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Existence and uniqueness theorem for slant immersions in Sasakian-space-forms

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Abstract. In this paper, we present the existence and uniqueness theorems for slant immersions into Sasakian-space-forms. By applying the first result, we prove several existence theorems for slant submanifolds. In particular, we prove the existence theorems for three-dimensional slant submanifolds with prescribed mean curvature or with prescribed scalar curvature.

0. Introduction

Slant immersions in complex geometry were defined by B.-Y. Chen as a natural generalization of both holomorphic and totally real immersions [3]. In a recent paper ([7]), A. Lotta has introduced the notion of slant immersion of a Riemannian manifold into an almost contact metric manifold. In [8], he has obtained examples of slant submanifolds in the Sasakian-space-form $\mathbb{R}^{2m+1}$ as the leaves of a harmonic Riemannian three-dimensional foliation. On the other hand, in [2], we have also studied and characterized slant submanifolds of K-contact and Sasakian manifolds. In particular, we have paid special attention to three-dimensional slant submanifolds.

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The purpose of the present paper is to establish a general existence and uniqueness theorem for slant immersions in Sasakian-space-forms, which is similar to the result presented by B.-Y. Chen and L. Vrancken for complex-space-forms in [5]. By applying the existence theorem, we prove that there exist infinitely many three-dimensional proper slant submanifolds with prescribed mean curvature (or with prescribed scalar curvature). In [2], we have given examples of slant submanifolds in $\mathbb{R}^{2m+1}$ with its usual Sasakian structure. It is well known that this manifold is a Sasakian-space-form with constant ϕ-sectional curvature -3. In this paper, we show that there are ample examples of proper slant submanifolds in Sasakian-space-forms with constant ϕ-sectional curvature c, for any $c < -3$.

In Section 1 we review basic formulas and definitions for almost contact metric manifolds and their submanifolds, which we shall use later. We also review the definition and some properties given in [2], [7]. Moreover, we develop the ground work which will allow us to present the existence and uniqueness theorems in Section 2. In Section 3, we show the applications of the main theorem.

1. Preliminaries

Let $(\widetilde{M}, g)$ be an odd-dimensional Riemannian manifold and denote by $T\widetilde{M}$ the Lie algebra of vector fields in $\widetilde{M}$. Let ϕ be a $(1,1)$ tensor field, ξ a global unit vector field (*structure vector field*), and η a 1-form on $\widetilde{M}$. If we have $\phi^2 X = -X + \eta(X)\xi$, $g(X,\xi) = \eta(X)$ and $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$, for any $X, Y \in T\widetilde{M}$, then $\widetilde{M}$ is said to have an almost contact metric structure (ϕ, ξ, η, g) and it is called an *almost contact metric manifold*. Let Φ denote the *fundamental* 2-*form* in $\widetilde{M}$, given by $\Phi(X,Y) = g(X, \phi Y)$ for all $X, Y \in T\widetilde{M}$. If $\Phi = \mathrm{d}\eta$, then $\widetilde{M}$ is said to be a *contact metric manifold*. Moreover, the contact metric structure is called a K-*contact structure* if

$$\widetilde{\nabla}_X \xi = -\phi X, \tag{1.1}$$

for any $X \in T\widetilde{M}$, where $\widetilde{\nabla}$ denotes the Levi–Civita connection of $\widetilde{M}$.

The structure of $\widetilde{M}$ is said to be *normal* if $[\phi,\phi] + 2\mathrm{d}\eta \otimes \xi = 0$, where $[\phi,\phi]$ is the Nijenhuis torsion of ϕ. A *Sasakian manifold* is a normal contact metric manifold. Every Sasakian manifold is a K-contact manifold. It is well-known that an almost contact metric manifold is a Sasakian manifold if and only if $(\widetilde{\nabla}_X \phi)Y = g(X,Y)\xi - \eta(Y)X$, for any $X, Y \in T\widetilde{M}$.

Given a Sasakian manifold $\widetilde{M}$, a plane section π in $T_p\widetilde{M}$ is called a *ϕ-section* if it is spanned by X and ϕX, where X is a unit tangent vector field orthogonal to ξ. The sectional curvature $K(\pi)$ of a ϕ-section π is called *ϕ-sectional curvature*. If a Sasakian manifold $\widetilde{M}$ has constant ϕ-sectional curvature c, $\widetilde{M}$ is called a *Sasakian-space-form*. It can be shown that $\mathbb{R}^{2m+1}$ with its usual Sasakian structure is a Sasakian-space-form with $c = -3$. Moreover, if we denote the usual contact metric structure on $\mathbb{S}^{2m+1}$ by (ϕ, ξ, η, g) and we consider the deformed structure given by the $\mathcal{D}$-homothetic deformation

$$\phi^* = \phi, \quad \xi^* = \frac{1}{a}\xi, \quad \eta^* = a\eta, \quad g^* = ag + a(a-1)\eta \otimes \eta, \tag{1.2}$$

where a is a positive constant, then $\mathbb{S}^{2m+1}$ with this structure is a Sasakian-space-form with $c = 4/a - 3 > -3$. Given a simply connected bounded domain B^m in $\mathbb{C}^m$ and a negative constant k, a different method can be followed to endow $B^m \times \mathbb{R}$ with a Sasakian structure with constant ϕ-sectional curvature $c = k - 3 < -3$ (see [1, 10]). Actually, it was proved by S. Tanno in [10] that these three types of model spaces are unique up to isomorphisms, where an isomorphism means a diffeomorphism which maps the structure tensors into the corresponding structure tensors, and so, they represent every Sasakian-space-form.

