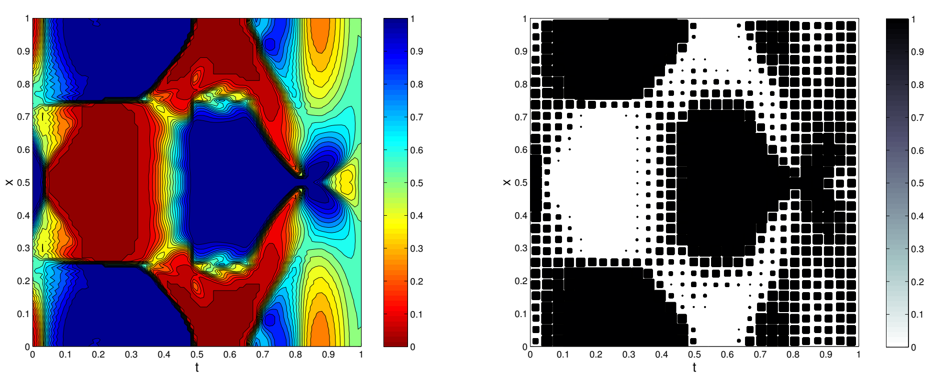



*Figura 1.4: Ejemplo 2:* $T = 1$, $(\alpha, \beta) = (1, 2)$ *-* $V_\alpha = 0{,}5$ $s^{pen}_{30,30}$ *-* $\widetilde{I}(s^{lim}) \approx 4{,}7584$*-* $\widetilde{I}(s^{lim}) \approx 5{,}62$, *$\tilde{N}$=30.*

En la Figura 1.4 mostramos, en la izquierda, los iso-valores de la densidad óptima para $(\alpha, \beta) = (1, 2)$ - $V_\alpha = 0{,}5$, $T = 1$, y, como condición inicial la misma que en el Ejemplo 1. En este caso el dominio de diseño completo es $Q = (0,1) \times (0,1)$, el cual es dividido en 900 subdominios, i.e., $\tilde{N} = 30$, y en el lado derecho del dibujo, mostramos la función característica asociada con la densidad penalizada.

El análisis de este problema está hecho con detalles en el Capítulo 2 de esta tesis, donde calcula explícitamente la "cuasiconvexificación restringida", los laminados que la recuperan, y se muestra el esquema numérico. Este capítulo corresponde con el trabajo [37].

### 1.3.2 El caso estacionario/elíptico tridimensional

Previo al estudio del problema de diseño óptimo hiperbólico bidimensional, analizaremos el problema elíptico tridimensional. Este problema ha sido estudiado con profundidad en el caso bidimensional desde el punto de vista de la homogeneización ([1]) y también usando la reformulación variacional ([56]), pero este problema ha sido mucho menos estudiado en el caso tridimensional ([7]). Es precisamente la situación espacial la dificultad principal en este caso. En [7], el autor realiza un interesante trabajo para el problema elíptico 3d desde una perspectiva variacional, introduciendo una nueva noción de convexidad apropiada, asociada con pares de gradientes y campos tridimensionales con divergencia nula ([6]), y entonces es capaz de calcular el problema relajado. Nosotros proponemos otra alternativa, usando la caracterización de los campos tridimensionales con divergencia nula por los potenciales de Clebsh. Con nuestra estrategia intentaremos seguir los pasos usados para el caso bidimensional ([56],[2]), pero es claro que el uso de los potenciales de Clebsh introduce una cierta no linealidad al problema lo cual debe ser tenido en cuenta.

Consideramos $\Omega \subset \mathbb{R}^3$ el dominio de diseño acotado y simplemente conexo. El problema consiste en minimizar

$$I(\chi) = \int_\Omega a(x,\chi(x))|\nabla u(x) - F(x)|^2 dx$$

donde $u$ es la única solución de

$$\begin{aligned} -\operatorname{div}\left((\alpha\chi + \beta(1-\chi))\nabla u\right) = g \quad &\text{en} \quad \Omega, \\ u = u_0 \quad &\text{en} \quad \partial\Omega, \end{aligned} \tag{1.4}$$

$$\int_\Omega \chi(x)dx \leq t_0|\Omega|$$

con $t_0 \in (0,1)$ fijo y las funciones $a$, $F$, $g$ y $u_0$ son conocidas.

Nuevamente procedemos a analizar la versión variacional del problema, que en este caso se escribe como

$$\min_{U} \quad \hat{I}\ (U) = \int_{\Omega} W(\nabla U(x))dx$$

sujeto a

$$\begin{array}{c} U \in H^1(\Omega)^3,\ U^{(1)} = u_0 \text{ en } \partial\Omega \\ \int_\Omega V(\nabla U(x))dx \leq t_0|\Omega|, \end{array}$$

donde los funcionales $W$ y $V$ están definidos en la sección previa.

La ecuación de estado 1.4 puede ser simplificada al caso homogéneo, introduciendo una adecuada función $G$, tal que

$$div(G) = g.$$

La principal dificultad inducida por el carácter espacial del problema, la no linealidad de las variedades $\Lambda_\gamma$ se puede resolver usando apropiadamente la continuidad débil de los menores,

$$\Lambda_{\gamma,x} = \{A \in M^{3\times 3} : \gamma A^{(1)} - A^{(2)} \times A^{(3)} + G(x) = 0\}$$

**Superar esta no linealidad y las dificultades asociadas es el principal punto de nuestro análisis.** Y de este forma calcular una relajación completa del problema. Ponemos

$$\begin{aligned} \psi(t,F) = \quad & \alpha\beta(t\alpha + (1-t)\beta)|F^{(1)}|^2 + \\ & \Big[((1-t)\alpha + t\beta)(|F^{(2)} \times F^{(3)}|^2 + |G|^2 - 2(F^{(2)} \times F^{(3)}) \cdot G)\Big] + \\ & [2\alpha\beta + t(1-t)(\beta-\alpha)^2](F^{(1)} \cdot G - \det F), \end{aligned}$$

$$h(x) = \beta a_\alpha(x) - \alpha a_\beta(x),$$

$$\varphi(t,F) = \begin{cases} \dfrac{h(x)}{t\beta(\beta-\alpha)^2}(\beta^2|F^{(1)}|^2 + |F^{(2)} \times F^{(3)}|^2 + |G|^2 - 2\beta \det F + \\ \qquad 2(\beta F^{(1)} - F^{(2)} \times F^{(3)}) \cdot G) + \dfrac{a_\beta(x)}{\beta}(\det F - G \cdot F^{(1)}) \\ \qquad\qquad \text{si } h(x) \geq 0, \psi(t,F) \leq 0, \\ \dfrac{-h(x)}{(1-t)\alpha(\beta-\alpha)^2}(\alpha^2|F^{(1)}|^2 + |F^{(2)} \times F^{(3)}|^2 + |G|^2 \\ \qquad -2\alpha \det F + 2(\alpha F^{(1)} - F^{(2)} \times F^{(3)}) \cdot G) + \\ \qquad \dfrac{a_\alpha(x)}{\alpha}(\det F - G \cdot F^{(1)}), \\ \qquad\qquad \text{si } h(x) \leq 0, \psi(t,F) \leq 0, \\ +\infty \qquad\qquad c.c. \end{cases}$$

Probaremos que el problema variacional

$$\min_{(t,U)} \int_\Omega \varphi(t(x), \nabla U(x))dx$$

sujeto a

$$U \in H^1(\Omega)^3,\ U^{(1)} = u_0 \text{ en } \partial\Omega,$$

$$\psi(t(x), \nabla U(x)) \leq 0,$$

$$0 \leq t(x) \leq 1,$$

$$\int_\Omega t(x)dx \leq t_0|\Omega|,$$

es una relajación del problema de diseño original, en el sentido explicado en el siguiente teorema

**Teorema 4** *Este último problema variacional es equivalente al (una relajación del) problema de diseño óptimo tridimensional en el sentido que*

*a) el ínfimo de ambos problemas coincide,*

*b) existen soluciones óptimas del problema relajado,*

*c) estas soluciones codifican (en el sentido de las medidas de Young) la microestructura óptima del problema de diseño óptimo original.*

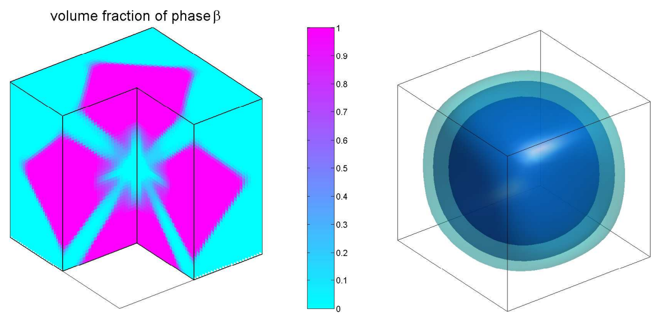
El análisis de este problema esta hecho con detalles en el Capítulo 3 de este tesis, y es el asunto principal en el artículo [36].

Siguiendo la idea en la subsección previa, de explorar la problema relajado, podemos rehacer un análisis similar del problema relajado para intentar simplificarlo. Este trabajo fue llevado a cabo por A. Donoso ([22]) siguiendo el resultado análogo de relajación para el problema tridimensional dado por J.C. Bellido en [7]. The Figure 1.5 es una simulación numérica presentada en este trabajo con los siguientes datos: $g = \exp -100|x - \frac{1}{2}|^2$, $(\beta/\alpha = 2$, $u_0 = 0$ y $t_0 = 0{,}6$

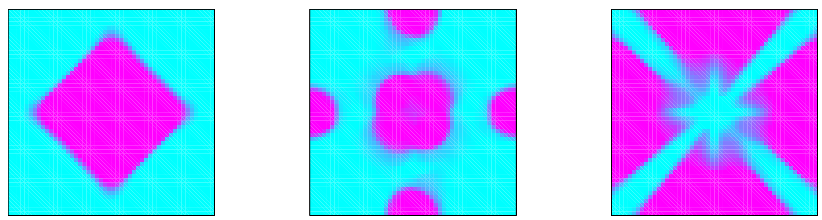
### 1.3.3 El caso dinámico/hiperbólico bidimensional

Usando los resultados obtenidos en las secciones anteriores, aquí analizaremos un problema más interesante y realista: el diseño dinámico de un placa (o lámina). Entenderemos este problema como una mezcla entre los dos previos in el sentido que, tenemos un problema hiperbólico como en el primer problema, but este es un problema tridimensional, pues tenemos una variable temporal y dos espaciales, con todas las dificultades que introduce el carácter espacial del problema.

Consideramos $\Omega \subset \mathbb{R}^2$ un abierto, acotado y simplemente conexo, y sea $T > 0$ fijo, el problema de diseño óptimo consiste en minimizar




(a) Sección (b) Iso-superficies de temperatura



(c) nivel-1 (d) nivel-12 (d) nivel-25

*Figura 1.5: simulación del problema de diseño óptimo tridimensional [22] (cortesia de A. Donoso.)*

$$I(\chi) = \int_0^T \int_\Omega u_t^2(t,x) + \big(a_\alpha(t,x)\chi + (1-\chi)a_\beta(t,x)\big)|\nabla u_x(t,x)|^2 dxdt$$

donde $u$ es la única solución de

$$\begin{aligned} u_{tt} - \mathrm{div}_x([\alpha\chi + \beta(1-\chi)]\nabla_x u) = 0 \quad &\text{in} \quad (0,T)\times\Omega, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) \quad &\text{in} \quad \Omega, \\ u(t,x) = f(t), \quad &\text{in} \quad [0,T]\times\partial\Omega, \end{aligned}$$

$$\int_\Omega \chi(t,x)dx \le V_\alpha|\Omega|, \quad \forall t \in [0,T].$$

y las funciones $a$, $u_0$, $u_1$ y $f$ son conocidas y la función $a$ satisface ciertas cotas.

De nuevo usamos la reformulación variacional previa. Entonces este problema es equivalente al problema variacional vectorial no convexo:

$$\min_U \quad \hat{I}\ (U) = \int_0^T \int_\Omega W(t,x,\nabla U(t,x))dxdt$$

sujeto a

$$\begin{aligned} U = (U^{(1)}, U^{(2)}, U^{(3)}) \in H^1((0,T)\times\Omega)^3, \\ U^{(1)}(0,x) = u_0(x),\ U_t^{(1)}(0,x) = u_1(x) \quad \text{en} \quad \Omega \\ U^{(1)}(t,x) = f(t,x) \quad \text{en} \quad [0,T]\times\partial\Omega \\ \int_\Omega V(t,x,\nabla U(t,x))dx \le V_\alpha|\Omega| \ \ \forall t\in[0,T], \end{aligned}$$

donde las funciones $W$ y $V$ están definidas de forma análoga como hemos descito anteriormente.

Ponemos

$$L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$z_\alpha = \frac{1}{s(\beta-\alpha)} L(\beta F^{(1)} + F^{(2)} \times F^{(3)}),$$

$$z_\beta = \frac{-1}{(1-s)(\beta-\alpha)} L(\alpha F^{(1)} + F^{(2)} \times F^{(3)}),$$

y la función

$$\psi(F,s) = \det F - F_{11}^2 + s\alpha|z_\alpha|^2 + (1-s)\beta|z_\beta|^2.$$

Consideramos el problema variacional

$$\min_{U,s} \int_0^T \int_\Omega \varphi(t,x,\nabla U(t,x),s(t,x))dxdt$$

sujeto a

$$\begin{aligned} 0 \le s(t,x) \le 1, \int_\Omega s(t,x)\,dx \le V_\alpha|\Omega| \ \ \forall t\in[0,T], \\ U\in H^1([0,T]\times\Omega)^3, \ \ (\nabla U^{(1)})_1 = (\nabla U^{(2)}\times\nabla U^{(3)})_1, \\ U^{(1)}(0,x) = u_0(x),\ U_t^{(1)}(0,x) = u_1(x) \text{ en } \Omega, \\ U^{(1)}(t,x) = f(t,x) \text{ en } [0,T]\times\partial\Omega, \end{aligned}$$

donde $\varphi(t,x,F,s)$ es dada por

$$\varphi(t,x,F,s) = \begin{cases} \dfrac{1}{\beta}((\beta+a_\beta)|F_{11}|^2 - a_\beta \det F + sh|z_\alpha|^2) \\ \qquad\qquad \text{if } h(t,x)\geq 0, \psi(F,s)\leq 0, \\ \dfrac{1}{\alpha}((\alpha+a_\alpha)|F_{11}|^2 - \det F - (1-s)h|z_\beta|^2) \\ \qquad\qquad \text{if } h(t,x)\leq 0, \psi(F,s)\leq 0, \\ \det F + s(\alpha+a_\alpha)|z_\alpha|^2 + (1-s)(\beta+\beta)|z_\beta|^2 \\ \qquad\qquad \text{if } \psi(F,s)\geq 0, \end{cases}$$

para toda $F$ tal que verifica $F_{11} = (F^{(2)} \times F^{(3)})_1$.

**Teorema 5** *El problema variacional descrito anteriormente es una relajación del problema de optimización original en el sentido que*
*a) el ínfimo de ambos problemas coincide;*
*b) existen soluciones óptimas del problema relajado;*
*c) estas soluciones codifican (en el sentido de las medidas de Young) la microestructura óptima del problema de diseño óptimo original.*

Más aún, podemos proporcionar explícitamente las microestructuras óptimas.

**Teorema 6** *Las medidas de Young óptimas asociadas al problema relajado son siempre laminados que podemos dar de forma completamente explícita*

Todos estos cálculos están hechos con detalles en el Capítulo 4, donde calculamos explícitamente la "cuasiconvexificación restringida" del anterior problema variacional vectorial, y la fórmula explícita de los laminados óptimos. El material de este capítulo corresponde con el trabajo [38].

## 1.4 Conclusiones y futuros trabajos

El objetivo de esta tesis ha sido el análisis de algunos tipos de problemas de diseño óptimo tanto elípticos como hiperbólicos. En estos problemas de optimización estructural o problemas de control óptimo en los coeficientes, es usual la falta de soluciones clásicas. La teoría de la Homogeneización ha sido una herramienta muy importante en los procesos de relajación, cuyas relajaciones han sido hechas con una ampliación suficiente del conjunto de diseños admisibles a través del concepto de $G$-clausura. Nosotros utilizamos una estrategia alternativa, donde los problemas de diseño óptimo son transformados mediante una formulación equivalente, y donde el análisis ha sido hecho en un marco de trabajo variacional a través de técnicas variacionales. En este marco variacional, este proceso de relajación es llevado a cabo en términos de convexidad, cuasicnvexidad para problemas vectoriales, y las medidas de Young gradientes, que son una herramienta fundamental para calcular la envoltura cuasiconvexa, y las microestructuras óptimas asociadas a ésta. Una de las más importantes ventajas de este método es que evitamos el cálculo de la $G$-clausura, que puede resultar un problema difícil o imposible. Con este método variacional, conseguimos una relajación completa del problema original y las medidas de Young óptimas, las cuales son laminados que codifican la microestructura óptima. Con esto mostramos un poco de las grandes posibilidades que ofrece esta perspectiva variacional, y establecemos importantes resultados de relajación, incluso para problemas hiperbólicos, los cuales son bastante desconocidos en la teoría de problemas de diseño óptimo.

Por otra parte, también estamos interesados en el análisis numérico de este tipo de problemas de diseño óptimo hiperbólico. El problema relajado resulta un problema bastante complicado de tratar numéricamente debido a la presencia de gradientes, restricciones puntuales y porque toma el valor $+\infty$ en gran parte del dominio, entre otras más. Después de un profundo análisis del problema relajado, podemos conjeturar otra reformulación más sencilla, en la cual las microestructuras óptimas son siempre laminados de primer orden. Probamos analíticamente que el ínfimo de este último problema relajado coincide con el ínfimo del problema original, y más tarde, comprobamos la evidencia numérica que este ínfimo es realmente un mínimo. Si el análisis de problemas de diseño óptimo hiperbólicos es poco conocido, los aspectos numéricos mucho más. Analizamos problemas de control óptimo en los coeficientes, los cuales dependen del espacio y del tiempo. Precisamente esta propiedad de dependencia del tiempo del control (diseño), introduce ciertas inestabilidades en el esquema numérico, las cuales hemos solventado añadiendo un término dispersivo y de viscosidad nula.

Finalmente, nos gustaría dirigir en el futuro otros situaciones diferentes y mas complejas. En este sentido algunos posibles trabajos futuros pueden estar dirigidos hacia:

- Analizar el problema de diseño óptimo hiperbólico tridimensional. En este caso podríamos usar potenciales de Clebsh como en el caso bidimensional, o bien, usar los recientes resultados obtenidos en el trabajo [57], donde se propone una perspectiva variacional del problema independiente de la dimensión, usando medidas de Young asociadas a pares de sucesiones de gradientes y campos con divergencia nula.

- Analizar problemas de diseño óptimo en conductividad gobernados por leyes de diferente naturaleza, desde leyes parabólicas hasta no lineales, como por ejemplo, en problemas estacionarios donde la ecuación de estado podría ser un p-laplaciano. Analizar sistemas mas complejos, en el contexto de elasticidad (lineal) es también muy interesante.

- Probar la Conjetura 1, acorde con el resultado probado por Tartar y la evidencia numérica, que nos dice que esta conjetura sería realmente un teorema.

- Análisis de casos degenerados, i.e. cuando algunas de las constantes isótropas $\alpha$ o $\beta$ tienden a 0 ó $+\infty$, respectivamente. En esta dirección, un trabajo pionero ([9]) muestra algún comportamiento degenerado del sistema en el caso que $\alpha$ tiende a 0.

- Analizar problemas de diseño óptimo con conjuntos de diseños admisibles regulares, por ejemplo, diseños admisibles funciones continuas Hölder con gradiente acotado, y analizar el comportamiento asintótico cuando esta cota tiende a $+\infty$.

# Capítulo 2

# Optimal design under the one-dimensional wave equation

## 2.1 Introduction

Optimal design problems in conductivity and elasticity have been extensively studied from various perspectives. For the homogenization viewpoint, see [1]. For more simulation-oriented approaches, see [10, 21]. For treatments based on variational reformulations, see [56]. In many of these examples, the state equation is assumed to be isotropic. There has also been attempts to understand non-isotropic situations ([40] and references therein).

Suppose we choose two diagonal, non-isotropic, $2 \times 2$ matrices of the form

$$A_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}, \quad A_\beta = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix},$$

and consider the state equation

$$\text{div } ([\chi(x)A_\alpha + (1-\chi(x))A_\beta]\nabla u) = 0 \quad \text{in} \quad \Omega$$

where $\Omega \subset \mathbb{R}^2$ is a bounded, regular, simply-connected domain. It is easy to see that we can also write

$$\text{div } \left( \begin{pmatrix} 1 & 0 \\ 0 & \chi(x)\alpha + (1-\chi(x))\beta \end{pmatrix} \nabla u \right) = 0 \quad \text{in} \quad \Omega,$$

and even more so

$$\text{div } (u_{x_1}, [\chi(x)\alpha + (1-\chi(x))\beta]u_{x_2}) = 0.$$

In the case where $\Omega = (0,T) \times (0,1)$, and we take both $\alpha$ and $\beta$ negative, we see that we have a 1-d wave equation as state equation, and 2-d optimal designs can be interpreted as 1-d time-dependent optimal designs as in [33, 34]. For this reason, we change the notation and write $x(= x_2)$ for the spatial variable, $t(= x_1)$ for the time variable, and replace $\alpha, \beta$ by $-\alpha, -\beta$, respectively, so that we focus on such a wave equation. We also change accordingly the domain, and consider initial and boundary condition as is usual in hyperbolic problems. Yet notice that the non-isotropic elliptic example is contained in our analysis too.

Optimal control problems in the coefficients are known in the elliptic case, however they are much less known under hyperbolic laws. A pioneer work in this direction is [34], where the author analyzes the hyperbolic G-closure of this sort of optimal control problem, a general report on dynamic materials is given in [33]. On the other hand, an interesting analysis for optimal control problems under the wave equation in greater dimensions is shown in [13], where the control is a time dependent coefficient, and under other constraints on modes where there is vibration. In this sense another work in which the authors examine time-harmonic solutions of the wave equation is [8], where they prove a relaxation result for this problem and very interesting results of existence of classical solutions for some particular cases. In this paper, we would like to generalize the 1-d hyperbolic optimal control design problem with designs (coefficients) depending both on $x$ and $t$. In the wave equation literature, we can find a huge family of optimal control problems where the design variable is not in the highest derivative term. When the control term acts on the first order derivative in time, the term is known as a "damping" term. These problems are of a different nature physically as well as mathematically. Some relevant references in this topic are [14, 24, 27].

### 2.1.1 Problem Statement

We will thus study the following optimal design problem. We consider a design domain $\Omega = (0, 1) \subset \mathbb{R}$, a positive time $T > 0$, and a maximum of one material at our disposal $V_\alpha \in (0, 1)$. The optimal design problem consists in deciding, for each time $0 < t < T$, the best distribution in $\Omega$ of the two materials in order to minimize the time-dependent cost functional depending on the square of the gradient (with respect to both variables $(t, x)$) of the underlying state. More precisely, let us denote by $(P)$ the problem consists in minimizing

$$I(\chi) = \int_0^T \int_\Omega \left[ u_t^2(t,x) + a(t,x,\chi) u_x^2(t,x) \right] \, dx \, dt \tag{2.1}$$

where $u$ is the unique solution of

$$
\begin{aligned}
u_{tt} - div([\alpha\chi + \beta(1-\chi)]u_x) = 0 \quad &\text{in} \quad (0,T)\times(0,1),\\
u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) \quad &\text{in} \quad \Omega, && (2.2)\\
u(t,0) = f(t),\ u(t,1) = g(t) \quad &\text{in} \quad [0,T], && (2.3)
\end{aligned}
$$

and the functions $a, u_0, u_1, f$ and $g$ are known. The function $\chi \in L^\infty([0,T]\times\Omega;\{0,1\})$ is the design variable, and it indicates where we place the $\alpha$-material for each time $t$. Since $\chi$ is a binary variable $a(t,x,\chi) \in \{a(t,x,0), a(t,x,1)\}$ and we can write

$$a(t,x,\chi) = \chi(t,x)a_\alpha(t,x) + (1-\chi(t,x))a_\beta(t,x).$$

In addition, we make the assumption $0 < \alpha < \beta$, and

$$a_\alpha(t,x) + \alpha \geq 0, \quad a_\beta(t,x) + \beta \geq 0.$$

The amount of the $\alpha$-material is given, and therefore we have to enforce the volume constraint

$$\int_\Omega \chi(t,x)dx \leq V_\alpha|\Omega|, \quad \forall t \in [0,T].$$

The lack of classical solutions for this sort of problems is well understood (see. Theorem 11, [44]). In this sense we propose and analyze a relaxation of the problem.

Our approach is based on an equivalent variational reformulation of the original optimal design problem as a non-convex vector variational problem: as in other situations examined under this perspective [2, 56], we change a scalar problem with differential constraints by a vector variational problem with integral constraints (where the state equation is implicit in the new cost function). It is well-known that the non-existence of optimal solution for vector variational problem is nearly associated to the lack of quasiconvexity of the cost functional, in this sense we propose to analyze the "*constrained quasiconvexification*", for this last problem by using gradient Young measures as generalized solutions of variational problems, and to compute an explicit relaxation of the original optimal design problem in the form of a relaxed (quasiconvexificated) variational problem.

It is elementary to check (this is done with some detail in Section 2.2), the equivalence of our dynamic optimal design problem with the following non-convex, vector variational problem which we note by $(VP)$

$$\min_U \ \hat{I}\ (U) = \int_0^T \int_\Omega W(t,x,\nabla U(t,x))dxdt$$

subject to

$$
\begin{aligned}
&U = (U^{(1)}, U^{(2)}) \in H^1([0,T]\times\Omega)^2,\\
&U^{(1)}(0,x) = u_0(x),\ U_t^{(1)}(0,x) = u_1(x) \quad \text{in} \quad \Omega,\\
&U^{(1)}(t,0) = f(t),\ U^{(1)}(t,1) = g(t) \quad \text{in} \quad [0,T],\\
&\int_\Omega V(t,x,\nabla U(t,x))dx \le V_\alpha|\Omega| \quad \forall t\in[0,T].
\end{aligned}
$$

The two integrands involved are

$$
W(t,x,A) = \begin{cases} a_{11}^2 + a_\alpha(t,x)a_{12}^2, & \text{if} \quad A\in\Lambda_\alpha,\\ a_{11}^2 + a_\beta(t,x)a_{12}^2, & \text{if} \quad A\in\Lambda_\beta\setminus\Lambda_\alpha,\\ +\infty, & \text{else,}\end{cases}
$$

$$
V(t,x,A) = \begin{cases} 1, & \text{if } A\in\Lambda_\alpha,\\ 0, & \text{if } A\in\Lambda_\beta\setminus\Lambda_\alpha,\\ +\infty, & \text{else.}\end{cases}
$$

Here

$$
\Lambda_\gamma = \{A\in M^{2\times 2} : M_{-\gamma}A^{(1)} - RA^{(2)} = 0\}, \quad \gamma = \alpha,\beta,
$$

where $A^{(i)}$ is the $i$-th row of the matrix

$$
A = \begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}.
$$

Finally

$$
M_{-\gamma} = \begin{pmatrix} 1 & 0\\ 0 & -\gamma\end{pmatrix}, \quad R = \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}.
$$

### 2.1.2 Results Statement

To write down an explicit relaxation, put

$$
h(t,x) = \beta a_\alpha(t,x) - \alpha a_\beta(t,x)
$$

and for

$$
F = \begin{pmatrix} F_{11} & F_{12}\\ F_{21} & F_{22}\end{pmatrix}, \quad s\in\mathbb{R},
$$

set

$$
\psi(F,s) = F_{12}F_{21} + \frac{\alpha}{s(\beta-\alpha)^2}(\beta F_{12}+F_{21})^2 + \frac{\beta}{(1-s)(\beta-\alpha)^2}(\alpha F_{12}+F_{21})^2.
$$

Consider the variational problem, which we note $(RP)$

$$
\min_{U,s}\int_0^T\int_\Omega \varphi(t,x,\nabla U(t,x),s(t,x))dxdt
$$

subject to

$$
\begin{aligned}
&U \in H^1([0,T]\times\Omega)^2, \quad tr(\nabla U(t,x)) = 0,\\
&U^{(1)}(0,x) = u_0(x),\ U^{(1)}_t(0,x) = u_1(x) \quad \text{in} \quad \Omega,\\
&U^{(1)}(t,1) = f(t),\ U^{(1)}(t,0) = g(t) \quad \text{in} \quad [0,T],\\
&0 \le s(t,x) \le 1, \int_\Omega s(t,x)\,dx \le V_\alpha|\Omega| \ \ \forall t \in [0,T],
\end{aligned}
$$

where $\varphi(t,x,F,s)$ is given by

$$
\begin{cases}
\dfrac{h}{s\beta(\beta-\alpha)^2}(\beta^2|F_{12}|^2 + |F_{21}|^2 + 2\beta F_{12}F_{21}) + |F_{11}|^2 - \dfrac{a_\beta}{\beta}F_{12}F_{21} \\
\hfill \text{if } h(x,t) \ge 0, \psi(F,s) \le 0,\\
\dfrac{-h}{(1-s)\alpha(\beta-\alpha)^2}(\alpha^2|F_{12}|^2 + |F_{21}|^2 + 2\alpha F_{12}F_{21}) + |F_{11}|^2 - \dfrac{a_\alpha}{\alpha}F_{12}F_{21}, \\
\hfill \text{if } h(x,t) \le 0, \psi(F,s) \le 0,\\
-detF + \dfrac{1}{s(1-s)(\beta-\alpha)^2}\Big(\big((1-s)\beta^2(\alpha+a_\alpha) + s\alpha^2(\beta+a_\beta)\big)|F_{12}|^2 \\
\quad +\big((1-s)(\alpha+a_\alpha) + s(\beta+a_\beta)\big)|F_{21}|^2 + 2\big((\alpha+a_\alpha)\beta - sh\big)F_{12}F_{21}\Big) \\
\hfill \text{if } \psi(F,s) \ge 0.
\end{cases}
$$

$tr$ stands above for the trace of a matrix.

**Theorem 1** *The last variational problem ($RP$) is a relaxation of the initial optimization problem ($P$) in the sense that*

*a) the infima of both problems coincide;*

*b) there are optimal solutions for the relaxed problem;*

*c) these solutions codify ( in the sense of the Young measures) the optimal microstructures of the original optimal design problem.*

In addition, we can provide explicitly optimal microstructures.

**Theorem 2** *Optimal Young measures leading to the relaxed formulation are always laminates which can be given in a completely explicit form.*

The formulae for all of these laminates are given later at the end of Section 2.4.

