Sankhyā: The Indian Journal of Statistics
1998, Volume 60, Series B, Pt. 2, pp. 269-292

LOCAL INFLUENCE ON THE GENERAL LINEAR MODEL

By J.M. MUÑOZ-PICHARDO
J. MUÑOZ-GARCÍA
J.M. FERNÁNDEZ-PONCE
and
F. LÓPEZ-BLÁZQUEZ
Universidad de Sevilla, Spain

SUMMARY. We study the Local Influence on the General Linear Model with a perturbation scheme in the variance-covariance matrix of the random errors. The comparison of the results obtained and the conditional bias have been used to get local influence measures. Finally, we introduce the Local Influence Potential for an observation. This function describes the local influence which is exerted by the observation on the BLUE of any estimable linear function.

1. **Introduction**

Statistical models are usually aproximate descriptions of more complex processes, and because of this aproximation, considering perturbation is very important in the influence study. Generally speaking, the analysis of influence on a statistical model is considered as the study of variation that results from perturbing the problem formulation. Many diagnostic methods aimed at the study of the influence that individual observations play in determining a fitting model have been proposed in the recent years. Cook (1987) proposed a general formulation of this problem. Munoz-Pichardo *et al.* (1995) gave a new approach to the problem by defining the conditional bias.

Let $Y_1, \cdots, Y_n$ be a random sample of the random variable Y, let $T = T(Y_1, \cdots, Y_n)$ be a statistic defined on the sample, and let $y_1, \cdots, y_n$ be a realization of the sample. The conditional bias of T given the i-th observation, is defined as

$$\mathcal{S}[y_i; T] = E[T|Y_i = y_i] - E[T] \qquad \ldots (1)$$

Paper received. June 1995; revised May 1998.
AMS (1991) *subject classification.* 62J99
Key words and phrases. Conditional bias, general linear model, local influence analysis, local influence potential.

This concept is easily generalized to a set of observations:

$$\mathcal{S}[y_{i_1},\cdots,y_{i_m};T]=E\left[T|Y_{i_1}=y_{i_1},\cdots,Y_{i_m}=y_{i_m}\right]-E[T] \qquad \ldots(2)$$

and can be considered as a measure of the influence of the set of observations on T. Note that the conditinal bias is based on the decomposition of a statistic given by Efron and Stein (1981). This approach provides a punctual information about the influence which is performed by an observation or a set of observations. So, a local influence analysis is necessary to complement that information (see Cook (1987)). The concept of local influence was introduced by Cook (1986) who gave a very general method for assessing the influence of local perturbations in models. He suggested using the likehood displacement.

Billor and Loynes (1993) proposed an alternative measure of local influence. It is also based on the likehood displacement. Therefore, both approaches suppose a hypothesis on the underlying distribution.

In this paper, we develop the local influence analysis on the General Lineal Model (Kshirsagar (1983)) with the perturbation scheme on the variance-covariance matrix of the random errors (see Cook (1987)). We have studied it in two different ways. Firstly, the comparison of the results obtained in Section 3, and, afterwards the conditional bias on the perturbed model in Section 4. In Section 5, using the above results, we introduce the Local Influence Potential for an observation on the BLUE of any estimable linear function. In these two different studies, we do not presuppose a particular hypthesis on the distribution of the variables.

2. The General Linear Model: Conditional Bias

In this section we present several results about the conditional bias on the General Linear Model (GLM). For more details see Munoz-Pichardo *et al.* (1995). Finally we obtain the conditional bias of the perturbed GLM estimators under perturbation scheme proposed by Cook (1987).

2.1 *Conditional bias on the general linear model.* Consider the General Linear Model

$$\underline{Y}=\mathbf{X}\underline{\beta}+\underline{\varepsilon}, \quad E[\underline{\varepsilon}]=0, \quad var[\underline{\varepsilon}]=\sigma^2\mathbf{I}_n \quad (GLM)$$

with $\underline{Y}$ a random n-vector; $\mathbf{X}$ a known $n\times p$ matrix with rank $r(r\leq p<n)$; $\underline{\beta}$ a p-vector of unknown parameters and $\underline{\varepsilon}$ a n-vector that represents the non-observable random errors.

The vector of adjusted values is denoted by $\widehat{\underline{Y}}=\mathbf{V}\underline{Y}$ with

$$\mathbf{V}=\mathbf{X}\mathbf{S}^{-}\mathbf{X}^{'}=((v_{ij}))_{i,j=1,\cdots,n}$$

the prediction matrix, which is symmetric, idempotent, with rank r and unique for any $\mathbf{S}^{-}$, generalized inverse of $\mathbf{S}=\mathbf{X}^{'}\mathbf{X}$ (see Kshirsagar (1983)). The vector of residuals is $\underline{e} = \underline{Y} - \widehat{\underline{Y}} = \mathbf{M}\underline{\varepsilon}$, where $\mathbf{M} = \mathbf{I}_n - \mathbf{V}$, and the least squares estimators (LSE) of $\underline{\beta}$ and σ^2 are denoted by $\widehat{\underline{\beta}}$ and $\widehat{\sigma}^2$, respectively. On the other hand, the estimable linear parametric functions (e.l.f.) of $\underline{\beta}$ are $\Lambda\underline{\beta}$, where Λ is a qxp matrix, with $rank(\Lambda) = q$, so that $\Lambda\mathbf{S}^{-}\mathbf{S} = \Lambda$, and $\Lambda\widehat{\underline{\beta}}$ is the BLUE of $\Lambda\underline{\beta}$.

For this model, the following results are obtained (see Munoz-Pichardo *et al.* (1995)):

1.
$$\mathcal{S}\left[y_i; \Lambda\widehat{\underline{\beta}}\right] = \Lambda\mathbf{S}^{-}\underline{x}_i\left[y_i - \underline{x}_i^{'}\underline{\beta}\right] \in \mathcal{R}^q \qquad \ldots(3)$$

for any $\Lambda\underline{\beta}$ e.l.f., where $\underline{x}_i^{'}$ is the i-th row of $\mathbf{X}$.

2.
$$\mathcal{S}\left[y_i; \widehat{\sigma}^2\right] = \frac{1}{n-r}\left[1 - v_{ii}\right]\left\{\left[y_i - \underline{x}_i^{'}\underline{\beta}\right]^2 - \sigma^2\right\} \in \mathcal{R}. \qquad \ldots(4)$$

As estimators of the above expressions they propose.

$$\widehat{\mathcal{S}}[y_i; \Lambda\widehat{\underline{\beta}}] = \frac{e_i}{1 - v_{ii}}\Lambda\mathbf{S}^{-}\underline{x}_i \in \mathcal{R}^q \qquad \ldots(5)$$

$$\widehat{\mathcal{S}}[y_i; \widehat{\sigma}^2] = \frac{1}{n-r-1}\widehat{\sigma}^2\left[r_i^2 - 1\right] \in \mathcal{R} \qquad \ldots(6)$$

where
$$r_i = \frac{e_i}{\left[\widehat{\sigma}^2(1 - v_{ii})\right]^{1/2}} \qquad \ldots(7)$$

is the i-th internally studentized residual.

2.2. *Conditional bias on the perturbed general linear model.* Cook (1987) studied the local influence of the i-th observation on the Multiple Linear Regression Model by considering the perturbation produced by the lack of homocedasticity when that observation is assigned a weight ω. Following a similiar idea, consider the perturbed GLM

$$\underline{Y} = \mathbf{X}\underline{\beta} + \underline{\varepsilon}, \quad E[\underline{\varepsilon}] = 0, \quad var[\underline{\varepsilon}] = \sigma^2\mathbf{W} \quad GLM(\omega, i)$$

with
$$\mathbf{W} = diag[1, \cdots, 1/\omega, \cdots, 1], \quad \omega > 0.$$

That is, we shall weight the only one observation which is subject in the influence analysis. By means of the transformation

$$\mathbf{W}^{-1/2}\underline{Y} = \mathbf{W}^{-1/2}\mathbf{X}\underline{\beta} + \mathbf{W}^{-1/2}\underline{\varepsilon}, \qquad \ldots(8)$$

the generalized least squares (GLS) estimators of $\underline{\beta}$ for the model $GLM(\omega, i)$ are
$$\widehat{\underline{\beta}}_{\omega} = \mathbf{S}_{\omega}^{-}\mathbf{X}^{'}\mathbf{W}^{-1}\underline{Y} + (\mathbf{I}_p - \mathbf{H}_{\omega})\underline{\mathbf{z}}, \quad \underline{\mathbf{z}} \in \mathcal{R}^p \text{ arbitrary} \qquad \ldots(9)$$

$$\text{with } \mathbf{S}_\omega = \mathbf{X}^{'}\mathbf{W}^{-1}\mathbf{X} \quad \text{ and } \mathbf{H}_\omega = \mathbf{S}_\omega^{-}\mathbf{S}_\omega .$$

And an unbiased estimator of σ^2 is:

$$\widehat{\sigma}^2_\omega = \frac{1}{n-r}\left[\underline{Y} - \mathbf{X}\underline{\widehat{\beta}}_\omega\right]^{'}\mathbf{W}^{-1}\left[\underline{Y} - \mathbf{X}\underline{\widehat{\beta}}_\omega\right]. \qquad \dots(10)$$

Given that an objective of our study is to compare the results with those obtained in the unperturbed model GLM, we study the estimability in GLM(ω, i) of the estimable linear functions (e.l.f) for GLM. From now on, we will assume that $v_{ii} \neq 1$. The value of v_{ii} can attain its absolute maximun of 1 only if $v_{ij} = 0$ for $j \neq i$. In this situation, $\widehat{y}_i = y_i$ and the i-th observation will be fitted exactly. This situation will rarely occur in practice. Under this conditions, any $\underline{\lambda}^{'}\underline{\beta}$ e.l.f. in **G.L.M.** , is also e.l.f. in GLM(ω, i) (see Theorem 1 in the Appendix), and $\underline{\lambda}^{'}\underline{\widehat{\beta}}_\omega$ is its BLUE (see Theorem 2 in the Appendix).