We denote by $\widetilde{M}^{2m+1}(c)$ the complete simply-connected Sasakian-space-form with dimension $2m+1$ and constant ϕ-sectional curvature c. The curvature tensor $\widetilde{R}$ of $\widetilde{M}^{2m+1}(c)$ is given by

$$\begin{aligned} \widetilde{R}(X,Y)Z = {}& \frac{c+3}{4}(g(Y,Z)X - g(X,Z)Y) + \frac{c-1}{4}(\eta(X)\eta(Z)Y \\ & - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ & + \Phi(Z,Y)\phi X - \Phi(Z,X)\phi Y + 2\Phi(X,Y)\phi Z), \end{aligned} \tag{1.3}$$

for any $X, Y, Z \in T\widetilde{M}$. For more details and background, we refer to the standard reference [1].

Now, let M be a submanifold immersed in $(\widetilde{M}, \phi, \xi, \eta, g)$. We also denote by g the induced metric on M. Let TM be the Lie algebra of vector fields in M and $T^{\perp}M$ the set of all vector fields normal to M. Denote by ∇

the Levi–Civita connection of M. Then, the Gauss–Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \quad \widetilde{\nabla}_X V = -A_V X + D_X V,$$

for any $X, Y \in TM$ and any $V \in T^\perp M$, where D is the connection in the normal bundle, σ is the second fundamental form of M and A_V is the Weingarten endomorphism associated with V.

Denote by R the curvature tensor of M and by R^D the curvature tensor of the normal connection D. Then the *equation of Gauss* and the *equation of Ricci* are given respectively by

$$\begin{aligned} \widetilde{R}(X,Y;Z,W) &= R(X,Y;Z,W) + g(\sigma(X,Z),\sigma(Y,W)) \\ &\quad - g(\sigma(X,W),\sigma(Y,Z)), \end{aligned} \tag{1.4}$$

$$R^D(X,Y;U,V) = \widetilde{R}(X,Y;U,V) + g([A_U, A_V](X), Y), \tag{1.5}$$

for any $X, Y, Z, W \in TM$ and any $U, V \in T^\perp M$.

For the second fundamental form σ, we define the covariant derivative $\overline{\nabla}\sigma$ of σ with respect to the connection on $TM \oplus T^\perp M$ by

$$(\overline{\nabla}_X \sigma)(Y,Z) = D_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{1.6}$$

for any $X, Y, Z \in TM$. The *equation of Codazzi* is given by

$$(\widetilde{R}(X,Y)Z)^\perp = (\overline{\nabla}_X \sigma)(Y,Z) - (\overline{\nabla}_Y \sigma)(X,Z), \tag{1.7}$$

for any $X, Y, Z \in TM$, where $(\widetilde{R}(X,Y)Z)^\perp$ denotes the normal component of $\widetilde{R}(X,Y)Z$.

For any $X \in TM$ and any $V \in T^\perp M$, we write

$$\phi X = TX + NX, \quad \phi V = tV + nV, \tag{1.8}$$

where TX (resp. tV) is the tangential component of ϕX (resp. ϕV) and NX (resp. nV) is the normal component of ϕX (resp. ϕV).

From now on, we suppose that the structure vector field ξ is tangent to M. Hence, if we denote by $\mathcal{D}$ the orthogonal distribution to ξ in TM, we can consider the orthogonal direct decomposition $TM = \mathcal{D} \oplus \langle \xi \rangle$.

In particular, from (1.1), (1.4) and (1.8) we obtain $\nabla_X \xi = -TX$ and $\sigma(X, \xi) = -NX$.

For each nonzero vector X tangent to M at p, such that X is not proportional to ξ_p, we denote by $\theta(X)$ the Wirtinger angle of X, that is, the angle between ϕX and T_pM. Then, M is said to be *slant* ([7]) if the Wirtinger angle $\theta(X)$ is a constant, which is independent of the choice of $p \in M$ and $X \in T_pM$, linearly independent from ξ_p. The Wirtinger angle θ of a slant immersion is called the *slant angle* of the immersion. Invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \pi/2$ respectively. A slant immersion which is neither invariant nor anti-invariant is called a *proper* slant immersion.