The main new contribution here is therefore to understand the character of the hyperbolic state law, and the differences it introduces with respect to the better known elliptic case. Some of these differences are related to the fact that the manifolds $\Lambda_\gamma$ are two 2-dimensional subspaces whose intersection is another 1-dimensional manifold. Moreover there are rank-one connections within those manifolds. An interesting consequence is that the relaxed integrand is finite everywhere (except for the condition involving the trace) in contrast with the elliptic case where the relaxed integrand is finite only in a certain (quasi)convex subset. An important issue is that optimal Young measures gives us the necessary information about the behavior of minimizing sequences of the original optimal design problem.

A subsequent important step, which we hope to address in the near future, is to explore the relaxed problem in some particular cases, like the ones described in Section 2.5, with the objective of producing numerical simulations of the optimal time-dependent structures [39]. For some particular cases in the (static) elliptic case, it has been shown that a simple relaxation consists in replacing the original discrete design variable $\chi \in L^\infty(\Omega, \{0,1\})$ by its convex envelop $s \in L^\infty(\Omega, [0,1])$. For the (dynamic) hyperbolic case with $a_\alpha = 1$, $a_\beta = 1$ (corresponding to the full quadratic case), some numerical experiments (see Section 2.7) suggest that the above assertion is true. In this regard, we establish the following conjecture (discussed in Section 2.6).

**Conjecture 1** *Suppose the coefficients $a_\alpha = 1$, $a_\beta = 1$. The optimization problem $(\widetilde{RP})$*

$$\min_s \int_0^T \int_\Omega u_t^2(t,x) + u_x^2(t,x)\, dx\, dt$$

*where $u$ is the unique solution of*

$$\begin{aligned} u_{tt} - div([\alpha s + \beta(1-s)]u_x) = 0 \quad & in \quad (0,T)\times(0,1), \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) \quad & in \quad \Omega, \\ u(t,0) = f(t),\ u(t,1) = g(t) \quad & in \quad [0,T], \end{aligned}$$

*is equivalent to the original optimal design problem $(P)$ in the sense that*

*a) the infima of both problems coincide, i.e., $\text{ínf}(\widetilde{RP}) = \text{ínf}(P)$;*

*b) the above optimal design problem $(\widetilde{RP})$ admits optimal solutions ;*

*c) these solutions ( in the sense of Young measures) show that the optimal microstructures are first order laminates with normal $n = (0,1)$ and volume fraction $s$.*

The paper is organized as follows. In Section 2.2, we describe in more detail the equivalent variational reformulation as well as a general relaxation result when integrands are not continuous and may take on infinite values abruptly. As there is nothing new here compared to other previous works in the elliptic case, our description is rather a remainder included here for the sake of completeness. Sections 2.3 and 2.4 are technical in nature but interesting, as we first compute a lower bound of the *constrained quasiconvexification* (Section 2.3), by using in a fundamental way the weak continuity of the determinant. Section 2.4 is concerned with the search for laminates furnishing the precise value of the lower bound in an attempt to show equality of the three convex hulls (poly-, quasi- and rank-one convex hulls), as it is standard in this kind of calculation. In Section 2.5, we show some particular examples of this relaxation for different and interesting choices of the coefficients $a_\alpha, a_\beta$. Finally, in Section 2.6 we analyze the relaxed problem and propose a simpler relaxation, numerically solved in Section 2.7 by using a gradient descent method.

## 2.2 Reformulation and relaxation

The lack of classical solution of the original optimal design problem is well-established. We propose to reformulate the problem as a vector variational problem to which we apply suitable tools to study its relaxation. We follow a similar approach to the one in [2, 56].

Under the hypothesis of simple-connectedness of $\Omega$ (an interval), there exists a potential $v \in H^1((0,T) \times \Omega)$ such that the state equation

$$-div(u_t(t,x), -[\alpha\chi(t,x) + \beta(1-\chi(t,x))]u_x(t,x)) = 0 \quad \text{in } [0,T] \times \Omega$$

where the div operator is consider now with respect to the variables $t$ and $x$, then the above formula is equivalent to the pointwise constraint

$$\begin{pmatrix} u_t(t,x) \\ -[\alpha\chi(t,x) + \beta(1-\chi(t,x))]u_x(t,x) \end{pmatrix} = R\nabla v(x,t) \quad a.e.\ (t,x) \in [0,T] \times \Omega \tag{2.4}$$

where $R$ is the counterclockwise $\pi/2$-rotation in the space-time plane.

It is clear that we can identify the design variable $\chi$ with the vector field $U = (u,v)$, and conversely, a pair $U = (u,v)$ which verifies (2.4) determines a characteristic function $\chi$, so that we can consider the new design variable $U = (U^{(1)}, U^{(2)}) = (u,v)$, where $U : \mathbb{R}^2 \to \mathbb{R}^2$ and $\nabla U(t,x) \in \mathbb{R}^{2\times 2}$.

Therefore, by using the above statement and the notation in Section 2.1 (Introduction), it is easy to check that the original optimal design problem $(P)$ is equivalent to the variational problem $(VP)$.

We have recast our optimal design problem as a typical variational problem. We see that it is a non-convex vector problem that we are going to analyze by seeking its relaxation. We use Young measures as a main tool in the computation of the suitable density for the relaxed problem. In this sense, we can rely on the following result of relaxation [2] whose main idea is a useful tool in other different places [2, 56, 55].

We note the initial condition (2.2) by I.C., the boundary condition (2.3) by B.C. and put

$$
\begin{aligned}
m = \text{ínf}\Big\{ \int_\Omega \int_0^T & W(t,x,\nabla U(t,x))dtdx : \\
& U \in H^1((0,T)\times\Omega)^2, U^{(1)} \text{ satisfies the B.C. and the I. C.,} \\
& \qquad \int_\Omega V(t,x,\nabla U(t,x))dx \le V_\alpha |\Omega|, \forall t \in [0,T]\Big\}.
\end{aligned}
$$

We know [2] that

$$
\begin{aligned}
m \ge \bar{m} = \text{ínf}\Big\{ \int_\Omega \int_0^T & CQW(t,x,\nabla U(t,x), s(t,x))dtdx : \\
& U \in H^1([0,T]\times\Omega)^2, U^{(1)} \text{ satisfies the B.C. and the I. C.,} \\
& \qquad 0 \le s(t,x) \le 1, \int_\Omega s(t,x)dx \le V_\alpha |\Omega|, \forall t \in [0,T]\Big\},
\end{aligned}
$$

where $CQW(t,x,F,s)$ is defined by,

$$
CQW(t,x,F,s) = \text{ínf}\Big\{ \int_{M^{2\times 2}} W(t,x,A)d\nu(A) : \nu \in \mathcal{A}(F,s)\Big\}
$$

with

$$
\begin{aligned}
\mathcal{A}(F,s) = \Big\{ & \nu : \nu \text{ is a homogeneous } H^1\text{-Young measure,} \\
& F = \int_{M^{2\times 2}} A d\nu(A), \int_{M^{2\times 2}} V(t,x,A)d\nu(A) = s\Big\}.
\end{aligned} \tag{2.5}
$$

Notice that the previous inequality will be an equality when W is a Carathéodory function with appropriate growth constrains. However, in our situation it is still possible to prove this equality despite the fact that $W$ is not a Carathéodory function. Let us consider the following minimization problem

$$
\tilde{m} = \text{ínf}\Big\{ \int_\Omega \int_0^T \int_{M^{2\times 2}} W(t,x,A)d\nu_{t,x}(A)dtdx : \nu \in \mathcal{B}(B.C., I.C., t_0)\Big\}
$$

where

$$
\begin{aligned}
\mathcal{B}(B.C., I.C., V_\alpha) = \Big\{ & \nu : H^1\text{-Young meas., } supp(\nu_{t,x}) \subset \Lambda_\alpha \cup \Lambda_\beta, \\
& \exists U \in H^1([0,T]\times\Omega)^2, U^{(1)} \text{ satisfies the I.C. and B.C.,} \\
& \qquad \nabla U(t,x) = \int_{M^{2\times 2}} A d\nu_{t,x}(A), \\
& \qquad \int_\Omega \int_{M^{2\times 2}} V(t,x,A)d\nu_{t,x}(A)dx \le V_\alpha |\Omega|, \forall t \in [0,T]\Big\}.
\end{aligned}
$$

We have the following result.

**Teorema 7** *([2]) The equalities*

$$m = \bar{m} = \tilde{m}$$

*hold. Moreover, for each measure* $\nu \in \mathcal{B}(B.C., I.C., V_\alpha)$ *such that* $supp(\nu_{x,t}) \subset \Lambda_\alpha \cup \Lambda_\beta$ *a.e.* $(t,x) \in [0,T] \times \Omega$*, there exists a sequence* $\{\nabla U_k\}$ *such that,*

*i)* $U_k \in (H^1([0,T] \times \Omega))^2$, $U^{(1)}$ *satisfies the I.C. and B.C.,* $\{|\nabla U_k|^2\}$ *is equi-integrable,*

*ii)* $\nabla U_k(t,x) \in \Lambda_\alpha \cup \Lambda_\beta$, *a.e.* $(t,x) \in [0,T] \times \Omega \; \forall k$, $\int_\Omega V(t,x,\nabla U_k(t,x))dx \leq V_\alpha |\Omega|, \; \forall t \in [0,T] \forall k$

*iii)* $\lim_{k\to\infty} \int_0^T \int_\Omega W(t,x,\nabla U_k(t,x))dxdt = \int_0^T \int_\Omega \int_{M^{2\times 2}} W(t,x,A)d\nu_{t,x}(A)dxdt$

## 2.3 The lower bound: polyconvexification.

We would like to compute explicitly the *constrained quasiconvexification* defined as

$$CQW(t,x,F,s) = \inf \left\{ \int_{M^{2\times 2}} W(t,x,A)d\nu(A) : \nu \in \mathcal{A}(F,s) \right\}$$

where $\mathcal{A}(F,s)$ is given in (2.5). Since the variable $(t,x) \in [0,T] \times \Omega$ can be consider as a parameter, we drop this dependence to simplify the notation. In this form, the constrained quasiconvexification can be expressed as

$$\inf_\nu \left\{ \int_{M^{2\times 2}} W(A)d\nu(A) : F = \int_{M^{2\times 2}} A d\nu(A), \int_{M^{2\times 2}} V(A)d\nu(A) = s, \; \forall t \in [0,T] \right\} \tag{2.6}$$

with $\nu$ a homogeneous $H^1$-Young measure with $supp(\nu) \subset \Lambda_\alpha \cup \Lambda_\beta$.

For $(F,s)$ (and $(t,x)$) fixed, we are going to compute the value of (2.6), i.e. $CQW(t,x,F,s)$. The main difficulty here is that we do not know explicitly the set of the admissible measures, which we note as $\mathcal{A}$. We propose the following strategy. Consider two classes of family of probability measures $\mathcal{A}_*, \mathcal{A}^*$ such that

$$\mathcal{A}_* \subset \mathcal{A} \subset \mathcal{A}^*.$$

We first calculate the minimum over the greater class of probability measures $\mathcal{A}^*$, and then we check that the optimal value is attained by at least one measure over the narrower class $\mathcal{A}_*$. This fact tells us that the optimal value so achieved is the same in $\mathcal{A}$, and hence we will have in fact computed the exact value $CQW(t,x,F,s)$.

Following [56], we choose $\mathcal{A}^*$ as the set of polyconvex measures, which are not necessarily gradient Young measures, and therefore obtain a lower bound ( the (constrained) polyconvexification). The main property of these measures is that they commute with the determinant. This constraint can be imposed in a more-or-less manageable way. We also choose $\mathcal{A}_*$ as the class of *laminates* which is a subclass of the gradient Young measures. By working with this class, we would get an upper bound (the (constrained) rank-one convexification).

The *polyconvexification* $CPW(F,s)$ can be computed through the following optimization problem

$$\min_{\nu} \int_{M^{2\times 2}} W(A)d\nu(A)$$

where,

$$\begin{aligned} \nu \in \mathcal{A}(F,s) = \Big\{ \nu : \nu &\text{ is a homogeneous Young measure,} \\ &\text{which commutes with det, } F = \int_{M^{2\times 2}} A d\nu(A), \qquad (2.7) \\ &\int_{M^{2\times 2}} V(A)d\nu(A) = s \Big\}. \qquad (2.8) \end{aligned}$$

From (2.8) we have the following decomposition

$$\nu = s\nu_\alpha + (1-s)\nu_\beta, \quad supp(\nu_\gamma) \subset \Lambda_\gamma, \ \gamma = \alpha, \beta,$$

and therefore, from (2.7)

$$F = s\int_{\Lambda_\alpha} A d\nu_\alpha(A) + (1-s)\int_{\Lambda_\beta} A d\nu_\beta(A). \qquad (2.9)$$

If we put

$$F_\gamma = \int_{\Lambda_\gamma} A d\nu_\gamma(A), \quad \gamma = \alpha, \beta,$$

we have $F_\gamma \in \Lambda_\gamma$ for $\gamma = \alpha, \beta$, so from this property and (2.9), we have a non-compatible system on $F_\gamma$ unless

$$F_{11} + F_{22} = 0, \text{ i.e. } tr(F) = 0.$$

Let us suppose henceforth that this compatibility condition holds. This condition lets us simplify the problem from $2\times 2$ matrices to 3-d vectors, using the identification

$$F = \begin{pmatrix} \boldsymbol{x} & \boldsymbol{y} \\ \boldsymbol{z} & -\boldsymbol{x} \end{pmatrix} \longleftrightarrow (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}).$$

Therefore the manifolds $\Lambda_\gamma$ can be rewritten as

$$\Lambda_\gamma = \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \mathbb{R}^3 : \boldsymbol{z} + \gamma \boldsymbol{y} = 0\}.$$

In this way, the above system does not uniquely determined its solution. Indeed

$$F_\alpha = (\lambda, y_\alpha, -\alpha y_\alpha),\ F_\beta = (\frac{x - s\lambda}{1-s}, y_\beta, -\beta y_\beta)$$

where

$$y_\alpha = \frac{1}{s(\beta - \alpha)}(\beta \boldsymbol{y} + \boldsymbol{z}), \qquad y_\beta = \frac{-1}{(1-s)(\beta-\alpha)}(\alpha \boldsymbol{y} + \boldsymbol{z})$$

and $\lambda \in \mathbb{R}$. We can check that if $A = (a_1, a_2, a_3) \in \Lambda_\gamma$ with $\gamma = \alpha, \beta$, then

$$\det A = -a_1^2 + \gamma a_2^2,$$

and by using the important constraint about the commutativity with det, we know that

$$\begin{aligned}
\det F =& \int_{\mathbb{R}^3} \det A d\nu(A) \\
=& s \int_{\mathbb{R}^3} \det A d\nu_\alpha(A) + (1-s) \int_{\mathbb{R}^3} \det A d\nu_\beta(A) \\
=& -\int_{\mathbb{R}} a_1^2 d\nu^{(1)}(A) + s\alpha \int_{\mathbb{R}} a_2^2 d\nu_\alpha^{(2)}(A) + (1-s)\beta \int_{\mathbb{R}} a_2^2 d\nu_\beta^{(2)}(A)
\end{aligned}$$

where $\nu_\gamma^i$ designates the projection of $\nu_\gamma$ onto the $i$-th component.

On the other hand, we can write the cost functional in the form

$$\begin{aligned}
\int_{\mathbb{R}^3} W(A) d\nu(A) = \int_{\mathbb{R}^3} a_1^2 d\nu(A) + s a_\alpha \int_{\mathbb{R}^3} a_2^2 d\nu_\alpha(A) \\
+ (1-s) a_\beta \int_{\mathbb{R}^3} a_2^2 d\nu_\beta(A)
\end{aligned}$$

so if we put

$$S_1 = \int_{\mathbb{R}^3} a_1^2 d\nu(A), \quad S_\gamma = \int_{\Lambda_\gamma} a_2^2 d\nu_\gamma(A), \text{ with } \gamma = \alpha, \beta,$$

and use Jensen's inequality, we have the constraints

$$S_1 \geq \boldsymbol{x}^2, \qquad S_\gamma \geq y_\gamma^2 \qquad \gamma = \alpha, \beta.$$

By using the notation just introduced, the above inequalities and the constraint on the determinant, the constrained polyconvexification is given by the following linear programming problem

$$\underset{(S_1, S_\gamma, x_\gamma)}{\text{minimize}}\ S_1 + s a_\alpha S_\alpha + (1-s) a_\beta S_\beta$$

subject to,

$$\begin{aligned}
-\det F = S_1 - s\alpha S_\alpha - (1-s)\beta S_\beta, \\
S_1 \geq \boldsymbol{x}^2, S_\gamma \geq y_\gamma^2, \text{ with } \gamma = \alpha, \beta,
\end{aligned}$$

We can eliminate $S_1$, by replacing its value from the equality constraint in the cost functional. By so doing, only the variables $(S_\alpha, S_\beta)$ occur, with inequality constraints (see Figure 2.1 for a geometrical interpretation of the progamming problem). It is easy to solve this problem. Under the conditions

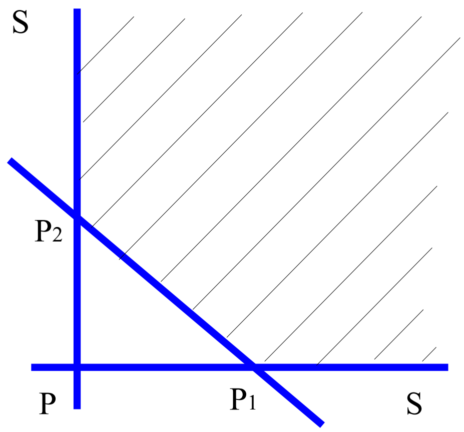



*Figura 2.1: New mathematical programming problem.*

$a_\alpha \geq -\alpha$ and $a_\beta \geq -\beta$, the optimal value depends on the relative position of the oblique line and the $P$ point. Namely, the optimal solution can be attained on $P, P_1$ or $P_2$.

We put the function

$$\psi(F,s) = \boldsymbol{yz} + \frac{\alpha}{s(\beta-\alpha)^2}(\beta\boldsymbol{y}+\boldsymbol{z})^2 + \frac{\beta}{(1-s)(\beta-\alpha)^2}(\alpha\boldsymbol{y}+\boldsymbol{z})^2,$$

the optimal value is

$$\begin{aligned}
&\frac{h}{s\beta(\beta-\alpha)^2}(\beta^2\boldsymbol{y}^2+\boldsymbol{z}^2+2\beta\boldsymbol{yz}) + \boldsymbol{x}^2 - \frac{a_\beta}{\beta}\boldsymbol{yz} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{if } h(x,t)\geq 0, \psi(F,s)\leq 0,\\
&\frac{-h}{(1-s)\alpha(\beta-\alpha)^2}(\alpha^2\boldsymbol{y}^2+\boldsymbol{z}^2+2\alpha\boldsymbol{yz}) + \boldsymbol{x}^2 - \frac{a_\alpha}{\alpha}yz,\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{if } h(x,t)\leq 0, \psi(F,s)\leq 0,\\
&-detF + \frac{1}{s(1-s)(\beta-\alpha)^2}\Big(\big((1-s)\beta^2(\alpha+a_\alpha)+s\alpha^2(\beta+a_\beta)\big)\boldsymbol{y}^2\\
&\quad +\big((1-s)(\alpha+a_\alpha)+s(\beta+a_\beta)\big)\boldsymbol{z}^2 + 2\big((\alpha+a_\alpha)\beta - s\beta\big)\boldsymbol{yz}\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \psi(F,s)\geq 0.
\end{aligned}$$

In addition, the optimal value is attained on

$$P_1 : S_\alpha = y_\alpha^2 \text{ and } S_1 = \boldsymbol{x}^2 \quad \text{if } h(x,t) \geq 0, \psi(F,s) \leq 0, \tag{2.10}$$

$$P_2 : S_\beta = y_\beta^2 \text{ and } S_1 = \boldsymbol{x}^2 \quad \text{if } h(x,t) \leq 0, \psi(F,s) \leq 0, \tag{2.11}$$

$$P : S_\alpha = y_\alpha^2 \text{ and } S_\beta = y_\beta^2 \quad \text{if } \psi(F,s) \geq 0. \tag{2.12}$$

Therefore we have an explicit computation of the constrained polyconvexification given by

$$CPW(F,s) = \begin{cases} \dfrac{h}{s\beta(\beta-\alpha)^2}(\beta^2 y^2 + z^2 + 2\beta\boldsymbol{yz}) + \boldsymbol{x}^2 - \dfrac{a_\beta}{\beta}\boldsymbol{yz} \\ \qquad\qquad \text{if } h(x,t) \geq 0, \psi(s,F) \leq 0, tr(F) = 0, \\ \dfrac{-h}{(1-s)\alpha(\beta-\alpha)^2}(\alpha^2\boldsymbol{y}^2 + \boldsymbol{z}^2 + 2\alpha\boldsymbol{yz}) + \boldsymbol{x}^2 - \dfrac{a_\alpha}{\alpha}\boldsymbol{yz}, \\ \qquad\qquad \text{if } h(x,t) \leq 0, \psi(s,F) \leq 0, tr(F) = 0, \\ \dfrac{1}{s(1-s)(\beta-\alpha)^2}\Big(\big((1-s)\beta^2(\alpha+a_\alpha) + s\alpha^2(\beta+a_\beta)\big)\boldsymbol{y}^2 \\ \qquad + \big((1-s)(\alpha+a_\alpha) + s(\beta+a_\beta)\big)\boldsymbol{z}^2 \\ \qquad + 2\big((\alpha+a_\alpha)\beta - s\beta\big)\boldsymbol{yz}\Big) - \det F \\ \qquad\qquad \text{if } \psi(s,F) \geq 0, tr(F) = 0, \\ +\infty \qquad\qquad \text{if } tr(F) \neq 0 \end{cases}$$

We claim that in fact this is an exact value. This amounts to finding laminates which yield this same optimal value.

## 2.4 Optimal microstructures: laminates

We have the lower bound given by the polyconvexification, and we will show that this bound is in fact attained. To this end, we seek an optimal microstructure (a laminate) whose second moments recover the value of the bound.

We try to find $\nu = s\nu_\alpha + (1-s)\nu_\beta$, a laminate with $supp(\nu_\gamma) \subset \Lambda_\gamma$, $\gamma = \alpha, \beta$, $s \in (0,1)$, and first moment $F$. We have different optimal conditions depending of the sign of $\psi$ and $h$, and we analyze different cases accordingly.

### 2.4.1 Case $\psi \geq 0$

We start with the case when $\psi(F,s) \geq 0$ holds. In this case the optimal conditions (2.12) tell us that

$$S_{\alpha,2} = y_\alpha^2, \quad S_{\beta,2} = y_\beta^2$$

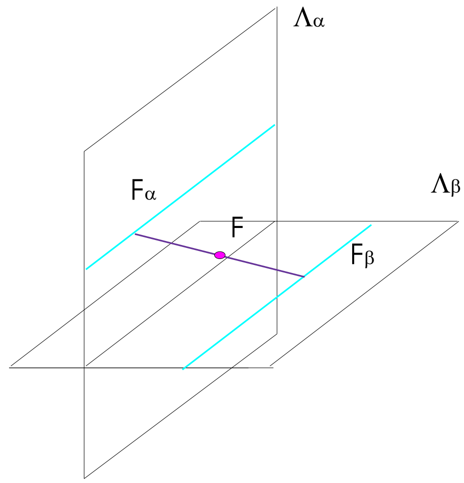



*Figura 2.2: Infinite decompositions of $F$.*

and therefore, by the strict convexity of the square function, we can deduce that

$$\nu_{\gamma}^{(2)} = \delta_{y_\gamma}, \gamma = \alpha, \beta.$$

Hence

$$F_\alpha = (\lambda, y_\alpha, -\alpha y_\alpha), \;\; F_\beta = \left(\frac{\boldsymbol{x} - s\lambda}{1-s}, y_\beta, -\beta y_\beta\right), \tag{2.13}$$

with $\lambda \in \mathbb{R}$ arbitrary. This means that for every $\lambda \in \mathbb{R}$ we can decompose $F$ as a convex combination of two matrices in $\Lambda_\alpha, \Lambda_\beta$ respectively, and satisfying the volume constraint, see Figure 2.2.

The next step is to check that there exist some $\lambda \in \mathbb{R}$ such that $rank(F_\alpha - F_\beta) = 1$. After some algebra, we can write

$$rank(F_\alpha - F_\beta) = 1 \Leftrightarrow C_{F,s}(\lambda) = 0$$

where

$$C_{F,s}(\lambda) = -detF - s(\lambda^2 - \alpha y_\alpha^2) - (1-s)((\frac{F_{11} - s\lambda}{1-s})^2 - \beta y_\beta^2)$$

is a second degree polynomial on $\lambda$. It is easy to check that the discriminant of $C_{F,s}$ is $\psi(F,s)$, and so that their roots are

$$\lambda_i = x + (-1)^i \sqrt{\frac{1-s}{s}\psi(F,s)} \quad i = 1,2.$$

Therefore for all pair $(F,s)$ such that $\psi(F,s) \geq 0$, there exist two first order laminate

$$\nu = s\delta_{F_{\alpha,i}} + (1-s)\delta_{F_{\beta,i}} \quad i = 1,2$$

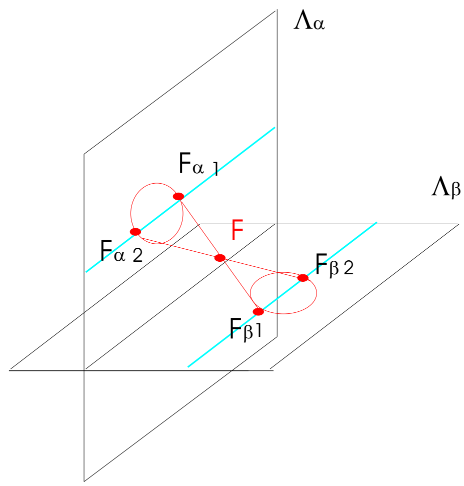



*Figura 2.3: Two first order laminates*

where

$$F_{\alpha,i} = \begin{pmatrix} \lambda_i & F_{\alpha,12} \\ -\alpha F_{\alpha,12} & -\lambda_i \end{pmatrix}, \quad F_{\beta,i} = \begin{pmatrix} \frac{x-s\lambda_i}{1-s} & F_{\beta,12} \\ -\beta F_{\beta,12} & -\frac{x-s\lambda_i}{1-s} \end{pmatrix}$$

and they provide the optimal value of the polyconvexification.

Thanks to the spatial identification $F = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$, we can observe the above computations from a geometric point of view (see Figure 2.3). For any matrix $F = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ the determinant is $\det F = -(\boldsymbol{x}^2 + \boldsymbol{y}\boldsymbol{z})$, this means that for any matrix $F$ there exist a cone $\{\boldsymbol{x}^2 + \boldsymbol{y}\boldsymbol{z} = 0\}$ of rank one directions through this matrix. From optimality conditions we obtain an explicit identification of $F_\gamma$ $\gamma = \alpha, \beta$, up to the first component (2.13), which let us a degree of freedom in the search of the optimal decomposition. Geometrically, we can observe that the intersection between the manifolds $\Lambda_\gamma$ and the rank one cone are ellipses, whose intersection with the admissible $F_\gamma$ are two points $F_{\gamma,i}$, $\gamma = \alpha, \beta$ and $i = 1, 2$.

### 2.4.2 Case $\psi \leq 0$

We study now the other case, $\psi(F, s) < 0$. In this situation, we have two different optimal conditions depending of the sign of $h$. We treat the case $h \geq 0$. The other case is similar.

From the optimal condition for this case (2.10), we have

$$S_{\alpha,2} = y_\alpha^2, \quad S_1 = \boldsymbol{x}^2$$

and by using similar arguments as above, we can deduce

$$\nu_\alpha^{(2)} = \delta_{y_\alpha}, \quad \nu^{(1)} = \delta_x$$

where

1. $\nu_\alpha = \delta_{F_\alpha}$ with
$$F_\alpha = (\boldsymbol{x}, y_\alpha, -\alpha y_\alpha), \tag{2.14}$$

2. by using that $F$ is the first moment of $\nu$, there exists a unique decomposition
$$F = sF_\alpha + (1-s)F_\beta$$
with $F_\gamma \in \Lambda_\gamma, \quad \gamma = \alpha, \beta$ where $F_\alpha$ is of the form just indicated, and
$$F_\beta = (\boldsymbol{x}, y_\beta, -\beta y_\beta). \tag{2.15}$$

Consider a pair $(F,s)$ such that $\psi(F,s) < 0$. After an elementary manipulation, we get

$$\begin{gathered}\psi(F,s) \le 0 \Longleftrightarrow \\ -(\beta-\alpha)^2 \boldsymbol{y}\boldsymbol{z} s^2 + \Big(\alpha\beta(\alpha-\beta)\boldsymbol{y}^2 + (\beta-\alpha)\boldsymbol{z}^2 + (\beta-\alpha)^2\boldsymbol{y}\boldsymbol{z}\Big)s \\ +\Big(\alpha\beta^2\boldsymbol{y}^2 + \alpha\boldsymbol{z}^2 + 2\alpha\beta\boldsymbol{y}\boldsymbol{z}\Big) \le 0.\end{gathered}$$

Let $P_F(s)$ be this second degree polynomial in $s$ for fixed $F$. The set where $\psi(F,s) \le 0$ is the set where $P_F$ has solutions in $[0,1]$, and $s$ lies between those two solutions. There exist real solutions if the discriminant is nonnegative, and, in addition, it is easy to check that $P_F(0), P_F(1)$ are positive[1] if $F \notin \Lambda_\alpha \cup \Lambda_\beta$. Therefore there are positive solutions if $P_F$ is decreasing at 0.

After some algebra the discriminant is

$$\begin{aligned} g(F) = \alpha^2\beta^2\boldsymbol{y}^4 + \boldsymbol{z}^4 + (\alpha^2 + 4\beta\alpha + \beta^2)\boldsymbol{y}\boldsymbol{z} \\ +2\alpha\beta\boldsymbol{y}^3\boldsymbol{z} + 2(\alpha+\beta)\boldsymbol{z}^3\boldsymbol{y} \ge 0,\end{aligned}$$

and the decreasing condition

$$h(F) = (\alpha+\beta)\boldsymbol{y}\boldsymbol{z} + \alpha\beta\boldsymbol{y}^2 + \boldsymbol{z}^2 \le 0.$$

Therefore the set of pairs $(F,s)$ where $\psi(F,s) \le 0$ can be described as

$$\{(F,s) \in M^{2\times 2} \times \mathbb{R} : g(F) \ge 0,\ h(F) \le 0,\ s \in (r_1, r_2)\}$$

where

$$r_i = \frac{1}{2} - \frac{1}{2(\beta-\alpha)\boldsymbol{y}\boldsymbol{z}}\Big(\alpha\beta\boldsymbol{y}^2 - \boldsymbol{z}^2 + (-1)^i\sqrt{g(F)}\Big) \quad i = 1,2.$$

[1] $P_F(0) = \alpha|\beta\boldsymbol{y} + \boldsymbol{z}|^2, P_F(1) = \beta|\alpha\boldsymbol{y} + \boldsymbol{z}|^2$

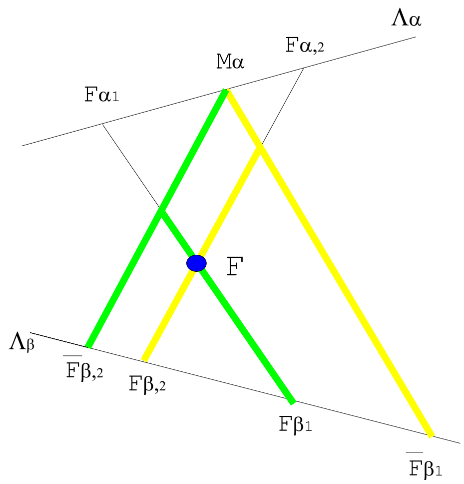



*Figura 2.4: Second order laminates.*

We thus have a characterization of the set $\psi(F,s) \leq 0$. We now look for rank-one connections between both manifolds.