In short, given $\Lambda\underline{\beta}$ an e.l.f. in GLM, then

$$\Lambda\underline{\widehat{\beta}}_\omega = \Lambda\underline{\widehat{\beta}} + \frac{\omega - 1}{1 + (\omega - 1)v_{ii}}\Lambda\mathbf{S}^{-}\underline{x}_i\left[Y_i - \underline{x}_i^{'}\underline{\widehat{\beta}}\right] \qquad \dots(11)$$

is unique for any $\underline{\widehat{\beta}}_\omega$ GLSE for GLM(ω, i) and it is BLUE of $\Lambda\underline{\beta}$.

In the following, we obtain the conditional bias of the B.L.U.E.of the estimable linear functions in GLM(ω, i).

THEOREM 2.1. *In the model GLM(ω, i) the conditional bias of $\Lambda\underline{\widehat{\beta}}_\omega$ given the i-th observation y_i is*

$$\mathcal{S}_\omega^{(i)}\left[y_i; \Lambda\underline{\widehat{\beta}}_\omega\right] = \frac{\omega}{1 + (\omega - 1)v_{ii}}\Lambda\mathbf{S}^{-}\underline{x}_i\left[y_i - \underline{x}_i^{'}\underline{\beta}\right] \in \mathcal{R}^q \qquad \dots(12)$$

where Λ is a $q \times p$ matrix, such that rank$(\Lambda) = q$ and $\Lambda\beta$ an e.l.f. in GLM.

PROOF. From (11)

$$E\left[\Lambda\underline{\widehat{\beta}}_\omega | Y_i = y_i\right] = E\left[\Lambda\underline{\widehat{\beta}} | Y_i = y_i\right] + \frac{\omega - 1}{1 + (\omega - 1)v_{ii}}\Lambda\mathbf{S}^{-}\underline{x}_i E\left[Y_i - \underline{x}_i^{'}\underline{\widehat{\beta}} | Y_i = y_i\right] \qquad \dots(13)$$

As $\Lambda\underline{\beta}$ and $\underline{x}_i^{'}\underline{\widehat{\beta}}$ are linearly estimable in **GLM**

$$E\left[\underline{x}_i^{'}\underline{\widehat{\beta}} | Y_i = y_i\right] = \underline{x}_i^{'}\underline{\beta} + v_{ii}\left[y_i - \underline{x}_i^{'}\underline{\beta}\right] \qquad \dots(14)$$

$$E\left[\Lambda\underline{\widehat{\beta}} | Y_i = y_i\right] = \Lambda\underline{\beta} + \Lambda\mathbf{S}^{-}\underline{x}_i\left[y_i - \underline{x}_i^{'}\underline{\beta}\right] \qquad \dots(15)$$

Now, substituting (14) and (15) in (13), the result follows easily. □

COROLLARY 2.2.

$$\mathcal{S}_\omega^{(i)}\left[y_i; \Lambda\underline{\widehat{\beta}}_\omega\right] = \frac{\omega}{1 + (\omega - 1)v_{ii}}\mathcal{S}\left[y_i; \Lambda\underline{\widehat{\beta}}\right] \quad \forall\omega > 0 \qquad \dots(16)$$

In the above expression, we can see the proportionality between the conditional bias of BLUE of an e.l.f. in the model GLM and the model GLM(ω, i). The ratio

$$\frac{\omega}{1+(\omega-1)v_{ii}}$$

plays an important role in the local influence analysis (see Section 5). Also, we can observe that the ratio not depends on the e.l.f. It is a function based on the weight ω which is assigned to the observation under study and depends on the only one i-th diagonal element of the prediction matrix $\mathbf{V}$.

The estimation of conditional bias proposed by Munoz-Pichardo *et al.* (1995) for this model is

$$\widehat{\mathcal{S}}_{\omega}^{(i)}\left[y_i;\Lambda\underline{\widehat{\beta}}_{\omega}\right]=\Lambda\underline{\widehat{\beta}}_{\omega}-\left[\Lambda\underline{\widehat{\beta}}_{\omega}\right]_{(i)}=\Lambda\underline{\widehat{\beta}}_{\omega}-\Lambda\underline{\widehat{\beta}}_{(i)} \qquad \ldots(17)$$

From (11) and the expression (see Munoz-Pichardo *et al.* (1995))

$$\Lambda\underline{\widehat{\beta}}_{(i)}=\Lambda\underline{\widehat{\beta}}-\frac{e_i}{1-v_{ii}}\Lambda\mathbf{S}^{-}\underline{x}_i \qquad \ldots(18)$$

we have

$$\widehat{\mathcal{S}}_{\omega}^{(i)}\left[y_i;\Lambda\underline{\widehat{\beta}}_{\omega}\right]=\frac{\omega}{1+(\omega-1)v_{ii}}\frac{e_i}{1-v_{ii}}\Lambda\mathbf{S}^{-}\underline{x}_i\in\mathcal{R}^q \qquad \ldots(20)$$

Therefore,

$$\widehat{\mathcal{S}}_{\omega}^{(i)}\left[y_i;\Lambda\underline{\widehat{\beta}}_{\omega}\right]=\frac{\omega}{1+(\omega-1)v_{ii}}\widehat{\mathcal{S}}\left[y_i;\Lambda\underline{\widehat{\beta}}\right]\in\mathcal{R}^q \qquad \ldots(21)$$

Like the conditional bias, the estimators of conditional bias are proportionals with the same proportionality ratio. In the following theorem, we obtain the conditional bias of $\widehat{\sigma}^2_{\omega}$.

THEOREM 2.3. *In the model GLM(ω,i) the conditional bias of $\widehat{\sigma}^2_{\omega}$ given the i-th observation y_i is*

$$\mathcal{S}_{\omega}^{(i)}\left[y_i;\widehat{\sigma}^2_{\omega}\right]=\frac{1-v_{ii}}{n-r}\frac{1}{1+(\omega-1)v_{ii}}\left[\omega\left[y_i-\underline{x}_i^{'}\underline{\beta}\right]^2-\sigma^2\right]\in\mathcal{R} \qquad \ldots(21)$$

By the transformation used in (8) we obtain

$$\underline{Z}=\mathbf{T}\underline{\beta}+\underline{\delta},\quad E\left[\underline{\delta}\right]=\theta,\quad Var\left[\underline{\delta}\right]=\sigma^2\mathbf{I}_n$$

with $\underline{Z}=\mathbf{W}^{-1/2}\underline{Y}$, $\mathbf{T}=\mathbf{W}^{-1/2}\mathbf{X}$ and $\underline{\delta}=\mathbf{W}^{-1/2}\underline{\varepsilon}$. By (4),

$$\mathcal{S}_{\omega}^{(i)}\left[y_i;\widehat{\sigma}^2_{\omega}\right]=\mathcal{S}\left[z_i;\widehat{\sigma}^2_{\omega}\right]=\frac{1}{n-r}\left[1-v_{ii}^*\right]\left\{\left[z_i-\underline{t}_i^{'}\underline{\beta}\right]^2-\sigma^2\right\} \qquad \ldots(22)$$

with v_{ii}^* the i-th diagonal element of the matrix $\mathbf{T}\left[\mathbf{T}^{'}\mathbf{T}\right]^{-}\mathbf{T}^{'}$. From (12),

$$v_{ii}^*=\frac{\omega v_{ii}}{1+(\omega-1)v_{ii}}$$

Moreover, $z_i = \omega^{-1/2} y_i$ and $\underline{t}_i = \omega^{-1/2} \underline{x}_i$. Substituting these expressions in (22) the result follows easily. □

An obvious estimator of the conditional bias $\mathcal{S}_\omega^{(i)}\left[y_i; \widehat{\sigma}_\omega^2\right]$ is

$$\widehat{\mathcal{S}}_\omega^{(i)}\left[y_i; \widehat{\sigma}_\omega^2\right] = \widehat{\sigma}_\omega^2 - \left[\widehat{\sigma}_\omega^2\right]_{(i)} = \widehat{\sigma}_\omega^2 - \widehat{\sigma}_{(i)}^2 \qquad \ldots (23)$$

Applying theorem 3 in the appendix and the following equality (see Cook and Weisberg (1982))

$$\widehat{\sigma}_{(i)}^2 = \frac{1}{n-r-1}\left[(n-r)\widehat{\sigma}^2 - \frac{e_i^2}{(1-v_{ii})}\right] \qquad \ldots (24)$$

we obtain

$$\widehat{\mathcal{S}}_\omega^{(i)}\left[y_i; \widehat{\sigma}_\omega^2\right] = \frac{\widehat{\sigma}^2}{n-r-1}\left\{\left[1 + \frac{n-r-1}{n-r}\frac{(\omega-1)(1-v_{ii})}{1+(\omega-1)v_{ii}}\right] r_i^2 - 1\right\}. \qquad \ldots (25)$$

3. Local Influence in GLM under the Comparative Scheme

In this section, measures of the local influence for the BLUE of an e.l.f. in GLM and for unbiased estimator of the variance are obtained, using the most clasical technique in the influence analysis: the comparison of results.

