Collect. Math. **49**, 2–3 (1998), 239–256

Combinatorics of syzygies for semigroup algebras

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Dedicated to the memory of Professor Fernando Serrano

Abstract

We describe how the graded minimal resolution of certain semigroup algebras is related to the combinatorics of some simplicial complexes. We obtain characterizations of the Cohen-Macaulay and Gorenstein conditions. The Cohen-Macaulay type is computed from combinatorics. As an application, we compute explicitly the graded minimal resolution of monomial both affine and simplicial projective surfaces.

Introduction

The motivation for this paper is to study the relationships between the generators of the ideals defining monomial varieties, i.e., affine varieties parameterized by monomial equations. More precisely, one wants to study the minimal resolution of the

* Partially supported by DGICYT PB94-1435 and partially supported by Junta de Andalucía Ayuda a Grupos 1144.

† Supported by DGICYT PB94 1111-C02-01.

algebra of a finitely generated semigroup viewed as a module over a polynomial ring. The module structure corresponds to the choice of a generator system for the semigroup.

It is well known that such defining ideals (also called toric ideals, see for instance [16]) are generated by binomials and that one can derive some combinatorial methods to construct minimal systems of binomial generators ([10], [9], [2], [13], [11], [6], [7], [3], [14], [16]). One can see (see [5] and 2.1 below), how the graded minimal resolution of the semigroup algebra is related to the combinatorics of some simplicial complexes associated to the semigroup elements for commutative cancellative finitely generated semigroups S with $S \cap (-S) = \{0\}$. In particular, when the algebra is Cohen-Macaulay (see [15], [17] and 2.2 below for characterization of that condition) the Cohen-Macaulay type can be computed from combinatorics (2.2) and the Gorenstein case can be characterized by a symmetry property on S (2.3) similar to the well known case for numerical semigroups due to Kunz ([12]). Methods to compute the homology of these complexes are developed in [4] using further combinatorics.

In section 3, we give, in terms of the homology of above simplicial complexes, an explicit construction of the minimal graded resolution of the algebra of such a semigroup. Thus, one concludes that, in practice, combinatorics can be applied to construct minimal systems of generators not only for the ideal but also for modules of higher order syzygies. Such a construction was first considered and applied in [1].

In section 4, we apply the above construction to compute explicitly the graded minimal resolution of monomial both affine and simplicial projective surfaces.

1. The minimal resolution

Let S denote a commutative semigroup with a zero element $0 \in S$. The associated abelian group is a pair $(G(S), i)$ where $G(S)$ is an abelian group and $i : S \to G(S)$ a semigroup homomorphism such that, for any other such pair (H, j) one has a unique group homomorphism $\varphi : G(S) \to H$ such that $\varphi \circ i = j$. The associated abelian group $G(S)$ exists and it is unique disregarding isomorphism, and is a finitely generated group if S is a finitely generated semigroup. The map i is injective if and only if S is cancellative, i.e., if $m + n = m + n'$, $m, n, n' \in S$, implies $n = n'$. Equivalently, S is cancellative if and only if it is isomorphic to an additive subsemigroup of some abelian group.

For the purpose of this paper, we will say that a semigroup S is combinatorially finite (c.f.) if for any $m \in S$ there are only a finite number of expressions of type $m = m_1 + \cdots + m_q$ with $q \in \mathbb{N}$ and $m_i \in S - \{0\}$. Proposition 1.1 in [3] characterizes the property c.f. for finitely generated cancellative semigroups, and provides the Nakayama lemma for S-graded modules (Proposition 1.4 in [3]).

Assume S is cancellative and combinatorially finite, and let $A = \oplus_{m\in S} A_m$ be a commutative ring graded over S. Let $P = \oplus_{m\in S-\{0\}} A_m$ be its irrelevant ideal. Thus, if $N = \oplus_{m\in S} N_m$ is a graded A-module and G is a subset of homogeneous elements in N, then G is a system of generators of the module N if and only if the classes module PN of the elements of G are a system of generators of the $A_0 = A/P$-module N/NP. In particular, when A_0 is a field, the minimal sets of homogeneous generators of N all have the same cardinality and are exactly those subsets G giving rise to bases of the vector space N/PN.