Now, suppose M is θ-slant in $\widetilde{M}^{2m+1}(c)$. Then, for any $X, Y \in TM$, we have (cf. [2]):

$$T^2 X = -\cos^2\theta(X - \eta(X)\xi), \tag{1.9}$$

$$g(TX, Y) + g(X, TY) = 0, \tag{1.10}$$

$$(\nabla_X T)Y = t\sigma(X, Y) + A_{NY}X + g(X, Y)\xi - \eta(Y)X, \tag{1.11}$$

$$D_X(NY) - N(\nabla_X Y) = n\sigma(X, Y) - \sigma(X, TY). \tag{1.12}$$

If $\theta \neq 0$, we will denote, for each $X \in TM$,

$$X^* = \frac{1}{\sin\theta} NX. \tag{1.13}$$

We define the symmetric bilinear TM-valued form α on M given by

$$\alpha(X, Y) = t\sigma(X, Y), \tag{1.14}$$

for any $X, Y \in TM$. In particular, it is easy to prove that, for any $X \in TM$,

$$\alpha(X, \xi) = \sin^2\theta(X - \eta(X)\xi). \tag{1.15}$$

Equations (1.8), (1.13) and (1.14) imply:

$$\phi\alpha(X, Y) = T\alpha(X, Y) + \sin\theta\,\alpha^*(X, Y). \tag{1.16}$$

Moreover, (1.8) and (1.14) imply

$$\phi\sigma(X, Y) = \alpha(X, Y) + \beta^*(X, Y), \tag{1.17}$$

where β is a symmetric bilinear $\mathcal{D}$-valued form on M. From (1.16) and (1.17), we have

$$-\sigma(X,Y) = T\alpha(X,Y) + (\sin\theta)\alpha^*(X,Y) + \phi\beta^*(X,Y), \tag{1.18}$$

since $\eta(\sigma(X,Y)) = 0$. It is easy to see that:

$$\phi\beta^*(X,Y) = -(\sin\theta)\beta(X,Y) - (T\beta(X,Y))^*. \tag{1.19}$$

Thus, from (1.18) and (1.19) it follows that $\beta(X,Y) = (\csc\theta)\cdot T\alpha(X,Y)$ and $\sigma(X,Y) = -(\csc\theta)\alpha^*(X,Y)$. This second formula is equivalent to:

$$\sigma(X,Y) = \csc^2\theta(T\alpha(X,Y) - \phi\alpha(X,Y)). \tag{1.20}$$

Given that $g(A_{NY}X,Z) = -g(\alpha(X,Z),Y)$ for any $X,Y,Z \in TM$, we obtain from (1.11) and (1.14):

$$\begin{aligned} g((\nabla_X T)Y,Z) &= g(\alpha(X,Y),Z) - g(\alpha(X,Z),Y) \\ &\quad + g(X,Y)\eta(Z) - g(X,Z)\eta(Y). \end{aligned} \tag{1.21}$$

For a θ-slant submanifold in $\widetilde{M}^{2m+1}(c)$ with $\theta \neq 0$, (1.3), (1.6), (1.8), (1.9)–(1.12), (1.14) and (1.20) imply that the equations of Gauss (1.4) and Codazzi (1.7) of M in $\widetilde{M}^{2m+1}(c)$ are given respectively by

$$\begin{aligned} R(X,Y;Z,W) &= \csc^2\theta(g(\alpha(X,W),\alpha(Y,Z)) - g(\alpha(X,Z),\alpha(Y,W))) \\ &\quad + \frac{c+3}{4}(g(X,W)g(Y,Z) - g(X,Z)g(Y,W)) \\ &\quad + \frac{c-1}{4}(\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) \\ &\quad + \eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(W)g(Y,Z) \\ &\quad + g(TX,W)g(TY,Z) - g(TX,Z)g(TY,W) \\ &\quad + 2g(X,TY)g(TZ,W)), \end{aligned} \tag{1.22}$$

$$
\begin{aligned}
&(\nabla_X\alpha)(Y,Z)-g(\alpha(Y,Z),TX)\xi\\
&\quad+\csc^2\theta\{T\alpha(X,\alpha(Y,Z))+\alpha(X,T\alpha(Y,Z))\}\\
&\quad+(\sin^2\theta)\frac{c-1}{4}\{g(X,TY)(Z-\eta(Z)\xi)+g(X,TZ)(Y-\eta(Y)\xi)\}\\
&=(\nabla_Y\alpha)(X,Z)-g(\alpha(X,Z),TY)\xi\\
&\quad+\csc^2\theta\{T\alpha(Y,\alpha(X,Z))+\alpha(Y,T\alpha(X,Z))\}\\
&\quad+(\sin^2\theta)\frac{c-1}{4}\{g(Y,TX)(Z-\eta(Z)\xi)+g(Y,TZ)(X-\eta(X)\xi)\}.
\end{aligned}
\tag{1.23}
$$