From (14)

$$\Lambda\underline{\widehat{\beta}} - \Lambda\underline{\widehat{\beta}}_\omega = \frac{\omega-1}{1+(\omega-1)v_{ii}} e_i \Lambda \mathbf{S}^- \underline{x}_i \in \mathcal{R}^q \qquad \ldots (26)$$

with Λ a $q \times p$ matrix such $rank(\Lambda) = q$ and $\Lambda\underline{\beta}$ linearly estimable in GLM. So, in order to characterize the local influence, a metric of generalized distance type will be applied, according to the characterization given by Barnett (1976). Hence, given a symmetric, positive definite matrix $\mathbf{Q}$ and a positive scalar c, we define the $(\mathbf{Q}, c)$-norm of a vector $\underline{X}$ as

$$||\underline{X}||_{(\mathbf{Q},c)} = \frac{1}{c}\underline{X}'\mathbf{Q}\underline{X} \qquad \ldots (27)$$

Considering the matrix $\mathbf{Q} = \left[\Lambda \mathbf{S}^- \Lambda'\right]^{-1}$, and an adequate choice of the scalar c, we can establish the following norms:

(I) For $c_1 = q\widehat{\sigma}^2$, we denote

$$D_i[\omega, \Lambda\underline{\widehat{\beta}}] = ||\Lambda\underline{\widehat{\beta}} - \Lambda\underline{\widehat{\beta}}_\omega||_{(\mathbf{Q},c_1)}$$

(II) For $c_2 = \widehat{\sigma}_{(i)}^2$, we denote

$$W_i[\omega, \Lambda\underline{\widehat{\beta}}] = ||\Lambda\underline{\widehat{\beta}} - \Lambda\underline{\widehat{\beta}}_\omega||_{(\mathbf{Q},c_2)}$$

(III) For $c_3 = \frac{r}{n-r}\widehat{\sigma}^2_{(i)}$, we denote

$$C_i[\omega, \Lambda\underline{\widehat{\beta}}] = ||\Lambda\underline{\widehat{\beta}} - \Lambda\underline{\widehat{\beta}}_\omega||_{(\mathbf{Q},c_3)}$$

These three norms are non-negative functions of ω. Note that for $\omega = 1$ the three norms are zero. They can be expressed as:

$$D_i[\omega, \Lambda\underline{\widehat{\beta}}] = D_i[\Lambda\underline{\widehat{\beta}}]\left[\frac{(\omega-1)(1-v_{ii})}{1+(\omega-1)v_{ii}}\right]^2 \quad \forall\omega > 0 \qquad \ldots(28)$$

$$W_i[\omega, \Lambda\underline{\widehat{\beta}}] = W_i[\Lambda\underline{\widehat{\beta}}]\left[\frac{(\omega-1)(1-v_{ii})}{1+(\omega-1)v_{ii}}\right]^2 \quad \forall\omega > 0 \qquad \ldots(29)$$

$$C_i[\omega, \Lambda\underline{\widehat{\beta}}] = C_i[\Lambda\underline{\widehat{\beta}}]\left[\frac{(\omega-1)(1-v_{ii})}{1+(\omega-1)v_{ii}}\right]^2 \quad \forall\omega > 0 \qquad \ldots(30)$$

with

$$D_i[\Lambda\underline{\widehat{\beta}}] = \frac{1}{q(1-v_{ii})}\underline{x}_i^{'}\mathbf{S}^{-}\Lambda^{'}\left[\Lambda\mathbf{S}^{-}\Lambda^{'}\right]^{-1}\Lambda\mathbf{S}^{-}\underline{x}_i r_i^2 \qquad \ldots(31)$$

$$W_i[\Lambda\underline{\widehat{\beta}}] = \frac{1}{(1-v_{ii})}\underline{x}_i^{'}\mathbf{S}^{-}\Lambda^{'}\left[\Lambda\mathbf{S}^{-}\Lambda^{'}\right]^{-1}\Lambda\mathbf{S}^{-}\underline{x}_i t_i^2 \qquad \ldots(32)$$

$$C_i[\Lambda\underline{\widehat{\beta}}] = \frac{n-r}{r(1-v_{ii})}\underline{x}_i^{'}\mathbf{S}^{-}\Lambda^{'}\left[\Lambda\mathbf{S}^{-}\Lambda^{'}\right]^{-1}\Lambda\mathbf{S}^{-}\underline{x}_i t_i^2 \qquad \ldots(33)$$

where

$$r_i = \frac{e_i}{[\widehat{\sigma}^2(1-v_{ii})]^{1/2}} \text{ and } t_i = \frac{e_i}{[\widehat{\sigma}^2_{(i)}(1-v_{ii})]^{1/2}} \qquad \ldots(34)$$

are the i-th internally and externally studentezed residuals, respectively. $D_i[\Lambda\underline{\widehat{\beta}}]$, $W_i[\Lambda\underline{\widehat{\beta}}]$ and $C_i[\Lambda\underline{\widehat{\beta}}]$ are measures of influence named D_i-distance, W_i-distance and C_i-distance associated with y_i, respectively. These distances have been recently proposed by Munoz-Pichardo *et al.* (1995), as a generalization for the Cook's distance [Cook and Weisberg (1982)], Welsch-Kuh's distance [Belsey et al (1980)] and modified Cook's distance [Atkinson (1982)] respectively.

As functions of ω, these distances satisfy:

(P0)

$$D_i[1, \Lambda\underline{\widehat{\beta}}] = W_i[1, \Lambda\underline{\widehat{\beta}}] = C_i[1, \Lambda\underline{\widehat{\beta}}] = 0$$

(P1)

$$\lim_{\omega\to 0} D_i[\omega, \Lambda\underline{\widehat{\beta}}] = D_i[\Lambda\underline{\widehat{\beta}}];$$

$$\lim_{\omega\to 0} W_i[\omega, \Lambda\underline{\widehat{\beta}}] = W_i[\Lambda\underline{\widehat{\beta}}];$$

$$\lim_{\omega\to 0} C_i[\omega, \Lambda\underline{\widehat{\beta}}] = C_i[\Lambda\underline{\widehat{\beta}}]$$

(P2)

$$\lim_{\omega\to\infty} D_i[\omega, \Lambda\underline{\widehat{\beta}}] = \left[\frac{1-v_{ii}}{v_{ii}}\right]^2 D_i[\Lambda\underline{\widehat{\beta}}] \qquad \ldots(35)$$

$$\lim_{\omega\to\infty} W_i[\omega, \Lambda\underline{\widehat{\beta}}] = \left[\frac{1-v_{ii}}{v_{ii}}\right]^2 W_i[\Lambda\underline{\widehat{\beta}}] \qquad \ldots(36)$$

$$\lim_{\omega\to\infty} C_i[\omega, \Lambda\underline{\widehat{\beta}}] = \left[\frac{1-v_{ii}}{v_{ii}}\right]^2 C_i[\Lambda\underline{\widehat{\beta}}] \qquad \ldots(37)$$

(P3)

$$\text{If } D_i[\Lambda\underline{\widehat{\beta}}] = D_j[\Lambda\underline{\widehat{\beta}}] \text{ and } v_{ii} > v_{jj} \text{ then } D_i[\omega, \Lambda\underline{\widehat{\beta}}] < D_j[\omega, \Lambda\underline{\widehat{\beta}}] \forall\omega > 0.$$

$$\text{If } W_i[\Lambda\underline{\widehat{\beta}}] = W_j[\Lambda\underline{\widehat{\beta}}] \text{ and } v_{ii} > v_{jj} \text{ then } W_i[\omega, \Lambda\underline{\widehat{\beta}}] < W_j[\omega, \Lambda\underline{\widehat{\beta}}] \forall\omega > 0.$$

$$\text{If } C_i[\Lambda\underline{\widehat{\beta}}] = C_j[\Lambda\underline{\widehat{\beta}}] \text{ and } v_{ii} > v_{jj} \text{ then } C_i[\omega, \Lambda\underline{\widehat{\beta}}] < C_j[\omega, \Lambda\underline{\widehat{\beta}}] \forall\omega > 0.$$

From (**P1**), it can be seen the three functions coincide with the influence measures in the limit, respectively. Property (**P3**) leads us to conclude that given two observations with the same influence, the observation with a smaller diagonal element in the prediction matrix has potentially larger influence. For instance in the multi-way classification ANOVA, the diagonal elements of **V** are the inverses of the sample sizes in each cell, therefore two observations in the same cell with equal influence have equal local influence. Other comparative studies takes into account residuals, diagonal elements of the prediction matrix and distances. Although, it is easier looking at the analysis of plots of those observations with larger influence.

Regarding the unbiased estimator of the variance, from (63), we have

$$\widehat{\sigma}^2 - \widehat{\sigma}^2_\omega = -\frac{e_i^2}{n-r}\frac{\omega-1}{1+(\omega-1)v_{ii}} \quad \omega \in (0,\infty) \qquad \ldots(38)$$

can be considered as a function of ω, and will be denoted as $SD_i(\omega)$. This function is non-negative in $(0,1)$, and is zero for $\omega = 1$. It is interesting to study the behavior of $SD_i(\omega)$ in a neighborhood of $\omega = 1$. The slope of the tangent to the curve in $\omega = 1$ is

$$\frac{d}{d\omega}SD_i(\omega)\,|_{\omega=1} = -\frac{e_i^2}{(n-r)} \qquad \ldots(39)$$

From this we conclude that the larger the absolute value of the slope, the larger is the local influence of the i-th observation on $\widehat{\sigma}^2$.

4. Local Influence and Conditional Bias in GLM.

In this section, we develop the study of local influence in GLM through the conditional bias. Therefore, we do not need to use any comparative scheme such as in the above section.

Given that $\widehat{\mathcal{S}}_{\omega}^{(i)}\left[y_i;\Lambda\underline{\widehat{\beta}}_{\omega}\right]$ is a q-dimensional vector, we can use the norms proposed in Section 3. So, if we consider the matrix $\mathbf{Q}=\left[\Lambda\mathbf{S}^{-}\Lambda^{'}\right]^{-1}$, the following measures of local influence appear naturally:

(I) For $c_1 = q\widehat{\sigma}^2$

$$\widetilde{D}_i[\omega;\Lambda\underline{\widehat{\beta}}] = ||\widehat{\mathcal{S}}_{\omega}^{(i)}\left[y_i;\Lambda\underline{\widehat{\beta}}_{\omega}\right]||_{(\mathbf{Q},c_1)} \quad \forall\omega>0. \qquad \ldots(40)$$

(II) For $c_2 = \widehat{\sigma}^2_{(i)}$

$$\widetilde{W}_i[\omega;\Lambda\underline{\widehat{\beta}}] = ||\widehat{\mathcal{S}}_{\omega}^{(i)}\left[y_i;\Lambda\underline{\widehat{\beta}}_{\omega}\right]||_{(\mathbf{Q},c_2)} \quad \forall\omega>0. \qquad \ldots(41)$$

(III) For $c_3 = \dfrac{r}{n-r}\widehat{\sigma}^2_{(i)}$

$$\widetilde{D}_i[\omega;\Lambda\underline{\widehat{\beta}}] = ||\widehat{\mathcal{S}}_{\omega}^{(i)}\left[y_i;\Lambda\underline{\widehat{\beta}}_{\omega}\right]||_{(\mathbf{Q},c_3)} \quad \forall\omega>0. \qquad \ldots(42)$$

