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A note on the homotopical characterization of $\mathbb{R}^n$

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Abstract

This note gives conditions which assure that the one-point compactification of an open manifold is a manifold. This result is used to show that the homotopical characterization of $\mathbb{R}^n$ ($n \geq 4$) can be derived from the Poincaré Conjecture.

The first example of a contractible open 3-manifold which is not the Euclidean space is due to J.H.C. Whitehead (see [20] for examples in dimensions ≥ 3). Therefore, further homotopical conditions are needed in order to characterize Euclidean spaces among contractible open manifolds.

The following homotopical characterization of Euclidean spaces is due to L. Siebenmann ($n \geq 5$) and M.H. Freedman ($n = 4$):

Theorem A ([5], [16])

Let X be a contractible open topological n-manifold ($n \geq 4$). Then X is 1-LC at ∞ if and only if X is homeomorphic to $\mathbb{R}^n$.

We recall that a neighbourhood of infinity (∞) in a Hausdorff space X is a subset N such that $\overline{X - N}$ is compact. In this note we shall deal with locally compact separable metric spaces. For these spaces we may find a decreasing sequence $\{U_i\}$ of neighbourhoods of ∞ such that $\overline{X - U_i} \subseteq \mathrm{int}(\overline{X - U_{i+1}})$. Such a sequence is called a system of ∞-neighbourhoods.

The space X is said to be 1-LC at ∞ if for any neighbourhood U of ∞ there exists a smaller neighbourhood V of ∞ such that any loop in V is nullhomotopic in U.

It is interesting to point out that the Poincaré Conjecture can be derived from the characterization of $\mathbb{R}^n$ by using basic facts from Algebraic Topology. That is, Theorem A yields

Theorem A' ([13], [5])

Let Σ^n be a closed topological n-manifold ($n \geq 4$) homotopically equivalent to S^n. Then Σ^n is homeomorphic to S^n.

Indeed, given $p \in \Sigma$ the open manifold $\Sigma - \{p\}$ is simply connected by Van Kampen's Theorem, and $\tilde{H}_*(\Sigma - \{p\})$ is trivial by Mayer-Victoris arguments. Then $\Sigma - \{p\}$ is contractible by the Whitehead Theorem. Since $\Sigma - \{p\}$ is 1-LC at ∞, $\Sigma - \{p\} \sim \mathbb{R}^n$ by Theorem A, and so $\Sigma^n \sim S^n$.

In this note we show that the converse A') $\Rightarrow$ A) also holds. In order to prove it, we shall give sufficient conditions which assure that the one-point compactification X^+ of an open manifold X is a manifold. We shall prove

Lemma B

Let X be an open homologically trivial n-manifold such that pro$-\pi_1(X)$ is semistable. Then X^+ is an n-homology sphere. Furthermore, if X is 1-LC at ∞ (i.e. pro$-\pi_1(X)$ is trivial), X^+ is a topological manifold. If in addition, $\pi_1(X) = 0$ (i.e. X is contractible) X^+ has the same homotopy type as S^n.

Remark 1. (i) A first version of Theorem A had been previously proven by E. Luft ([10]) for a simply connected at ∞ open n-manifold X ($n \geq 5$). That is, X admits a system of simply connected ∞-neighbourhoods. In dimensions ≥ 5, Theorem A' was already achieved as a corollary of Luft's Theorem in [10, §4]. The Poincaré Conjecture for topological n-manifolds ($n \geq 5$) had been originally proven by M. Newman ([13]).

(ii) The first antecedent of Luft's Theorem and Theorem A is the Stallings Theorem for simply connected at ∞ open PL-manifolds of dimension ≥ 5 ([7, I.1]). The Stallings Theorem gave a proof of the Poincaré Conjecture for PL-manifolds different to the original proof due to S. Smale ([18]). See [7, I.1.4].

(iii) In dimensions ≥ 5, the hypotheses of Theorem A can be actually reduced to the hypotheses of the Stallings Theorem. Indeed, the Kirby-Siebenmann obstruction $o(X) \in H^4(X;\mathbb{Z}_2)$ vanishes for any contractible open topological n-manifold X ($n \geq 5$), therefore X always admits a structure of PL-manifold (see [9]). On the other hand, by [17, 3.10] it is known that 1-LC at ∞ condition is actually equivalent to 1-connectedness at ∞ for PL-manifolds of dimension ≥ 5. Similarly for Luft's Theorem.