We would like to write

$$F = rB_\alpha + (1-r)B_\beta$$

with $r \in (0,1)$, $B_\gamma \in \Lambda_\gamma$, $(B_\gamma)_1 = x$, $\gamma = \alpha, \beta$, and $rank(B_\alpha - B_\beta) = 1$.

On the one hand,

$$\left.\begin{array}{l} B_\gamma \in \Lambda_\gamma \\ (B_\gamma)_1 = \boldsymbol{x} \end{array}\right\} \Rightarrow B_\gamma = (\boldsymbol{x}, y_\gamma, -\gamma y_\gamma) \quad \gamma = \alpha, \beta.$$

The constraint on the vanishing determinant can be rewritten, after some manipulation, as

$$P_F(r) = 0,$$

whose roots are $r_i$. We can therefore guarantee that there exist two rank-one directions between $\Lambda_\alpha$ and $\Lambda_\beta$ with barycenter $F$.

We are now in a position to find an optimal second order laminate which recovers the lower bound given by the polyconvexification. We take $\nu_\alpha = \delta_{F\alpha}$ and $\nu_\beta$ as a convex combination of two Dirac masses supported in the $\beta$ manifold (see Figure 2.4).

Put

$$F_{\beta,i} = (\boldsymbol{x}, y_{\beta,i}, -\beta y_{\beta,i})$$

with

$$y_{\beta,i} = \frac{-1}{(1-r_i)(\beta-\alpha)}(\alpha\boldsymbol{y} + \boldsymbol{z}), \quad i = 1, 2.$$

Since $r_1 \leq s \leq r_2$, it is clear that $y_\beta$ is a convex combination of $F_{\beta,i}, i = 1, 2$.

If we consider $\bar{F}_{\beta,i} = F_\alpha + l_i(F_{\beta,i} - F_{\alpha,i})$ with $l_i$ such that $\bar{F}_{\beta,i} \in \Lambda_\beta$, and take

$$l_i = \frac{r_i}{s}, \quad \rho_{i,j} = \frac{(1-r_j)(r_i - s)}{r_i - r_j}, \quad \tau_{i,j} = \frac{(r_j - s)(r_i - 1)}{r_j(1+r_i) + s(1-r_j)},$$

we can define the second-order laminate with support on $\Lambda_\alpha \cup \Lambda_\beta$, barycenter $F$, and mass in $\Lambda_\alpha$ equal to $s$, by putting

$$\nu_{i,j} = \tau_{i,j}\delta_{F_{\beta,i}} + (1-\tau_{i,j})(\rho_{i,j}\delta_{\bar{F}_{\beta,j}} + (1-\rho_{i,j})\delta_{F_\alpha})$$

with $i, j \in \{1, 2\}$, $i \neq j$ where,

$$\det(\bar{F}_{\beta,j} - F_\alpha) = 0$$

and

$$\det(F_{\beta,i} - (\rho_{i,j}\bar{F}_{\beta,j} + (1-\rho_{i,j})F_\alpha)) = 0.$$

Again, using the spatial identification we can interpret geometrically the above analytical computations. We lost the degree of freedom of the first component of the matrices $F_\alpha$ and $F_\beta$, since these matrices are explicitly determined by (2.14) and (2.15), and their first component is $\boldsymbol{x}$ in both cases. This fact lets us simplify the spatial situation to a 2-d case in the plane determined by the first component equal to $\boldsymbol{x}$. The intersection between the manifolds $\Lambda_\gamma$ - the cone of rank one directions - and that reduces to two matrices in each manifold, which we noted by $F_{\gamma,i}$. From these matrices connected by rank one directions we can obtain a second order laminate with volume fraction $s$ on $\Lambda_\alpha$ and $(1-s)$ on $\Lambda_\beta$. This construction is shown in the Figure 2.4, where the spatial situation is reduced to the plane of the first component equal to $\boldsymbol{x}$. A similar result holds for the other point where the optimal value is attained ($h(x,s) \leq 0$).

We summarize all of these computations of optimal laminates leading to the relaxed integrand $\varphi$.

When $\psi(F,s) \geq 0$ there exist two optimal first-order laminates leading to the value of the relaxed integrand $\varphi$

$$\nu = s\delta_{F_{\alpha,i}} + (1-s)\delta_{F_{\beta,i}} \qquad i = 1, 2$$

where,

$$F_{\alpha,i} = \begin{pmatrix} \lambda_i & F_{\alpha,12} \\ -\alpha F_{\alpha,12} & -\lambda_i \end{pmatrix}, \quad F_{\beta,i} = \begin{pmatrix} \frac{F_{11}-s\lambda_i}{1-s} & F_{\beta,12} \\ -\beta F_{\beta,12} & -\frac{F_{11}-s\lambda_i}{1-s} \end{pmatrix}$$

with

$$\lambda_i = F_{11} + (-1)^i\sqrt{\frac{1-s}{s}\psi(F,s)} \quad i = 1, 2,$$

$$F_{\alpha,12} = \frac{1}{s(\beta-\alpha)}(\beta F_{12} + F_{21}), \quad F_{\beta,12} = \frac{-1}{(1-s)(\beta-\alpha)}(\alpha F_{12} + F_{21}).$$

When $\psi(F,s) \leq 0$ and $h(x,t) \geq 0$, there exist two optimal second-order laminates

$$\nu_{i,j} = \tau_{i,j}\delta_{F_{\beta,i}} + (1-\tau_{i,j})(\rho_{i,j}\delta_{\bar{F}_{\beta,j}} + (1-\rho_{i,j})\delta_{F_\alpha})$$

with $i, j \in \{1,2\}$, $i \neq j$ where the scalars are

$$\rho_{i,j} = \tfrac{(1-r_j)(r_i-s)}{r_i-r_j}, \quad \tau_{i,j} = \tfrac{(r_j-s)(r_i-1)}{r_j(1+r_i)+s(1-r_j)}$$

and the matrices are

$$F_\alpha = \begin{pmatrix} F_{11} & F_{\alpha,12} \\ -\alpha F_{\alpha,12} & -F_{11} \end{pmatrix}, \quad F_{\beta,i} = \begin{pmatrix} F_{11} & F_{\beta,12,i} \\ -\beta F_{\beta,12,i} & -F_{11} \end{pmatrix}$$

with

$$F_{\beta,12,i} = \frac{-1}{(1-r_i)(\beta-\alpha)}(\alpha F_{12} + F_{21}), \quad i = 1,2$$

$$r_i \frac{1}{2} - \frac{1}{2(\beta-\alpha)F_{12}F_{21}}\left(\alpha\beta|F_{12}|^2 - |F_{21}|^2 + (-1)^i\sqrt{g(F)}\right) \quad i = 1,2$$

$$\bar{F}_{\beta,i} = F_\alpha + l_i(F_{\beta,i} - F_{\alpha,i}), \quad l_i = \frac{r_i}{s}.$$

Similarly, when $\psi(F,s) \leq 0$ and $h(x,t) \leq 0$, the optimal microstructure is another second-order laminate given by

$$\nu_{i,j} = \tau_{i,j}\delta_{F_{\alpha,i}} + (1-\tau_{i,j})(\rho_{i,j}\delta_{\bar{F}_{\alpha,j}} + (1-\rho_{i,j})\delta_{F_\beta})$$

with $i, j \in \{1,2\}$, $i \neq j$ where the scalars are

$$\rho_{i,j} = \tfrac{r_j(r_i-s)}{r_i-r_j}, \quad \tau_{i,j} = \tfrac{(s-r_j)r_i}{r_i(r_j-1)+r_j(1-s)},$$

and the matrices involved are

$$F_\beta = \begin{pmatrix} F_{11} & F_{\beta,12} \\ -\beta F_{\beta,12} & -F_{11} \end{pmatrix}, \quad F_{\alpha,i} = \begin{pmatrix} F_{11} & F_{\alpha,12,i} \\ -\beta F_{\alpha,12,i} & -F_{11} \end{pmatrix}$$

with

$$F_{\alpha,12,i} = \frac{1}{r_i(\beta-\alpha)}(\beta F_{12} + F_{21}), \quad i = 1,2$$

$$\bar{F}_{\alpha,i} = F_\beta - l_i(F_{\beta,i} - F_{\alpha,i}), \quad l_i = \frac{1-r_i}{1-s}.$$

## 2.5 Some particular examples

In this section we would like to emphasize some particular examples where, by using Theorem 1, we can compute explicitly the relaxed cost functional.

**Example 1 -** An interesting and academic example case is the compliance (which means the the work done by the loads), for which we take $a_\alpha(t,x) = \alpha$, $a_\beta(t,x) = \beta$ so that $h \equiv 0$, the cost functional can be written as

$$\int_0^T \int_\Omega \left[u_t^2(t,x) + (\alpha\chi + \beta(1-\chi))u_x^2(t,x)\right] dxdt,$$

and the *constrained quasiconvexification* is

$$\varphi(F,s) \begin{cases} F_{11}^2 - F_{12}F_{21} & \text{if } \psi(s,F) \le 0, \\ -\det F + \dfrac{1}{s(1-s)(\beta-\alpha)^2}\Big(2\alpha\beta(s\alpha + (1-s)\beta)\Big)|F_{12}|^2 & \\ \qquad +2\big((1-s)\alpha + s\beta\big)|F_{21}|^2 + 4\alpha\beta F_{12}F_{21}\Big) & \\ & \text{if } \psi(s,F) \ge 0. \end{cases}$$

**Example 2 -** Another interesting case result when we take $a(t,x,\chi) = 1$, the most simple quadratic cost function. In this case the relaxed cost functional is

$$\int_0^T \int_\Omega \left[u_t^2(t,x) + u_x^2(t,x)\right] dxdt,$$

and therefore $a_\alpha(t,x) = a_\beta(t,x) = 1$. Hence

$$h(t,x) = \beta - \alpha,$$

and the *constrained quasiconvexification* simplifies to

$$\varphi(F,s) = \begin{cases} \dfrac{1}{s\beta(\beta-\alpha)}(s\beta(\beta-\alpha)|F_{11}|^2 + \beta^2|F_{12}|^2 + |F_{21}|^2 & \\ +(s\alpha + \beta(2-s))F_{12}F_{21}), & \text{if } \psi(s,F) \le 0, \\ \dfrac{1}{s(1-s)(\beta-\alpha)^2}\Big((1-s)\beta^2(\alpha+1) + s\alpha^2(\beta+1)\Big)|F_{12}|^2 & \\ +\big((1-s)\alpha + s\beta + 1\big)|F_{21}|^2 + 2\big(\beta(1-s) + \alpha(s+\beta)\big)F_{12}F_{21}\Big) & \\ \qquad -\det F, & \text{if } \psi(s,F) \ge 0. \end{cases} \tag{2.16}$$

**Example 3 -** The last case lies in the border line for our computations to be valid. We take $a_\alpha(t,x) = -\alpha$ and $a_\beta(t,x) = -\beta$ so that $h$ identically vanishes. The cost functional is

$$\int_0^T \int_\Omega \left[u_t^2(t,x) - (\alpha\chi + \beta(1-\chi))u_x^2(t,x)\right] dxdt,$$

and for this case the relaxed integrand surprisingly is -det (recall the restriction on the trace)

$$\varphi(F, s) = F_{11}^2 + F_{12}F_{21} = -\det F.$$

## 2.6 Analysis of $(RP)$ in the quadratic case

In this section we would like to analyze for the quadratic case, the relaxed problem obtained in the last sections. We will analyze the problem $(RP)$ where the cost density is given in the previous section by (2.16).

From the previous sections we know that this problem admits optimal solutions, and moreover we know that such optimal solutions are first or second-order laminates depending on the sign of the function $\psi$. An important fact which we can observe is that all functions involved are quadratic in the vector gradient variable and therefore regular, but the presence of gradients and the pointwise constraint make the problem difficult to analyze.

One common way is look at optimality conditions introducing several multipliers to keep track of restrictions, but this makes the problem more difficult in the sense that we have to solve a system of partial differential equations. In this sense we follow a similar strategy as in [21]. The pointwise constraint given by $\psi$ depends only on the variables $F_{12}, F_{21}$ , therefore the way is to find some relationship between these two variables, it is elementary to check the following result :

**Lema 1** *For fixed s, the optimal solution of the quadratic, mathematical programming problem*

$$\textit{Minimize in } F_{(21)} : \varphi(F, s)$$

*occurs when*

$$(\alpha s + \beta(1 - s))F_{12} + F_{21} = 0.$$

*In addition, the associated optimal microstructures are first-order laminates with volume fraction s for the $\alpha$-material and orientation of layers always vertical (along the time axis). Having in mind the trace condition $F_{11}+F_{22} = 0$ the optimal value is*

$$F_{11}^2 + F_{12}^2. \tag{2.17}$$

The idea is then to replace the complicated cost function $\varphi$ by the expression (2.17) and then to minimize under the constraints

$$(\alpha s + \beta(1 - s))F_{12}(t, x) + F_{21}(t, x) = 0, \qquad F_{11}(t, x) + F_{22}(t, x) = 0$$

i.e.

$$\begin{pmatrix} F_{11}(t, x) \\ -[\alpha s(t, x) + \beta(1 - s(t, x))]F_{12}(t, x) \end{pmatrix} = TF^{(2)}(x, t), a.e.(t, x) \in [0, T] \times \Omega$$

which is equivalent to

$$div \begin{pmatrix} F_{11}(t,x) \\ -[\alpha s(t,x) + \beta(1 - s(t,x))]F_{12}(t,x) \end{pmatrix} = 0$$

Therefore we can write the minimization problem in terms of the original variable $U^{(1)} = u$ leading to the new relaxed problem (stated in Conjecture 1): $(\widetilde{RP})$

$$\min_{s} \int_0^T \int_\Omega u_t^2(t,x) + u_x^2(t,x)\, dx\, dt$$

where $u$ is the unique solution of

$$\begin{aligned} u_{tt} - div([\alpha s + \beta(1-s)]u_x) = 0 \quad &\text{in} \quad (0,T) \times (0,1), \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) \quad &\text{in} \quad \Omega, \\ u(t,0) = f(t),\ u(t,1) = g(t) \quad &\text{in} \quad [0,T], \end{aligned}$$

This new problem may be seen as the continuous version of the original design problem in which the function $\chi(x,t)$ is replaced by the continuous function $s(x,t)$. We cannot prove directly that the above problem admits optimal solutions, tough we claim, by our conjecture that it indeed does because of the particular form of the problem and not as a consequence of general results. We support numerically our conclusion in the next section. Analytically, we can assert the following result.

**Lema 2** *The equalities*

$$\text{ínf}(P) = \text{ínf}(\widetilde{RP}) = \text{mín}(RP)$$

*hold.*

**Proof.** It is easy to see that

$$\text{ínf}(P) \geq \text{ínf}(\widetilde{RP})$$

and

$$\text{ínf}(\widetilde{RP}) \geq \text{mín}(RP)$$

and using the relaxation Theorem 7

$$\text{ínf}(P) = \text{mín}(RP)$$

holds, therefore we have all equalities. ■

## 2.7 Numerical simulations

We address the numerical resolution of the problem stated in Conjecture 1 for $a_\alpha(t,x) = 1$ and $a_\beta(t,x) = 1$:

$$(\widetilde{RP}) : \quad \inf_{s \in S_{V_\alpha}} \widetilde{I}(s) = \int_0^T \int_\Omega (u_t^2 + a(t,x,s)u_x^2)\, dxdt,$$

with $a(t,x,s) = sa_\alpha(t,x) + (1-s)a_\beta(t,x)$ and

$$S_{V_\alpha} = \{s \in L^\infty((0,T)\times\Omega;[0,1]), \int_\Omega s(t,x)dx \le V_\alpha|\Omega|, \forall t \in [0,T]\},$$

and $u = u(s)$ is the unique solution of

$$\begin{aligned} & -div(u_t, -[s\alpha + (1-s)\beta]u_x) = 0 && \text{in } (0,T)\times\Omega, \\ & u(t,0) = f(t),\ u(t,1) = g(t) && \text{in } (0,T), \\ & u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) && \text{in } \Omega. \end{aligned} \tag{2.18}$$

We first describe the algorithm of minimization and then present some numerical experiments.

### 2.7.1 Algorithm of minimization

We briefly present the resolution of the relaxed problem $(\widetilde{RP})$ using a gradient descent method. In this respect, we compute the first variation of the cost function. In order to simplify the expression, we take $f \equiv 0$ and $g \equiv 0$.

For any $\eta \in \mathbb{R}^+$, $\eta << 1$, and any $s_1 \in L^\infty((0,T)\times\Omega)$, we associate with the perturbation $s^\eta = s + \eta s_1$ of $s$ the derivative of $\widetilde{I}$ with respect to $s$ in the direction $s_1$ as follows :

$$\frac{\partial \widetilde{I}(s)}{\partial s} \cdot s_1 \lim_{\eta\to 0} \frac{\widetilde{I}(s+\eta s_1) - \widetilde{I}(s)}{\eta}.$$

We obtain the following result.

**Teorema 8** *If $(u_0,u_1) \in (H^2(\Omega)\cap H_0^1(\Omega)) \times H_0^1(\Omega)$, then the derivative of $\widetilde{I}$ with respect to $s$ in any direction $s_1$ exists and takes the form*

$$\frac{\partial \widetilde{I}(s)}{\partial s} \cdot s_1 = \int_0^T \int_\Omega s_1 \Big( (a_\alpha - a_\beta)u_x^2 + (\alpha - \beta)u_x p_x \Big)\, dxdt \tag{2.19}$$

*where $u$ is the solution of (2.18) and $p$ is the solution in $C^1([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$ of the adjoint problem*

$$\begin{cases} div(p_t, -[s\alpha + (1-s)\beta]p_x) = div(u_t, a(t,x,s)u_x) & in \quad (0,T)\times\Omega, \\ p = 0 & on \quad (0,T)\times\partial\Omega, \\ p(T,x) = 0, \quad p_t(T,x) = u_t(T,x) & in \quad \Omega. \end{cases} \tag{2.20}$$

*Sketch of the proof.* Let us explain briefly how we obtain the expression (2.19). We introduce the lagrangian

$$\mathcal{L}(s,\phi,\psi) = \int_0^T\int_\Omega (\phi_t^2 + a(t,x,s)\phi_x^2)\,dxdt \\ + \int_0^T\int_\Omega \Big[\phi_{tt} - div([\alpha s + \beta(1-s)]\phi_x)\Big]\psi\,dxdt$$

for any $s \in L^\infty((0,T)\times\Omega)$, $\phi \in C([0,T];H^2(\Omega)\cap H_0^1(\Omega))\cap C^1([0,T];H_0^1(\Omega))$ and $\psi \in C([0,T];H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))$ and then write formally that

$$\frac{d\mathcal{L}}{ds}\cdot s_1 = \frac{\partial}{\partial s}\mathcal{L}(s,\phi,\psi)\cdot s_1 + < \frac{\partial}{\partial \phi}\mathcal{L}(s,\phi,\psi), \frac{\partial \phi}{\partial s}\cdot s_1 > \\ + < \frac{\partial}{\partial \psi}\mathcal{L}(s,\phi,\psi), \frac{\partial \psi}{\partial s}\cdot s_1 > .$$

The first term is

$$\frac{\partial}{\partial s}\mathcal{L}(s,\phi,\psi)\cdot s_1 = \int_0^T\int_\Omega s_1\Big((a_\alpha - a_\beta)\phi_x^2 + (\alpha-\beta)\phi_x\psi_x\Big)\,dxdt \tag{2.21}$$

for any $s,\phi,\psi$ whereas the third term is equal to zero if $\phi = u$ solution of (2.18). We then determine the solution $p$ so that, for all $\phi \in C([0,T];H^2(\Omega)\cap H_0^1(\Omega))\cap C^1([0,T];H_0^1(\Omega))$, we have

$$< \frac{\partial}{\partial \phi}\mathcal{L}(s,\phi,p), \frac{\partial \phi}{\partial s}\cdot s_1 >= 0,$$

which leads to the formulation of the adjoint problem (2.20). Next, writing that $\widetilde{I}(s) = \mathcal{L}(s,u,p)$, we obtain (2.19) from (2.21). ■

In order to take into account the volume constraint on $s$, we introduce the Lagrange multiplier function $\gamma \in L^\infty((0,T);\mathbb{R})$ and the functional

$$\widetilde{I}_\gamma(s) = \widetilde{I}(s) + \int_0^T \gamma(t)\int_\Omega s(t,x)dxdt.$$

Using Theorem 8, we obtain that the derivative of $J_\gamma$ is

$$\frac{\partial \widetilde{I}_\gamma(s)}{\partial s}\cdot s_1 = \int_0^T\int_\Omega s_1((a_\alpha - a_\beta)u_x^2 + (\alpha-\beta)u_x p_x)\,dxdt + \int_0^T \gamma(t)\int_\Omega s_1 dxdt$$

which permits to define the following descent direction :

$$s_1(x,t) = -((a_\alpha - a_\beta)u_x^2 + (\alpha-\beta)u_x p_x + \gamma(t)), \quad \forall x \in \Omega, \forall t \in (0,T). \tag{2.22}$$

Consequently, for any function $\eta \in L^\infty(\Omega\times(0,T),\mathbb{R}^+)$ with $||\eta||_{L^\infty(\Omega\times(0,T))}$ small enough, we have $\widetilde{I}_\gamma(s+\eta s_1) \le \widetilde{I}_\gamma(s)$. The multiplier function $\gamma$ is then determined in order that, for any function $\eta \in L^\infty(\Omega \times (0,T), \mathbb{R}^+)$, $||s+\eta s_1||_{L^1(\Omega)} = V_\alpha|\Omega|$ leading to, for all $t \in (0,T)$,

$$\gamma(t) = \frac{\left(\int_\Omega s(t,x)dx - V_\alpha|\Omega|\right) - \int_\Omega \eta(t,x)\Big((a_\alpha - a_\beta)u_x^2 + (\alpha-\beta)u_x p_x\Big)\,dx}{\int_\Omega \eta(t,x)dx}. \tag{2.23}$$

At last, the function $\eta$ is chosen so that $s+\eta s_1 \in [0,1]$, for all $x \in \Omega$ and $t \in (0,T)$. A simple and efficient choice consists in taking $\eta(t,x) = \varepsilon s(t,x)(1-s(t,x))$ for all $x \in \Omega$ and $t \in (0,T)$ with $\varepsilon$ small real positive. Consequently, the descent algorithm to solve numerically the relaxed problem $(\widetilde{RP})$ may be structured as follows :

Let $\Omega \subset \mathbb{R}^N$, $(u_0,u_1) \in (H^2(\Omega)\cap H^1_0(\Omega)) \times H^1_0(\Omega)$, $L \in (0,1)$, $T>0$, and $\varepsilon < 1$, $\varepsilon_1 << 1$ be given.

- Initialization of the density function $s^0 \in L^\infty(\Omega; ]0,1[)$;
- For $k \geq 0$, iteration until convergence (i.e.$|\widetilde{I}(s^{k+1}) - \widetilde{I}(s^k)| \leq \varepsilon_1 |\widetilde{I}(s^0)|$) as follows :
  - Computation of the solution $u_{s^k}$ of (2.18) and then the solution $p_{s^k}$ of (2.20), both corresponding to $s = s^k$.
  - Computation of the descent direction $s_1^k$ defined by (2.22) where the multiplier $\gamma^k$ is defined by (2.23).
  - Update the density function in $\Omega$:

    $$s^{k+1} = s^k + \varepsilon s^k(1-s^k)s_1^k$$

    with $\varepsilon \in \mathbb{R}^+$ small enough in order to ensure the decrease of the cost function and $s^{k+1} \in L^\infty(\Omega \times (0,T), [0,1])$.

### 2.7.2 Numerical experiments in the quadratic case

In this section, we present some numerical simulations for $\Omega = (0,1)$ in the quadratic case, i.e, $(a_\alpha, a_\beta) = (1,1)$. On a numerical viewpoint, we highlight that the numerical resolution of the descent algorithm is *a priori* difficult because the descent direction (2.22) depends on the derivative of $u$ and $p$, both solution of a wave equation with space and time coefficients only in $L^\infty((0,T)\times\Omega; \mathbb{R}^\star_+)$. To the knowledge of the authors, there does not exist any numerical analysis for this kind of equation. We use a $C^0$-finite element approximation for $u$ and $p$ with respect to $x$ and a finite difference centered approximation with respect to $t$. Moreover, we add a vanishing viscosity and dispersive term of the type $\epsilon^2 \mathrm{div}([s\alpha + (1-s)\beta]u_{xtt})$ with $\epsilon$ of order of $h$ - the space discretization parameter. This term has the effect to regularize the descent term (2.22) and to lead to a convergent algorithm. At last, this provides an implicit and unconditionally stable scheme, consistent with (2.18) and (2.20), and of order two in time and space.

In the sequel, we treat the following two simple and smooth initial conditions on $\Omega = (0, 1)$:

- **Case 1** : $u_0(x) = \sin(\pi x), \quad u_1(x) = 0 \quad ;$
- **Case 2** : $u_0(x) = \exp^{-80(x-0,5)^2}, \quad u_1(x) = 0,$

and we discuss the results with respect to the value of $\alpha, \beta$ and $V_\alpha$. Results are obtained with $h = dt = 10^{-2}$, $\varepsilon_1 = 10^{-5}$, $s^0(t, x) = V_\alpha$ on $[0, T] \times \Omega$ and $\varepsilon = 10^{-2}$ (see the algorithm).

### Case 1

We first consider the case 1, with $T = 2$ and $(\alpha, \beta) = (1, 1{,}1)$. Figure 2.5 depicts the iso-values of the optimal limit density $s^{lim}$ (obtained at the convergence of the descent algorithm) for $V_\alpha = 0{,}3$ (top the figure) and $V_\alpha = 0{,}5$ (bottom of the figure) respectively. For these values of $\alpha$ and $\beta$, we observe that the limit densities are "almost" characteristics functions taking either the value 0 or the value 1. As a consequence the relaxed problem $(\widetilde{RP})$ coincides with the original one $(P)$ which is well-posed in the class of characteristics function. This suggests that Conjecture 1 is true in this case. Moreover, we observe that the limit densities are independent of the choice of the initialization $s^0$. This suggests that $\widetilde{I}$ admits a unique minimum.

Figure 2.6 represent the corresponding evolution of the energy $E(t) = 1/2 \int_\Omega (y_t^2 + y_x^2) dx$ with respect to $t$. The fact that, the coefficients of the state equation depend on the time variable (furthermore on space) is the main reason because the corresponding system is not conservative nor strictly dissipative in general. However, for a long time period, the optimization procedure with respect to $s$ leads to a damping mechanism (proportional to the gap $\beta - \alpha$).

Results are qualitatively different if we now consider a larger gap $\beta - \alpha$. Figures 2.7 and 2.8 represent the result obtained with $(\alpha, \beta) = (1, 6)$. We observe that the limit densities are no more characteristic functions and take value in $(0, 1)$. This clearly indicates that the original problem may be no well-posed and justify the search of a relaxed formulation. We also observe that this property depends also on the value of $V_\alpha$: for $V_\alpha$ or $1 - V_\alpha$ arbitrarily small, numerical simulation leads to bi-valued limit densities. Furthermore, in the case $(\alpha, \beta) = (1, 6)$, we observe Figure 2.15 the strong damping mechanism of the optimal distribution and explain why, for $t$ sufficiently large, the value of the cost function is less sensitive to the density $s$ (i.e. for $t$ large, the variations of $s$ with respect to $x$ and $t$ are low).

We have also observed that as soon as the gap is large enough, the limit of the density depends on the initialization $s^0$ highlighting the existence of

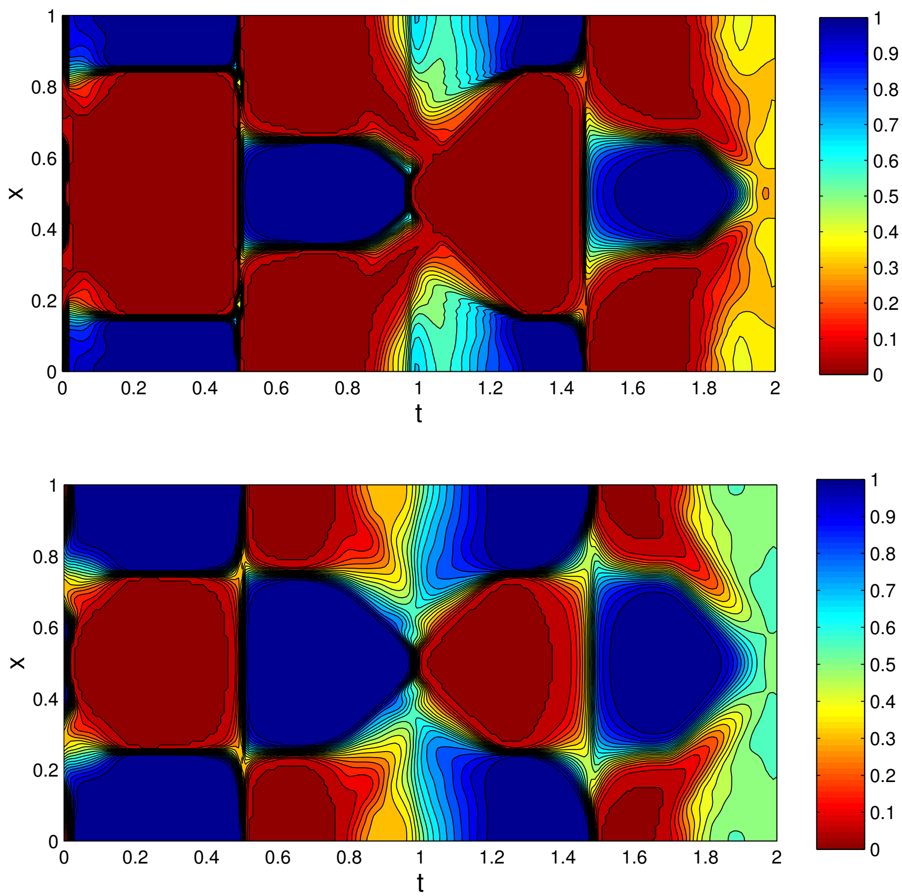



*Figura 2.5:* **Case 1** - $T = 2$, $(\alpha, \beta) = (1, 1{,}1)$ - *Iso-values of the limit density - Top : $V_\alpha = 0{,}3$ - $\tilde{I}(s^{lim}) \approx 9{,}7451$ - Bottom: $V_\alpha = 0{,}5$ - $\tilde{I}(s^{lim}) \approx 9{,}5613$.*

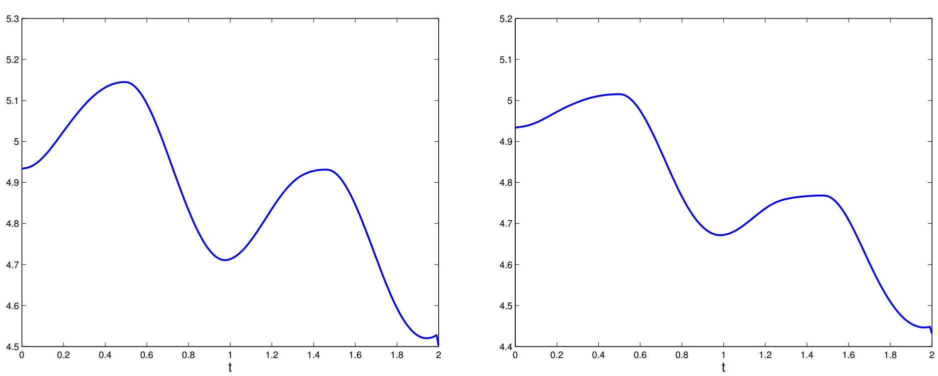



*Figura 2.6:* **Case 1** - $T = 2$, $(\alpha, \beta) = (1, 1{,}1)$ - $E(t)$ *vs.* $t$ - *Left :* $V_\alpha = 0{,}3$ - $\widetilde{I}(s^{lim}) \approx 9{,}7451$ - *Right :* $V_\alpha = 0{,}5$ - $\widetilde{I}(s^{lim}) \approx 9{,}5613$.