These three norms are non-negative functions of ω. They can be expressed as:

$$\widetilde{D}_i[\omega;\Lambda\underline{\widehat{\beta}}] = D_i[\Lambda\underline{\widehat{\beta}}]\left[\frac{\omega}{1+(\omega-1)v_{ii}}\right]^2 \quad \omega>0. \qquad \ldots(43)$$

$$\widetilde{W}_i[\omega;\Lambda\underline{\widehat{\beta}}] = W_i[\Lambda\underline{\widehat{\beta}}]\left[\frac{\omega}{1+(\omega-1)v_{ii}}\right]^2 \quad \omega>0. \qquad \ldots(44)$$

$$\widetilde{C}_i[\omega;\Lambda\underline{\widehat{\beta}}] = C_i[\Lambda\underline{\widehat{\beta}}]\left[\frac{\omega}{1+(\omega-1)v_{ii}}\right]^2 \quad \omega>0. \qquad \ldots(45)$$

In these expressions, the factor $\omega/\left[1+(\omega-1)v_{ii}\right]$ appears too. The local influence measures are functions based on ω which coincide with the square of that factor, except constants. Also, these constants are established by the influence measures of the observation under study. These functions have the following properties:

(P0')

$$\widetilde{D}_i[1;\Lambda\underline{\widehat{\beta}}] = D_i[\Lambda\underline{\widehat{\beta}}];\ \ \widetilde{W}_i[1;\Lambda\underline{\widehat{\beta}}] = W_i[\Lambda\underline{\widehat{\beta}}];\ \ \widetilde{C}_i[1;\Lambda\underline{\widehat{\beta}}] = C_i[\Lambda\underline{\widehat{\beta}}]$$

(P1')

$$\lim_{\omega\to 0}\widetilde{D}_i[\omega;\Lambda\underline{\widehat{\beta}}]=0;\ \lim_{\omega\to 0}\widetilde{W}_i[\omega;\Lambda\underline{\widehat{\beta}}]=0;\ \lim_{\omega\to 0}\widetilde{C}_i[\omega;\Lambda\underline{\widehat{\beta}}]=0.$$

(P2')

$$\lim_{\omega\to+\infty}\widetilde{D}_i\left[\omega;\Lambda\underline{\widehat{\beta}}\right]=\left[\frac{1}{v_{ii}}\right]^2 D_i[\Lambda\underline{\widehat{\beta}}] \qquad \ldots(46)$$

$$\lim_{\omega\to+\infty}\widetilde{W}_i[\omega;\Lambda\underline{\widehat{\beta}}]=\left[\frac{1}{v_{ii}}\right]^2 W_i[\Lambda\underline{\widehat{\beta}}] \qquad \ldots(47)$$

$$\lim_{\omega\to+\infty}\widetilde{C}_i[\omega;\Lambda\underline{\widehat{\beta}}]=\left[\frac{1}{v_{ii}}\right]^2 C_i[\Lambda\underline{\widehat{\beta}}] \qquad \ldots(48)$$

(P3') If $D_i[\Lambda\underline{\widehat{\beta}}]=D_j[\Lambda\underline{\widehat{\beta}}]$ and $v_{ii}>v_{jj}$ then

$$\widetilde{D}_i[\omega;\Lambda\underline{\widehat{\beta}}]<\widetilde{D}_j[\omega;\Lambda\underline{\widehat{\beta}}]\quad\forall\omega>0.$$

If $W_i[\Lambda\underline{\widehat{\beta}}]=W_j[\Lambda\underline{\widehat{\beta}}]$ and $v_{ii}>v_{jj}$ then

$$\widetilde{W}_i[\omega;\Lambda\underline{\widehat{\beta}}]<\widetilde{W}_j[\omega;\Lambda\underline{\widehat{\beta}}]\quad\forall\omega>0.$$

If $C_i[\Lambda\underline{\widehat{\beta}}]=C_j[\Lambda\underline{\widehat{\beta}}]$ and $v_{ii}>v_{jj}$ then

$$\widetilde{C}_i[\omega;\Lambda\underline{\widehat{\beta}}]<\widetilde{C}_j[\omega;\Lambda\underline{\widehat{\beta}}]\quad\forall\omega>0.$$

According to $(\mathbf{P0'})$, for $\omega=1$, that is to say when both models, GLM and GLM(ω,i) coincide, these measures give us the value of measure of influence. In addition, these functions are increasing, this is intuitively correct because as ω increases, the relative weight of the i-th observation increases. The property $(\mathbf{P3'})$ leads us to analogous conclusions to those obtained in Section 3, when we discussed $(\mathbf{P3})$.

The study and evaluation of these functions are of great interest, fundamentally in a neighborhood of $\omega=1$. The slope in this point gives us information about the potential influence of the observations. In fact, we can easily obtain

$$\frac{d}{d\omega}\widetilde{D}_i\left[\omega;\Lambda\underline{\widehat{\beta}}\right]\Big|_{\omega=1}=2[1-v_{ii}]D_i[\Lambda\underline{\widehat{\beta}}] \qquad \ldots(49)$$

$$\frac{d}{d\omega}\widetilde{W}_i\left[\omega;\Lambda\underline{\widehat{\beta}}\right]\Big|_{\omega=1}=2[1-v_{ii}]W_i[\Lambda\underline{\widehat{\beta}}] \qquad \ldots(50)$$

$$\frac{d}{d\omega}\widetilde{C}_i\left[\omega;\Lambda\underline{\widehat{\beta}}\right]\Big|_{\omega=1}=2[1-v_{ii}]C_i[\Lambda\underline{\widehat{\beta}}] \qquad \ldots(51)$$

Then, the slope depends on the influence of the observation and is decreasing as function of the i-th diagonal element of the prediction matrix.

Now, we carry out the study of the local influence on $\widehat{\sigma}^2$ by means of the conditional bias. In section 2.2 we obtained an estimation of conditional bias $\mathcal{S}_\omega^{(i)}\left[y_i;\widehat{\sigma}_\omega^2\right]$

$$\widehat{\mathcal{S}}_\omega^{(i)}\left[y_i;\widehat{\sigma}_\omega^2\right] = SMED_i + \frac{1}{n-r}\,\frac{\omega-1}{1+(\omega-1)v_{ii}}e_i^2 \qquad \ldots(52)$$

where

$$SMED_i = \frac{1}{n-r-1}\widehat{\sigma}^2\left[r_i^2-1\right] \in \mathcal{R} \qquad \ldots(53)$$

$SMED_i$ is a measure of influence on $\widehat{\sigma}^2$ proposed by Munoz-Pichardo *et al.* (1995). $\widehat{\mathcal{S}}_\omega^{(i)}\left[y_i;\widehat{\sigma}_\omega^2\right]$ is a function of ω that we will be denoted by $SMED_i(\omega)$ and it verifies:

(Q0)

$$SMED_i[1] = SMED_i$$

(Q1)

$$\lim_{\omega\to 0} SMED_i[\omega] = SMED_i - \frac{e_i^2}{(n-r)(1-v_{ii})}$$

(Q2)

$$\lim_{\omega\to\infty} SMED_i[\omega] = SMED_i - \frac{e_i^2}{(n-r)v_{ii}}$$

Similarly, the behaviour of $SD_i(\omega)$ in a neighbourhood of $\omega = 1$ must be studied. The slope of the curve shows that the growth is

$$SMED_i^{'}(1) = \frac{e_i^2}{n-r}. \qquad \ldots(54)$$

Therefore, except in the sign, we get the same slope as for $SD_i(\omega)$ in $\omega = 1$. This last slope depends on the i-th residual and the sample size.

5. Local Influence Potential

Until now, we have proposed local influence measures of the $i-th$ observation on the BLUE of an e.l.f. $\Lambda\underline{\beta}$, in the model GLM. The expressions (16) and (20) provide us the relationship between the conditional bias in the model GLM (ω, i) and the postulated model GLM, and among their estimators, respectively. Both relations are in proportion with the ratio

$$\alpha_i(\omega) = \frac{\omega}{[1+(\omega-1)v_{ii}]} \qquad \ldots(55)$$

In the same way, in the expressions (43), (44) and (45), we get the direct proportion between $\widetilde{D}_i\left[\omega;\Lambda\underline{\widehat{\beta}}\right]$ and $D_i[\Lambda\underline{\widehat{\beta}}]$, $\widetilde{W}_i\left[\omega;\Lambda\underline{\widehat{\beta}}\right]$ and $W_i[\Lambda\underline{\widehat{\beta}}]$, and $\widetilde{C}_i\left[\omega;\Lambda\underline{\widehat{\beta}}\right]$ and $C_i[\Lambda\underline{\widehat{\beta}}]$, respectively. The proportion ratio is $[\alpha_i(\omega)]^2$, in the three relationships. Analogously, the relationship between $D_i\left[\omega;\Lambda\underline{\widehat{\beta}}\right]$ and $D_i[\Lambda\underline{\widehat{\beta}}]$, $W_i\left[\omega;\Lambda\underline{\widehat{\beta}}\right]$ and $W_i[\Lambda\underline{\widehat{\beta}}]$, and $C_i\left[\omega;\Lambda\underline{\widehat{\beta}}\right]$ and $C_i[\Lambda\underline{\widehat{\beta}}]$, conserves the same proportion ratios $[1-\alpha_i(\omega)]^2$ (see (28), (29) and (30)).

The following result potencies the importance of the proportion ration $\alpha_i(\omega)$ and $(1-\alpha_i(\omega))$ in the study of the local influence. We denote by $GLM_{(i)}$ the resulting linear model with i-th observation omitted.