In the following section we show how equations (1.9), (1.10), (1.15), (1.21), (1.22) and (1.23) allow us to establish the existence theorem for slant immersions into Sasakian-space-forms. We will also need Theorem 1 of [6] (which was previously proved in [11]). We recall its formulation:

Theorem 1.1. *Let S be a manifold with complete connection $\overline{D}$ with parallel torsion and curvature tensors. Let M be a simply connected manifold and E a vector bundle with connection $\overline{D}$ over M having the algebraic structure $(\overline{R},\overline{T})$ of S. Let $F : TM \to E$ be a vector bundle homomorphism satisfying equations*

$$\overline{D}_V F(W)-\overline{D}_W F(V)-F([V,W])=\overline{T}(F(V),F(W)),$$

$$\overline{D}_V\overline{D}_W U-\overline{D}_W\overline{D}_V U-\overline{D}_{[V,W]}U=\overline{R}(F(V),F(W))U,$$

for any sections V, W of TM and U of E. Then there exists a smooth map $f : M \to S$ and a parallel bundle isomorphism $\overline{\Phi} : E \to f^*TS$ preserving $\overline{T}$ and $\overline{R}$ such that $df = \overline{\Phi}\circ F$. If S is simply connected, then f is unique up to affine diffeomorphisms of S.

2. Existence and uniqueness theorems

We have the following existence and uniqueness theorems for slant immersions:

Theorem 2.1 (Existence)**.** *Let c and θ be two constants with $0 < \theta \le \pi/2$ and M a simply-connected Riemannian manifold with dimension $m+1$ and metric tensor g. Suppose that there exist a unit global vector field ξ*

on M, an endomorphism T of the tangent bundle TM and a symmetric bilinear TM-valued form α on M such that for $X,Y,Z,W \in TM$, we have

$$T(\xi)=0, \quad g(\alpha(X,Y),\xi)=0, \quad \nabla_X\xi=-TX, \tag{2.1}$$

and the equations (1.9), (1.10), (1.15), (1.21), (1.22) *and* (1.23) *are satisfied, where η denotes the dual* 1*-form of ξ. Then, there exists a θ-slant immersion from M into $\widetilde{M}^{2m+1}(c)$ whose second fundamental form σ is given by:*

$$\sigma(X,Y)=\csc^2\theta(T\alpha(X,Y)-\phi\alpha(X,Y)). \tag{2.2}$$

Proof. Let c, θ, M, ξ, T and α be in the above conditions. Denote by $\mathcal{D}$ the orthogonal distribution to ξ on M and consider the Whitney sum $TM\oplus\mathcal{D}$. For each $X\in TM$, we identify $(X,0)$ with X. In particular, we identify $\widehat{\xi}=(\xi,0)$ with ξ. Moreover, we denote $(0,Z)$ by Z^* for each $Z\in\mathcal{D}$.

Let $\widehat{g}$ be the product metric on $TM\oplus\mathcal{D}$. Hence, if we denote by $\widehat{\eta}$ the dual 1-form of $\widehat{\xi}$, then $\widehat{\eta}(X,Z)=\eta(X)$, for any $X\in TM$ and any $Z\in\mathcal{D}$.

Let $\widehat{\phi}$ be the endomorphism on $TM\oplus\mathcal{D}$ defined by

$$\begin{aligned}\widehat{\phi}(X,0)&=(TX,\sin\theta(X-\eta(X)\xi)),\\ \widehat{\phi}(0,Z)&=(-(\sin\theta)Z,-TZ),\end{aligned} \tag{2.3}$$

for any $X\in TM$ and $Z\in\mathcal{D}$. Then, we have $\widehat{\phi}^2(X,0)=-(X,0)+\widehat{\eta}(X,0)\widehat{\xi}$ and, similarly, $\widehat{\phi}^2(0,Z)=-(0,Z)$. Thus $\widehat{\phi}^2(X,Z)=-(X,Z)+\widehat{\eta}(X,Z)\widehat{\xi}$ for any $X\in TM$ and any $Z\in\mathcal{D}$. By using (1.9), (1.10) and (2.3), it is easy to check that $(\widehat{\phi},\widehat{g},\widehat{\xi},\widehat{\eta})$ is an almost contact metric structure on $TM\oplus\mathcal{D}$.

Now we define A, σ and D by

$$A_{Z^*}X = \csc\theta\{(\nabla_XT)Z-\alpha(X,Z)-g(X,Z)\xi\}, \tag{2.4}$$

$$\sigma(X,Y)=-(\csc\theta)\alpha^*(X,Y), \tag{2.5}$$

$$\begin{aligned}D_XZ^* &= (\nabla_XZ-\eta(\nabla_XZ)\xi)^*\\ &\quad+\csc^2\theta\{(T\alpha(X,Z))^*+\alpha^*(X,TZ)\},\end{aligned} \tag{2.6}$$

for any $X, Y \in TM$ and any $Z \in \mathcal{D}$. It is easy to verify that each A_{Z^*} is an endomorphism on TM, σ is a $(\mathcal{D})^*$-valued symmetric bilinear form on TM and D is a metric connection of the vector bundle $(\mathcal{D})^*$ over M.