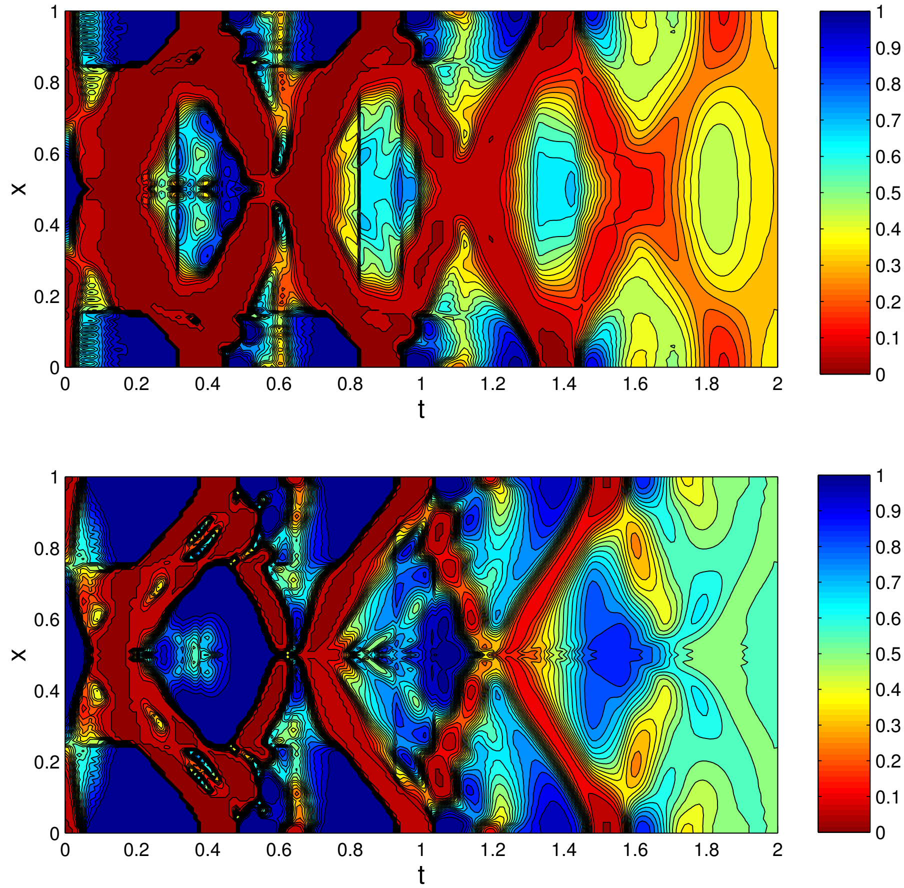



*Figura 2.7:* **Case 1** - $T = 2$, $(\alpha, \beta) = (1, 6)$ - *Iso-values of the limit density - Top :* $V_\alpha = 0{,}3$ - $\tilde{I}(s^{lim}) \approx 7{,}9567$ - *Bottom:* $V_\alpha = 0{,}5$ - $\tilde{I}(s^{lim}) \approx 6{,}1439$.

several infima for $\widetilde{I}$. We found that the choice $s^0$ constant on $(0, T) \times \Omega$ - which have the advantage to not favor any distribution between $\alpha$ and $\beta$ - leads to the lowest value of $\widetilde{I}(s^{lim})$. Moreover, for this choice, the algorithm appears robust, stable and convergent with respect to the discretization parameters $h$ and $\Delta t$. Under theses circumstances, we suspect that the infimum of $(\widetilde{RP})$ (see Lemma 2) is in fact a minimum.

Remark that the relaxation analysis and the results presented in the previous sections are unchanged if we consider the weaker volume constraint:

$$\int_0^T \int_\Omega s(t, x) dx dt \leq V_\alpha |\Omega| T. \tag{2.24}$$

Figure 2.9 depict the limit densities for $V_\alpha = 0{,}5$ for $(\alpha, \beta) = (1, 1{,}1)$ (Top) and $(\alpha, \beta) = (1, 6)$ (Bottom) respectively. Once again, in the first case, the density is a characteristic function whereas in the second case, the density takes values in $(0, 1)$. Furthermore, as expected, these densities leads to a

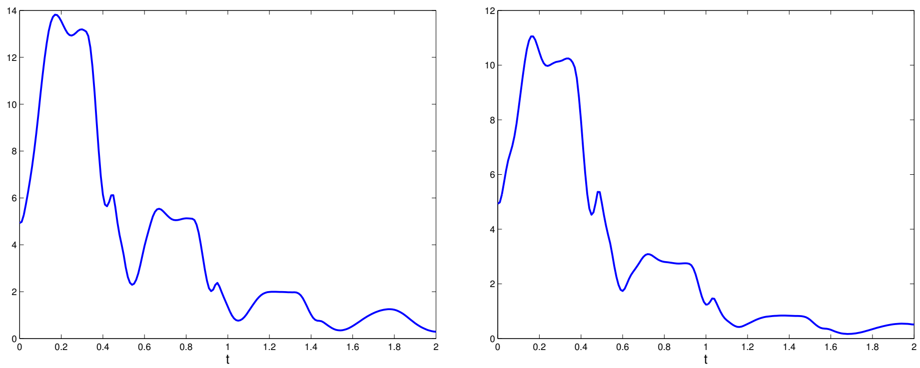



*Figura 2.8:* **Case 1** - $T = 2$, $(\alpha,\beta) = (1,2)$ - $E(t)$ vs. $t$ - *Left* : $V_\alpha = 0{,}3$ - $\widetilde{I}(s^{lim}) \approx 7{,}9567$ - *Right* : $V_\alpha = 0{,}5$ - $\widetilde{I}(s^{lim}) \approx 6{,}1439$.

better distribution of materials: we obtain $\widetilde{I}(s^{lim}) \approx 9{,}2147$ and $\widetilde{I}(s^{lim}) \approx 4{,}3109$ respectively (to be compared with $\widetilde{I}(s^{lim}) \approx 9{,}5613$ and $\widetilde{I}(s^{lim}) \approx 6{,}1439$ for the initial volume constraint $\int_\Omega s(t,x)dx \leq V_\alpha|\Omega|$, for all $t$).

### Case 2

Similarly, we present some results for the case 2. Similarly to the first case, the optimal density takes value in $(0,1)$ if and only if the gap $\beta-\alpha$ is large enough. The pictures also clearly highlight that the optimal distribution is related to the propagation of the components of the solution on the cylinder $(0,T)\times(0,1)$. For this case, we observe that the two volume constraint give similar results on the density and the optimal cost (see Figures 2.10 and 2.14 and 2.16 ).

### Construction of a characteristic density associated to $s^{lim}$

In the case where the optimal density $s^{lim}$ is not in $L^\infty((0,T)\times\Omega;\{0,1\})$, one may associate to $s^{lim}$ a characteristic function $s^{pen} \in L^\infty((0,T)\times\Omega;\{0,1\})$ whose cost $\widetilde{I}(s^{pen})$ is arbitrarily near from $\widetilde{I}(s^{lim})$. Following [43], one may proceed as follows: we first decompose the cylinder $(0,T)\times\Omega$ into $M\times N$ cells such that $(0,T)\times\Omega = \cup_{i=1,M}[t_i,t_{i+1}]\times\cup_{j=1,N}[x_j,x_{j+1}]$. Then, we associate to each cell the mean value $m_{i,j}\in[0,1]$ defined by

$$m_{i,j} = \frac{1}{(t_{i+1}-t_i)(x_{j+1}-x_j)}\int_{t_i}^{t_{i+1}}\int_{x_j}^{x_{j+1}} s^{lim}(t,x)dx\,dt$$

At last, we define the function $s^{pen}_{M,N}$ in $L^\infty([0,T]\times\Omega)$ by

$$s^{pen}_{M,N}(t,x) = \sum_{i=1}^{M}\sum_{j=1}^{N}\chi_{[t_i,(1-\sqrt{m_{i,j}})t_i+\sqrt{m_{i,j}}t_{i+1}]\times[x_j,(1-\sqrt{m_{i,j}})x_j+\sqrt{m_{i,j}}x_{j+1}]}(t,x).$$

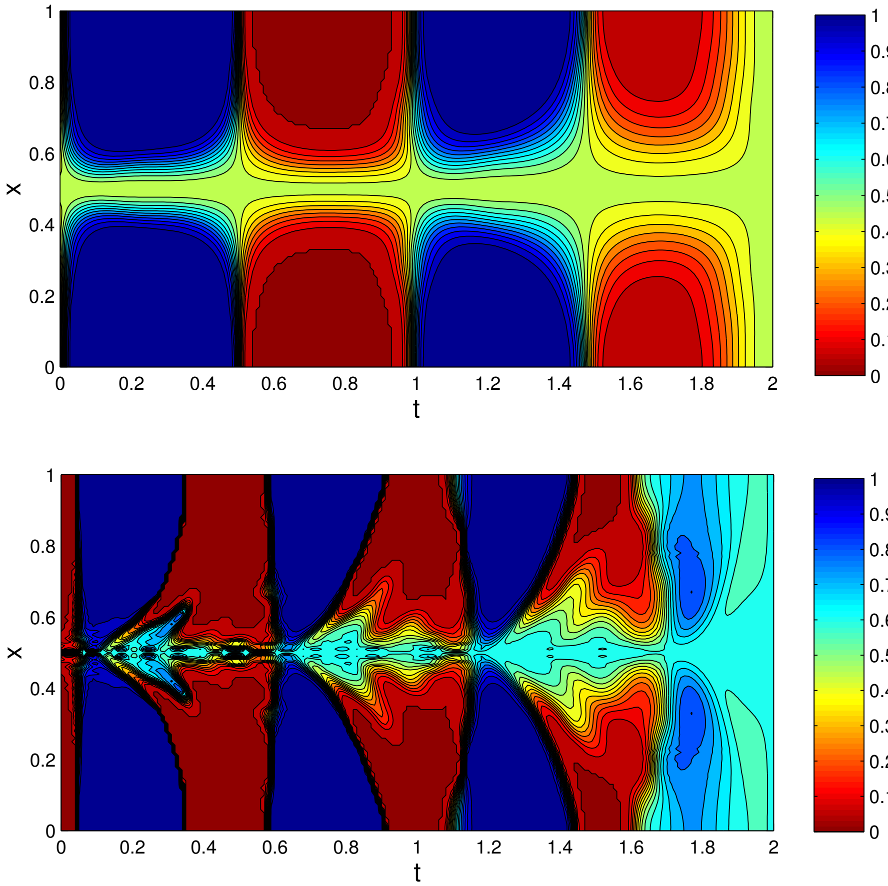



*Figura 2.9:* **Case 1** *with the volume constraint (2.24) -* $T = 2, V_\alpha = 0{,}5$ *- Iso-value of the limit density - Top :* $(\alpha, \beta) = (1, 1{,}1)$ $\widetilde{I}(s^{lim}) \approx 9{,}2147$ *- Bottom :* $(\alpha, \beta) = (1, 6)$ *-* $\widetilde{I}(s^{lim}) \approx 4{,}3109$.

We easily check that $||s^{pen}_{M,N}||_{L^1((0,T)\times\Omega)} = ||s^{lim}||_{L^1((0,T)\times\Omega)}$, for all $M, N > 0$. Thus, the bi-valued function $s^{pen}_{M,N}$ takes advantage of the information codified in the density $s^{lim}$.

In order to illustrate this, we consider the case 1 with $T = 1$, $(\alpha, \beta) = (1, 2)$ and $V_\alpha = 0{,}5$. Figure 2.17 depicts the corresponding optimal density $s^{lim}$ and the associated function $s^{pen}_{M,N}$ for $M = N = 30$. We obtain $\widetilde{I}(s^{lim}) = 4{,}7584$ and $\widetilde{I}(s^{pen}_{30,30}) = 5{,}62$ respectively. By letting $M$ and $N$ go to infinity, we expect to converge to the value $\widetilde{I}(s^{lim})$ and then construct a minimizing sequence of domains $\omega_{M,N}$ such that $\chi_{\omega_{\infty,\infty}}$ be the infimum for $I$ (see Table 2.1). We refer to [39] for more example.

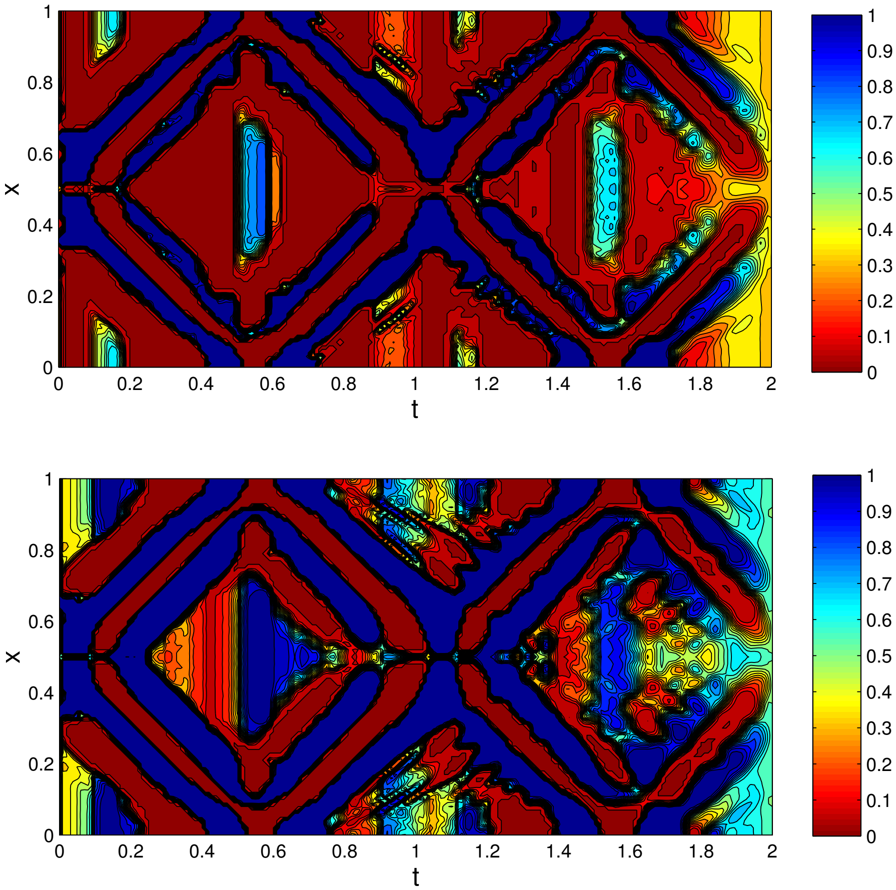



*Figura 2.10:* **Case 2** - $T = 2$, $(\alpha, \beta) = (1, 1{,}1)$ - *Iso-value of the limit density - Top : $V_\alpha = 0{,}3$ - $\widetilde{I}(s^{lim}) \approx 15{,}8839$ - Bottom : $V_\alpha = 0{,}5$ - $\widetilde{I}(s^{lim}) \approx 15{,}48$.*

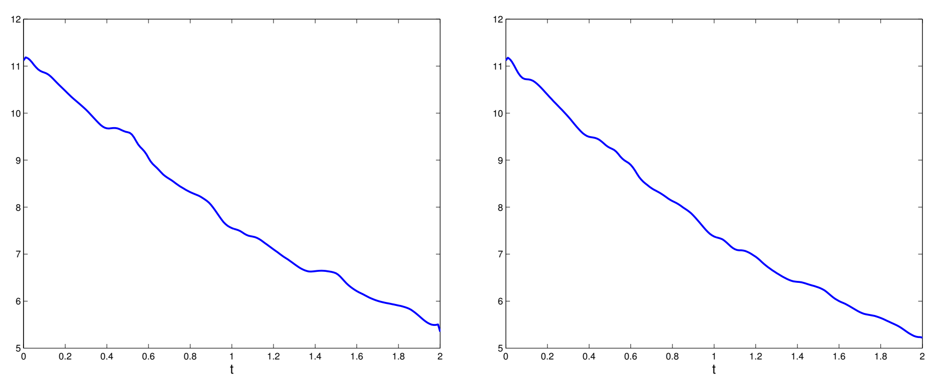



*Figura 2.11:* **Case 2** - $T = 2$, $(\alpha, \beta) = (1, 1{,}1)$ - *$E(t)$ vs. $t$ - Left : $V_\alpha = 0{,}3$ - $\widetilde{I}(s^{lim}) \approx 15{,}8839$ - Right : $V_\alpha = 0{,}5$ - $\widetilde{I}(s^{lim}) \approx 15{,}48$.*

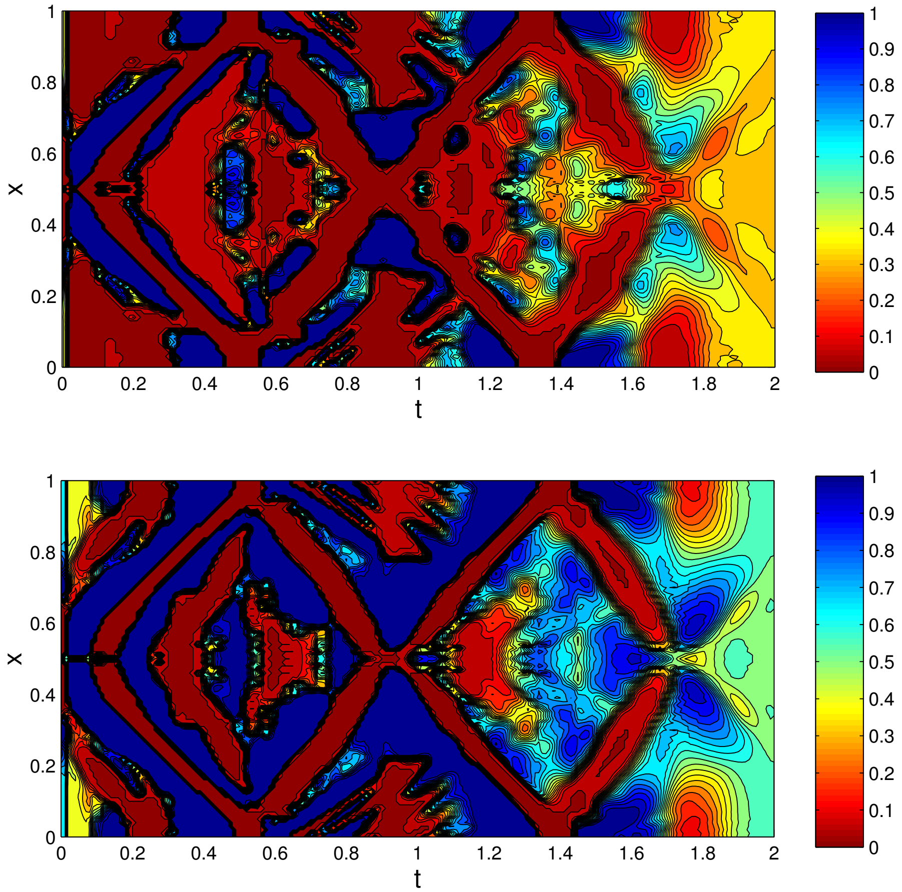



*Figura 2.12:* **Case 2** - $T = 2$, $(\alpha, \beta) = (1, 2)$ - *Iso-value of the limit density -Top :* $V_\alpha = 0{,}3$ - $\widetilde{I}(s^{lim}) \approx 6{,}0610$ *Bottom:* $V_\alpha = 0{,}5$ - $\widetilde{I}(s^{lim}) \approx 5{,}5857$.

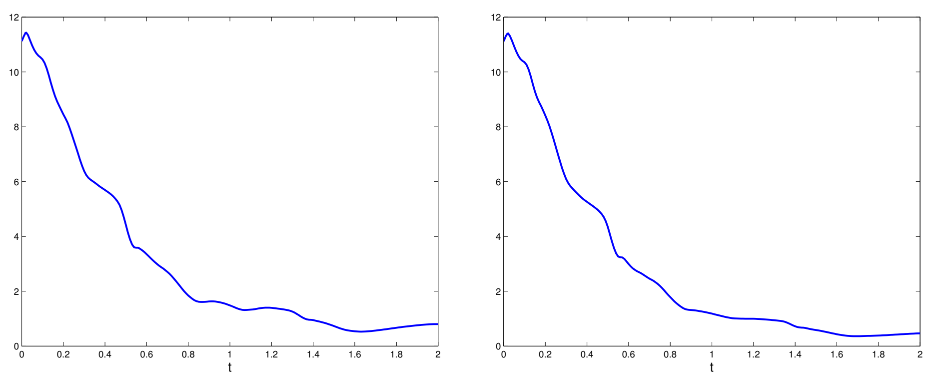



*Figura 2.13:* **Case 2** - $T = 2$, $(\alpha, \beta) = (1, 2)$ - $E(t)$ *vs.* $t$ - *Left :* $V_\alpha = 0{,}3$ - $\widetilde{I}(s^{lim}) \approx 6{,}0610$ - *Right :* $V_\alpha = 0{,}5$ - $\widetilde{I}(s^{lim}) \approx 5{,}5857$.

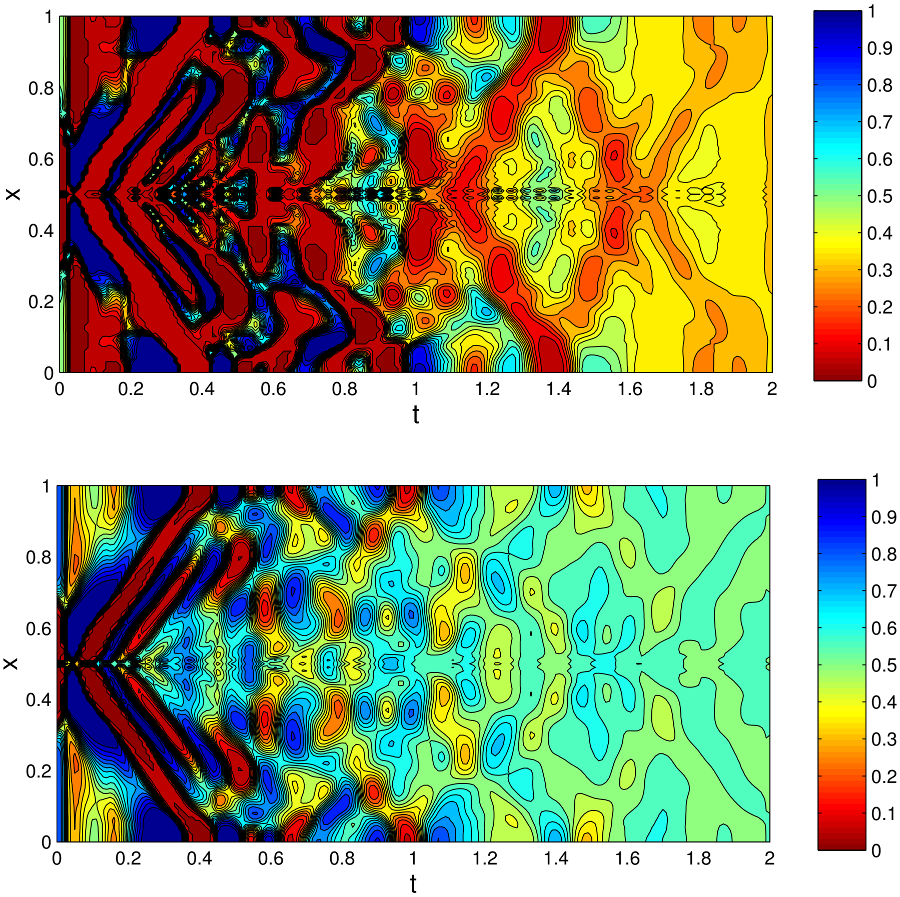



*Figura 2.14:* **Case 2** *- $T = 2$, $(\alpha, \beta) = (1, 6)$ - Iso-value of the limit density - Top: $V_\alpha = 0{,}3$ - $\widetilde{I}(s^{lim}) \approx 4{,}7856$ Bottom : $V_\alpha = 0{,}5$ - $\widetilde{I}(s^{lim}) \approx 4{,}5414$.*

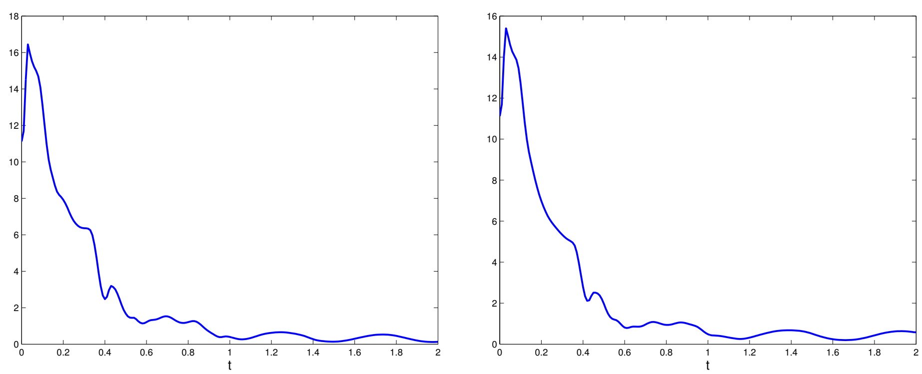



*Figura 2.15:* **Case 2***- $T = 2$, $(\alpha, \beta) = (1, 6)$ - $E(t)$ vs. $t$ - Left : $V_\alpha = 0{,}3$ - $\widetilde{I}(s^{lim}) \approx 4{,}7856$ - Right : $V_\alpha = 0{,}5$ - $\widetilde{I}(s^{lim}) \approx 4{,}5414$.*

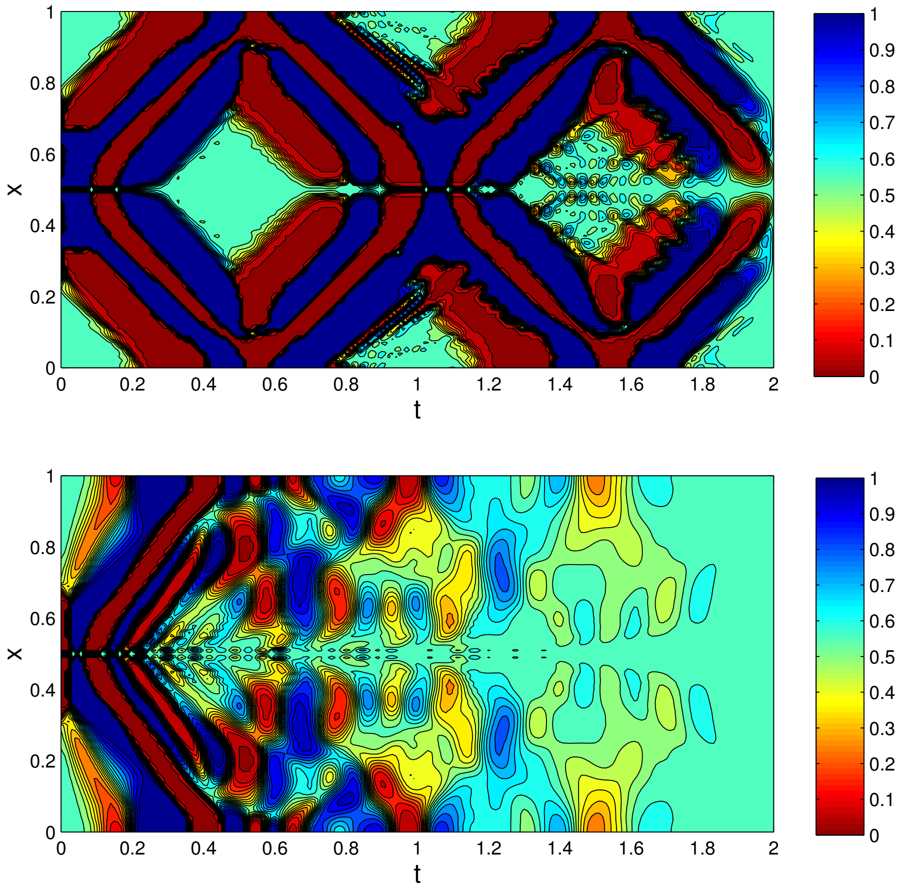



*Figura 2.16:* **Case 2** *with the volume constraint (2.24) -* $T = 2, V_\alpha = 0{,}5$ *- Iso-value of the limit density - Top :* $(\alpha, \beta) = (1, 1{,}1)$ $\widetilde{I}(s^{lim}) \approx 7{,}8740$ *- Bottom :* $(\alpha, \beta) = (1, 10)$ *-* $\widetilde{I}(s^{lim}) \approx 4{,}8414$.