From now on, S denote a semigroup that is: finitely generated, combinatorially finite, cancellative, and commutative (f.g.c.f.c.c. in short). Let us fix a system of generators $n_1, ..., n_r$ for S with $n_i \in S - \{0\}$. Let also fix a commutative field k.

Associated with this situation one has the algebra of S, i.e. the vector space

$$R = \bigoplus_{m\in S} R_m, \quad R_m := k\{m\}\,,$$

endowed with a multiplication which is k-linear and such that $\{m\}\cdot\{n\} := \{m + n\}$ for the symbols $\{m\}$, $\{n\}$ of $m, n \in S$. The choice of generators provides an S-graduation over the polynomial algebra in r indeterminates $A := k[X_1, ..., X_r]$, assigning the weight n_i to variable X_i. That is,

$$A = \bigoplus_{m\in S} A_m,$$

where A_m is the vector subspace of A generated by all the monomials $X_1^{l_1} \cdots X_r^{l_r}$ with $\sum_{i=1}^{r} l_i n_i = m$. The condition c.f. means precisely that the vector spaces A_m are finitely dimensional. One has the surjective S-graded k-algebra homomorphism

$$\varphi_0 : A \to R$$

which takes X_i to the symbol $\{n_i\}$, giving on R an S-graded A-module.

By the S-graded Nakayama lemma and using recurrence, one constructs S-graded k-algebra homomorphisms

$$\varphi_{j+1} : A^{b_{j+1}} \to A^{b_j}$$

corresponding to a choice of a minimal set of homogeneous generators for the module $N_j := \ker(\varphi_j)$ ($b_0 = 1$, N_0 is the ideal $\ker(\varphi_0)$ which will be denoted by I). Here, if m_i is the degree of the i-th generator for N_j, then, the grading on $A^{b_{j+1}}$ has as homogeneous elements the b_{j+1}-tuples whose entries are homogeneous elements in A the i-th one being of degree $m - m_i$ for each i.

Thus, one gets a minimal free S-graded resolution for the A-module R of type

$$\cdots \to A^{b_{j+1}} \overset{\varphi_{j+1}}{\to} A^{b_j} \to \cdots \to A^{b_2} \overset{\varphi_2}{\to} A^{b_1} \overset{\varphi_1}{\to} A \overset{\varphi_0}{\to} R \to 0$$

where $b_{j+1} := \sum_{m \in S} \dim_k V_j(m)$ with $V_j(m) := (N_j)_m/(PN_j)_m$.

The above summation is finite by the noetherian property on A. The dimension of the vector space $V_j(m)$ can be interpreted as the number of generators of degree m in a minimal system of generators of j-th syzygy module N_j.

The Auslander-Buchbaum theorem guarantees that $b_j = 0$ for $j > p = r - \text{depth}_A R$ and $b_p \neq 0$. The depth of R is bounded by its dimension as k-algebra, this dimension being nothing but the rank of the abelian group $G(S)$. The bound is reached exactly when R is Cohen-Macaulay. On the other hand, if $S \neq \{0\}$ the condition c.f. yields a $\text{depth}_A R$ of at least 1. We will assume $S \neq \{0\}$ throughout the rest of the paper.

2. Simplicial complexes and Koszul homology

Let S be a f.g.c.f.c.c. semigroup and $n_1, ..., n_r$ a system of generators of S with $n_i \in S - \{0\}$. Set $\Lambda := \{1, ..., r\}$ and for each subset $F \subset \Lambda$, $n_F := \sum_{i \in F} n_i$, ($n_\emptyset = 0$). For each $m \in S$ one has an abstract simplicial complex (subcomplex of the simplex of parts of Λ) given by