Let $\widehat{\nabla}$ denote the connection on $TM \oplus \mathcal{D}$ induced from equations (2.4)–(2.6). Then, from (1.9), (1.15), (2.1) and (2.3), given $X, Y \in TM$ and $Z \in \mathcal{D}$, we have:

$$(\widehat{\nabla}_{(X,0)}\widehat{\phi})(Y,0) = \widehat{g}((X,0),(Y,0))\widehat{\xi} - \widehat{\eta}(Y,0)\ (X,0),$$
$$(\widehat{\nabla}_{(X,0)}\widehat{\phi})(0,Z) = 0.$$

Let R^D denote the curvature tensor associated with the connection D on $(\mathcal{D})^*$, i.e. $R^D(X,Y)Z^* = D_X D_Y Z^* - D_Y D_X Z^* - D_{[X,Y]}Z^*$, for any $X, Y \in TM$ and any $Z \in \mathcal{D}$. Then, by virtue of (1.9), (1.10), (1.15), (1.23), (2.1), (2.6) and a simple computation, we may obtain:

$$\begin{aligned} R^D(X,Y)Z^* &= (R(X,Y)Z - \eta(R(X,Y)Z)\xi)^* \\ &+ \Big\{\frac{c-1}{4}T[g(Y,TZ)X - g(X,TZ)Y - 2g(X,TY)Z] \\ &+ \frac{c-1}{4}[g(Y,T^2Z)(X - \eta(X)\xi) \\ &- g(X,T^2Z)(Y - \eta(Y)\xi) - 2g(X,TY)TZ] \\ &+ \csc^2\theta[(\nabla_X T)\alpha(Y,Z) - (\nabla_Y T)\alpha(X,Z) - \eta(\nabla_X T\alpha(Y,Z))\xi \\ &+ \eta(\nabla_Y T\alpha(X,Z))\xi - \alpha(X,(\nabla_Y T)Z) + \alpha(Y,(\nabla_X T)Z)]\Big\}^*. \end{aligned} \tag{2.7}$$

Also, (1.21), (2.1), (2.4) and (2.5) yield, for any $X, Y \in TM$ and any $Z, W \in \mathcal{D}$:

$$\begin{aligned} \sin^2\theta g([A_{Z^*}, A_{W^*}]X, Y) &= g((\nabla_Y T)Z, (\nabla_X T)W) \\ &- g((\nabla_X T)Z, (\nabla_Y T)W) + g((\nabla_X T)Z, \alpha(Y,W)) \\ &+ g((\nabla_Y T)W, \alpha(X,Z)) - g((\nabla_Y T)Z, \alpha(X,W)) \\ &- g((\nabla_X T)W, \alpha(Y,Z)) \\ &+ g(\alpha(X,W), \alpha(Y,Z)) - g(\alpha(X,Z), \alpha(Y,W)) \\ &+ (1 - 2\cos^2\theta)(g(X,W)g(Y,Z) - g(X,Z)g(Y,W)). \end{aligned} \tag{2.8}$$

From (1.10) we have:

$$g(\alpha(Y,Z),TW)+g(T\alpha(Y,Z),W)=0. \tag{2.9}$$

By taking the derivative of (2.9) with respect to X and using (1.10), we find that:

$$g(\alpha(Y,Z),(\nabla_X T)W)+g((\nabla_X T)\alpha(Y,Z),W)=0. \tag{2.10}$$

Moreover, by virtue of (1.10) we obtain:

$$\begin{aligned} g((\nabla_X T)Z,(\nabla_Y T)W) &= g(\alpha(Y,W),(\nabla_X T)Z) \\ &-g(\alpha(Y,(\nabla_X T)Z),W)+\cos^2\theta g(X,Z)g(Y,W). \end{aligned} \tag{2.11}$$

Hence, by applying (2.7), (2.8), (2.10), (2.11) and a direct computation, we get:

$$\begin{aligned} &g(R^D(X,Y)Z^*,W^*)-g([A_{Z^*},A_{W^*}]X,Y) \\ &=\frac{c-1}{4}\{\sin^2\theta(g(X,W)g(Y,Z)-g(X,Z)g(Y,W))-2g(X,TY)g(TZ,W)\}. \end{aligned} \tag{2.12}$$