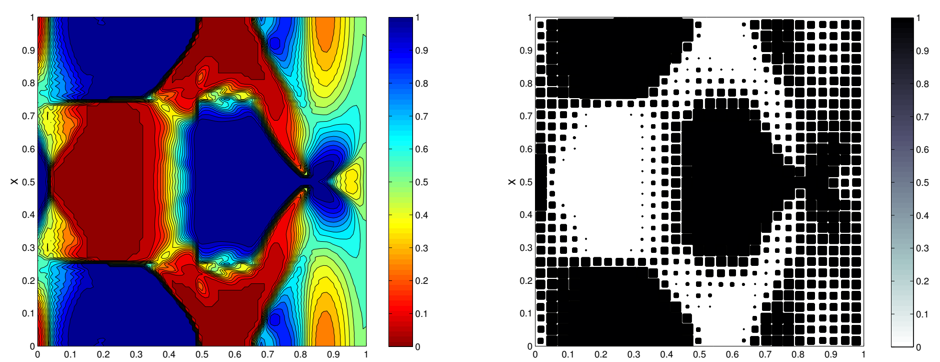



*Figura 2.17: Example 2:* $T = 1$, $(\alpha, \beta) = (1, 2)$ *-* $V_\alpha = 0{,}5$ $s^{pen}_{30,30}$ *-* $\widetilde{I}(s^{lim}) \approx 4{,}7584$*-* $\widetilde{I}(s^{lim}) \approx 5{,}62$, $\tilde{N}$*=30.*

| $M = N$ | 10 | 20 | 30 | 40 | 50 |
|---|---|---|---|---|---|
| $\widetilde{I}(s^{pen}_{M,N})$ | 7.45 | 6.21 | 5.62 | 5.09 | 4.93 |

*Cuadro 2.1: Case 1 - $T = 1$ - $(\alpha, \beta) = (1, 3)$ - $V_\alpha = 0{,}5$ - Values of the cost function* $\widetilde{I}(s^{pen}_{M,N})$

# Chapter 3

# Quasiconvexification in 3-d for a variational reformulation of an optimal design problem in conductivity

## 3.1 Introduction

In this paper, we will study a typical optimal design problem in conductivity, which consists in looking for the optimal distribution of two different conducting materials with isotropic constants $\alpha$ and $\beta$ $(0 < \alpha < \beta)$ on a domain $\Omega \subset \mathbb{R}^3$, such that it minimizes a certain functional cost which depends on the underlying electric field of the state equation in the form

$$I(\chi) = \int_\Omega a(x, \chi(x))|\nabla u(x) - F(x)|^2 dx \tag{3.1}$$

where $u$ is the unique solution of

$$\begin{aligned} -\text{ div }((\alpha\chi + \beta(1-\chi))\nabla u) = g \quad &\text{in} \quad \Omega, \\ u = u_0 \quad &\text{on} \quad \partial\Omega, \end{aligned} \tag{3.2}$$

and the functions $a$, $F$, $g$ and $u_0$ are known. The function $\chi \in L^\infty(\Omega, \{0,1\})$ is the design variable and it indicates where we place the $\alpha$-material. The amount of $\alpha$-material is given, and therefore we have to enforce the volume constraint

$$\int_\Omega \chi(x)dx \le t_0|\Omega| \tag{3.3}$$

with $t_0 \in (0,1)$ fixed.

In short form, the optimal design problem is to

$$\min_{\chi} I(\chi)$$

under

$$\begin{aligned} -\operatorname{div}\left((\alpha\chi+\beta(1-\chi))\nabla u\right) = g \quad &\text{in} \quad \Omega, \\ u = u_0 \quad &\text{on} \quad \partial\Omega, \end{aligned}$$

where the admissible set of $\chi$'s is the set of the characteristic functions over $\Omega$ under the volume constraint

$$\frac{1}{|\Omega|}\int_{\Omega}\chi(x)dx \leq t_0.$$

This problem has been extensively studied in the two-dimensional case ([1],[2], [56], [55]), and it has also been examined in the three-dimensional case ([7]). In this paper, we would like to pursue a more direct analysis of the three-dimensional situation in order to deal with a more natural generalization, as compared to the treatment in ([7]), of the two-dimensional case. In doing so, we will have to overcome a certain non-linear structure which does not have a parallelism in the 2-D case ([15]). This is the result of using Clebsch potentials, a suitable characterization of three-dimensional divergence-free vector fields.

It is well understood the lack of classical solutions for this type of problems. This is the reason why we reformulate the problem through relaxation techniques([52],[17],[23]). The theory of homogenization is an important tool which introduces new types of composites as structural elements through the concepts of H-convergence or G-convergence ([1],[63],[64],[32]). The theory of homogenization is especially useful when we work with non-explicit dependence on the flux $\nabla u(x)$.

Our strategy is directed towards the understanding and computation of the constrained quasiconvexification of a certain integrand which is obtained as a result of a suitable variational reformulation of the problem ([56], [54], [55], [52]).

Notice, to begin with, that

$$a|\nabla u - F|^2 = a|\nabla u|^2 - 2a\nabla u\cdot F + a|F|^2,$$

and that the second part $-2a\nabla u \cdot F + a|F|^2$ is linear in $\nabla u$, therefore it suffices to study the case $F \equiv 0$. The optimal design problem we will treat will be

$$\min_{\chi\in L^\infty(\Omega,\{0,1\})} I(\chi) = \int_{\Omega} a(x,\chi(x))|\nabla u(x)|^2 dx$$

subject to,

$$\begin{aligned} -\text{ div }((\alpha\chi+\beta(1-\chi))\nabla u) = g \quad & \text{in} \quad \Omega \\ u = u_0 \quad & \text{on} \quad \partial\Omega \\ \textstyle\int_\Omega \chi(x)dx \leq t_0|\Omega|. & \end{aligned}$$

We want to reformulate this problem in a different form. We take $G \in H^1(\Omega)$ such that

$$\text{div }(G) = g \text{ in } \Omega,$$

so that the state equation can be rewritten as

$$-\text{ div }((\alpha\chi+\beta(1-\chi))\nabla u + G(x)) = 0 \text{ in } \Omega \subset \mathbb{R}^3. \tag{3.4}$$

The treatment of this equation is the main novelty with respect to the two-dimensional case in which the divergence-free fields are characterized as the counterclockwise $\pi/2$-rotation of the gradients of scalar functions, while in the three-dimensional case the situation is more complex, we will use a characterization by "Clebsch potentials". There are different results in which n-dimensional divergence-free vector fields can been represented in terms of (n-1) arbitrary functions. For the three dimensional case, if $F \in \mathbb{R}^3$ with $\text{div}(F) = 0$, then there exist Clebsch potentials $v, w$ such that $F = \nabla v \times \nabla w$. The proof of this result is beyond the scope of the present paper and we refer [31], [51]. [65], [58]. In fact, looking at the specialized literature, is seems that this representation en terms of the Clebsch potentials is not always valid ([26]). But this is a fine point for experts which we have avoided. We simply use this representation in the sequel.

Therefore using the Clebsch potentials, we have that (3.4) can be replaced by,

$$(\alpha\chi(x)+\beta(1-\chi(x)))\nabla u(x) - \nabla v(x) \times \nabla w(x) + G(x) \quad = \quad 0 \quad \text{ in } \Omega.$$

We can therefore use $(u, v, w)$ as new design variables provided they satisfy the following pointwise constraint,

$$\left.\begin{array}{l} \alpha\nabla u(x) - \nabla v(x) \times \nabla w(x) + G(x) = 0 \\ \text{ó} \\ \beta\nabla u(x) - \nabla v(x) \times \nabla w(x) + G(x) = 0 \end{array}\right. \quad \text{a.e. } x \in \Omega. \tag{3.5}$$

It is clear that we can identify the design variable $\chi$ with the vector $(u, v, w)$, and conversely a vector $(u, v, w)$ which verify (3.5) with a design variable $\chi$. We consider then the new design variable $U = (U^{(1)}, U^{(2)}, U^{(3)}) = (u, v, w)$, where $U : \mathbb{R}^3 \to \mathbb{R}^3$ and $\nabla U(x) \in \mathbb{R}^{3\times3}$.

Let $\Lambda_{\gamma,x}$ be the (non-linear) manifold,

$$\Lambda_{\gamma,x} = \{A \in M^{3\times3} : \gamma A^{(1)} - A^{(2)} \times A^{(3)} + G(x) = 0\}$$

where $A^{(i)}$ is the $i$-th row of the matrix A.

Put $a_\alpha(x) = a(x,1)$, $a_\beta(x) = a(x,0)$ and

$$h(x) = \beta a_\alpha(x) - \alpha a_\beta(x),$$

and set

$$W(x,A) = \begin{cases} a_\alpha(x)|A^{(1)}|^2, & \text{if } A \in \Lambda_{\alpha,x}, \\ a_\beta(x)|A^{(1)}|^2, & \text{if } A \in \Lambda_{\beta,x} \setminus \Lambda_{\alpha,x}, \\ +\infty, & \text{else,} \end{cases}$$

$$V(x,A) = \begin{cases} 1, & \text{if } A \in \Lambda_{\alpha,x}, \\ 0, & \text{if } A \in \Lambda_{\beta,x} \setminus \Lambda_{\alpha,x}, \\ +\infty, & \text{else.} \end{cases}$$

It is clear that the original optimal design problem is equivalent to the non-convex vector variational problem

$$\min_U \quad \hat{I}\ (U) = \int_\Omega W(x, \nabla U(x))dx$$

subject to

$$\begin{gathered} U \in H^1(\Omega)^3,\ U^{(1)} = u_0 \text{ on } \partial\Omega \\ \textstyle\int_\Omega V(x,\nabla U(x))dx \le t_0|\Omega|. \end{gathered}$$

$$\begin{aligned} \psi(t,F) = \quad & \alpha\beta(t\alpha + (1-t)\beta)|F^{(1)}|^2 + \\ & \Big[((1-t)\alpha + t\beta)(|F^{(2)} \times F^{(3)}|^2 + |G|^2 - 2(F^{(2)} \times F^{(3)}) \cdot G)\Big] + \\ & [2\alpha\beta + t(1-t)(\beta-\alpha)^2](F^{(1)} \cdot G - \det F) \end{aligned}$$

$$\varphi(t,F) = \begin{cases} \dfrac{h(x)}{t\beta(\beta-\alpha)^2}(\beta^2|F^{(1)}|^2 + |F^{(2)} \times F^{(3)}|^2 + |G|^2 - 2\beta \det F + \\ \qquad 2(\beta F^{(1)} - F^{(2)} \times F^{(3)}) \cdot G) + \dfrac{a_\beta(x)}{\beta}(\det F - G \cdot F^{(1)}) \\ \qquad\qquad \text{if } h(x) \ge 0, \psi(t,F) \le 0, \\ \dfrac{-h(x)}{(1-t)\alpha(\beta-\alpha)^2}(\alpha^2|F^{(1)}|^2 + |F^{(2)} \times F^{(3)}|^2 + |G|^2 \\ \qquad -2\alpha \det F + 2(\alpha F^{(1)} - F^{(2)} \times F^{(3)}) \cdot G) + \\ \qquad \dfrac{a_\alpha(x)}{\alpha}(\det F - G \cdot F^{(1)}), \\ \qquad\qquad \text{if } h(x) \le 0, \psi(t,F) \le 0, \\ +\infty \qquad\qquad \textit{else}. \end{cases}$$

We will show that the variational problem

$$\min_{(t,U)} \int_\Omega \varphi(t(x), \nabla U(x))dx$$

subject to

$$U \in H^1(\Omega)^3,\ U^{(1)} = u_0 \text{ on } \partial\Omega,\ \psi(t(x), \nabla U(x)) \leq 0,$$

$$0 \leq t(x) \leq 1,\ \int_\Omega t(x)dx \leq t_0|\Omega|,$$

is a relaxation of the original optimal design problem, in the sense explained in the next theorem. This result is the main objective of this work.

**Theorem 4** *This final variational problem is equivalent to ( a relaxation for ) the original optimal design problem ( determined by (3.1), (3.2), (3.3) ) in the sense that*

*a) the infima of both problems coincide,*

*b) there are optimal solutions for the relaxed problem,*

*c) these solutions codify ( in the sense of the Young measure) the optimal microstructures of the original optimal design problem.*

For the particular case in which we take $a_\alpha(x) = a_\beta(x) = 1$ and $G = 0$, the above formulae simplify to

$$h(x) = \beta - \alpha,$$

$$\begin{aligned}\psi(t, F) = \alpha\beta(t\alpha + (1-t)\beta)|F^{(1)}|^2 + ((1-t)\alpha + t\beta)|F^{(2)} \times F^{(3)}|^2 \\ -(2\alpha\beta + t(1-t)(\beta-\alpha)^2)\det F\end{aligned}$$

$$\varphi(t, F) = \begin{cases} \dfrac{1}{t\beta(\beta-\alpha)}(\beta^2|F^{(1)}|^2 + |F^{(2)} \times F^{(3)}|^2 - 2\beta\det F) + \dfrac{1}{\beta}\det F \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \psi(t, F) \leq 0, \\ +\infty \qquad\qquad\qquad\qquad\qquad\qquad\quad \textit{else}. \end{cases}$$

The main new contribution here is to understand how the non-linear character of the manifolds $\Lambda_{\gamma,x}$ above does not in fact interfere with the analogous computations for the 2-D situation. This is so because this non-linearity is intimately connected to the weak continuity of minors.

The work is organized as follows. We begin by studying a relaxation of the original problem which is given by the *constrained quasionvexification*. Next, we compute explicitly this relaxation by firstly computing a lower bound (*polyconvexification*), and then seeking a laminate that recovers this *polyconvexification* thus showing that the bound is optimal.

## 3.2 Relaxation

We have recast our optimal design problem as a typical variational problem. We see that it is a non-convex vector problem that we are going to analyze by seeking its relaxation. We use Young measures as a main tool in the computation of the suitable density for the relaxed problem. We are going to follow the same plan that in the two-dimensional case ([2],[56], [55]).

Put

$$m = \inf \Big\{ \int_\Omega W(x, \nabla U(x))dx : U \in H^1(\Omega)^3, U^{(1)} - u_0 \in H_0^1(\Omega), \int_\Omega V(x, \nabla U(x))dx = t_0|\Omega| \Big\}.$$

We know ([2]) that

$$m \geq \bar{m} = \inf \Big\{ \int_\Omega CQW(x, \nabla U(x), t(x))dx : U \in H^1(\Omega)^3, U^{(1)} - u_0 \in H_0^1(\Omega), 0 \leq t(x) \leq 1, \int_\Omega t(x)dx = t_0|\Omega| \Big\}$$

where $CQW(x, F, t)$ is defined by,

$$CQW(x, F, t) = inf \Big\{ \int_{M^{3\times 3}} W(x, A)d\nu(A) : \nu \in \mathcal{A}(F, t) \Big\}$$

with

$$\mathcal{A}(F, t) = \Big\{ \nu : \nu \text{ is a homogeneous } H^1\text{-Young measure}, F = \int_{M^{3\times 3}} Ad\nu(A), \int_{M^{3\times 3}} V(x, A)d\nu(A) = t \Big\}. \tag{3.6}$$

Notice that the previous inequality will be an equality when W is a Carathéodory function with appropriate growth constrains ([52]). However, in our situtation it is still possible to prove this equality despite the fact that $W$ is not a Carathéodory function. Let us consider the following minimization problem

$$\tilde{m} = \inf \Big\{ \int_\Omega \int_{M^{3\times 3}} W(x, A)d\nu_x(A)dx : \nu \in \mathcal{B}(u_0, t_0) \Big\}$$

where

$$\mathcal{B}(u_0, t_0) = \Big\{ \nu : H^1\text{-Young meas.}, supp(\nu_x) \subset \Lambda_\alpha \cup \Lambda_\beta, \exists U \in H^1(\Omega)^3, U^{(1)} - u_0 \in H_0^1(\Omega), \int_\Omega \int_{M^{3\times 3}} V(x, A)d\nu_x(A)dx = t_0|\Omega|, \nabla U(x) = \int_{M^{3\times 3}} Ad\nu_x(A) \Big\}.$$

We have the following result.

**Teorema 9** *The equalities*

$$m = \bar{m} = \tilde{m}$$

*hold. Moreover, for each measure* $\nu \in \mathcal{B}(u_0, t_0)$ *such that* $supp(\nu_x) \subset \Lambda_\alpha \cup \Lambda_\beta$ *a.e.* $x \in \Omega$, *there exists a sequence* $\{\nabla U_k\}$ *such that,*

*i)* $U_k \in (H^1(\Omega))^3,\ U_k^{(1)} - u_0 \in H_0^1(\Omega),\ \{|\nabla U_k|^2\}$ *is equi-integrable,*

*ii)* $\nabla U_k(x) \in \Lambda_\alpha \cup \Lambda_\beta,$ *a.e.* $x \in \Omega\ \forall k,\ \int_\Omega V(x, \nabla U_k(x))dx = t_0,\ \forall k$

*iii)* $\lim_{k\to\infty} \int_\Omega W(x, \nabla U_k(x))dx = \int_\Omega \int_{M^{3\times 3}} W(x, A)d\nu_x(A)dx$

**Proof.** It is enough to generalize to the three-dimensional case the proof used in ([2]) for the two-dimensional case. This is a straightforward generalization.

## 3.3 Constrained quasiconvexification

We would like to compute explicitly the *constrained quasionvexification* defined as

$$CQW(x, F, t) = inf\Big\{ \int_{M^{3\times 3}} W(x, A)d\nu(A) : \nu \in \mathcal{A}(F, t)\Big\}$$

where $\mathcal{A}(F, t)$ is given in (3.6). This constrained quasiconvexification can be expressed as

$$\inf_\nu \Big\{ \int_{M^{3\times 3}} W(x, A)d\nu(A) : F = \int_{M^{3\times 3}} Ad\nu(A), \int_{M^{3\times 3}} V(x, A)d\nu(A) = t\Big\} \tag{3.7}$$

with $\nu$ a homogeneous $H^1$-Young measure with $supp(\nu) \subset \Lambda_\alpha \cup \Lambda_\beta$.

For $(F, t)$ (and $x$) fixed, we are going to calculate the value in (3.7), i.e. $CQW(x, F, t)$. The main difficulty here is that we don't know explicitly the set of the admissible measures, which we note as $\mathcal{A}$. The plan to follow will be similar to the two-dimensional case. The first step is to calculate the minimum over a greater class of probability measures $\mathcal{A}^* \supset \mathcal{A}$ where $\mathcal{A}^*$ is the set of all polyconvex measures. In this way we obtain a lower bound ( the (constrained) polyconvexification). Once this bound is computed, we search a measure over a narrower class of measures ( the *laminates*) which will tell us that the bound is attained, so that we will have in fact computed the exact value $CQW(x, F, t)$.

The *polyconvexification* $CPW(x, F, t)$ can be computed through the following optimization problem

$$\min_\nu \int_{M^{3\times 3}} W(x, A)d\nu(A)$$

subject to

$$\nu = t\nu_\alpha + (1-t)\nu_\beta, \text{ commutes with all (some) minors}$$

$$supp(\nu_\gamma) \subset \Lambda_\gamma, \ \gamma = \alpha, \beta,$$

$$F = t\int_{\Lambda_\alpha} A d\nu_\alpha(A) + (1-t)\int_{\Lambda_\beta} A d\nu_\beta(A).$$

It is clear that the integral constraints have been incorporated in the decomposition of the measure $\nu$. Let us first examine the constraints. We introduce the following variables

$$S_\gamma = \int_{R^3} |\lambda|^2 d\nu_\gamma^{(1)}(\lambda), \quad \text{with } \gamma = \alpha, \beta, \tag{3.8}$$

where $\nu_\gamma^{(1)}$ is the probability measure resulting from the projection of $\nu_\gamma$ onto the first row. On the other hand, we put

$$F_\gamma = \int_{\Lambda_\gamma} A d\nu_\gamma(A) \quad \text{for } \gamma = \alpha, \beta.$$

From the fact that the measure $\nu = t\nu_\alpha + (1-t)\nu_\beta$ commutes with the determinant, it is clear that

$$A \in \Lambda_\gamma \Rightarrow \left\{ \begin{array}{l} \gamma A^{(1)} - A^{(2)} \times A^{(3)} + G = 0 \\ \det A = A^{(1)} \cdot (A^{(2)} \times A^{(3)}) \end{array} \right\} \Rightarrow \det A = \gamma |A^{(1)}|^2 + F^{(1)} \cdot G, \tag{3.9}$$

thanks to the commutations with the minors. In particular, applied to the determinant, it leads to

$$\det F = t\int_{\Lambda_\alpha} \det A d\nu_\alpha(A) + (1-t)\int_{\Lambda_\beta} \det A d\nu_\beta(A).$$

Keeping in mind (3.8) and (3.9), we have that

$$\det F = t\alpha S_\alpha + (1-t)\beta S_\beta + F^{(1)} \cdot G.$$

The components of $F^{(2)} \times F^{(3)}$ are the second order minors which have been computed using the second and third row of the matrix $F$, and again the commutation with $\nu$ yields

$$F^{(2)} \times F^{(3)} = \int_{R^{3\times 3}} A^{(2)} \times A^{(3)} d\nu(A) =$$

$$t\alpha \int A^{(1)} d\nu_\alpha + (1-t)\beta \int A^{(1)} d\nu_\beta + G = t\alpha F_\alpha^{(1)} + (1-t)\beta F_\beta^{(1)} + G.$$

Moreover

$$F = tF_\alpha + (1-t)F_\beta \Rightarrow F^{(1)} = tF_\alpha^{(1)} + (1-t)F_\beta^{(1)}.$$

Using the last equalities, we can deduce

$$\left\{\begin{array}{l} F_\alpha^{(1)} = \frac{1}{t(\beta-\alpha)}(\beta F^{(1)} - F^{(2)} \times F^{(3)} + G) \\ \\ F_\beta^{(1)} = \frac{-1}{(1-t)(\beta-\alpha)}(\alpha F^{(1)} - F^{(2)} \times F^{(3)} + G). \end{array}\right. \tag{3.10}$$

On the other hand, by the Jensen's inequality

$$S_\gamma \geq \Big| \int_{R^3} \lambda d\nu_\gamma^{(1)}(\lambda) \Big|^2 = |F_\gamma^{(1)}|^2, \text{ with } \gamma = \alpha, \beta,$$

and bearing in mind (3.10), we can write

$$t^2(\beta-\alpha)^2 S_\alpha \geq \beta^2|F^{(1)}|^2 + |F^{(2)} \times F^{(3)}|^2 + |G|^2 - \\ 2\beta \det F + 2(\beta F^{(1)} - F^{(2)} \times F^{(3)}) \cdot G,$$

$$(1-t)^2(\beta-\alpha)^2 S_\beta \geq \alpha^2|F^{(1)}|^2 + |F^{(2)} \times F^{(3)}|^2 + |G|^2 - \\ 2\alpha \det F + 2(\alpha F^{(1)} - F^{(2)} \times F^{(3)}) \cdot G.$$

The cost functional can be rewriten in terms of the $S_\gamma$ variables as follows

$$ta_\alpha S_\alpha + (1-t)a_\beta S_\beta.$$

Hence, we can rewrite the original optimization problem as a mathematical programming problem

$$\underset{(S_\alpha, S_\beta)}{\text{minimize}} \quad ta_\alpha S_\alpha + (1-t)a_\beta S_\beta$$

subject to

$$\det F = t\alpha S_\alpha + (1-t)\beta S_\beta$$

$$\beta^2|F^{(1)}|^2 + |F^{(2)} \times F^{(3)}|^2 + |G|^2 - 2\beta \det F + \\ 2(\beta F^{(1)} - F^{(2)} \times F^{(3)}) \cdot G - t^2(\beta-\alpha)^2 S_\alpha \leq 0,$$

$$\alpha^2|F^{(1)}|^2 + |F^{(2)} \times F^{(3)}|^2 + |G|^2 - 2\alpha \det F + \\ 2(\alpha F^{(1)} - F^{(2)} \times F^{(3)}) \cdot G - (1-t)^2(\beta-\alpha)^2 S_\beta \leq 0,$$

where the parameters $\alpha, \beta, a_\alpha, a_\beta$ are part of the data set of the original problem, and the variables $t$, $F$ (and $x$) are fixed.

The first issue about this mathematical programming problem is to compute the admissible set for the variables $(S_\alpha, S_\beta)$. This is determined by the intersection of two semi-planes and one line; therefore the admissible set will

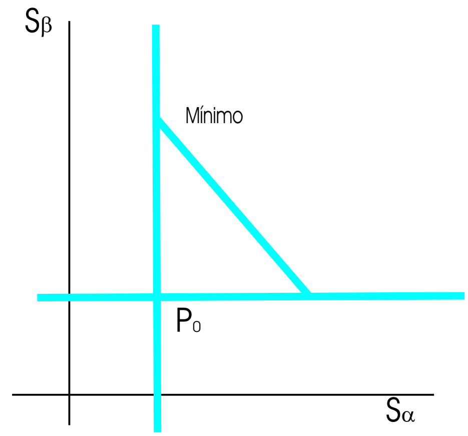



*Figure 3.1: Mathematical programming problem*

be the segment of the line within the two semi-planes. This is easy to see geometrically in Figure 3.1.

The admissible set will be non-empty when the point of intersection where the two inequality constraints become equalities ($P_0$ in Figure 3.1) is under the line represented by the equality constraint. This amounts to

$$\begin{aligned} &\alpha\beta(t\alpha + (1-t)\beta)|F^{(1)}|^2 + \\ ((1-t)\alpha + t\beta)(|F^{(2)} &\times F^{(3)}|^2 + |G|^2 - 2(F^{(2)} \times F^{(3)}) \cdot G) \\ &+(2\alpha\beta + t(1-t)(\beta-\alpha)^2)(F^{(1)} \cdot G - \det F) \leq 0. \end{aligned} \tag{3.11}$$

The second issue is to decide the point(s) where the minimum value is attained. It is clear that the optimal point depends of the coefficients $a_\alpha, a_\beta$. We have previously determined that the admissible set (when it is non-empty) is a segment, and the functional cost is a linear functional. Therefore the minimum value will be attained at one of the extreme points of the segment, or become constant over all of the segment, depending on the particular values of the parameters $a_\alpha, a_\beta$. Let

$$h(x) = \beta a_\alpha(x) - \alpha a_\beta(x)$$

It is easy to compute that the minimum value, which depends on the sign of the function $h(x)$, therefore assuming that (3.11) holds,

$$CPW(x,t,F) = \frac{h(x)}{t\beta(\beta-\alpha)^2}(\beta^2|F^{(1)}|^2 + |F^{(2)} \times F^{(3)}|^2 + |G|^2$$
$$-2\beta \det F + 2(\beta F^{(1)} - F^{(2)} \times F^{(3)}) \cdot G) + \frac{a_\beta(x)}{\beta}(\det F - G \cdot F^{(1)})$$
$$\text{if } h(x) \geq 0,$$
$$CPW(x,t,F) = \frac{-h(x)}{(1-t)\alpha(\beta-\alpha)^2}(\alpha^2|F^{(1)}|^2 + |F^{(2)} \times F^{(3)}|^2 + |G|^2$$
$$-2\alpha \det F + 2(\alpha F^{(1)} - F^{(2)} \times F^{(3)}) \cdot G) + \frac{a_\alpha(x)}{\alpha}(\det F - G \cdot F^{(1)}),$$
$$\text{if } h(x) \leq 0.$$

This lower bound will become an exact value for $CQW(x,t,F)$ if these extreme points can be attained as the second moments of some measures $\nu_\alpha$, $\nu_\beta$ (according to (3.8)) and that the convex combination $t\nu_\alpha + (1-t)\nu_\beta$ is a laminate.

We explicitly find such measures in the case $h(x) \geq 0$, where the extreme point is attained when

$$\beta^2|F^{(1)}|^2 + |F^{(2)} \times F^{(3)}|^2 + |G|^2$$
$$-2\beta \det F + 2(\beta F^{(1)} - F^{(2)} \times F^{(3)}) \cdot G - t^2(\beta-\alpha)^2 S_\alpha = 0.$$

In this case we have

$$S_\alpha = \int_{R^3} |A^{(1)}|^2 d\nu_\alpha(A) = |F_\alpha^{(1)}|^2,$$

and bearing in mind that the functional $\int_{\Lambda_\alpha} |A^{(1)}|^2 d\nu_\alpha(A)$ is strictly convex, we can deduce that $\nu_\alpha = \delta_{G_\alpha}$ for $G_\alpha \in \Lambda_\alpha$ and $G_\alpha^{(1)} = F_\alpha^{(1)}$.

Let the functions $g, h : \mathbb{R}^{3\times 3} \to \mathbb{R}$ be defined as

$$g(F) = \alpha^2\beta^2|F^{(1)}|^4 + |F^{(2)} \times F^{(3)}|^4 + (\alpha^2 + 6\alpha\beta + \beta^2)(\det F - G \cdot F^{(1)})^2$$
$$-2\alpha\beta|F^{(1)}|^2|F^{(2)} \times F^{(3)} - G \cdot F^{(1)}|^2 - 2\alpha\beta(\alpha+\beta)|F^{(1)}|^2(\det F - G \cdot F^{(1)})$$
$$-2(\alpha+\beta)|F^{(2)} \times F^{(3)} - G \cdot F^{(1)}|^2(\det F - G \cdot F^{(1)}),$$

$$h(F) = (\alpha+\beta)(\det F - G \cdot F^{(1)}) - \alpha\beta|F^{(1)}|^2 - |F^{(2)} \times F^{(3)} - G|^2.$$

**Lema 3** *Let $F \notin \Lambda_\alpha \cup \Lambda_\beta$ and such that $g(F) \geq 0$ and $h(F) \geq 0$. Then, there exist $a \in \mathbb{R}^3$, $(r,s) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}^+$ such that,*

$$F + \begin{pmatrix} a \\ ra \\ sa \end{pmatrix} \in \Lambda_\alpha, \;\; F - \lambda \begin{pmatrix} a \\ ra \\ sa \end{pmatrix} \in \Lambda_\beta. \tag{3.12}$$

**Proof.** Conditions (3.12) can be written as

$$\left\{\begin{array}{l} \alpha a - sF^{(2)} \times a - ra \times F^{(3)} = -(\alpha F^{(1)} - F^{(2)} \times F^{(3)} + G), \\ \beta a - sF^{(2)} \times a - ra \times F^{(3)} = \frac{1}{\lambda}(\beta F^{(1)} - F^{(2)} \times F^{(3)} + G), \end{array}\right. \tag{3.13}$$

where we have

$$a = \frac{1}{(\beta - \alpha)}[(\alpha F^{(1)} - F^{(2)} \times F^{(3)} + G) + \frac{1}{\lambda}(\beta F^{(1)} - F^{(2)} \times F^{(3)} + G)].$$

Thus the above system has solutions if and only if

$$a \cdot (\alpha F^{(1)} - F^{(2)} \times F^{(3)} + G + \alpha a) = 0. \tag{3.14}$$

The necessity is elementary while for the sufficiency simply notice that any vector $(\alpha F^{(1)} - F^{(2)} \times F^{(3)} + G + \alpha a)$ orthogonal to $a$, can be decomposed as a linear combination of the basis $\{F^{(2)} \times a, a \times F^{(3)}\}$.