Theorem 5.1. *Let $\Lambda\underline{\beta}$ be an e.l.f. on the GLM. We suppose that $\Lambda\underline{\widehat{\beta}}$, $\Lambda\underline{\widehat{\beta}}_\omega$ and $\Lambda\underline{\widehat{\beta}}_{(i)}$ are the BLUE on the models GLM, GLM(ω,i) and $GLM_{(i)}$, respectively. Then, it is verified:*

$$\mathcal{S}_\omega^{(i)}\left[y_j;\Lambda\underline{\widehat{\beta}}_\omega\right]=\alpha_i(\omega)\mathcal{S}\left[y_j;\Lambda\underline{\widehat{\beta}}\right]+(1-\alpha_i(\omega))\mathcal{S}\left[y_j;\Lambda\underline{\widehat{\beta}}_{(i)}\right] \qquad \ldots(56)$$

$$\forall\omega>0 \text{ and } \forall j=1,\cdots,n$$

Proof. Munoz-Pichardo *et al.* (1995) obtained

$$\Lambda\underline{\widehat{\beta}}_{(i)}=\Lambda\underline{\widehat{\beta}}-\frac{1}{1-v_{ii}}\Lambda\mathbf{S}^-\underline{x}_i'\left[Y_i-\underline{x}_i\underline{\widehat{\beta}}\right] \qquad \ldots(57)$$

¿From the expressions (15) and (61), we get

$$\Lambda\underline{\widehat{\beta}}_\omega=\alpha_i(\omega)\Lambda\underline{\widehat{\beta}}+(1-\alpha_i(\omega))\Lambda\underline{\widehat{\beta}}_{(i)} \qquad \ldots(58)$$

And the result is immediately obtained. □

When $\omega\in(0,1)$, the proportion ratio verifies $0<\alpha_i(\omega)<1$, therefore, $\mathcal{S}_\omega^{(i)}\left[y_j;\Lambda\underline{\widehat{\beta}}_\omega\right]$ can be considered as a convex linear combination between $\mathcal{S}\left[y_j;\Lambda\underline{\widehat{\beta}}\right]$ and $\mathcal{S}\left[y_j;\Lambda\underline{\widehat{\beta}}_{(i)}\right]$ for $\omega\in(0,1)$. Their coefficients does not depend on the j-th observation and on the e.l.f. $\Lambda\underline{\beta}$. Therefore, $\alpha_i(\omega)$ is the proportion of the $\mathcal{S}_\omega^{(i)}\left[y_j;\Lambda\underline{\widehat{\beta}}_\omega\right]$ in the direction of $\mathcal{S}\left[y_j;\Lambda\underline{\widehat{\beta}}\right]$ for any j-th observation. That is to say, $\alpha_i(\omega)$ is the explained proportion by $\mathcal{S}\left[y_j;\Lambda\underline{\widehat{\beta}}\right]$ on $\mathcal{S}_\omega^{(i)}\left[y_j;\Lambda\underline{\widehat{\beta}}_\omega\right]$. Analogously, $[1-\alpha_i(\omega)]$ is the proportion in the direction of $\mathcal{S}\left[y_j;\Lambda\underline{\widehat{\beta}}_{(i)}\right]$.

Consequently, we can interpret $\alpha_i(\omega)$ as the explained proportion of the influence of any observation on the BLUE for any e.l.f. $\Lambda\underline{\beta}$ in the GLM(ω,i) by the influence of that observation on the BLUE in the GLM. Similarly, we can also interpret $[1-\alpha_i(\omega)]$. Since the proportion ratios between the distances are the squares of $\alpha_i(\omega)$ and $[1-\alpha_i(\omega)]$, we can define $[\alpha_i(\omega)]^2$ as the Local Influence Potential of the i-th observation:

$$LIP_i(\omega) = \left[\frac{\omega}{1+(\omega-1)v_{ii}}\right]^2 \qquad \ldots(59)$$

This function of the ω perturbation describes the performed local influence of an observation on the BLUE on any e.l.f. in GLM (see Fig. 1).

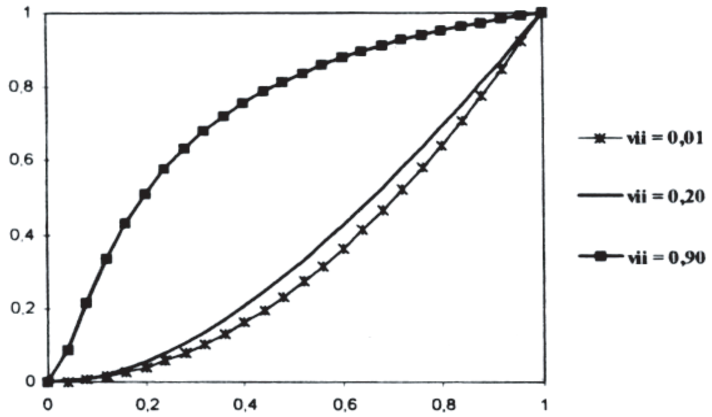


Figure 1. Local Influential Potential. $LIP_i(w)$

6. Examples

In this section, we present two examples to illustrate some of the proposed local influence measures. Firstly, we include a real-life example where observations have equal magnitude of influence but different local influence. This example justifies the application of the proposed process. Finally, we include an artificial example presenting the different possible cases which are described in the paper.

6.1. *Example 1: Cloud seeding data set.* Cook and Weisberg (1982) analyzed different models using the cloud seeding data set (Woodley *et al.*, 1977). This data set consists of 24 observations with the following variables:

$Y \equiv$ *response variable,* the amount of rain (in $m^3 \times 10^7$).

$C \equiv$ *echo coverage,* per cent cloud cover in the experiment area.

$P \equiv$ *prevetness,* total rainfall in the target area 1 hour before seeding (in $m^3 \times 10^7$).

$E \equiv$ *echo motion,* a classification indicating a moving radar echo (1) or a stationary radar echo (2).

$A \equiv$ *action,* a classification indication seeding (1) or no seeding (0).

$S - N_e \equiv$ suitability for seeding.

$T \equiv$ number of days after the first day of the experiment (June $1, 1975 = 0$).

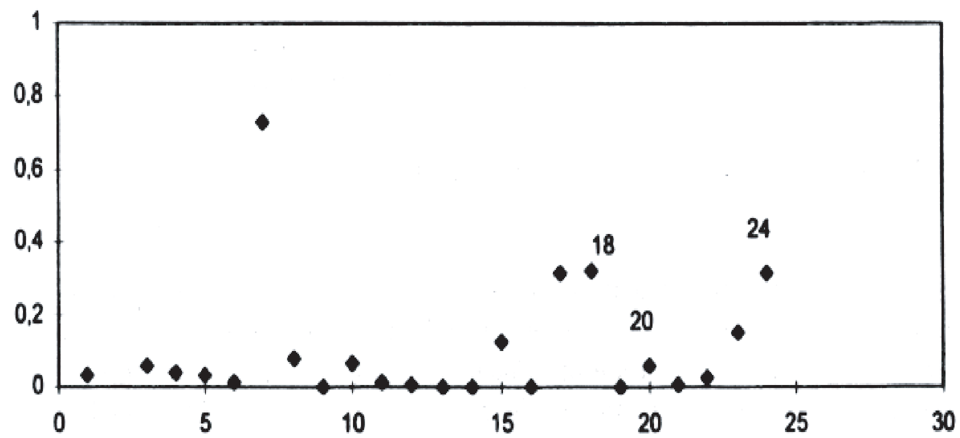
Cook and Weisberg (1982, pp 126) used the following linear model:

$$Y^{1/3} = \beta_0 + \beta_1 A + \beta_2 T + \beta_3 (S - N_e) + \beta_4 C + \beta_5 P^{1/3} + \beta_6 E$$

$$+\beta_{13}(A \times (S - N_e)) + \beta_{14}(A \times C) + \beta_{15}(A \times P^{1/3}) + \beta_{16}(A \times E) + \varepsilon$$

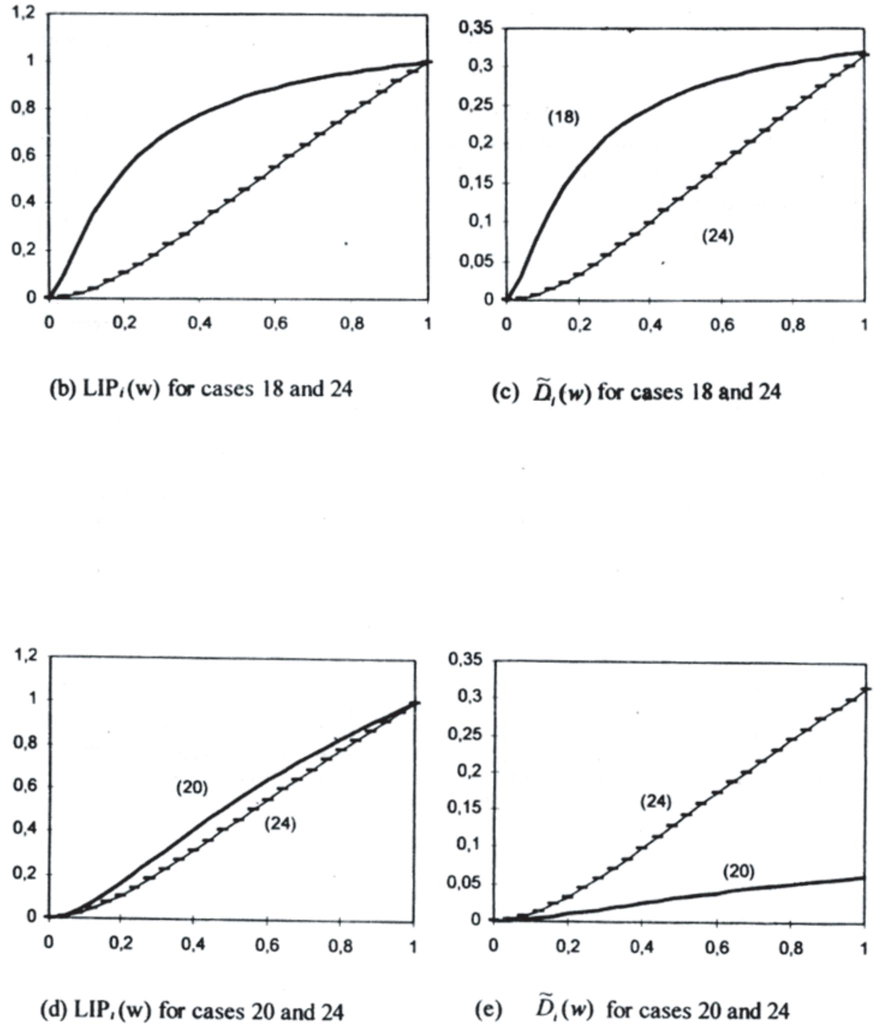
Using the procedures which appear in this paper, we can study the influence analysis on the vector of regression coefficients. The results are shown in Fig. 2.
Basically, we can get the following conclusions:

i) In Fig 2a, cases 18 and 24 exert similar influence on the vector of regression coefficients, but the local influence potential is very different for each one (see Fig 2b). Therefore, case 18 is locally more influential than case 24 (see Fig 2c).