Equations (1.3), (1.9), (1.10) and (2.12) imply that (M,A,D) satisfies the equation of Ricci (1.5) for a $(m+1)$-dimensional θ-slant submanifold in $\widetilde{M}^{2m+1}(c)$. Also, (1.22) and (1.23) imply that (M,σ) satisfies the equations of Gauss and Codazzi for a θ-slant submanifold in $\widetilde{M}^{2m+1}(c)$. Hence, the vector bundle $TM\oplus\mathcal{D}$ over M equipped with the product metric, the shape operator A, the second fundamental form σ and the connections D and $\widehat{\nabla}$ satisfy the structure equations of $(m+1)$-dimensional θ-slant submanifolds in $\widetilde{M}^{2m+1}(c)$. Therefore, if we put $S=\widetilde{M}^{2m+1}(c)$, $E=TM\oplus\mathcal{D}$, $\overline{D}=\widehat{\nabla}$ and $F:TM\to E:X\mapsto(X,0)$, then assumptions of Theorem 1.1 verify given that E has the algebraic structure of $\widetilde{M}^{2m+1}(c)$ as we have indicated above. Then, we know that there exists a θ-slant immersion of M into $\widetilde{M}^{2m+1}(c)$ with (2.2) as its second fundamental form, A as its shape operator and D as its normal connection. □

The following result gives sufficient conditions to obtain the uniqueness of a slant immersion.

Theorem 2.2 (Uniqueness). *Let $x^1, x^2 : M \to \widetilde{M}^{2m+1}(c)$ be two slant immersions, with slant angle θ $(0 < \theta \leq \pi/2)$, of a connected Riemannian manifold M, with dimension $m+1$, into the Sasakian-space-form $\widetilde{M}^{2m+1}(c)$. Let σ^1 and σ^2 denote the second fundamental forms of x^1 and x^2 respectively. Suppose that there is a vector field $\bar{\xi}$ on M such that $x^i_{*p}(\bar{\xi_p}) = \xi_{x^i(p)}$, for any $i = 1, 2$ and any $p \in M$ and that*

$$g(\sigma^1(X,Y), \phi x^1_* Z) = g(\sigma^2(X,Y), \phi x^2_* Z),$$

for all vector fields X, Y, Z tangents to M. Suppose also that we have one of the following conditions:

i) $\theta = \pi/2$,

ii) *there exists a point p of M such that $T_1 = T_2$ on p,*

iii) $c \neq 1$.

Then, there exists an isometry φ of $\widetilde{M}^{2m+1}(c)$ such that $x^1 = \varphi \circ x^2$.

Proof. This proof works like that of the Uniqueness Theorem in the Kaehlerian case (see [4], [5]) by choosing $\bar{\xi}$ in the initial orthonormal frame on TM. Nevertheless, calculations are longer. □

3. Applications and examples

Let $\psi = \psi(x)$, $\psi_i = \psi_i(x)$, $i = 1, \ldots, 3$, be four functions defined on an open interval containing 0. Let c and θ be two constants with $0 < \theta \leq \pi/2$. Now, put:

$$f(x) = \exp\left(\int \psi_3(x)dx\right). \tag{3.1}$$

Let M be a simply-connected open neighborhood of the origin $(0,0,0) \in \mathbb{R}^3$. We define

$$\eta = dz + 2(\cos\theta) f(x) y dx \tag{3.2}$$

and we consider on M the warped metric:

$$g = \eta \otimes \eta + (dx \otimes dx + f^2(x) dy \otimes dy). \tag{3.3}$$

Let

$$e_1 = \frac{\partial}{\partial x} - 2(\cos\theta)f(x)y\frac{\partial}{\partial z}, \quad e_2 = \frac{1}{f}\frac{\partial}{\partial y}, \quad \xi = \frac{\partial}{\partial z}.$$

Then, it is easy to check that $\{e_1, e_2, \xi\}$ is a local orthonormal frame field of TM and that η is the dual 1-form of ξ. Moreover, we have:

$$\begin{aligned} &\nabla_{e_1}e_1 = 0, && \nabla_{e_1}e_2 = \cos\theta\xi, && \nabla_{e_1}\xi = -\cos\theta e_2,\\ &\nabla_{e_2}e_1 = \psi_3 e_2 - \cos\theta\xi, && \nabla_{e_2}e_2 = -\psi_3 e_1, && \nabla_{e_2}\xi = \cos\theta e_1,\\ &\nabla_{\xi}e_1 = -\cos\theta e_2, && \nabla_{\xi}e_2 = \cos\theta e_1, && \nabla_{\xi}\xi = 0. \end{aligned}$$

We define the tensor ϕ given by $\phi e_1 = e_2$, $\phi e_2 = -e_1$ and $\phi\xi = 0$, and a symmetric bilinear TM-valued form α on M by:

$$\begin{aligned} &\alpha(e_1, e_1) = \psi e_1 + \psi_1 e_2, \qquad \alpha(e_1, e_2) = \psi_1 e_1 + \psi_2 e_2,\\ &\alpha(e_2, e_2) = \psi_2 e_1 - \psi_1 e_2,\\ &\alpha(e_1, \xi) = \sin^2\theta e_1, \qquad \alpha(e_2, \xi) = \sin^2\theta e_2, \qquad \alpha(\xi, \xi) = 0. \end{aligned} \tag{3.6}$$

It is easy to prove that (M, ϕ, ξ, η, g) is an almost contact metric manifold with $(\nabla_X\phi)Y = \cos\theta(g(X, Y)\xi - \eta(Y)X)$, for any $X, Y \in TM$. If we put $T = \cos\theta\ \phi$, then (M, g, ξ, T, α) satisfies equations (1.9), (1.10), (1.15), (1.21) and (2.1).