It is elementary to check that (3.14) is equivalent to $S_F(1/\lambda) = 0$ with

$$\begin{aligned} S_F(x) = x^2\beta|\alpha F^{(1)} - F^{(2)} \times F^{(3)} + G|^2 \\ +x(\alpha+\beta)(\alpha F^{(1)} - F^{(2)} \times F^{(3)} + G) \cdot (\beta F^{(1)} - F^{(2)} \times F^{(3)} + G) \\ +\alpha|\beta F^{(1)} - F^{(2)} \times F^{(3)} + G|^2. \end{aligned}$$

$S_F$ is a second degree polynomial that will have real roots if its discriminant is non negative, i.e., $g(F) \geq 0$. On the other hand, it is easy to check that $S_F(0) > 0$ and therefore there will exist positive solutions if $S_F$ is decreasing in 0, i.e., $h(F) \geq 0$. It is easy to check that these conditions are the hypotheses of the lemma.

There exist then two solutions, namely

$$\begin{aligned} \frac{1}{\lambda_i} = \frac{1}{2(\beta\alpha^2|F^{(1)}|^2 + \beta|F^{(2)} \times F^{(3)} - G|^2 - \alpha(\det F - G \cdot F^{(1)}))} \\ \Big((\beta+\alpha)((\alpha+\beta)(\det F - G \cdot F^{(1)})) - \alpha\beta|F^{(1)}|^2 + |F^{(2)} \times F^{(3)} - G|^2) \\ +(-1)^i(\beta-\alpha)\sqrt{g(F)}\Big) \end{aligned}$$

and the corresponding $(r_i, s_i)$ and $a_i$ with $i = 1, 2$.

□

Let us put $P_F(t)$ to designate

$$\begin{aligned} (\beta-\alpha)^2(\det F - G \cdot F^{(1)})t^2 + \Big(\beta|\alpha F^{(1)} - F^{(2)} \times F^{(3)} + G|^2 \\ -\alpha|\beta F^{(1)} - F^{(2)} \times F^{(3)} + G|^2 - (\beta-\alpha)^2(\det F - G \cdot F^{(1)}))\Big)t \\ +\alpha|\beta F^{(1)} - F^{(2)} \times F^{(3)} + G|^2. \end{aligned}$$

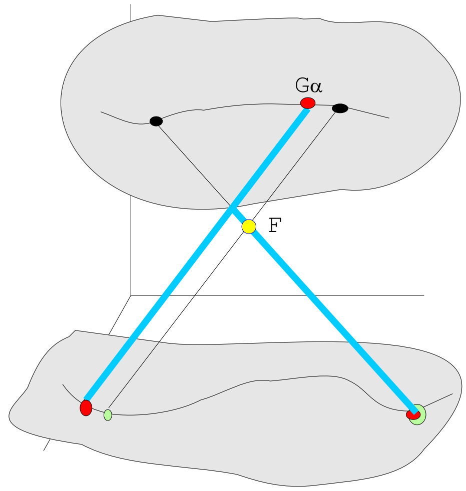



*Figure 3.2: Spatial situation*

After some additional algebraic manipulations, one can show that condition (3.11) is equivalent to $P_F(t) \leq 0$. Moreover, it is elementary to check that $P_F(t) = S_F(\frac{t}{1-t})$, and therefore the conditions which guarantee that $t \in (0,1)$ are $g(F) \geq 0$ and $h(F) \geq 0$. Let us put $t_i$, $i = 1, 2$ the two roots of $P_F$,

$$t_i = \frac{1}{2} + \frac{1}{2(\beta-\alpha)(\det F) - G \cdot F^{(1)}} \Big[ \alpha\beta |F^{(1)}|^2 - |F^{(2)} \times F^{(3)} - G|^2 \\ +(-1)^i \sqrt{g}(F) \Big] i = 1, 2.$$

It is clear that these remarks imply that the set where $CPW$ is finite can be described as the pairs $(t, F)$ such that

$$g(F) \geq 0, \;\; h(F) \geq 0, \;\; t \in [t_1, t_2]. \tag{3.15}$$

To summarize, we have that for a pair $(t, F)$ verifying (3.15), by Lemma 3, we can guarantee that there exist exactly two first-order laminates supported in $\Lambda_\alpha \cup \Lambda_\beta$ and barycenter $F$; from here we can obtain a second-order laminate which attains the optimal value of $CPW$. We can see the geometrical situation in Figure 3.2.

We are now going to work in the plane determined by $F$ and the two rank-ones directions. We seek a matrix $M_\alpha$ such that,

$$M_\alpha \in \Lambda_\alpha$$
$$M_\alpha^{(1)} = \frac{1}{t(\beta-\alpha)}(\beta F^{(1)} - F^{(2)} \times F^{(3)}). \tag{3.16}$$

A matrix $M$ belongs to that plane if

$$M = F + \sigma A_1 + \mu A_2 \tag{3.17}$$

where

$$A_i = \begin{pmatrix} a_i \\ r_i a_i \\ s_i a_i \end{pmatrix} \quad i = 1, 2,$$

are the rank-one directions determined in Lemma 3 and $(\sigma, \mu) \in R^2$ are arbitrary.

When we impose to a matrix in the plane (3.17) that its first row be given by (3.16), we find a unique pair $(\sigma^*, \mu^*)$ given by

$$\sigma^* = \frac{\lambda_1(t + \lambda_2 t - \lambda_2)}{t(\lambda_1 - \lambda_2)},$$

$$\mu^* = \frac{-\lambda_2(t + \lambda_1 t - \lambda_1)}{t(\lambda_1 - \lambda_2)}.$$

Note that $\mu^* = 1 - \sigma^*$. Then the issue is to check if $M_\alpha = F + \sigma^* A_1 + \mu^* A_2$ belongs to $\Lambda_\alpha$.

We take a matrix $M$ in the plane (3.17) and force that $M$ to belong to $\Lambda_\alpha$. Having in mind (3.13),

$$s_i F^{(2)} \times a_i + r_i a_i \times F^{(3)} = \alpha F^{(1)} - F^{(2)} \times F^{(3)} + G + \alpha a_i \quad \text{with } i = 1, 2,$$

$$\left.\begin{array}{l} M = F + \sigma A_1 + \mu A_2 \\ M \in \Lambda_\alpha \end{array}\right\} \Leftrightarrow$$

$$\begin{array}{l}
\alpha(F^{(1)} + \sigma a_1 + \mu a_2) + G = \\
(F^{(2)} + \sigma r_1 a_1 + \mu r_2 a_2) \times (F^{(3)} + \sigma s_1 a_1 + \mu s_2 a_2) = F^{(2)} \times F^{(3)} \\
+ \sigma(s_1 F^{(2)} \times a_1 + r_1 a_1 \times F^{(3)}) + \mu(s_2 F^{(2)} \times a_2 + r_2 a_2 \times F^{(3)}) = \\
\{\text{using (3.13)}\} \\
F^{(2)} \times F^{(3)} + (\sigma + \mu)(\alpha F^{(1)} - F^{(2)} \times F^{(3)} + G) + \alpha(\sigma a_1 + \mu a_2) \Leftrightarrow \\
(1 - \sigma - \mu)(\alpha F^{(1)} - F^{(2)} \times F^{(3)} + G) = 0.
\end{array}$$

Then a matrix $M = F + \sigma A_1 + \mu A_2$ belongs to $\Lambda_\alpha$, if and only if $\mu = 1 - \sigma$. From this we can deduce that the intersection between $\Lambda_\alpha$ and the plane determined by the rank-one directions is a linear manifold, and most important, $M_\alpha = F + \sigma^* A_1 + \mu^* A_2$ belongs to $\Lambda_\alpha$, since $\mu^* = 1 - \sigma^*$. Therefore, the characterization of this intersection can be written in the form

$$M = F + \sigma A_1 + (1 - \sigma) A_2 = \sigma F_{\alpha,1} + (1 - \sigma) F_{\alpha,2},$$

where $F_{\alpha,i} = F + A_i$, $i = 1, 2$, are the intersection between the rank-one directions and the $\Lambda_\alpha$ manifold (the black points in Figure 4.1).

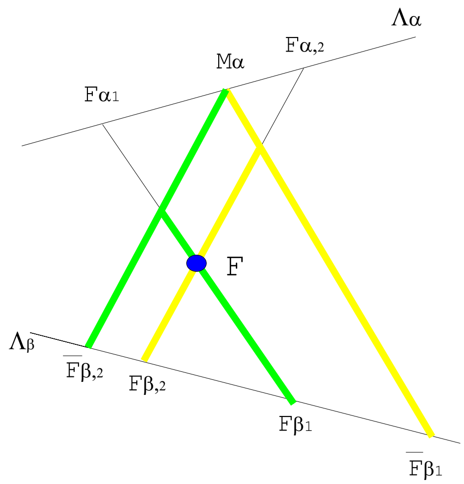



*Figure 3.3: Plane of rank-one directions*

In a similar way, we can show that the intersection between the rank-one directions and the $\Lambda_\beta$ manifold is again another linear manifold determined by

$$M = F + \sigma A_1 + (1-\sigma)A_2 = \sigma F_{\beta,1} + (1-\sigma)F_{\beta,2}$$

where $F_{\beta,i} = F + \lambda_i A_i$, $i = 1, 2$, are the intersection between the rank-one directions and the $\Lambda_\beta$ manifold (the green points in Figure 3.2).

Then we have $\nu_\alpha = \delta_{M_\alpha}$ such that $supp(\nu_\alpha) \subset \Lambda_\alpha$. If we seek $\nu_\beta$ as a convex combination of two Dirac masses with support in the $\Lambda_\beta$ manifold, we may produce a second-order laminate supported in $\Lambda_\alpha \cup \Lambda_\beta$. The situation in the plane can be drawn as in Figure 3.3 where laminates are shown with green and yellow colors.

Let $\bar{F}_{\beta,i} = M_\alpha + l_i(F_{\beta,i} - F_{\alpha,i})$. It is easy to check that

$$\bar{F}_{\beta,i} = l_i^* F_{\beta,1} + (1-l_i^*)F_{\beta,2}$$

with

$$l_i^* = \frac{\lambda_i - t(\lambda_2+1)}{t(\lambda_1-\lambda_2)}.$$

To sum up, if we consider the matrices,

$$\begin{aligned}
F_{\alpha,i} = F + A_i,\ F_{\beta,i} = F - \lambda_i A_i \quad &\text{with} \quad A_i = \begin{pmatrix} a_i \\ r_i a_i \\ s_i a_i \end{pmatrix}, i = 1, 2,\\
M_\alpha = \sigma^* F_{\alpha,1} + (1-\sigma^*)F_{\alpha,2} \quad &\text{with} \quad \sigma^* = \tfrac{\lambda_1(t+\lambda_2 t-\lambda_2)}{t(\lambda_1-\lambda_2)},\\
\bar{F}_{\beta,i} = l_i^* F_{\beta,1} + (1-l_i^*)F_{\beta,2} \quad &\text{with} \quad l_i^* = \tfrac{\lambda_i - t(\lambda_2+1)}{t(\lambda_1-\lambda_2)},
\end{aligned}$$

and the scalars

$$\rho_{i,j} = \tfrac{t-\lambda_i(1-t)}{\lambda_j-\lambda_i} \quad \tau_{i,j} = \tfrac{t-\lambda_j(1-t)}{t(\lambda_i+1)-\lambda_j},$$

we can define the second-order laminate with support on $\Lambda_\alpha \cup \Lambda_\beta$, barycenter $F$, and mass in $\Lambda_\alpha$ equal to $t$, by putting

$$\nu_{i,j} = \tau_{i,j}\delta_{F_{\beta,i}} + (1-\tau_{i,j})(\rho_{i,j}\delta_{\bar{F}_{\beta,j}} + (1-\rho_{i,j})\delta_{M_\alpha})$$

with $i, j \in \{1, 2\}$, $i \neq j$ where,

$$\det(F_{\beta,j} - M_\alpha) = 0$$

and

$$\det(F_{\beta,i} - (\rho_{i,j}\bar{F}_{\beta,j} + (1-\rho_{i,j})M_\alpha)) = 0.$$

A similar result holds for the other point where the optimal value is attained.

This finishes the computation of $CQW(x, t, F)$ and the proof of Theorem 4.

We would like to thank C.Barbarosie for bringing to our attention the references related to Clebsch potentials.

# Chapter 4

# Dynamic materials for an optimal design problem under the two-dimensional wave equation

## 4.1 Introduction

In this paper, we will study the following optimal design problem. We consider a design domain $\Omega \subset \mathbb{R}^2$, a positive time $0 < T$, and the amount of one material at our disposal $V_\alpha \in (0, 1)$. The optimal design problem consists in deciding, for each time $0 < t < T$, the best distribution in $\Omega$ of the two materials in order to minimize the time-dependent cost functional depending on the square of the gradient (with respect to all variables $(t, x)$) of the underlying state. In precise terms, the problem consists in minimizing

$$I(\chi) = \int_0^T \int_\Omega \left[u_t^2(t,x) + a(t,x,\chi)|\nabla_x u(t,x)|^2\right] \, dx \, dt$$

where $u$ is the unique solution of

$$\begin{aligned} u_{tt} - div_x([\alpha\chi + \beta(1-\chi)]\nabla_x u) = 0 \quad &\text{in} \quad (0,T)\times\Omega, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) \quad &\text{in} \quad \Omega, \qquad (4.1) \\ u(t,x) = f(t,x), \quad &\text{in} \quad [0,T]\times\partial\Omega, \qquad (4.2) \end{aligned}$$

and the functions $a$ $u_0$, $u_1$ and $f$ are known and satisfying the typical compatibility condition for the hyperbolic equations we are considering. The function $\chi \in L^\infty([0,T]\times\Omega; \{0,1\})$ is the design variable, and it indicates where we place the $\alpha$-material for each time $t$. Since $\chi$ is a binary variable, $a(t,x,\chi) \in \{a(t,x,1), a(t,x,0)\}$ and we can write

$$a(t,x,\chi) = \chi(t,x)a_\alpha(t,x) + (1-\chi(t,x))a_\beta(t,x).$$

In addition, we make the assumption

$$a_\alpha(t,x) + \alpha \geq 0, \quad a_\beta(t,x) + \beta \geq 0.$$

The amount of the $\alpha$-material is given, and therefore we have to enforce the volume constraint

$$\int_\Omega \chi(t,x)dx \leq V_\alpha|\Omega|, \quad \forall t \in [0,T].$$

The lack of classical solutions for this sort of problems is well understood ([44]). In general, this is also the situation for our problem governed by a wave equation. The theory of homogenization is an important tool which introduces new types of composites as structural elements through the concepts of H-convergence or G-convergence ([1]). The theory of homogenization is especially useful when we work with non-explicit dependence on the flux $\nabla u(x)$. There exist some works where the hyperbolic problem is treated. The characterization of a "hyperbolic $G - clousure$" is apparently still unknown ([34], [33]). In [33] the author proves that in the one spatial dimension the $G - clousure$ has the special property on the det, which we assert is true for the two spatial time. In greater dimensions the hyperbolic case has been studied for more simple models in which the design variable depend on the time variable only ([13]). In this work we study a more general case within time-space designs.

Our approach is based on an equivalent variational reformulation of the original optimal design problem as a non-convex vector variational situation ([17]). This is indeed a clear indication that there might not be optimal designs. As in other situations examined under this perspective ([2], [56]), we propose to analyze the "*constrained quasiconvexification*" for this last problem by using gradient Young measures as generalized solutions of variational principles, and compute an explicit relaxation of the original optimal design problem in the form of a relaxation.

It is elementary to check (see details in Section 4.2), the equivalence of our dynamic optimal design problem with the following non-convex, vector variational problem

$$\underset{U}{min} \quad \hat{I}\ (U) = \int_0^T \int_\Omega W(t,x,\nabla U(t,x))dxdt$$

subject to

$$\begin{aligned}
U = (U^{(1)}, U^{(2)}, U^{(3)}) \in H^1([0,T]\times\Omega)^3,& \\
U^{(1)}(0,x) = u_0(x),\ U_t^{(1)}(0,x) = u_1(x) \quad &\text{in} \quad \Omega, \\
U^{(1)}(t,x) = f(t,x) \quad &\text{in} \quad [0,T]\times\partial\Omega, \\
\int_\Omega V(t,x,\nabla U(t,x))dx \leq V_\alpha|\Omega| \quad \forall t \in [0,T],&
\end{aligned}$$

where the two integrands involved are

$$W(t,x,A) = \begin{cases} a_{11}^2 + a_\alpha(t,x)[a_{12}^2 + a_{13}^2], & \text{if } A \in \Lambda_\alpha \cup \Lambda_\beta, \\ a_{11}^2 + a_\beta(t,x)[a_{12}^2 + a_{13}^2], & \text{if } A \in \Lambda_\beta \setminus \Lambda_\alpha, \\ +\infty, & \text{else,} \end{cases}$$

$$V(t,x,A) = \begin{cases} 1, & \text{if } A \in \Lambda_\alpha, \\ 0, & \text{if } A \in \Lambda_\beta \setminus \Lambda_\alpha, \\ +\infty, & \text{else.} \end{cases}$$

Here

$$\Lambda_\gamma = \{A \in M^{3\times 3} : M_{-\gamma} A^{(1)} - A^{(2)} \times A^{(3)} = 0\}, \quad \gamma = \alpha, \beta,$$

where $A^{(i)}$ is the $i$-th row of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

and

$$M_{-\gamma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\gamma & 0 \\ 0 & 0 & -\gamma \end{pmatrix}.$$

The aim of this paper is to find an explicit relaxation of this variational problem which may eventually be used to perform numerical simulations of the optimal dynamical profiles of the original optimal design problem. To this end, put

$$F = \begin{pmatrix} F^{(1)} \\ F^{(2)} \\ F^{(3)} \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}, \text{ and } L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$z_\alpha = \frac{1}{s(\beta-\alpha)} L(\beta F^{(1)} + F^{(2)} \times F^{(3)}),$$

$$z_\beta = \frac{-1}{(1-s)(\beta-\alpha)} L(\alpha F^{(1)} + F^{(2)} \times F^{(3)})$$

Set

$$\psi(F,s) = \det F - F_{11}^2 + s\alpha|z_\alpha|^2 + (1-s)\beta|z_\beta|^2, \tag{4.3}$$

$$h(t,x) = \beta a_\alpha(t,x) - \alpha a_\beta(t,x).$$

Consider the variational problem

$$\min_{U,s} \int_0^T \int_\Omega \varphi(t,x,\nabla U(t,x), s(t,x))\,dxdt$$

subject to

$$\begin{aligned} 0 \le s(t,x) \le 1, \int_\Omega s(t,x)\,dx \le V_\alpha|\Omega| \quad \forall t \in [0,T], \\ U \in H^1([0,T]\times\Omega)^3, \quad (\nabla U^{(1)})_1 = (\nabla U^{(2)} \times \nabla U^{(3)})_1, \\ U^{(1)}(0,x) = u_0(x),\ U_t^{(1)}(0,x) = u_1(x) \text{ in } \Omega, \\ U^{(1)}(t,x) = f(t,x) \text{ in } [0,T]\times\partial\Omega, \end{aligned}$$

where $\varphi(t,x,F,s)$ is given by

$$\varphi(t,x,F,s) = \begin{cases} \frac{1}{\beta}((\beta+a_\beta)|F_{11}|^2 - a_\beta \det F + sh|z_\alpha|^2) \\ \qquad\qquad \text{if } h(t,x) \ge 0, \psi(F,s) \le 0, \\ \frac{1}{\alpha}((\alpha+a_\alpha)|F_{11}|^2 - \det F - (1-s)h|z_\beta|^2) \\ \qquad\qquad \text{if } h(t,x) \le 0, \psi(F,s) \le 0, \\ \det F + s(\alpha+a_\alpha)|z_\alpha|^2 + (1-s)(\beta+\beta)|z_\beta|^2 \\ \qquad\qquad \text{if } \psi(F,s) \ge 0, \end{cases} \tag{4.4}$$

for every $F$ such that verifies $F_{11} = (F^{(2)} \times F^{(3)})_1$.

**Theorem 5** *The above variational problem is a relaxation of the initial optimization problem in the sense that*

*a) the infima of both problems coincide;*

*b) there are optimal solutions for the relaxed problem;*

*c) these solutions codify ( in the sense of Young measures) the optimal microstructures of the original optimal design problem.*

In addition, we can provide explicitly optimal microstructures.

**Theorem 6** *Optimal Young measures leading to the relaxed formulation are always laminates which can be given in a completely explicit form.*

The formulae for all of these laminates are given later at the end of Section 4.5.

The main new contribution here is therefore to understand the character of the hyperbolic state law, and the differences it introduces with respect to the better known elliptic case. On the other hand, the most important difference with the one-dimensional case (treated in [37]) is the non-linear character introduced by the characterization of the divergence-free vector fields in terms of Clebsch potentials: the manifolds $\Lambda_\gamma$ are two non-linear six-dimensional manifold whose intersection is another non-linear four-dimensional manifold. Moreover, there are rank-one connections within those manifolds. An interesting consequence of this is that the relaxed integrand is finite everywhere (except for the compatibility condition $(\nabla U^{(1)})_1 = (\nabla U^{(2)} \times \nabla U^{(3)})_1$) in contrast with the elliptic case where the relaxed integrand is finite only in a certain (quasi)convex subset.

A subsequent important step, which we hope to address in the near future, is to explore the relaxed problem in some particular cases, like the ones described in the last section, with the objective of producing numerical simulations of the optimal time-dependent structures.

The paper is organized as follows. In Section 4.2, we describe in more detail the equivalent variational reformulation as well as a general relaxation result when integrands are not continuous and take on infinite values abruptly. As there is nothing new here compared to other previous works in the elliptic case and the hyperbolic one-dimensional case, our description is rather a remainder included here for the sake of completeness. Sections 4.3 and 4.4 are technical in nature but interesting. We study the original optimal design problem when the function $a$ is identically to 1. We first compute a lower bound for the *constrained quasiconvexification* (Section 4.3), by using in a fundamental way not only the weak continuity of the determinant but also of some 2×2 minors. Section 4.4 is concerned with the search for laminates furnishing the precise value of the lower bound in an attempt to show equality of the three convex hulls (poly-, quasi- and rank-one convex hulls), as it is standard in this kind of calculation. In Section 4.5 we generalize the results in Sections 4.3 and 4.4 to a general function $a$. Finally, in Section 4.6 we show some particular examples of this relaxation for different and interesting choices of the coefficient $a$.

## 4.2 Variational reformulation and relaxation

We know the lack of classical solution of the original optimal design problem. In this sense we propose to reformulate the problem as a vector variational problem to which we can apply suitable tools to study its relaxation. We follow the technique as in [36].

The treatment of this state equation is more delicate than the two-dimensional case, in which the divergence-free fields are characterized as the counterclockwise $\pi/2$-rotation of the gradients of scalar functions, while in the three-dimensional case the situation is more complex. We will use a characterization by "Clebsch potentials". There are different results in which n-dimensional divergence-free vector fields can be represented in terms of (n-1) arbitrary functions. For the three dimensional case, if $F \in \mathbb{R}^3$ with $\mathrm{div}(F) = 0$, then there typically exist Clebsch potentials $v, w$ such that $F = \nabla v \times \nabla w$. The proof of this result is beyond the scope of the present paper and we refer to [31], [51], [65], [58]. In fact, looking at the specialized literature, it seems that this representation in terms of the Clebsch potentials is not always valid ([26]). But this is a fine point for experts which we have avoided. We simply use this representation in the sequel.

There exists two potentials $v, w \in H^1((0,T)\times\Omega)$ such that,

$$-div\Big(u_t(t,x), -[\alpha\chi(t,x)+\beta(1-\chi(t,x))]\nabla_x u(t,x)\Big) = 0 \text{ in } [0,T]\times\Omega$$

is equivalent to the pointwise constraint

$$\begin{pmatrix} u_t(t,x) \\ -(\alpha\chi(t,x)+\beta(1-\chi(t,x)))\nabla_x u(t,x) \end{pmatrix} = \nabla v(t,x)\times\nabla w(t,x) \tag{4.5}$$

a.e. $(t,x)\in[0,T]\times\Omega$, and $v,w\in H^1([0,T]\times\Omega)$. Set

$$\Lambda_\gamma = \{A \in M^{3\times3} : M_{-\gamma}A^{(1)} - A^{(2)}\times A^{(3)} = 0\}, \quad \gamma=\alpha,\beta,$$

where $A^{(i)}$ is the $i$-th row of the matrix A, and the matrices $M_{-\gamma}$ are defined as in the Introduction.

It is clear that we can identify the design variable $\chi$ with the vector field $U=(u,v,w)$ and conversely a triplet $U=(u,v,w)$ which verifies (4.5), with a characteristic function $\chi$, so that, we can consider the new design variable $U=(U^{(1)},U^{(2)},U^{(3)})=(u,v,w)$, where $U:\mathbb{R}^3\to\mathbb{R}^3$ and $\nabla U(t,x)\in\mathbb{R}^{3\times3}$.

Therefore by using the above statement and the notation in the Introduction. It is easy to check that the original optimal design problem is equivalent to the variational problem

$$\min_U \quad \hat{I}\ (U) = \int_0^T\int_\Omega W(t,x,\nabla U(t,x))dxdt$$

subject to

$$\begin{aligned} U=(U^{(1)},U^{(2)},U^{(3)}) &\in H^1((0,T)\times\Omega)^3, \\ U^{(1)}(0,x)=u_0(x),\ U_t^{(1)}(0,x) &= u_1(x) \quad \text{in} \quad \Omega \\ U^{(1)}(t,x) &= f(t,x) \quad \text{in} \quad [0,T]\times\partial\Omega \\ \int_\Omega V(t,x,\nabla U(t,x))dx &\le V_\alpha|\Omega| \quad \forall t\in[0,T]. \end{aligned}$$

We have recast our optimal design problem as a typical variational problem. We see that it is a non-convex vector problem that we are going to analyze by seeking its relaxation. We use Young measures as a main tool in the computation of the suitable density for the relaxed problem. In this sense, we have the following relaxation result ([37],[36]) whose main idea is a useful tool in other different situations ([2],[56]). The full relaxation for the above reformulation can be expressed as follows

$$\text{Minimize in } \nu \qquad \int_\Omega \int_0^T \int_{M^{3\times 3}} W(t,x,A)d\nu_{t,x}(A)dtdx$$

subject to

$$\nu \text{ is a } H^1((0,T)\times\Omega)\text{-gradient Young measure, supported in } \Lambda_\alpha \cup \Lambda_\beta,$$
$$U^{(1)} \text{ satisfies the I.C. and B.C.,}$$
$$\nabla U(t,x) = \int_{M^{3\times 3}} A d\nu_{t,x}(A)$$
$$\int_\Omega \int_{M^{3\times 3}} V(t,x,A)d\nu_{t,x}(A)dx = V_\alpha|\Omega|, \forall t \in [0,T].$$

Here I.C. and B.C. stand for initial condition (4.1), and boundary condition (4.2), respectively.

We would like to analyze the above generalized variational problem. We note that

$$s(t,x) = \int_{M^{3\times 3}} V(t,x,A)d\nu_{t,x}(A),$$

is the volume fraction associated with the $\alpha$-material. Relaxation can be established in terms of $(s,U)$

$$\underset{(s,U)}{Minimize} \qquad \int_\Omega \int_0^T CQW(t,x,\nabla U(t,x), s(t,x))dtdx$$

subject to,

$$U^{(1)} \text{ satisfies the I.C. and B.C.,}$$
$$\int_\Omega s(t,x)dx = V_\alpha|\Omega|, \forall t \in [0,T].$$

where $CQW$ is called *constrained quasiconvexification* of the functional $W$ and it is defined as

$$CQW(t,x,A,s) = \inf_\nu \Big\{ \int_{M^{3\times 3}} W(t,x,A)d\nu(A) : F = \int_{M^{3\times 3}} A d\nu(A), \\ \int_{M^{3\times 3}} V(t,x,A)d\nu(A) = s,\ \forall t \in [0,T] \Big\}$$

with $\nu$ is a $H^1$-gradient Young measure with $supp(\nu) \subset \Lambda_\alpha \cup \Lambda_\beta$.

In the following sections, we compute this relaxed integrand.

## 4.3 Polyconvexification.

We would like to start by computing a lower bound of the "*constrained quasiconvexification*" when the function $a$ is identically equal to 1. In the above context, and dropping the time-space dependence (it can be consider as a parameter) this density is defined as

$$\inf_{\nu}\Big\{\int_{M^{3\times 3}} W(A)d\nu(A) : F = \int_{M^{3\times 3}} Ad\nu(A),\\ \int_{M^{3\times 3}} V(A)d\nu(A) = s\ \forall t\in[0,T]\Big\}$$

with $\nu$, the class of $H^1$-Young measures with $supp(\nu)\subset\Lambda_\alpha\cup\Lambda_\beta$.

The main difficulty here is that we do not know the gradient Young measure set over which we have to minimize, as in other similar situations ([7], [37], [36], [56], [54]). First, we will study a minimum problem over a broader class of probability measures. We take the polyconvex measures whose main property is the commutativity of the minors. Once this minimum (lower bound) is computed, we will check that optimal measures in the polyconvexification are indeed laminates, and therefore gradient Young measure which give us the "*quasiconvexification*".

We compute the "*constrained polyconvexification*" which is given by the following minimization problem

$$CPW(F,s) = \min_{\nu}\int_{M^{3\times 3}} W(A)d\nu(A)$$

where,

$$\nu\in\mathcal{A}(F,s) = \Big\{\nu : \nu \text{ is a homogeneous Young measure,}$$

$$\text{which commutes with the minors, } F = \int_{M^{3\times 3}} Ad\nu(A), \tag{4.6}$$

$$s = \int_{M^{3\times 3}} V(A)d\nu(A)\Big\}. \tag{4.7}$$

From the volume constraint (4.7), we can deduce

$$\nu = s\nu_\alpha + (1-s)\nu_\beta, supp(\nu_\gamma)\subset\Lambda_\gamma,\ \gamma=\alpha,\beta,$$

and using the first moment (4.6) and the above decomposition for $\nu$,

$$F = s\int_{\Lambda_\alpha} Ad\nu_\alpha(A) + (1-s)\int_{\Lambda_\beta} Ad\nu_\beta(A). \tag{4.8}$$

We note by $F_\gamma$ the first moment of $\nu_\gamma$

$$F_\gamma = \int_{\Lambda_\gamma} Ad\nu_\gamma(A),\quad \gamma=\alpha,\beta.$$

An important difference with respect to the one-dimensional case is found here. Because of the non-linear character of the manifold $\Lambda_\gamma$. One can check that $F_\gamma \notin \Lambda_\gamma$, in general.