(a) Index plot of D-distance (*)

Figure 2 (Contd.)



(b) $\mathrm{LIP}_i(w)$ for cases 18 and 24

(c) $\widetilde{D}_i(w)$ for cases 18 and 24

(d) $\mathrm{LIP}_i(w)$ for cases 20 and 24

(e) $\widetilde{D}_i(w)$ for cases 20 and 24

Figure 2. Several plots for influence analysis of Cloud Seeding data set
(*) It has been omitted case 2, with $D_i = 4.55$

ii) Case 20 does not exert influence on the considered BLUE, but the local influence potential is very similar to the local influence potential for the case 24 (see Fig. 2d). However, we can observe that the local influence is significantly different for each one of them (see Fig. 2e).

Finally, we can conclude that it is convenient to complete the influence analysis with the local influence analysis because it can provide an interesting additional information (see cases 18 vs 24). Although, they can coincide sometimes (see cases 20 vs 24).

Table 6.1. Artificial data set.

Group	X	Z	V	Y
1	3,40	16,08	7,34	69,87
1	2,02	30,16	3,03	76,47
1	2,54	24,78	14,55	60,34
1	5,39	20,14	15,71	50,27
1	2,32	25,90	1,22	78,31
1	2,51	30,65	14,10	51,60
1	4,06	18,34	9,37	79,06
1	3,06	13,34	10,04	123,34
1	1,88	15,92	12,05	44,51
1	3,98	21,23	12,45	34,45
1	3,48	16,60	1,57	70,84
1	2,33	9,19	18,42	23,71
1	2,88	16,70	17,10	56,89
1	4,13	21,51	22,28	33,03
1	4,03	11,20	13,37	42,87
2	0,87	16,39	6,91	39,31
2	4,70	22,99	15,83	59,34
2	2,55	18,46	23,73	4,71
2	3,46	21,89	-1,21	101,23
2	2,91	23,72	13,73	61,25
2	1,59	3,38	2,80	26,93
2	3,87	18,31	5,21	42,28
2	2,52	17,46	9,75	29,00
2	2,09	21,66	10,76	45,76
2	1,01	20,85	8,21	45,19
2	3,76	26,09	1,91	64,18
2	2,58	9,06	6,41	62,21
2	2,16	18,86	17,42	30,38
2	2,13	22,00	10,39	42,97
2	0,95	22,83	23,03	3,17

Group	X	Z	V	Y
3	2,52	29,78	7,34	39,37
3	1,03	11,90	3,13	42,22
3	1,27	23,92	5,31	44,82
3	3,37	21,58	15,87	11,51
3	3,70	20,27	19,29	22,68
3	2,43	30,63	10,40	48,54
3	3,38	19,02	5,60	46,97
3	4,68	16,71	13,94	37,46
3	3,07	9,50	7,37	32,50
3	3,73	22,34	2,12	46,93
3	1,66	3,82	12,37	69,87
3	1,96	8,88	3,86	27,24
4	1,96	22,85	13,53	18,04
4	2,77	23,40	10,96	36,65
4	3,26	23,97	17,16	33,65
4	1,73	14,64	9,46	29,99
4	4,93	18,45	16,04	17,01
4	4,38	11,83	12,81	15,77
4	2,29	11,08	5,98	32,00
4	1,71	7,59	14,25	8,30
4	3,43	14,64	14,97	23,29
4	3,23	16,44	9,72	20,18
4	0,23	39,54	21,76	34,87
4	4,06	25,09	7,94	47,97
4	2,81	27,23	18,64	24,35

6.2. *Example 2: an artificial data set.* An artificial data set is given in Table 6.1. We study the analysis of covariance considering Y as response variable, $GROUP$ as factor variable, and X, V and Z as covariates. That is to say, the fitted model is:

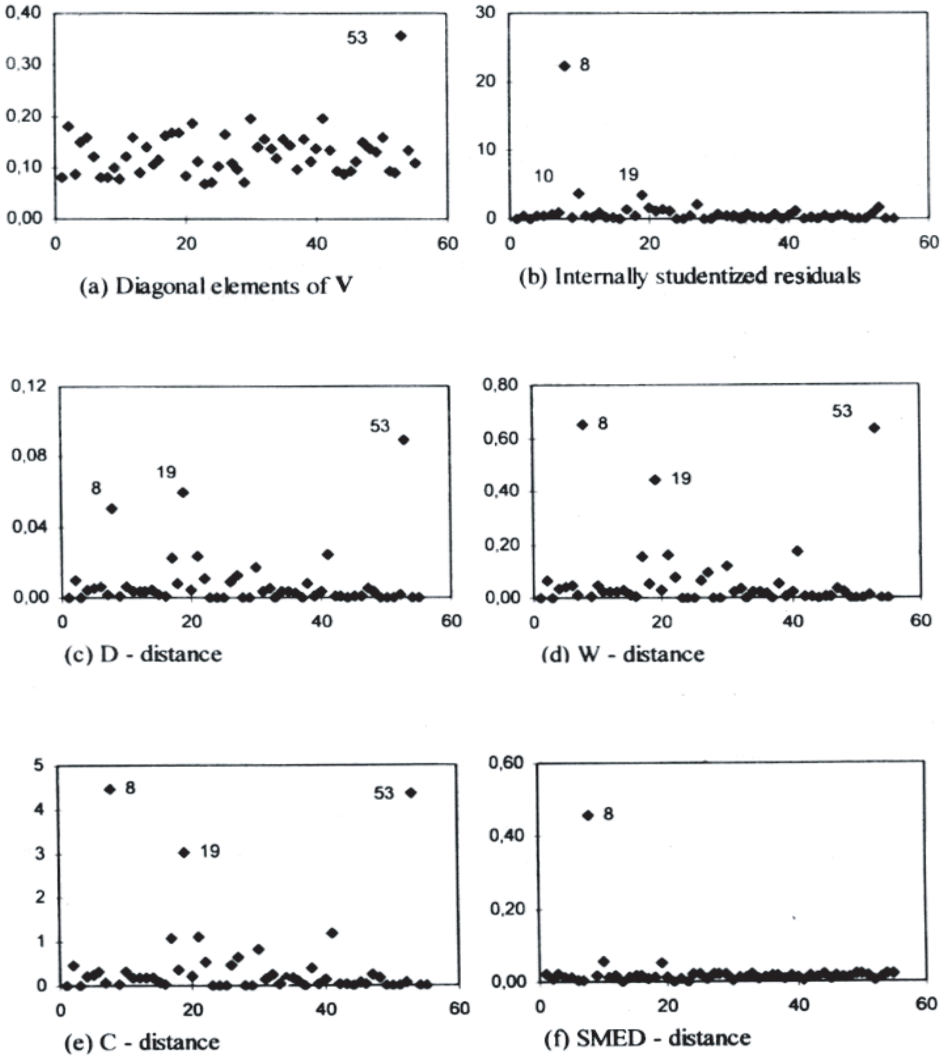


Figure 3. Index plots for influence analysis of artificial data set

$$y_{ij} = \mu + \alpha_i + \beta_1 x_{ij} + \beta_2 v_{ij} + \beta_3 z_{ij} + \varepsilon_{ij} \qquad \ldots(6.1)$$

$$E[\varepsilon_{ij}] = 0 \qquad Cov[\varepsilon_{ij}, \varepsilon_{rs}] = \begin{cases} 0 & (i,j) \neq (r,s) \\ \sigma^2 & (i,j) = (r,s) \end{cases}$$

where α_i is the effect of the i-th level of the factor group, and the subindex ij represents the value of the corresponding variable on the j-th case of the i-th group.

The BLUE of the vector $(\beta_1, \beta_2, \beta_3)'$ is $(3.323, -2.221, 1.070)$, and the unbiased estimator of σ^2 is 220.827. Several diagnostics measures resulting from fitting model (6.1) in both estimations have been computed and the corresponding index plots have been drawn in Figure 3.

Examination of the index plots indicates that

1. Case 53 can be considered as high leverage but not as an outlier.

2. Cases 8,10 and 19 can be regarded as outliers but not high leverage.

3. Cases 8,19 and 53 can be considered as influence observations on the BLUE of the estimable linear function. Moreover, case 8 is the only one observation that it can be regarded as influence observation on the unbiased estimator of the variance.

Moreover, the influence on the BLUE is very similar between cases 8 and 53. Particularly, $W_8 = 0.652$ and $W_{53} = 0.633$. However, when the different values in the diagonal elements of the prediction matrix, 0.08 and 0.35 respectively, the local influence potential of the case 53 is greater than case 8 (see Fig. 4a). Then, although they exert a similar influence on the BLUE considered, case 53 exerts locally more influence than case 8 (see Fig. 4b).

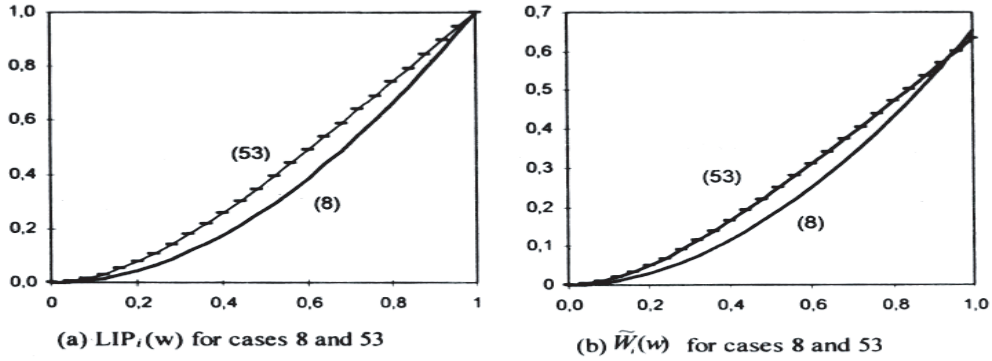


Figure 4. Comparative study for cases 8 and 53

If we observe Fig. 4a and Fig. 4b, we can deduce that they are similar. Then, it would be sufficient to study one of them but the plots of local influence potential and the measures of local influence can be very different. Such as we can observe in Fig. 5, the local influence measure $\tilde{W}_i(w)$ is represented for the observations 7 and 8. They have similar local influence potential because their diagonal elements in the prediction matrix are equal to 0.081, however the measures of the local influence are very different.