On the other hand, it can be proved that M satisfies condition (1.22) if and only if

$$\psi_3' = -\psi_3^2 - \csc^2\theta\{\psi\psi_2 - 2\psi_1^2 - \psi_2^2\} - \frac{c+3}{4}(1 + 3\cos^2\theta).$$

Furthermore, we can also see that M satisfies (1.23) if we have the following equations:

$$\psi_2' = (-2\psi_2 + \psi)\psi_3 - \csc\theta\cot\theta(\psi_2 + \psi)\psi_1,$$

$$\psi_1' = -3\psi_1\psi_3 + \csc\theta\cot\theta(\psi_2 + \psi)\psi_2 + 3\frac{c+3}{4}\sin^2\theta\cos\theta, \tag{3.6}$$

$$\psi_1' = -3\psi_1\psi_3 + \csc\theta\cot\theta(\psi_2 + \psi)\psi_2 - 3\frac{c+3}{4}\sin^2\theta\cos\theta. \tag{3.7}$$

But (3.6) and (3.7) hold simultaneously if and only if $(c+3)/4\sin^2\theta\cos\theta=0$. Since $0<\theta\leq\pi/2$, we know that $\sin^2\theta\neq 0$. Hence, it must be $c=-3$ or $\theta=\pi/2$. By applying Theorem 2.1, we obtain the following result:

Theorem 3.1. *Let $\psi=\psi(x)$ be a function defined on an open interval containing* 0 *and a_1, a_2, a_3, c,θ be five constants with $0<\theta\leq\pi/2$. Consider the system of first order ordinary differential equations*

$$y_1'=-3y_1y_3+\csc\theta\cot\theta(y_2+\psi)y_2,$$

$$y_2'=(-2y_2+\psi)y_3-\csc\theta\cot\theta(y_2+\psi)y_1,$$

$$y_3'=-y_3^2-\csc^2\theta(\psi y_2-2y_1^2-y_2^2),$$

with the initial conditions: $y_1(0)=a_1$, $y_2(0)=a_2$, $y_3(0)=a_3$. Let ψ_1, ψ_2 and ψ_3 be the components of the unique solution of this differentiable system on some open interval containing 0*. Let M be a simply-connected open neighborhood of the origin $(0,0,0)\in\mathbb{R}^3$, endowed with the metric given by* (3.1)–(3.3)*. Let α be the TM-valued form defined by* (3.4)–(3.5)*. Then, we have:*

i) *If $c=-3$, then there exists a θ-slant immersion from (M,g) into $\widetilde{M}^5(-3)$, whose second fundamental form is given by $\sigma(X,Y)=\csc^2\theta(T\alpha(X,Y)-\phi\alpha(X,Y))$.*

ii) *If $\theta=\pi/2$, then there exists an anti-invariant immersion from (M,g) into $\widetilde{M}^5(c)$, whose second fundamental form is given by $\sigma(X,Y)=-\phi\alpha(X,Y)$.*

We can obtain immediately from Theorem 3.1 the following existence result for three-dimensional slant submanifolds with prescribed scalar curvature or with prescribed mean curvature.

Corollary 3.2. *For a given constant θ with $0<\theta\leq\pi/2$ and a given function $F_1=F_1(x)$ (resp. $F_2=F_2(x)$), there exist infinitely many three-dimensional θ-slant submanifolds in $\widetilde{M}^5(-3)$ with F_1 (resp. F_2) as the prescribed scalar curvature (resp. mean curvature) function.*

Slant submanifolds with F_1 as the scalar curvature function can be obtained from Theorem 3.1, by putting $a_2\neq 0$ and choosing ψ to be a function satisfying $3\sin^2\theta\ F_1=\psi\psi_2-2\psi_1^2-\psi_2^2-\sin^2\theta\cos^2\theta$. On the other hand, it is enough to put $\psi = 3\sin\theta F_2 - \psi_2$ in order to obtain F_2 as the prescribed mean curvature function.

Clearly, we can obtain a similar result for anti-invariant submanifolds in $\widetilde{M}^5(c)$, for a given constant c.

The following proposition gives the first examples of slant submanifolds in a Sasakian-space-form with ϕ-sectional curvature $c \neq -3$.