Thus, we cannot follow the same path as in the one-dimensional case, in which we know explicitly the matrices $F_\gamma$. In this situation, we use the polyconvex character of $\nu$, which implies that $\nu$ commutes with all the minors, not just the determinat. The components of $F^{(2)} \times F^{(3)}$ are the second order minors which have been computed by using the second and third rows of the matrix $F$. The commutation with $\nu$ leads to

$$\begin{aligned} F^{(2)} \times F^{(3)} &= \int_{R^{3\times 3}} A^{(2)} \times A^{(3)} d\nu(A) \\ &= s \int M_{-\alpha} A^{(1)} d\nu_\alpha + (1-s) \int M_{-\beta} A^{(1)} d\nu_\beta \\ &= s M_{-\alpha} F_\alpha^{(1)} + (1-s) M_{-\beta} F_\beta^{(1)}. \end{aligned} \tag{4.9}$$

and from the first moment decomposition (4.8)

$$F = sF_\alpha + (1-s)F_\beta \Rightarrow F^{(1)} = sF_\alpha^{(1)} + (1-s)F_\beta^{(1)}.$$

From (4.9) and the above formula, we have in general an incompatible system on $F_\gamma^{(1)}$. The compatibility condition on the matrix F is

$$F_{11} = F_{22}F_{33} - F_{32}F_{23}.$$

For the rest of the paper we suppose that this compatibility condition is true. In this, case the system is a non-uniquely-determined system whose solutions are

$$F_\alpha^{(1)} = \begin{pmatrix} \lambda \\ z_\alpha \end{pmatrix} \text{ and } F_\beta^{(1)} = \begin{pmatrix} \frac{F_{11} - s\lambda}{1-s} \\ z_\beta \end{pmatrix}.$$

with $\lambda \in \mathbb{R}$, and where $z_\gamma$ are the vectors

$$z_\alpha = \frac{1}{s(\beta-\alpha)} L(\beta F^{(1)} + F^{(2)} \times F^{(3)}), = \frac{1}{s(\beta-\alpha)} \bar{z}_\alpha$$

and

$$z_\beta = \frac{-1}{(1-s)(\beta-\alpha)} L(\alpha F^{(1)} + F^{(2)} \times F^{(3)}), = \frac{-1}{(1-s)(\beta-\alpha)} \bar{z}_\beta.$$

One checks that if

$$A \in \Lambda_\gamma \Rightarrow \det A = A^{(1)} M_{-\gamma} A^{(1)},$$

and, by using the commutativity property of the determinant (minors) of the polyconvex measure $\nu$, we have that

$$\det F = \int_{M^{2\times 2}} \det A d\nu(A)$$
$$= s\int_{\Lambda_\alpha} \det A d\nu_\alpha(A) + (1-s)\int_{\Lambda_\beta} \det A d\nu_\beta(A)$$
$$= \int_{\mathbb{R}} a_{11}^2 d\nu(A) - s\alpha \int_{\mathbb{R}} a_{12}^2 + a_{13}^2 d\nu_\alpha^{(12)} + (1-s)\beta \int_{\mathbb{R}} a_{12}^2 + a_{13}^2 d\nu_\beta^{(12)} \quad (4.10)$$

where $\nu_\gamma^{1i}$ denotes the projection of $\nu_\gamma$ onto the $1i$-th component.

We introduce the new variables

$$1 = \int_{M^{3\times 3}} a_{11}^2 d\nu^{(11)}, \quad S_\gamma = \int_{\Lambda_\gamma} a_{12}^2 + a_{13}^2 d\nu_\gamma, \text{ with } \gamma = \alpha, \beta.$$

Then (4.10) can be rewritten as

$$\det F = S_1 - s\alpha S_\alpha - (1-s)\beta S_\beta.$$

On the other hand, by Jensen's inequality

$$S_1 \geq (F_{11})^2, \quad S_\gamma \geq (z_\gamma)^2 \quad \gamma = \alpha, \beta.$$

The cost functional can be rewritten in terms of the $S_\gamma$ variables

$$S_1 + sS_\alpha + (1-s)S_\beta.$$

Altogether, by using the notation and formulae so far introduced, we transform the calculation of the polyconvexification into the following linear mathematical programming problem

$$\underset{(S_1, S_\gamma, F_{\gamma,11})}{\text{minimize}} \; S_1 + sS_\alpha + (1-s)S_\beta$$

subject to

$$\det F = S_1 - s\alpha S_\alpha - (1-s)\beta S_\beta,$$
$$S_1 \geq (F_{11})^2, S_\gamma \geq (z_\gamma)^2, \text{ with } \gamma = \alpha, \beta,$$
$$F_{11} = sF_{\alpha,11} + (1-s)F_{\beta,11}.$$

The first important thing to realize is that there exit always optimal solutions (the admissible set is always non-empty). The second issue about this problem is to determine the point(s) where the optimal value is attained. To this end, we consider the function $\psi$ defined in (4.3).

There exist two different minimum points, depending on the sign of the function $\psi$. The minima are

$$\frac{1}{\beta}((\beta+1)|F_{11}|^2 - \det F + s(\beta-\alpha)|z_\alpha|^2) \qquad \text{if } \psi(F,s) \le 0,$$

and

$$\det F + s(\alpha+1)|z_\alpha|^2 + (1-s)(\beta+1)|z_\beta|^2 \qquad \text{if } \psi(F,s) \ge 0.$$

From the optimality condition, we can deduce that the minimum is attained on

$$S_{\alpha,2} = |z_\alpha|^2 \text{ and } S_1 = |F_{11}|^2 \text{ if } \psi(F,s) \le 0, \tag{4.11}$$

$$S_\alpha = |z_\alpha|^2 \text{ and } S_\beta = |z_\beta|^2 \text{ if } \psi(F,s) \ge 0. \tag{4.12}$$

Therefore the *constrained polyconvexification* is

$$CPW(F,s) = \begin{cases} \frac{1}{\beta}((\beta+1)|F_{11}|^2 - \det F + s(\beta-\alpha)|z_\alpha|^2) & \\ \qquad \text{if } \psi(F,s) \le 0, & \\ \det F + s(\alpha+1)|z_\alpha|^2 + (1-s)(\beta+1)|z_\beta|^2 & \\ \qquad \text{if } \psi(F,s) \ge 0. & \end{cases}$$

## 4.4 Upper bound: laminates

We alredy have a lower bound given by the polyconvexification. The next step is to check that this bound is recovered by at least one laminate. This tells us that the lower bound is optimal, and so $\varphi$ given in (4.4) is the *constrained quasiconvexification* sought.

We search for a laminate $\nu = s\nu_\alpha + (1-s)\nu_\beta$ with $supp(\nu_\gamma) \subset \Lambda_\gamma$, $s \in (0,1)$ and first moment $F$. We have different optimality conditions depending on the sign of $\psi$. We study the different cases, successively.

If $\psi(F,s) \ge 0$, the optimality condition (4.12) tells us

$$S_\alpha = |z_\alpha|^2 \text{ and } S_\beta = |z_\beta|^2.$$

We can deduce by using the strict convexity of the function $|\cdot|^2$ that

$$\nu_\gamma^{(1i)} = \delta_{z_{\gamma(i-1)}}, \text{ with } \gamma = \alpha, \beta \quad i = 2, 3.$$

therefore, we only know that

$$F_\gamma^{(1)} = (\lambda, z_\gamma)$$

with arbitrary $\lambda \in \mathbb{R}$.

We try to find two matrices $F_\gamma \in \Lambda_\gamma$ such that

$$rk(F_\alpha - F_\beta) = 1,$$

$$F = sF_\alpha + (1-s)F_\beta \quad \text{with } s \in (0,1).$$

In others words, we seek a matrix $A \in M^{3\times 3}$ with $rk(A) = 1$,

$$A = \begin{pmatrix} a \\ la \\ ma \end{pmatrix},$$

with $a \in \mathbb{R}^3$ and $(l,m) \in \mathbb{R}^2$, such that

$$F + A \in \Lambda_\alpha, \quad F - \frac{s}{1-s}A \in \Lambda_\beta. \tag{4.13}$$

The above condition (4.13) can be rewritten as

$$\begin{cases} M_{-\alpha}a - mF^{(2)} \times a - la \times F^{(3)} = -(M_{-\alpha}F^{(1)} - F^{(2)} \times F^{(3)}), \\ M_{-\beta}a - mF^{(2)} \times a - la \times F^{(3)} = \frac{1-s}{s}(M_{-\beta}F^{(1)} - F^{(2)} \times F^{(3)}). \end{cases} \tag{4.14}$$

From (4.14), we can deduce

$$(M_{-\beta} - M_{-\alpha})a = (M_{-\alpha}F^{(1)} - F^{(2)} \times F^{(3)}) + \frac{1-s}{s}(M_{-\beta}F^{(1)} - F^{(2)} \times F^{(3)}).$$

Hence

$$a = (\lambda, \frac{1}{(\beta-\alpha)s}L\big((s\alpha + (1-s)\beta)F^{(1)} - F^{(2)} \times F^{(3)}\big)).$$

On the other hand,

$$\left.\begin{array}{l} F + A \in \Lambda_\alpha \Rightarrow F^{(1)} + a = F_\alpha^{(1)} \\ F - \frac{s}{1-s}A \in \Lambda_\beta. \Rightarrow F^{(1)} - \frac{s}{1-s}a = F_\beta^{(1)} \end{array}\right\} \Rightarrow$$

$$a = (\lambda, (1-s)(z_\alpha - z_\beta)).$$

Moreover, the system (4.14) has solutions if and only if

$$a(M_{-\alpha}F^{(1)} - F^{(2)} \times F^{(3)} + M_{-\alpha}a) = 0,$$

and this equation is equivalent to

$$\lambda^2 - (1-s)^2\Big(\alpha|z_\alpha|^2 + |z_\beta|^2 - (\alpha+\beta)z_\alpha z_\beta\Big) = 0.$$

After some tedious computations, one can check that

$$\alpha|z_\alpha|^2 + |z_\beta|^2 - (\alpha+\beta)z_\alpha z_\beta = \psi(F,s),$$

and therefore, there exist two solutions

$$\lambda_i = (-1)^i (1-s)\sqrt{\psi(F,s)},$$

and two rank-one directions associated with them. If we put

$$a_i = (\lambda_i, (1-s)(z_\alpha - z_\beta),$$

its associated pair $(l_i, m_i)$, the matrices

$$A_i = \begin{pmatrix} a_i \\ l_i a_i \\ m_i a_i \end{pmatrix}$$

$$\begin{aligned} F_{\alpha,i} &= F + A_i \in \Lambda_\alpha, \\ F_{\beta,i} &= F - \tfrac{s}{1-s} A_i \in \lambda_\beta, \end{aligned}$$

are such that

$$F = sF_{\alpha,i} + (1-s)F_{\beta,i} \quad \text{and} \quad rk(F_{\alpha,i} - F_{\beta,i}) = 1,$$

and the optimal laminates are

$$\nu = s\delta_{F_{\alpha,i}} + (1-s)\delta_{F_{\beta,i}} \qquad i = 1, 2.$$

The next step consists in analyzing the case when $\psi(F,s) \le 0$. In this case the optimal condition (4.11) tells us

$$S_{\alpha,2} = |z_\alpha|^2 \text{ and } S_1 = |F_{11}|^2,$$

and again by using the strict convexity of $|\cdot|^2$, we conclude that

$$\nu^{(11)} = \delta_{F_{11}} \text{ and } \nu_\alpha^{(1i)} = \delta_{F_{\alpha,1i}}, \quad i = 2, 3,$$

i.e.

$$\nu_\alpha^{(1)} = \delta_{F_\alpha^{(1)}} \text{ with } F_\alpha^{(1)} = (F_{11}, z_\alpha).$$

Therefore we seek a laminate $\nu = s\delta_{M_\alpha} + (1-s)\nu_\beta$, where

$$\begin{aligned} &M_\alpha \in \Lambda_\alpha, \\ &M_\alpha^{(1)} = F_\alpha^{(1)} = (F_{11}, z_\alpha). \end{aligned} \tag{4.15}$$

We follow a similar argument as in [36].

Firstly, we analyze the set $\psi(F,s) \le 0$. We can check

$$\begin{aligned} &\psi(F,s) \le 0 \Leftrightarrow \\ -(\beta-\alpha)^2(\det F - F_{11}^2)s^2 + (\beta|\bar{z}_\beta|^2 &+ (\beta-\alpha)^2(\det F - F_{11}^2) - \alpha|\bar{z}_\alpha|^2)s \\ &+\alpha|\bar{z}_\alpha|^2 \le 0. \end{aligned}$$

Let $P_F(s)$ stand for the above second-degree polynomial. It is easy to check that for fixed $F$, the set where $\psi(F,s) \le 0$ is the set where $P_F(s)$ has a solution in the interval (0,1). Therefore we study this polynomial and consider the functions $g, h : \mathbb{R}^{3\times 3} \to \mathbb{R}$ defined as

$$g(F) = (\beta|\bar{z}_\beta|^2 + (\beta-\alpha)^2(\det F - F_{11}^2) - \alpha|\bar{z}_\alpha|^2)^2 \\ +4(\beta-\alpha)^2(\det F - F_{11}^2)\alpha|\bar{z}_\alpha|^2,$$

$$h(F) = \beta|\bar{z}_\beta|^2 + (\beta-\alpha)^2(\det F - F_{11}^2) - \alpha|\bar{z}_\alpha|^2,$$

and the scalars

$$s_i = \frac{1}{2} - \frac{1}{2(\beta-\alpha)(F_{11}^2 - \det F)}\Big[\alpha\beta(|F_{12}|^2 + |F_{13}|^2) \\ -(F^{(2)} \times F^{(3)})_2^2 - (F^{(2)} \times F^{(3)})_3^2 + (-1)^i\sqrt{g(F)}\Big] \quad i = 1,2.$$

The set where $\psi(F,s) \le 0$ can be represented by

$$\{(F,s) \in M^{3\times 3} \times \mathbb{R} : g(F) \ge 0,\ h(F) \le 0,\ s \in (s_1, s_2)\}. \tag{4.16}$$

The next step is to examine the rank-one connection between the two manifolds.

We seek two matrices $B_\gamma \in \Lambda_\gamma$ with $\gamma = \alpha, \beta$ such that

$$F = rB_\alpha + (1-r)B_\beta$$

with $r \in (0,1)$ and $rk(B_\alpha - B_\beta) = 1$.

The above statement is again equivalent to finding a rank one matrix $A$

$$A = \begin{pmatrix} a \\ la \\ ma \end{pmatrix}$$

with $a \in \mathbb{R}^3$ and $(l,m) \in \mathbb{R}^2$, such that

$$F + A \in \Lambda_\alpha, \quad F - \mu A \in \Lambda_\beta. \tag{4.17}$$

Having in mind the necessary condition (4.15), we can deduce that the first component of the vector $a$ has to vanish

$$a = (0, a_2, a_3).$$

We can establish the following result.

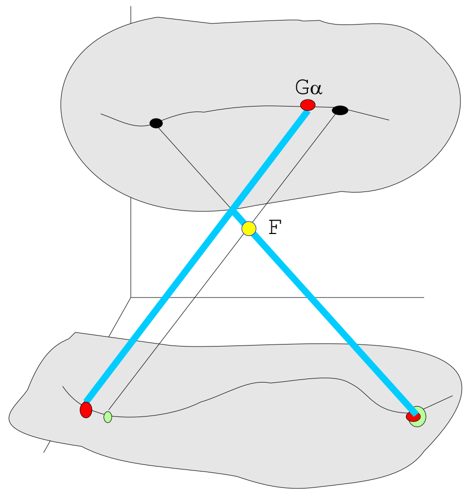



*Figure 4.1: Spatial situation*

**Lema 4** *([36]) Let $F \notin \Lambda_\alpha \cup \Lambda_\beta$ and such that $g(F) \geq 0$ and $h(F) \leq 0$. Then, there exist two vectors $a_i \in \mathbb{R}^3$, $(l_i, m_i) \in \mathbb{R}^2$ and $\mu_i \in \mathbb{R}^+$ i=1,2 such that,*

$$F + \begin{pmatrix} a_i \\ l_i a_i \\ m_i a_i \end{pmatrix} \in \Lambda_\alpha, \quad F - \mu_i \begin{pmatrix} a_i \\ l_i a_i \\ m_i a_i \end{pmatrix} \in \Lambda_\beta. \tag{4.18}$$

To summarize, keeping in mind Lemma 4 and the description (4.16) of the set $\psi(F, s) \leq 0$, we can assert that for every pair $(F, s)$ such that $\psi(F, s) \leq 0$, there exist two first-order laminates with barycenter $F$ and supported in $\Lambda_\alpha \cup \Lambda_\beta$. Hence, we propose to seek a second order laminate $\nu = s\delta_{M_\alpha} + (1-s)\nu_\beta$, where $\nu_\beta$ is a convex combination of two Dirac masses with support in $\Lambda_\beta$, and such that it verifies (see Figure 4.1)

$$M_\alpha \in \Lambda_\alpha,$$
$$M_\alpha^{(1)} = F_\alpha^{(1)} = (F_{11}, z_\alpha).$$

The two rank-one directions which we have found from Lemma 4 are very singular directions. The most important fact is that the intersection between the plane determined by the two rank-one connections, and $F$ (which we note by $\Pi_{rk1}$), and the manifolds $\Lambda_\gamma$ are linear manifolds (see details in [36]), i.e.

$$\Pi_{rk1} \cap \Lambda_\alpha = \sigma F_{\alpha,1} + (1-\sigma) F_{\alpha,2}, \quad \sigma \in \mathbb{R},$$

$$\Pi_{rk1} \cap \Lambda_\beta = \sigma F_{\beta,1} + (1-\sigma) F_{\beta,2}, \quad \sigma \in \mathbb{R},$$

where we consider the matrices

$$F_{\alpha,i} = F + A_i \quad F_{\beta,i} = F - \mu_i A_i$$

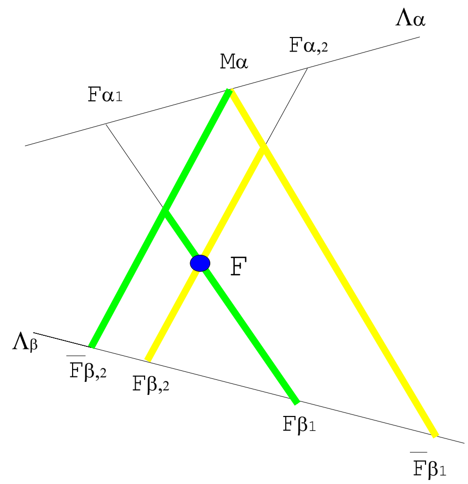



*Figure 4.2: Second order laminates*

for

$$A_i = \begin{pmatrix} a_i \\ l_i a_i \\ m_i a_i \end{pmatrix}, i = 1, 2.$$

By using in a fundamental way the linearity of the above intersection, we can assure that there exists a matrix $M_\alpha \in \Lambda_\alpha$ with $M_\alpha^{(1)} = F_\alpha^{(1)}$,

$$M_\alpha = \sigma^* F_{\alpha,1} + (1 - \sigma^*) F_{\alpha,2}$$

with

$$\sigma^* = \frac{\mu_1 (s - \lambda_2 (1 - s)}{s(\mu_1 - \mu_2)}.$$

In this situation, we can find two optimal second-order laminates (see Figure 4.2). Consider the matrices

$$\bar{F}_{\beta,i} = l_i^* F_{\beta,1} + (1 - l_i^*) F_{\beta,2}$$

with

$$l_i^* = \frac{\lambda_i - t(\lambda_2 + 1)}{t(\lambda_1 - \lambda_2)}.$$

If we take the matrices

$$F_{\alpha,i} = F + A_i, \; F_{\beta,i} = F - \mu_i A_i \quad \text{with} \quad A_i = \begin{pmatrix} a_i \\ l_i a_i \\ m_i a_i \end{pmatrix},$$

$$M_\alpha = \sigma^* F_{\alpha,1} + (1 - \sigma^*) F_{\alpha,2} \quad \text{with} \quad \sigma^* = \frac{\mu_1 (s + \mu_2 s - \mu_2)}{s(\mu_1 - \mu_2)},$$

$$\bar{F}_{\beta,i} = l_i^* F_{\beta,1} + (1 - l_i^*) F_{\beta,2} \quad \text{with} \quad l_i^* = \frac{\mu_i - s(\mu_2 + 1)}{s(\mu_1 - \mu_2)},$$

and the scalars

$$\rho_{i,j} = \tfrac{s-\mu_i(1-s)}{\mu_j-\mu_i}, \quad \tau_{i,j} = \tfrac{s-\mu_j(1-s)}{s(\mu_i+1)-\mu_j},$$

the optimal second-order laminates are

$$\nu_{i,j} = \tau_{i,j}\delta_{F_{\beta,i}} + (1-\tau_{i,j})(\rho_{i,j}\delta_{\bar{F}_{\beta,j}} + (1-\rho_{i,j})\delta_{M_\alpha})$$

with $i, j \in \{1,2\}$, $i \neq j$ where,

$$rk(F_{\beta,j} - M_\alpha) = 1$$

and

$$rk(F_{\beta,i} - (\rho_{i,j}\bar{F}_{\beta,j} + (1-\rho_{i,j})M_\alpha)) = 1.$$

## 4.5 A more general case

In this section we extend the previous results to a more general case. We consider a functional cost which can depend on the design. The optimal design problem consists in minimizing

$$I(\chi) = \int_0^T \int_\Omega u_t^2(t,x) + a(t,x,\chi)|\nabla_x u(t,x)|^2 dxdt$$

where $u$ is the unique solution of

$$\begin{aligned} -div(u_t, -(\alpha\chi + \beta(1-\chi))\nabla_x u) = 0 \quad &\text{in} \quad [0,T]\times\Omega \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) \quad &\text{in} \quad \Omega \\ u(t,x) = f(t,x), \quad &\text{in} \quad [0,T]\times\partial\Omega \end{aligned}$$

where the functions $a$, $u_0$, $u_1$, and $f$ are known. In particular, $a$ is a bounded function which verifies the following lower bound

$$a_\alpha(t,x) + \alpha \geq 0, \quad a_\beta(t,x) + \beta \geq 0,$$

where we denote

$$a_\alpha(t,x) = a(t,x,0) \ \text{ and } \ a_\beta(t,x) = a(t,x,1),$$

and respect the volume constraint

$$\int_\Omega \chi(x,t)dx \leq V_\alpha|\Omega|, \quad \forall t \in [0,T].$$

A analogous strategy to the previous sections (variational reformulation, polyconvexification, and search for laminates) again provides the "*constrained quasiconvexification*". It is given by the formula.

$$\bar{\varphi}(t,x,F,s)=\begin{cases}\dfrac{1}{\beta}((\beta+a_\beta)|F_{11}|^2-a_\beta\det F+sh|z_\alpha|^2)\\ \qquad\qquad \text{if } h(t,x)\geq 0, \psi(F,s)\leq 0,\\ \dfrac{1}{\alpha}((\alpha+a_\alpha)|F_{11}|^2-\det F-(1-s)h|z_\beta|^2)\\ \qquad\qquad \text{if } h(t,x)\leq 0, \psi(F,s)\leq 0,\\ \det F+s(\alpha+a_\alpha)|z_\alpha|^2+(1-s)(\beta+a_\beta)|z_\beta|^2\\ \qquad\qquad \text{if } \psi(F,s)\geq 0.\end{cases}$$

where

$$h(t,x)=\beta a_\alpha(t,x)-\alpha a_\beta(t,x),$$

The optimal condition for the generalized problem are

$$S_\alpha=|z_\alpha|^2 \text{ and } S_1=|F_{11}|^2 \text{ if } h(x,t)\geq 0, \psi(F,s)\leq 0,$$

$$S_\beta=|z_\beta|^2 \text{ and } S_1=|F_{11}|^2 \text{ if } h(x,t)\leq 0, \psi(F,s)\leq 0,$$

$$S_\alpha=|z_\alpha|^2 \text{ and } S_\beta=|z_\beta|^2 \text{ if } \psi(F,s)\geq 0.$$

Hence, the optimal laminates are as follows.

If $\psi(F,s)\geq 0$,

$$\nu=s\delta_{F_{\alpha,i}}+(1-s)\delta_{F_{\beta,i}} \qquad i=1,2$$

with

$$F_{\alpha,i}=F+A_i\in\Lambda_\alpha,\quad F_{\beta,i}=F-\tfrac{s}{1-s}A_i\in\Lambda_\beta,$$
$$A_i=\begin{pmatrix}a_i\\ l_ia_i\\ m_ia_i\end{pmatrix},$$
$$a_i=(\lambda_i,(1-s)(z_\alpha-z_\beta)),$$
$$\lambda_i=(-1)^i(1-s)\sqrt{\psi(F,s)}.$$

If $\psi(F,s)\leq 0$ and $h(t,x)\geq 0$,

$$\nu_{i,j}=\tau_{i,j}\delta_{F_{\beta,i}}+(1-\tau_{i,j})(\rho_{i,j}\delta_{\bar{F}_{\beta,j}}+(1-\rho_{i,j})\delta_{M_\alpha})$$

with $i,j\in\{1,2\}$, $i\neq j$ where the matrices are

$$F_{\alpha,i}=F+A_i,\ F_{\beta,i}=F-\mu_iA_i \quad \text{with} \quad A_i=\begin{pmatrix}a_i\\ l_ia_i\\ m_ia_i\end{pmatrix},$$
$$M_\alpha=\sigma^*F_{\alpha,1}+(1-\sigma^*)F_{\alpha,2} \quad \text{with} \quad \sigma^*=\tfrac{\mu_1(s+\mu_2s-\mu_2)}{s(\mu_1-\mu_2)},$$
$$\bar{F}_{\beta,i}=l_i^*F_{\beta,1}+(1-l_i^*)F_{\beta,2} \quad \text{with} \quad l_i^*=\tfrac{\mu_i-s(\mu_2+1)}{s(\mu_1-\mu_2)},$$

and the scalars

$$\rho_{i,j} = \tfrac{s-\mu_i(1-s)}{\mu_j-\mu_i}, \quad \tau_{i,j} = \tfrac{s-\mu_j(1-s)}{s(\mu_i+1)-\mu_j}.$$

If $\psi(F,s) \leq 0$ and $h(t,x) \leq 0$,

$$\nu_{i,j} = \tau_{i,j}\delta_{F_{\alpha,i}} + (1-\tau_{i,j})(\rho_{i,j}\delta_{\bar{F}_{\alpha,j}} + (1-\rho_{i,j})\delta_{M_\beta})$$

with $i,j \in \{1,2\}$, $i \neq j$ where,

$$M_\beta = \sigma^* F_{\beta,1} + (1-\sigma^*)F_{\beta,2} \quad \text{with} \quad \sigma^* = \tfrac{s-\mu_2(1-s)}{s(\mu_1-\mu_2)},$$

$$\bar{F}_{\alpha,i} = l_i^* F_{\alpha,1} + (1-l_i^*)F_{\alpha,2} \quad \text{with} \quad l_i^* = \tfrac{\mu_i s(\mu_2+1)+\mu_2}{s\mu_j(\mu_1-\mu_2)},$$

and the scalars

$$\rho_{i,j} = \tfrac{s\mu_j(s-\mu_j(1-s))}{s\mu_j+(1-s)\mu_i}, \quad \tau_{i,j} = \tfrac{s\mu_i(1+\mu_j)}{s(\mu_i(s+1)-\mu_i}.$$

## 4.6 Some examples

We would like to highlight some particular examples where, by using Theorem 5, we can compute explicitly the relaxed cost density.

**Example 1.** An interesting and very well-known case is the compliance, for which we take $a_\alpha(t,x) = \alpha$, $a_\beta(t,x) = \beta$ so that $h \equiv 0$, and the cost functional can be written as

$$\int_0^T \int_\Omega \left[u_t^2(t,x) + (\alpha\chi + \beta(1-\chi))|\nabla_x u(t,x)|^2\right] dxdt,$$

where the *constrained quasiconvexification* is

$$\varphi(F,s) = \begin{cases} 2F_{11}^2 - \det F & \text{if } \psi(s,F) \leq 0, \\ -\det F + \dfrac{1}{s(1-s)(\beta-\alpha)^2}\Big(2\alpha\beta(s\alpha+(1-s)\beta)\Big)|F_{12}|^2 & \\ \qquad +2\Big((1-s)\alpha + s\beta\Big)|F_{21}|^2 + 4\alpha\beta F_{12}F_{21}\Big) & \\ & \text{if } \psi(s,F) \geq 0. \end{cases}$$

**Example 2.** The last case lies in the border line for our computations to be valid. We take $a_\alpha(t,x) = -\alpha$ and $a_\beta(t,x) = -\beta$ so that $h$ identically vanishes. The cost functional is

$$\int_0^T \int_\Omega \left[u_t^2(t,x) - (\alpha\chi + \beta(1-\chi))u_x^2(t,x)\right] dxdt,$$

and for this case the relaxed integrand surprisingly is -det

$$\varphi(F,s) = -\det F.$$

# Appendix A

# General Results

In this appendix we would like to provide in a concise form the basic results stated in this thesis. We include a brief description of the homogenization method and the $\Gamma-$convergence concept. Finally, we recall some classical results of the Calculus of Variations and Young measures. The proofs of this last part can be found in [52]

## A.1 Homogenization and Γ-convergence

From the homogenization theory we obtain the $G-$convergence notion, which was introduced by S. Spagnolo ([59]) and generalized by F. Murat and L. Tartar ([48]) with the $H-$convergence. This concept can be given in a very general framework, but it is a very useful tool for applications to structural optimization, where it lets us take limits in a partial differential equation in a very small scale, and obtain information in a macroscopic level from the properties of microscopically heterogeneous media. In this sense the homogenization theory can be defined as a theory for "averaging partial differential equations". The theory of homogenization has been the main tool to analyze theoretical and numerically existence issues in optimal design problems. In this context the concept of $G-closure$ is crucial. We recall the concept of $H-$convergence,

**Definición 1** *A sequence of coercive, and with coercive inverse, symmetric matrices $\{A_\epsilon\}_\epsilon$ in $L^\infty(\Omega)$ is said to $H-$ converge to a homogenized matrix $A^\star \in L^\infty(\Omega)$, if for any $g \in H^{-1}(\Omega)$ and $u_0 \in H^1(\Omega)$ the sequence $u_\epsilon$ of solutions of*

$$\begin{cases} -\ div\ A_\epsilon(x)\nabla u_\epsilon(x) = g(x) & in\ \Omega \\ \\ u_\epsilon = u_0 & on\ \partial\Omega. \end{cases}$$

*verifies*

$$\begin{cases} u_\epsilon \rightharpoonup u \text{ } weakly \text{ } in \text{ } H^1(\Omega) \\ A_\epsilon \nabla u_\epsilon \rightharpoonup A^\star \nabla u \text{ } weakly \text{ } in \text{ } L^2(\Omega) \end{cases}$$

*where u is the unique solution of the homogenized equation*

$$\begin{cases} -\ div\ A^\star(x)\nabla u(x) = g(x) & in\ \Omega \\ u = u_0 & on\ \partial\Omega. \end{cases}$$

We are interested in two-phase composite materials, which are obtained mixing only two different phases. These materials are characterized by the properties of its two constituents $A^1$ and $A^2$, its volume fraction $\theta$, and $(1-\theta)$ respectively (with $\theta \in (0,1)$) and the geometric arrangement. These materials are obtained as the limit of a sequence of characteristic function $\chi_\epsilon$, which represent one of the two phases (for instance $A_1$), with volume fraction $\theta$. Then the composite material is identified by the sequence of matrices

$$A_\epsilon = \chi_\epsilon A^1 + (1-\chi_\epsilon)A^2. \tag{A.1}$$

Therefore the homogenized matrix of a two-phase composite material obtained from the materials $A^1$ and $A^2$, with volume fraction $\theta$ and $(1-\theta)$, respectively, is defined as

$$A^\star \text{ equal to the } H-\text{limit of } A_\epsilon$$

with

$$\chi_\epsilon \overset{\star}{\rightharpoonup} \theta \text{ } weakly \text{ } \star \text{ } in \text{ } L^\infty(\Omega;[0,1]).$$

The natural issue is the determination of what is the possible set of the homogenized matrices $A^\star$, if we know the two phases $A_1$ and $A_2$, and the volume fraction $\theta$. This is called $G-$closure problem. This is a difficult problem which is solved explicitly in few cases. For instance when mixing two isotropic conductors under elliptic laws. It was solved by Murat and Tartar ([49], [62]), but for two isotropic conductors under hyperbolic laws is still unknown. Some partial results are given by optimal bounds which are recovered by homogenized tensors.