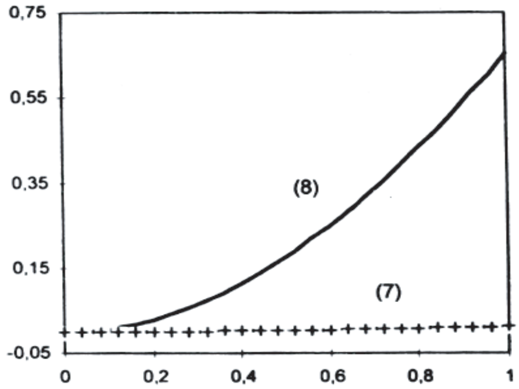


Figure 5. $\tilde{W}_i(w)$ for cases 7 and 8

7. Conclusion

The Local Influence Analysis can be considered as a complementary technique to the influence analysis. The diagnostics measures directed to study the influence which is performed by an observation, or a subset, are not sufficient. Particularly, using the properties (**P3**) y (**P3**$'$) we deduce that two observations with the same influence can perform potentially different influences. Then, it is necessary to complement the influence analysis with the local influence analysis. In this paper, we propose different techniques for linear models which can be fitted by the GLM. The proposed techniques do not presuppose hypothesis on the underlying distribution and we get them from two different ways.

Firstly, we have followed the general scheme of comparison among results which was proposed by Cook (1987). So, the functions $D_i[\omega, \Lambda\underline{\widehat{\beta}}], W_i[\omega, \Lambda\underline{\widehat{\beta}}]$ and $C_i[\omega, \Lambda\underline{\widehat{\beta}}]$ were proposed to Local Influence Analysis on the BLUE of an e.l.f. $\Lambda\underline{\beta}$, and $SD_i(\omega)$ for the unbiased estimator of the variance. Afterwards, we used the conditional bias in which possible arbitrary comparative scheme is not used. In this case, the proposed techniques appear to estimate the deviations which are given by perturbation on the expectation of the statistic. So, the functions $\widetilde{D}_i\ [\omega, \Lambda\underline{\widehat{\beta}}], \widetilde{W}_i\ [\omega, \Lambda\underline{\widehat{\beta}}], \widetilde{C}_i\ [\omega, \Lambda\underline{\widehat{\beta}}]$, and $SMED_i(\omega)$ are proposed.

Although both approaches can be used to study the Local Influence Analysis, we think that the second one is prefereble because:

1. The measures based on the conditional bias have the proportion ratio $LIP_i(\omega)$. That is to say, the square of the proportion of the influence which is explained by the influence on the postulated model, while the rest of measures have the square of the proportion of the influence which is explained by influence on the model $\mathrm{GLM}_{(i)}$.

2. The plots for the distances versus to ω provide easier views. In the following table we show this idea, particularly for $\widetilde{D}_i\ [\omega, \Lambda\underline{\widehat{\beta}}]$ and $D_i\ [\omega, \Lambda\underline{\widehat{\beta}}]$.

Table 7.1

	ω	$D_i\ [\omega, \Lambda\underline{\widehat{\beta}}]$	$\widetilde{D}_i\ [\omega, \Lambda\underline{\widehat{\beta}}]$
ABSENCE	$\omega \to 0$	$D_i\ [\Lambda\underline{\widehat{\beta}}]$	0
	$\omega \in (0,1)$	*decreasing*	*increasing*
PRESENCE	$\omega = 1$	0	$D_i\ [\Lambda\underline{\widehat{\beta}}]$
	$\omega > 1$	*increasing*	*increasing*

When the relative weight of the observation is increasing, a larger local influence in the analysis is provided, which is seen better in $\widetilde{D}_i\ [\omega, \Lambda\underline{\widehat{\beta}}]$ than in $D_i\ [\omega, \Lambda\underline{\widehat{\beta}}]$.

Appendix

LEMMA 1. *Let* $\mathbf{A}$ *be a* $p \times p$ *matrix and* $\mathbf{L}$ *be a* $m \times p$ *matrix* $(m \leq p)$, *such that belongs to the row space generated by the matrix* $\mathbf{A}$. *If* $\gamma \in \mathcal{R}, \gamma \neq 0$, *and the inverse of the matrix* $[\mathbf{I} + \gamma \mathbf{L}\mathbf{A}^{-}\mathbf{L}']$ *exits, then the matrices*

$$\left[\mathbf{A}^{-} - \gamma \mathbf{A}^{-}\mathbf{L}'[\mathbf{I} + \gamma \mathbf{L}\mathbf{A}^{-}\mathbf{L}']^{-1}\mathbf{L}\mathbf{A}^{-}\right]$$

are generalized inverses of $[\mathbf{A} + \gamma \mathbf{L}'\mathbf{L}]$.

PROOF. The proof is straightforward considering the following equalities (see Pringle and Rayner (1971))

$$\mathbf{L}\mathbf{A}^{-}\mathbf{A} = \mathbf{L}, \qquad \mathbf{A}\mathbf{A}^{-}\mathbf{L}^{'} = \mathbf{L}^{'}$$

□

THEOREM 1. *If* $v_{ii} \neq 1$, *then*

1. $\mathbf{X}\mathbf{S}_{\omega}^{-}\mathbf{X}^{'}$ *is unique for any choice of* $\mathbf{S}_{\omega}^{-}$.

2. If $\underline{\lambda}^{'}\underline{\beta}$ *is an e.l.f. for GLM, then* $\underline{\lambda}^{'}\mathbf{H}_{\omega} = \underline{\lambda}^{'}$.

3. If $\underline{\lambda}^{'}\underline{\beta}$ *is an e.l.f. for GLM and* $\underline{\widehat{\beta}}_{\omega}$ *and* $\underline{\widehat{\beta}}_{\omega}^{*}$ *any two GLS estimators of* $\underline{\beta}$ *for GLM(ω,i), then* $\underline{\lambda}^{'}\underline{\widehat{\beta}}_{\omega} = \underline{\lambda}^{'}\underline{\widehat{\beta}}_{\omega}^{*}$. *Moreover,*

$$\underline{\lambda}^{'}\underline{\widehat{\beta}}_{\omega} = \underline{\lambda}^{'}\underline{\widehat{\beta}} - \frac{\omega-1}{1+(\omega-1)v_{ii}}\underline{\lambda}^{'}\mathbf{S}^{-}\underline{x}_i\left[y_i - \underline{x}_i^{'}\underline{\widehat{\beta}}\right] \qquad \ldots(60)$$

PROOF. 1. By means of Lemma 1 we calculate $\mathbf{S}_{\omega}^{-}$. As

$$\mathbf{S}_{\omega} = \mathbf{X}^{'}\mathbf{W}^{-1}\mathbf{X} = \mathbf{X}^{'}\mathbf{X} + (\omega-1)\underline{x}_i\underline{x}_i^{'} \qquad \ldots(61)$$

then, taking $\mathbf{L} = \underline{x}_i$, $\gamma = \omega - 1$, $\mathbf{A} = \mathbf{S} = \mathbf{X}^{'}\mathbf{X}$ and given that $[\mathbf{I} + \gamma\mathbf{L}\mathbf{A}^{-}\mathbf{L}^{'}] = 1 + (\omega-1)v_{ii}$ is positive for any $\omega > 0$, we have

$$\mathbf{S}_{\omega}^{-} = \mathbf{S}^{-} - \frac{(\omega-1)}{1+(\omega-1)v_{ii}}\mathbf{S}^{-}\underline{x}_i\underline{x}_i^{'}\mathbf{S}^{-} \qquad \ldots(62)$$

Then, using the uniqueness of $\mathbf{V}, \mathbf{X}\mathbf{S}_{\omega}^{-}\mathbf{X}^{'}$ is unique for any choice of the generalized inverse of $\mathbf{S}_{\omega}$.

2. From (62)

$$\begin{aligned}
\underline{\lambda}^{'}\mathbf{H}_{\omega} &= \underline{\lambda}^{'}\left[\mathbf{S}^{-} - \frac{(\omega-1)}{1+(\omega-1)v_{ii}}\mathbf{S}^{-}\underline{x}_i\underline{x}_i^{'}\mathbf{S}^{-}\right]\left[\mathbf{S} + (\omega-1)\underline{x}_i\underline{x}_i^{'}\right] \\
&= \underline{\lambda}^{'}\left[\mathbf{H} + \left\{(\omega-1) - \frac{(\omega-1)}{1+(\omega-1)v_{ii}} - \frac{(\omega-1)^2 v_{ii}}{1+(\omega-1)v_{ii}}\right\}\mathbf{S}^{-}\underline{x}_i\underline{x}_i^{'}\right] \\
&= \underline{\lambda}^{'}\mathbf{H} = \underline{\lambda}^{'}.
\end{aligned}$$