Proposition 3.3. *For each given constant θ with $0 < \theta < \pi/2$, there exist three-dimensional θ-slant submanifolds in $\widetilde{M}^5(-7)$ with nonzero constant mean curvature and constant negative scalar curvature.*

Proof. For a given constant θ with $0 < \theta < \pi/2$, we choose two nonzero constants β, γ such that

$$\beta^2 + \gamma^2 = 4\cos^2\theta. \tag{3.8}$$

Let a, b, c be constants defined by:

$$a_1 = -\sin^2\theta\sec^3\theta\left(\frac{1}{4}\beta^3 - \frac{3}{2}\beta\cos^2\theta + \frac{6}{\beta}\cos^4\theta\right), \tag{3.9}$$

$$a_2 = \gamma\sin^2\theta\sec^3\theta\left(\frac{1}{4}\beta^2 - \cos^2\theta\right), \tag{3.10}$$

$$a_3 = -\beta\sin^2\theta\sec^3\theta\left(\frac{1}{4}\beta^2 - \frac{1}{2}\cos^2\theta + \frac{1}{2}\gamma^2\right). \tag{3.11}$$

Let M be $\mathbb{R}^3$. We define the 1-form η by $\eta = dz + 2\cos\theta e^{-\gamma x}dy$. We consider on M the metric g given by:

$$g = \eta\otimes\eta + (dx\otimes dx - \beta e^{-\gamma x}(dx\otimes dy + dy\otimes dx) + (\beta^2+\gamma^2)e^{-2\gamma x}dy\otimes dy).$$

Put:

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{1}{\gamma}\left(\beta\frac{\partial}{\partial x} + e^{\gamma x}\frac{\partial}{\partial y} - 2\cos\theta\frac{\partial}{\partial z}\right), \quad \xi = \frac{\partial}{\partial z}. \tag{3.12}$$

Then, e_1, e_2, ξ form an orthonormal frame field for (M, g) and η is the dual 1-form of ξ. We can obtain:

$$\begin{aligned}
&\nabla_{e_1}e_1 = \beta e_2, && \nabla_{e_1}e_2 = -\beta e_1\cos\theta\xi, && \nabla_{e_1}\xi = -\cos\theta e_2,\\
&\nabla_{e_2}e_1 = -\gamma e_2 - \cos\theta\xi, && \nabla_{e_2}e_2 = \gamma e_1, && \nabla_{e_2}\xi = \cos\theta e_1,\\
&\nabla_{\xi}e_1 = -\cos\theta e_2, && \nabla_{\xi}e_2 = \cos\theta e_1, && \nabla_{\xi}\xi = 0.
\end{aligned}$$

By virtue of (3.8), this implies that the scalar curvature of M is given by $\tau = -\cos^2\theta < 0$.

We define a TM-valued symmetric bilinear form α on M by:

$$\alpha(e_1,e_1) = a_1e_1 + a_2e_2, \qquad \alpha(e_1,e_2) = -a_2e_1 + a_3e_2, \tag{3.13}$$
$$\alpha(e_2,e_2) = a_3e_1 - a_2e_2,$$

$$\alpha(e_1,\xi) = \sin^2\theta e_1, \qquad \alpha(e_2,\xi) = \sin^2\theta e_2, \tag{3.14}$$
$$\alpha(\xi,\xi) = 0.$$

Let T be the endomorphism on TM defined by $Te_1 = \cos\theta e_2$, $Te_2 = -\cos\theta e_1$ and $T\xi = 0$. Then by (3.8)–(3.14) and a very long computation, we may check that (M,ξ,T,α) satisfies the nine conditions stated in Theorem 2.1 for $c = -7$. Hence, this implies that there exists a θ-slant immersion from (M,g) into $\widetilde{M}^5(-7)$, whose second fundamental form is given by $\sigma = \csc^2\theta(T\alpha - \phi\alpha)$.

Since θ and β are constants such that $0 < \theta < \pi/2$ and $\beta \neq 0$, the obtained proper slant submanifolds have nonzero constant mean curvature and constant negative scalar curvature. □

Remark 3.4. During a personal conversation, D. E. Blair pointed out to the second author that, by combining Proposition 3.3 and a $\mathcal{D}$-homothetic deformation, we may also obtain the following theorem.

Theorem 3.5. *Let c, θ be two constants with $c < -3$ and $0 < \theta < \pi/2$. Then, there exist three-dimensional θ-slant submanifolds in a Sasakian-space-form with constant ϕ-sectional curvature c.*

Proof. For a given constant θ with $0 < \theta < \pi/2$, we choose a θ-slant submanifold M of $\widetilde{M}^5(-7)$, given by Proposition 3.3.

Now, for any $c < -3$, we consider the constant $a = -4/(c+3) > 0$ and the $\mathcal{D}$-homothetic deformation given by (1.2). Then, from Lemmas 2.1 and 6.1 of [9], we know that, with this change, $\widetilde{M}^5(-7)$ becomes a Sasakian-space-form with constant ϕ-sectional curvature $-(4/a) - 3 = c$.

Finally, it is easy to prove that a $\mathcal{D}$-homothetic deformation maps slant submanifolds into slant submanifolds. □

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