A microstructure with special relevance in relaxation process are the laminated composites. In two phase materials we can arrange the materials in the form of a strip with a specific size and direction. It is made in a microscale level. This microstructure is called laminated (see Figure A.1). Moreover we can iterate this process by putting a laminate within a laminate. It is called a second-order laminate (see Figure A.2), and in this form we can iterate successively and obtain order-p laminates, with p an integer number.

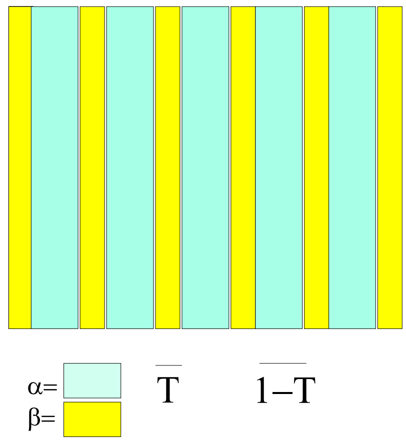



*Figure A.1: First order laminate.*

*Figure A.2: Microstructure in a Cu-Al-Ni single crystal (courtesy of C. Chu and R. D. James, Department of Aerospace Engineering and Mechanics, Minneapolis). The different phases can form striped structures only in definite directions, which can then be combined to form complex patterns.*

In this case, if we have a sequence of two-phase matrices as in Figure A.1, with volume fraction $\theta_i,\ i = 1, ..., p$ and $n_i,\ i = 1, ..., p$ the unit vectors of laminations direction, the homogenized matrix $A^\star$ is known and given by

$$(\prod_{i=1}^{q} \theta_i)(A_{\star,q} - B)^{-1} = (A - B)^{-1} + \sum_{j=1}^{q}((1-\theta_j)\prod_{i=1}^{j-1}\theta_i)\frac{n_j \otimes n_j}{Bn_j \cdot n_j}.$$

The relaxation process consist in replacing the original design variable $\chi$, by other generalized design which recover the behaviour of the minimizing sequences. Through the homogenization method, this set is determined by the pair $(\theta, A^\star)$, where $\theta$ is the weak limit of the minimizing sequence $\chi_\epsilon$, and $A^\star$, the homogenized tensor associated. Therefore the main problem or difficulty in this precess is that we need to know the $G-$closure associated with the problem, which is unknown in many cases. In particular, for the hyperbolic two-phase problem which we are interested to analyze.

Another important tool in minimization problems is the $\Gamma-$convergence concept. It was introduced by De Giorgi in the 1970's ([19],[20]). It is a concept which lets us know the asymptotic behaviour of families of minimizing problems. One can regard it as a variational convergence.

Finally, we would like to remark that $\Gamma-$convergence theory has an important relevance today in different contexts: homogenization, two phase transitions, dimension reduction, ... (see [11] for more examples), but specifically speaking the $\Gamma-$convergence concept is not built to analyze time-dependent problems. In particular in our case (for hyperbolic equations), where the functional may not be coercive.

We consider a family of integral functionals $F_\epsilon$ and consider the minimization problem

$$\min\,\{F_\epsilon(v) : v \in X_\epsilon\}. \tag{A.2}$$

In $\Gamma-$convergence we try to understand the behaviour of the solution of the minimizers for problem A.2, through the problem

$$\min\,\{F(v) : v \in X\}.$$

This new functional $F$ is the $\Gamma-$limit of $F_\epsilon$ (without lost generality we can assume $X_\epsilon = X,\ \forall\epsilon$). The definition of $\Gamma-$convergence is obtained in a natural way after to input a some requirement which gives a good compactness properties to this type of functionals.

One of the most important properties of $\Gamma-$convergence is the convergence of minimizers and minimun values. First of all, we assume that the family of functional are equi-coercive. We consider $\bar{v}$ a candidate to minimizer, and $\bar{v}_\epsilon$ a minimizing sequence associated within.

We would like to obtain a upper bound of F in the following way

$$\limsup_j \inf\{F_j(v) : v \in X\} \leq F(\bar{v})$$

which can be rephrased as follow: for all $v \in X$ <u>there exists a</u> sequence $(v_j)$ converging to v such that

$$\limsup_j F_j(v_j) \leq F(\bar{v}). \tag{A.3}$$

On the other hand, we need a lower bound which recover some lower semicontinuity

$$F(\bar{v}) \leq \liminf_j F_j(\bar{v}_j)$$

which can be deduced by this other more general condition: <u>for all</u> $(v_j)$ converging to a $v$ we have

$$F(v) \leq \liminf_j F_j(v_j) \tag{A.4}$$

Then a sequence of functionals $\{F_\epsilon\}$ $\Gamma-$converges to F if for all $v \in X$, (A.4) and (A.3) hold.

## A.2 The Direct Method in the Calculus of Variations

The classical approach to variational problems has two different forms: the indirect method, based in the Euler-Lagrange equation, and the Direct Method. The first one focuses on solving the first order optimality condition, (Euler-Lagrange equation), but the problem is that it can be a non-linear (system) equation, which can be difficult to solve and requires smoothness of integrands. In this sense the Direct Method of the Calculus of Variations is as more usual way to deal with variational problems. With the Direct Method we try to find minimizers directly without using necessary conditions. In this way the basic tool in this approach is the Weierstrass' theorem.

**Teorema 10** *We consider the variational principle*

$$\inf\{I(u) : u \in A\}$$

*where:*

1. *$A$ is a closed, convex in a reflexive Banach space $X$*

2. *$I : X \to \bar{\mathbb{R}}$ is weak lower semicontinuous (w.l.s.c.)*

3. *$I$ is coercive on $X$, i.e.,*

   $$I(u) \geq \alpha||u|| + \beta$$

   *for all $u \in X$ and some $\alpha > 0$ and $\beta \in \mathbb{R}$*

4. *there exists $\tilde{u} \in X$ with $I(\tilde{u}) < +\infty$.*

*Then there exists at least one solution $\bar{u} \in X$, i.e.*

$$I(\bar{u}) = \inf\{I(u) : u \in X\}$$

In this thesis, using the variational reformulation we are interested in problems of the form:

$$\inf\Big\{\int_\Omega \phi(x,y(x),\nabla y(x))dx : y-y_0\in W_0^{1,p}(\Omega;\mathbb{R}^m)\Big\}$$

If we assume some bound and coercivity properties, we have that all the hypotheses of Theorem 10 hold, up to 3), which is a difficult property to verify. Convexity conditions appear when this property is analized, and it is necessary to introduce some more general convexity notion.

**Teorema 11** *Let $m>1$, $n>1$ and $\phi$ be a Carathéodory function, which is quasiconvex in the gradient variable and such that*

$$c(|A|^p-1)\le \phi(x,u,A)\le h(x,u)(1+|A|^p),\qquad c>0,\qquad p>1$$

*with $h\in L^\infty_{loc}(\Omega\times\mathbb{R}^m)$ continuous on the second variable. Then the above variational problem admits at least one minimizer.*

**Definición 2** *Let $\phi:\mathbb{R}^{n\times m}\to\mathbb{R}$ be a function. $\phi$ is said to be a quasiconvex function if*

$$\phi(Y)\le\frac{1}{|\Omega|}\int_\Omega \phi(Y+\nabla u(x))dx$$

*for every $u\in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$, and every $Y\in\mathbb{R}^{n\times m}$ and every domain $\Omega$.*

This concept was introduced by Morrey at middle of the last century ([41]). In the scalar case this condition simplifies to the classical convexity; but in the vector case, it is a hard condition to check because of its non-local character (see [30]). In this sense necessary and sufficient conditions for quasiconvexity have been sought. These conditions are polyconvexity and rank one convexity.

**Definición 3** *$\phi:\mathbb{R}^{n\times m}\to\mathbb{R}$ is said to be a polyconvex function if*

$$\phi(A)=g(M(A)),$$

*for every matrix $A$, where $g$ is a convex function and $M(A)$ is the vector of all minors of $A$, given in some order.*

The polyconvexity is a property stronger than quasiconvexity, it was introduced by Morrey [42], and later used by Ball [4] in non-linear elasticity problems. The main motivation for this definition is due to the weak continuity of the minors ([17], [52]). A weaker condition is the rank one convexity. This definition is motivated after to analyzing the w.l.s.c. condition applied to laminate sequences ([17], [52]).

**Definición 4** $\phi : \mathbb{R}^{n\times m} \to \mathbb{R}$ *is said to be rank one convex function if*

$$\phi(\lambda A_1 + (1-\lambda)A_2) \leq \lambda\phi(A_1) + (1-\lambda)\phi(A_2)$$

*for every* $\lambda \in (0,1)$ *and* $A_1, A_2$ *matrices such that* $rank(A_1 - A_2) \leq 1$.

The last two definitions will be two important tools in our approach to analyze the optimal design problem. To sum up, we have following diagram

$$\phi \text{ convex } \Rightarrow \phi \text{ polyconvex } \Rightarrow \phi \text{ quasiconvex } \Rightarrow \phi \text{ rank one convex.}$$

As we told before, the main reason of the lack of solution is the lack of convexity (or quasiconvexity) of the problem. We propose to study a relaxation and compute the quasiconvexified problem. We have the following relaxation theorem established by Dacorogna ([17])

**Teorema 12** *Let* $\phi : \mathbb{R}^{m\times n} \to \mathbb{R}$ *be a Carathéodory function such that*

$$c(|A|^p - 1) \leq \phi(A) \leq C(1 + |A|^p), 1 < p < \infty, 0 < c < C.$$

*Then, for any given* $u_0 \in W^{1,p}(\Omega; \mathbb{R}^m)$*, the two numbers*

$$inf\{\int_\Omega \phi(\nabla u(x))dx : u - u_0 \in W_0^{1,p}(\Omega; \mathbb{R}^m)\}$$

$$inf\{\int_\Omega Q\phi(\nabla u(x))dx : u - u_0 \in W_0^{1,p}(\Omega; \mathbb{R}^m)\}$$

*are equal.*

Here $Q\phi$ is the quasiconvexification of $\phi$ defined as follow.

**Definición 5** *For a function* $\phi$*, its quasiconvexification or quasiconvex envelope is defined by*

$$Q\phi(Y) = \inf\{\frac{1}{|\Omega|}\int_\Omega \phi(Y + \nabla u(x))dx : u \in W_0^{1,\infty}(\Omega; \mathbb{R}^m)\}$$

*for all* $Y \in \mathbb{R}^{m\times n}$

## A.3 Young Measures

A very important tool in our analysis is the Parametrized or Young Measures. All the envelopes discussed in previous sections can be written in terms of Young measures.Moreover we can generalize some of these results in a more general context. For instance, Theorem 12 can be generalized to non-Carathéodory and non-bounded functions. Let us introduce this concept.

**Definición 6** *Let* $\nu = \{\nu_x\}_{x\in\Omega}$ *be a family of probability measures on a* $\sigma-$*algebra* $\Sigma$ *and* $f_j : \Omega \subset R^N \to R^m$ *a sequence of functions.* $\nu = \{\nu_x\}_{x\in\Omega}$ *is said to be a parametrized measure associated with the sequence* $\{f_j\}_{j\in N}$, *if supp*$(\nu_x) \subset \mathbb{R}^m$ *and they depend measurable on* $x \in \Omega$, *in the following sense, for any continuous* $\phi : R^m \to R$ *the function*

$$\bar{\phi}(x) = \int_{R^m} \phi(\lambda) d\nu_x(\lambda) = \langle \phi, \nu_x \rangle$$

*is measurable.*

*We say that* $\{\nu_x\}_{x\in\Omega}$ *is a homogeneous parametrized measure when the family of probability measures* $\{\nu_x\}_{x\in\Omega}$ *reduces to a unique single measure,* $\nu_x = \nu$ *a.e.* $x \in \Omega$.

There are a lot of applications where Young measures play an important role (see [52]). In particular, in our case the most useful property which we have used is that the Young measure can identify non-linear weak limits. If we consider $\{\phi(f_j)\}$ (where $\{f_j\}$ is the sequence associated with $\{\nu_x\}$) such that it converges weakly in $L^p(\Omega)$ (or weak$^\star$ in $L^\infty(\Omega)$), the weak limit can be represented by the function $\bar{\phi}$,

$$\lim_{j\to\infty} \int_\Omega \phi(f_j) g(x) dx = \int_\Omega g(x) \bar{\phi}(x) dx,$$

with

$$\bar{\phi}(x) = \int_{R^m} \phi(\lambda) d\nu_x(\lambda).$$

Another useful property of the Young measure is that it can capture the phenomena of hight oscillatory phenomena, which is associated with minimizing sequences when the integrand is not a (quasi)convex function.

We establish the Fundamental Theorem of Young measure, which lets us identify weak limits of a weakly convergent sequence composed with arbitrary function.

**Teorema 13** ***(Fundamental Young Measures Theorem)*** *Let* $\Omega \subset R^N$ *be a measureable set and* $f_j : \Omega \to R^m$ *be measurable functions such that*

$$\sup_j \int_\Omega g(|f_j|) dx < \infty$$

*where* $g : [0,\infty) \to [0,\infty]$ *is a continuous, nondecreasing function such that* $\lim_{t\to\infty} g(t) = \infty$. *Then there exists a subsequence, not relabeled, and a family of probability measures* $\nu = \{\nu_x\}$ *(the associated parametrized measure) such that whenever the sequence* $\{\psi(x, f_j)\}$ *is weakly convergent in* $L^1(\Omega)$ *for any Carathédory function* $\psi(x,\lambda) : \Omega \times \mathbb{R}^m \to \mathbb{R}^\star$, *the weak limit is the function*

$$\bar{\psi}(x) = \int_{R^m} \psi(\lambda) d\nu_x(\lambda) = \langle \psi, \nu_x \rangle$$

When we observe the reformulated variational problem, we realize that the problem is formulated in terms of gradients (the integrands $W$ and $V$ and the manifolds $\Lambda_\alpha$ and $\Lambda_\beta$ are defined in terms of the gradients of the designs $(u, v_1, v_n)$). Therefore Young measures associated with minimizing sequences are Gradient Young measure.

**Definición 7** *Let $\nu = \{\nu_x\}_{x\in\Omega}$ be a Young measure, we tell that $\nu$ is a Gradient (or $W^{1,p}$-) Young measure, if there exists $\{u_k\}_k$ a bounded sequence in $W^{1,p}(\Omega, \mathbb{R}^m)$ such that $\nabla u_k$ generates the Young measure $\{\nu_x\}_{x\in\Omega}$*

For our perspective, Gradient Young measures play a fundamental role, and therefore a deep knowledge of this tool will be useful to analyze our problem. A full characterization of the Gradient Young measure was given in a nice work by Kinderlehrer and Pedregal (see [28], [29]), and later Pedregal made a full analysis of Young measures and variational principles in [52], where we can find the proof of all the results of this Appendix connected to Young measures.

**Teorema 14** *Let $\{u_j\}$ be a bounded sequence in $W^{1,p}(\Omega)$, $p > 1$, and $\nu = \{\nu_x\}_{x\in\Omega}$ be a family of probability measures supported on $\mathbb{R}^{m\times N}$. $\nu$ is a Young measure generated by the gradient sequence $\{\nabla u_j\}$, if and only if,*

1. $\nabla u(x) = \int_{\mathbf{R}^{n\times m}} A d\nu_x(A)$ *for some $u \in W^{1,p}(\Omega)$;*

2. $\int_{\mathbf{R}^{n\times m}} \phi(A) d\nu_x(A) \geq \phi(\nabla u(x))$ *for a.e. $x \in \Omega$ and for any $\phi$ quasiconvex and with at most order $p$ polynomial growth.*

3. $\int_\Omega \int_{\mathbf{R}^{n\times m}} |A|^p d\nu_x(A) dx < \infty$.

From this important result we can deduce a characterization of the quasiconvexification or quasiconvex envelope of a function.

**Lema 5** *For a function $\varphi$, its quasiconvexification or quasiconvex envelope in terms of Young measure is,*

$$Q\varphi(Y) = \inf\{\int_{\mathbf{R}^{n\times m}} \varphi(A) d\nu(A) : \nu - W^{1,p} \textit{ hom. Young measure}, \int_{\mathbf{R}^{n\times m}} A d\nu(A) = Y\}$$

Once we have the quasiconvexification defined in terms of Young measures, it leads us to introduce the Young measure as generalized solutions. We know that under the lack of (quasi)convexity of the integrand minimizing sequences may be highly oscillating. In this sense, it suggest us the use of Young measures as generalized solutions, since Young measures contain information about the oscillating behavior of minimizing sequences.

If we consider the following variational principle

$$I(u) = \int_\Omega \phi(x, u(x), \nabla u(x))dx, \quad u \in u_0 + W^{1,p}(\Omega; \mathbb{R}^m)$$

where $\phi$ is assumed to verify certain bound, the generalized variational principle consists in

$$\min \tilde{I}(\mu) = \int_\Omega \int_{\mathbb{R}^{n\times m}} \phi(x, u(x), A)d\mu_x(A)dx,$$

with

$$\nabla u(x) = \int_{\mathbb{R}^{n\times m}} Ad\mu_x(A)$$

and $\{\mu_x\}_x \in \Omega$ a $W^{1,p}$-Young measure. In this way we have the following existence and relaxation theorem.

**Teorema 15** *There exists an admissible Young measure $\nu$ such that*

$$\tilde{I}(\nu) = \inf_\mu \tilde{I}(\mu).$$

*Moreover, under the assumptions of Theorem 12, the equalities*

$$\inf I(u) = \min \tilde{I}(\mu) = \min \bar{I}(u)$$

*hold, where $\bar{I}$ is the functional whose density is the classical quasiconvexification, and the optimal solution to this variational problem is $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ such that*

$$\nabla u(x) = \int_{\mathbb{R}^{n\times m}} Ad\nu_x(A)$$

*and,*

$$Q\phi(x, u(x), \nabla u(x)) = \int_{M^{n\times m}} \phi(x, u(x), A)d\nu_x(A) \quad a.e. x \in \Omega.$$

# Bibliography


[1] Allaire, G., *Shape optimization by the homogenization method*, Springer, 2002.

[2] Aranda, E. and Pedregal P., *Constrained envelope for a general class of design problems,* Disc. Cont. Dyn. Syst., Supplement Volume, (2003), 30-41.

[3] Ball, J., *A version of the Fundamental Theorem for Young Measures.* PDEs and Continuum Models of Phase Transitions. Lecture Notes and Physics vol. 344, eds. M. Rascle et al. Springer Verlag, 1989.

[4] Ball, J., *Convexity conditions and existence theorems in nonlinear elasticity.* Arch. Mech. Anal. 63 (1997), 337-403.

[5] Bellido, J.C., *Explicit quasiconvexification for some cost functionals depending on derivatives of the state in optimal design* Discrete Contin. Dyn. Syst. vol.**8** (2002), no. 4, 967-982.

[6] Bellido, J.C., *Análisis de algunos problemas de optimización no locales.* Tesis, Universidad de Sevilla, 2000.

[7] Bellido, J.C., *Explicit computation of the relaxed density coming from a three-dimensional optimal design problem*, Nonlinear Analysis **52** (2003), 1709-1726.

[8] Bellido, J.C. and Donoso, A., *On an optimal design problem in wave-propagation.* JOTA, vol. **133 (3)** (2007).

[9] Bellido, J.C., *On an optimal shape design problem in conduction*, ESAIM-COCV **12(4)**, (2006), 699-720 .

[10] Bends$\phi$e, MP. and Sigmundo, O., *Topology Optimization: Theory, Methods and Aplications.* Springer, Berlin, Heidelberg, New York, 2003.

[11] Braides, A. $\Gamma$-*convergence for beginngers.* Oxford University press, 2002.

[12] Casado, J. and Murat, F., In preparation.

[13] Chambolle, A. and Santosa, F., *Control of the wave equation by time-dependent coefficient*, ESAIM-COCV, vol (**8**), (2002) 375-392.

[14] Castro, C. and Cox, S., *Achieving arbitrarily large decay in the damped wave equation.* SIAM J. Control Optim. (**39** (6)), (2001), 1748-1755.

[15] Chipot, M., *Elements of Nolinear Analysis*, *Birkhäuser*, Basel, 2000.

[16] Dacorogna, B., *Weak continuity and weak lower semicontinuity for non-linear functionals*, vol.922, Springer Lectures Notes, 1982.

[17] Dacorogna, B., *Direct method in the calculus of variations*, Springer, 1989.

[18] Dal Maso, G., *An introduction to* $\Gamma$*-convergence*. Progress in Nonlinear Differential Equations and their Applications, vol. 8, *Birkhäuser*, Boston, 1993.

[19] De Giorgi, E., *sulla convergenza di alcune successione di integrali del tipo dell'area*. Rend. Mat. (6) **8** (1975), 277–294.

[20] De Giorgi, E., Franzoni, T. *Su un tipo di convergenza variazionale.* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **58** (1975), no. 6, 842–850.

[21] Donoso, A. and Pedregal, P., *Optimal design of 2-D conducting graded materials by minimizing quadratic funtionals in the field*, Struc. Multidisc Optim. vol. (**30**), (2005), 360-367.

[22] Donoso, A. *Numerical simulation in 3-D heat conduction: minimizing the quadratic mean temperature gradient*, SIAM Journal on Scientific Computing, vol. (**30**), 360-367, 2005.

[23] Ekeland, I. and Téman, R., *Convex Analisys and Variational Problems* SIAM, Philadelphia, 1999.

[24] Freitas, P., *Optimizing the rate of decay of solutions of the wave equation using genetic algorithms: a counterexample to the constant damping conjeture.* SIAM J. Control Optim, vol. (**37** (2)), (1999), 376-387.

[25] Grabosysky, Y., *Optimal design problems for two-phase conducting composites with weakly discontinous objective functionals*, Advanl. appl. Math, no 27 (2001), 683-704.

[26] Graham, C. R. and Henyey, F. S., *clebsch representation near points where the vorticity vanishes*, Physis of Fluids, **12**, no 4, (2000), 744-746.

[27] Hébrard, P. and Henrot, A., *Optimal shape and position of the actuators for the stabilization of a string.* System Control Lett., (**48**), (2003), 199-209

[28] Kinderlehrer, D., and Pedregal, P., *Characterization of Young measure generated by gradients.* Arch. Rational Mech. Anal. 115 (1991) 329-365.

[29] Kinderlehrer, D., and Pedregal, P., *Gradient Young measures generated by sequences in sobolev spaces.* J. Geom. Anal. 4 (1994) 59-90.

[30] Kristensen, J., *On the non-locality of quasiconvexity.*Ann- IHP, Ana. nonlin. 16, 1 (1999), 1-13.

[31] Kotiuga, P.R., *Clebsch potentials and the visualization of three-dimensional solenoidal vector fields*, IEEE Trans. Magnetics, **27**, no. 5, (1991), 3986-3989

[32] Lipton, R. and Velo, A.P., *Optimal design of gradient field with applications to electrostatics*, in *Nonlinear Partial Differential Equations and Their Applications*, College de France Seminar Volume XIV, (D. Cioranescu and J.L. Lions eds.), (2002).

[33] Lurie, K., *Control of the coefficients of linear hyperbolic equations via spatio-temporal composites.* In Homogenization, World Scientific, Singapore (V. Berdychevisky, V. Jikov and G. Papanicolau eds.), (1999), 285-315.

[34] Lurie, K., *Some new advances in the theory of dynamic materials*, Journal of Elasticity (**72**), (2003), 229-239.

[35] Maestre, F., *Optimal control in coefficients for a semilinear elliptic equations with piontwise control-state constraints.* Preprint.

[36] Maestre, F. and Pedregal, P., *Quasiconvexification in 3-D for a variational reformulation of an optimal design problem in conductivity*, Nonlinear Analysis, **64**, (2006), 1962-1976.

[37] Maestre, F. Münch, A. and Pedregal, P., *Optimal design under the one dimensional wave equation.* Preprint.

[38] Maestre, F. and Pedregal, P., *Dynamic materials for an optimal design problem under the two dimensional wave equation.* Preprint.

[39] Maestre, F. and Münch,A., In preparation.

[40] Milton, G.W., *The theory of composites*, Cambridge University Press, 2002

[41] Morrey, C.B., *Quasiconvexity and the lower semicontinuity of multiple integranls.* Pacific J. Math. 2, 25-53, 1952.

[42] Morrey, C.B., *Multiple integrals in the Calculus of Variations.* Springer-Verlag, 1966.

[43] Münch, A., Pedregal, P., Periago, F. *Optimal design of the damping set for the stabilization of the wave equation*, Journal of Differential Equation , 231(1), 331-358 (2006)

[44] Murat, F. *Contre-exemples pur divers problémes ou le contrôle intervient dans les coefficients*, Ann. Mat. Pura ed Appl. Serie 4 (1977) 112, 49-68.

[45] Murat, F. *Compasité par compensation.* Ann. Scuola Norm. Sup. Pisa Sci. Fis. Mat. 5 (1978), 489-507.

[46] Murat, F. *Compasité par compensation II.* Recent Methods in Nonlinear Analysis (Bologna, 1979), E. Giorgi, E. Magenes, and U. Mosco, Eds., Pitagora, 245-256.

[47] Murat, F. and Tartar, L., *Calculus of variations and homogenization.* Topics in the mathematical modelling of composites materials, (1997), A. Cherkaev and R. Kohn, Eds., Birhhaäuser, 139-173. (French version: F. Murat, *Calcul des variations et homogénéisation. Les mèthodes de l'homogeénéisation, thèorie et applications en physique.* Coll. Dir Etudes et REcherches EDF, Eyrolles, (1985) 319-369.)

[48] Murat, F. and Tartar, L., *H-convergence.* Topics in the mathematical modelling of composites materials, (1997), A. Cherkaev and R. Kohn, Eds., Birhhaäuser, 21-44. (French version: F. Murat, *H-convergence.* Mimeographed notes, Séminaire d'Analyse Fonctionnelle et Numérique de l'Université d'Alger,1978)

[49] Murat, F. and Tartar, L., *On the control of the coefficients in partial differential equations.* Topics in the mathematical modelling of composites materials, (1997), A. Cherkaev and R. Kohn, Eds., Birhhaäuser, 1-8. (French version: L. Tartar, *Problèmes de contrôle des coefficentes dans des équations aux dérivées partiales,* Lectures Notes in Economics and Mathematical Systems 107 (Control Theory, Numerical Methods and Computer Systems Modelling) Springer-Verlag, (1975) 420-426)

[50] Nesi, V., *Pollicristalline configuration that maximize electrical resistivity.* Journal of the mechanics and physics of solids, vol **39** 4, (1991), 525-542.

[51] Nigam, S.D., Usha R. and Swaminathan, k., *Divergence-free vectors fields*, Jour. Math. Phy. Sci. **14**, no. 5 (1980), 523-527

[52] Pedregal, P. *Parametrized Measure and Variationales Principles*, *Birkhäuser*, Basel, 1997.

[53] Pedregal, P. *Optimal design and constrained quasiconvexity* SIAM J. Math. Anal., **32** (2000) 854-869.

[54] Pedregal, P. *Constrained quasiconvexification of the square of the gradient of the state in optimal design*, Quart. Appl. Math., **62** , no 3 (2004), 459-470.

[55] Pedregal, P. *Optimal design in 2-D conductivity for quadratic functionals in the field*, Prc. Advanced NATO Conference in Non-linear homogenization, Varsovia, 2004

[56] Pedregal, P. *Vector variational problems and applications to optimal design*, ESAIM-COCV vol.15 (2005) 357-381.

[57] Pedregal, P. *Div-Curl Young measures and optimal design in any dimension,* Preprint.

[58] Shivamoggi, B.K., *Theorical Fluid Dynamics*, John Wiley and Sons, Inc., 1998

[59] Spagnolo, S., *Sulla convergenza di soluzioni di equazioni parabolique ed elliptiche.* Ann. Sc. Norm. Sup. di Pisa Cl. di Scienze vol. **22**, 577-597, 1968.

[60] Tartar, L. *Compensated compactness and applications to partial differeantial equations.* Nonlinear analysis and mechanics: Heriot-Watt Symposium (1979), R. Knops, Ed., vol 39, Pitman Res. Notes Math., pp 136-212.

[61] Tartar, L. *The compesated compactness method applied to systems of conservations laws.* Systems of nonlinear partial differeantial equations (1983), J. Ball, Ed., Dordrecht: D. Reidel, 263-285.

[62] Tartar, L. *Estimations fines des Coefficients Homogénéisés*, Ennio de Giorgi colloquium, P. Krée ed., Pitman Research Notes in Math., **125** (1985) 168-187.

[63] Tartar, L. *Remarks on optimal design problems, in Calculus of Variations, Homogenizacion and Continuum Mechasnics*, (G. Buttazzo, G. Bouchitte and P. Suquet, eds.), World Scientific, Singapore, (1994) 279-296.

[64] Tartar, L. *An introduction to the homogenization method in optimal design*, Springer Lecture Notes in Math. ,1740, (2000) 47-156.

[65] Yih, C.S., *Stream functions in three-dimensional flows*, La Houille Blanche no **3** (1957) 445-450.