3. As $\lambda^{'}\underline{\beta}$ is e.l.f. in GLM, by the previous result, we have

$$\underline{\lambda}^{'}\underline{\widehat{\beta}}_{\omega} = \underline{\lambda}^{'}\mathbf{S}_{\omega}^{-}\mathbf{X}^{'}\mathbf{W}^{-1}\underline{Y} \text{ and } \underline{\lambda}^{'}\underline{\widehat{\beta}}_{\omega}^{*} = \underline{\lambda}^{'}\mathbf{T}_{\omega}^{-}\mathbf{X}^{'}\mathbf{W}^{-1}\underline{Y}$$

where $\mathbf{S}_\omega^-$ y $\mathbf{T}_\omega^-$ are two generalized inverses of $\mathbf{S}_\omega$. Moreover, exists $\mathbf{u} \in \mathcal{R}^n$ such that $\underline{\lambda}' = \mathbf{u}'\mathbf{X}$ (see Kshirsagar, 1983), so

$$\underline{\lambda}'\underline{\widehat{\beta}}_\omega = \mathbf{u}'\mathbf{X}\mathbf{S}_\omega^-\mathbf{X}'\mathbf{W}^{-1}\underline{Y} \text{ and } \underline{\lambda}'\underline{\widehat{\beta}}_\omega^* = \mathbf{u}'\mathbf{X}\mathbf{T}_\omega^-\mathbf{X}'\mathbf{W}^{-1}\underline{Y}$$

which are equal by the uniqueness of $\mathbf{X}\mathbf{S}_\omega^-\mathbf{X}'$. On the other hand,

$$\begin{aligned}
\underline{\lambda}'\underline{\widehat{\beta}}_\omega &= \underline{\lambda}'\mathbf{S}_\omega^-\mathbf{X}'\mathbf{W}^{-1}\underline{Y} \\
&= \underline{\lambda}'\left[\mathbf{S}^- - \frac{(\omega-1)}{1+(\omega-1)v_{ii}}\underline{\lambda}'\mathbf{S}^-\underline{x}_i\underline{x}_i'\mathbf{S}^-\right]\left[\mathbf{X}'\underline{Y} - (\omega-1)\underline{x}_i Y_i\right] \\
&= \underline{\lambda}'\underline{\widehat{\beta}} - \frac{\omega-1}{1+(\omega-1)v_{ii}}\underline{\lambda}'\mathbf{S}^-\underline{x}_i\left[Y_i - \underline{x}_i'\underline{\widehat{\beta}}\right]
\end{aligned}$$

□

THEOREM 2. *If $\underline{\lambda}'\underline{\beta}$ is an e.l.f. in GLM, its BLUE in GLM(ω,i) is $\underline{\lambda}'\underline{\widehat{\beta}}_\omega$, for any $\underline{\widehat{\beta}}_\omega$ GLS estimators of $\underline{\beta}$.*

PROOF. $\underline{\lambda}'\underline{\widehat{\beta}}_\omega$ is unbiased

$$E\left[\underline{\lambda}'\underline{\widehat{\beta}}_\omega\right] = \underline{\lambda}'\mathbf{S}_\omega^-\mathbf{X}'\mathbf{W}^{-1}E\left[\underline{Y}\right] = \underline{\lambda}'\mathbf{S}_\omega^-\mathbf{S}_\omega\underline{\beta} = \underline{\lambda}'\underline{\beta}$$

On the other hand, if $\mathbf{a} \in \mathcal{R}^p$, such that $\mathbf{a}'\underline{Y}$ is unbiased of $\underline{\lambda}'\underline{\beta}$, that is to say $\mathbf{a}'\mathbf{X} = \underline{\lambda}'$, then

$$\begin{aligned}
Cov\left[(\mathbf{a}'\underline{Y} - \underline{\lambda}'\underline{\widehat{\beta}}_\omega), \underline{\lambda}'\underline{\widehat{\beta}}_\omega\right] &= \left[\mathbf{a}' - \underline{\lambda}'\mathbf{S}_\omega^-\mathbf{X}'\mathbf{W}^{-1}\right] Var(\underline{Y})\left[\underline{\lambda}'\mathbf{S}_\omega^-\mathbf{X}'\mathbf{W}^{-1}\right]' \\
&= \sigma^2\left[\mathbf{a}' - \underline{\lambda}'\mathbf{S}_\omega^-\mathbf{X}'\mathbf{W}^{-1}\right]\mathbf{X}\mathbf{S}_\omega^-\underline{\lambda} \\
&= \sigma^2\left[\mathbf{a}'\mathbf{X}\mathbf{S}_\omega^-\underline{\lambda} - \underline{\lambda}'\mathbf{S}_\omega^-\underline{\lambda}\right] \\
&= 0
\end{aligned}$$

Therefore,

$$Var\left[\mathbf{a}'\underline{Y}\right] = Var\left[\mathbf{a}'\underline{Y} - \underline{\lambda}'\underline{\widehat{\beta}}_\omega\right] + Var\left[\underline{\lambda}'\underline{\widehat{\beta}}_\omega\right] \geq Var\left[\underline{\lambda}'\underline{\widehat{\beta}}_\omega\right] \quad \forall \mathbf{a} \in \mathcal{R}^p$$

Equality is attained when $Var\left[\mathbf{a}'\underline{Y} - \underline{\lambda}'\underline{\widehat{\beta}}_\omega\right] = 0$, that is to say when

$$\mathbf{a}'\underline{Y} = \underline{\lambda}'\underline{\widehat{\beta}}_\omega$$

□

THEOREM 3. *In the model GLM(ω, i),*

$$\widehat{\sigma}^2_\omega = \widehat{\sigma}^2 + \frac{1}{n-r} \quad \frac{\omega - 1}{1 + (\omega - 1)v_{ii}} e_i^2 \quad \forall \omega > 0 \qquad \ldots (63)$$

and this estimation is independent of the choice of $\widehat{\underline{\beta}}_\omega$

PROOF. The vector of residuals for GLM(ω, i) is

$$\underline{Y} - \mathbf{X}\widehat{\underline{\beta}}_\omega = \underline{e} - \mathbf{X}\left[\widehat{\underline{\beta}} - \widehat{\underline{\beta}}_\omega\right] = \underline{e} - \frac{\omega - 1}{1 + (\omega - 1)v_{ii}} \mathbf{X}\mathbf{S}^- \underline{x}_i e_i$$

where $\underline{e}$ is the vector of residuals for GLM. Then

$$\begin{aligned}
&\left[\underline{Y} - \mathbf{X}\widehat{\underline{\beta}}_\omega\right]' \mathbf{W}^{-1} \left[\underline{Y} - \mathbf{X}\widehat{\underline{\beta}}_\omega\right] \\
&\quad = \underline{e}' \mathbf{W}^{-1} \underline{e} - \frac{2(\omega - 1)}{1 + (\omega - 1)v_{ii}} \underline{e}' \mathbf{W}^{-1} \mathbf{X}\mathbf{S}^- \underline{x}_i e_i \\
&\quad + \left[\frac{\omega - 1}{1 + (\omega - 1)v_{ii}}\right]^2 e_i^2 \underline{x}_i' \mathbf{S}^- \mathbf{X}' \mathbf{W}^{-1} \mathbf{X}\mathbf{S}^- \underline{x}_i \\
&\quad = \underline{e}'\underline{e} + (\omega - 1)e_i^2 - \frac{2(\omega - 1)}{1 + (\omega - 1)v_{ii}} (\omega - 1) e_i^2 v_{ii} \\
&\quad + \left[\frac{\omega - 1}{1 + (\omega - 1)v_{ii}}\right]^2 e_i^2 \underline{x}_i' \mathbf{S}^- \left[\mathbf{S} + (\omega - 1)\underline{x}_i \underline{x}_i'\right] \mathbf{S}^- \underline{x}_i \\
&= \underline{e}'\underline{e} + \left[(\omega - 1) - \frac{2(\omega - 1)^2 v_{ii}}{1 + (\omega - 1)v_{ii}} + \frac{(\omega - 1)^2 v_{ii}}{1 + (\omega - 1)v_{ii}}\right] e_i^2 \\
&= \underline{e}'\underline{e} + \frac{\omega - 1}{1 + (\omega - 1)v_{ii}} e_i^2
\end{aligned}$$

By means of Theorem 2, it can be easily shown that this estimation is unique for any choice of $\widehat{\beta}$. □

References

ATKINSON, A.C. (1982). Regression Diagnostics. Transformations and Constructed Variables. *Jour. Royal Statist. Soc.,* Ser. B. **44,** No. 1, 1-36.

BARNETT, V. (1976). The ordering of multivariate data. *Jour. Royal Statist. Soc.,* Ser. A. **139,** Part. 3, 318-344.

BELSEY, D.A. KUH, E. AND WELSCH, R.E. (1980).*Regression Diagnostics: Identifying Influential Data an Sources of Collinearity.* New York: John Wiley & Sons.

BILLOR, N. AND LOYNES, R.M. (1993) Local influence: A new approach.*Commun. Statist.-Theory Meth.,***22(6),** 1595-1611.

COOK, R.D. (1987). Influence assessment. *Journal of Applied Statistics,* **14,** no. 2, 117-132.

– – –– (1986). Assessment of Local Influence (with discussion). *Jour. Royal Statist. Soc.,* Ser. B., **48,** 133-169.

COOK, R.D, AND WEISBERG, S. (1982). *Residuals and Influence in Regression.*New York: Chapman and Hall.

EFRON, B. AND STEIN, C. (1981). The jackknife estimate of variance. *Ann. Statist.,* **9,** no. 3, 586-596.

KSHIRSAGAR, A.M. (1983). *A Course in Linear Models.* New York: Marcel Dekker, Inc.

MUNOZ-PICHARDO, J.M.; MUNOZ-GARCIA, J.; MORENO- REBOLLO, J.L. AND PINO-MEJIAS, R. (1995). A new apporach to the influence analysis on linear models. *Sankhyā,* ser. A. **57**, Pt. 3.

PRINGLE, R.M. AND RAYNER, A.A. (1971).*Generalized Inverse Matrices, with Applications to Statistics.* London. Charles Griffin an Co.

WOODLEY, W.L.; SIMPSON, J.; BIONDINI, R. AND BERKELEY, J. (1977) Rainfall results 1970-75: Florida Area Cumulus Experiment. Science, 195 (February 25), 735-42.

J.M. MUÑOZ-PICHARDO, J. MUÑOZ-GARCIA, J.M. FERNÁNDEZ-PONCE, F. LÓPEZ-BLÁZQUEZ.
DEPARTMENTO DE ESTADISTICA E I.O.
UNIVERSIDAD DE SEVILLA
C/. TARFIA S/N.
41008-SEVILLA
SPAIN.
e-mail :juanm@cica.es; joaquinm@cica.es; ferpon@cica.es; lopez@cica.es