Before proving Lemma B we give some notations and results.

If X is a space with one end (i.e. X has a system of ∞-neighbourhoods $\{U_i\}$ with U_i connected), we consider the inverse sequence (pro-group)

$$\text{pro}-\pi_k(X) = \left\{\pi_k(X) \leftarrow \pi_k(U_1) \leftarrow \pi_k(U_2) \leftarrow \cdots\right\}$$

where the bonding morphisms are induced by inclusions and changing of base points. In a similar way we can consider the Abelian pro-group

$$\text{pro}-H_k(X) = \left\{H_k(X) \leftarrow H_k(U_1) \leftarrow H_k(U_2) \leftarrow \cdots\right\}$$

They are called the k-th homotopy and homology pro-group of X, respectively.

In the category $\text{pro}-\mathcal{G}r$ of pro-groups and pro-morphisms we say that a pro-group $\underline{X}$ is semistable (stable) if $\underline{X}$ is isomorphic in $\text{pro}-\mathcal{G}r$ to a pro-group $\underline{Y}$ whose bonding morphisms are onto (isomorphisms). We refer the reader to [11] for details on the category $\text{pro}-\mathcal{G}r$.

In the proof of Lemma B we shall also use the groups $H_n^\infty(X)$ of locally finite cycles of X. Namely, the n-th homology of the complex $C_*^\infty = \{C_n^\infty(X)\}$ defined by the formal sums $\sum n_\sigma \sigma$, where σ is a singular simplex in X and n_σ is an integer such that the set $\{n_\sigma; \text{Im}(\sigma) \cap K \neq \emptyset, n_\sigma \neq 0\}$ is finite for any compact subset $K \subseteq X$.

If $C_*^e(X) = C_*^\infty(X)/C_*(X)$, where $C_*(X)$ is the singular chain complex of X, we have the long exact sequence

$$\cdots \to H_n(X) \to H_n^\infty(X) \to H_n^e(X) \to H_{n-1}(X) \to \cdots \tag{1}$$

The groups $H_*^e(X)$ are called the homology groups of X at ∞, and they are related to $\text{pro}-H_*(X)$ by the following exact sequence (see [6, 3.5.13])

$$0 \longrightarrow \varprojlim{}^1\left(\text{pro}-H_{n+1}(X)\right) \longrightarrow H_{n+1}^e(X) \longrightarrow \varprojlim\left(\text{pro}-H_n(X)\right) \longrightarrow 0 \tag{2}$$

The crucial point in the proof of Lemma B will be the following result ([1, 1.4] for $n \geq 5$ and [14, 2.5.1] for $n = 4$).

Theorem C

Let Y be a generalized n-manifold ($n \geq 4$) whose singular set $S(Y)$ is 1-LCC embedded in Y and $\dim(S(Y)) \leq 0$. Then Y is a topological manifold.

We recall that a locally compact separable metric space Y is said to be a generalized n-manifold if Y is a finite-dimensional ANR and if, for each $y \in Y$, $H_k(Y, Y - \{y\}; \mathbb{Z})$ is isomorphic to $H_k(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z})$ for all k. An n-homology sphere is a generalized manifold Y such that $H_*(Y;\mathbb{Z}) \sim H_*(S^n;\mathbb{Z})$.

We say that Y is 1-LCC at y provided that each neighbourhood U of y contains another neighbourhood V such that any loop in $V - \{y\}$ is nullhomotopic in $U - \{y\}$. Therefore Y is 1-LC at ∞ if and only if the one-point compactification Y^+ is 1-LCC at ∞.

Proof of Lemma B. Firstly, notice that X has one end by [7, 1.1.7].
a) X^+ is an ANR. Indeed, by using [4, 4.4], it is enough to check the stability of $\text{pro}-H_q(X)$ for all $q \geq 0$. Notice that the semistability of $\text{pro}-\pi_1(X)$ implies the nearly 1-movability condition in [4, 4.4].

We start with the Poincaré Duality isomorphism $H_q^\infty(X) \sim H^{n-q}(X)$ (see [11, III.11.2]). Thus, $H_n^\infty(X) \cong \mathbb{Z}$ and $H_q^\infty(X)$ is trivial otherwise. Now the exact sequence (1) yields $H_n^e(X) \cong \mathbb{Z}$ and $H_q^e(X) = 0$ if $q \neq n$. Using (2) we get $\varprojlim(\text{pro}-H_{n-1}(X)) \cong \mathbb{Z}$ and $\varprojlim^1(\text{pro}-H_q(X))$ and $\varprojlim(\text{pro}-H_m(X))$ are trivial if $m, q \geq 0$, $m \neq n-1$. We now can use [11, Th. 12, p. 175] and [11, Corol. 8, p. 177] to get

$$\text{pro}-H_{n-1}(X) \cong \mathbb{Z} \quad \text{and} \quad \text{pro}-H_q(X) \text{ trivial otherwise.} \tag{3}$$

b) X^+ is a generalized manifold. Using *a)*, it only remains to show

$$H_*(X^+, X^+ - \{\infty\}) \simeq H_*(\mathbb{R}^n, \mathbb{R}^n - \{0\}). \tag{4}$$

As X^+ is already an ANR, X^+ is locally contractible at $\infty \in X^+$ by [11, Th. 7, p. 40]. Therefore, if $\{U_i\}$ is an ∞-neighbourhood system of X, we may assume that $U_i' = U_i \cup \{\infty\}$ is contractible in U_{i-1}'. So, the pro-group $\{H_q(U_i')\}$ is trivial for all q, and the levelwise exact sequence in $\text{pro}-\mathcal{G}r$,

$$\longrightarrow \{H_q(U_i')\} \longrightarrow \{H_q(U_i', U_i)\} \longrightarrow \text{pro}-H_{q-1}(X) \longrightarrow \{H_{q-1}(U_i')\}$$

yields an isomorphism $\{H_q(U_i', U_i)\} \cong \text{pro}-H_{q-1}(X)$ for all q. Since the first pro-group is isomorphic to the constant pro-group $H_q(X^+, X^+ - \{\infty\})$ by excision, (4) follows from (3).

By *b)*, if X is 1-LC at ∞, then X^+ is a topological manifold by Theorem C.

c) X^+ is a homology sphere. In fact, as X is G-orientable for any Abelian group G, we have the isomorphisms

$$H_{n-q}(X;G) \cong H_c^q(X;G) \cong H^q(X^+;G) \tag{5}$$

where the former is the Poincaré Duality isomorphism and the latter is given in [8, 27.3] since $\check{H}^q(\{\infty\};G) = 0$ for each $q \neq 0$. Here $\check{H}^q$ denotes the Čech cohomology.

As a consequence of (5) we get $H^q(X^+;G) \sim H^q(S^n;G)$ for any Abelian group and any q. As an easy application of the Universal Coefficient Theorem (see [11, I.4.17]) we obtain that X^+ is a homology sphere.

Assume $\pi_1(X) = 0$. As X^+ is locally contractible at the point $\infty \in X^+$, there is a neighbourhood V of ∞ in X^+ such that $\pi_1(V) \to \pi_1(X^+)$ is trivial. Now $\pi_1(X^+) = 0$ as a consequence of the Van Kampen Theorem, and the homological Whitehead Theorem shows that X^+ is homotopically equivalent to S^n. □

Remark 2. If D^n is a Davis manifold ($n \geq 4$) (see [3]), it is known that pro$-\pi_1(D)$ is semistable but D is not 1-LC at infinity. Therefore, D^+ is a generalized homology sphere with $\{\infty\}$ as singular set.

Remark 3. *(i)* Although in dimension 3 the Poincaré Conjecture is still open, the statements of Theorem A and Theorem A' are equivalent in this dimension (see [19, Cor. 2]). In addition, D. Repovs has informed us that he has independently proven Lemma B in the case of 3-manifolds (see [15, Th. 3]).

(ii) The Kirby-Siebenmann obstruction is a basic tool in the proof of Theorem 1.4 in [1]. There is a proof of this result which does not use the Kirby-Siebenmann obstruction (see [2, VII.40.2]).

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