$$\Delta_m := \left\{ F \subset \Lambda \mid m - n_F \in S \right\}.$$

We will consider the reduced homology $\widetilde{H}.(\Delta)$ of the complexes Δ of $\mathcal{P}(\Lambda)$ with values in a field k (fixed from now on for the rest of the paper). To fix notations, write dim F =card $F-1$ for a face F and choose the orientation on each face F of Δ_m taken the elements of F in increasing order. Then, $\widetilde{C_j}(\Delta_m)$ is the k-vector space generated (freely) by the j-dimensional faces of Δ_m and, $\widetilde{H}_j(\Delta_m) \cong (\ker \delta_j)/(\text{Im } \delta_{j+1})$ where

$$\delta_j(F) = \sum_{\substack{F' \in \Delta_m \\ \dim F' = j-1}} \varepsilon_{F'F} \cdot F'$$

for a j-dimensional face F and $\varepsilon_{F'F} = 0, 1, -1$ are the coefficients given by the above choice of orientations. Notice that $\varepsilon_{F'F} = 0$ if and only if $F' \not\subset F$.

The choice of the generators in S gives us the sequences $X_1, ..., X_r$ of homogeneous elements of the S-graded algebra A, as well as the sequence of symbols $\{n_1\}, ..., \{n_r\}$ which are also homogeneous elements of R. Since $\underline{X} := (X_1, ..., X_r)$ is a regular sequence in A, then the Koszul complex for it is the exact sequence

$$0 \to \bigwedge^{r} A^r \to \cdots \to \bigwedge^{j+1} A^r \overset{\lambda_j}{\to} \bigwedge^{j} A^r \overset{\lambda_{j-1}}{\to} \cdots \to A^r \overset{\lambda_0}{\to} A \to k \to 0$$

which is S-graded of degree 0 if one gives to the element $e_{i_1} \wedge \cdots \wedge e_{i_j}$ ($\{e_i\}$ the standard basis of A^r) the degree $n_{i_1} + \cdots + n_{i_j}$.

Notice that, according to the choice of orientations made about the subsimplices of parts of Λ, if for $F = \{i_0 < \cdots < i_j\} \subset \Lambda$ one writes $e_F := e_{i_0} \wedge \cdots \wedge e_{i_j}$, then λ_j is given by

$$\lambda_j(e_F) = \sum_{\substack{F' \in \Delta_m \\ \dim F' = j-1}} \varepsilon_{F'F} \, \frac{X_F}{X_{F'}} \, e_{F'}$$

where X_F stands for $\prod_{i \in F} X_i$ ($X_\emptyset = 1$).

The Koszul homology of symbols $\{\underline{n}\} := (\{n_1\}, ..., \{n_r\})$ in R is related to the homology of Δ_m. Notice that, since $\{n_i\}$ is homogeneous, again the Koszul complex is S-graded and therefore one has a graded decomposition for the homology

$$K_j\big(\{\underline{n}\}, R\big) = \bigoplus_{m \in S} K_j\big(\{\underline{n}\}, R\big)_m$$

For each degree m, the definitions of Δ_m and the Koszul homology gives isomorphisms

$$K_j\big(\{\underline{n}\}, R\big)_m \cong \widetilde{H_j}\big(\Delta_m\big)$$

Theorem 2.1

Let S be a f.g.c.f.c.c. semigroup. Fix a system of generators $n_1, ..., n_r$ for S in $S - \{0\}$ and a commutative field k and consider the minimal S-graded resolution of R and the complexes Δ_m associated to the choice of generators. Then one has k-vector space isomorphisms

$$\widetilde{H}_j(\Delta_m) \cong V_j(m)$$

for every $m \in S$ and $j \geq 0$.

Proof. Both $R = A/I$ and $k = A/P$ are S-graded A-modules, so one has S-graded A-module isomorphisms $\mathrm{Tor}^j_A(R,k) \cong \mathrm{Tor}^j_A(k,R)$ for $j \geq 0$. In particular, degree by degree one has k-vector space isomorphisms $\mathrm{Tor}^j_A(R,k)_m \cong \mathrm{Tor}^j_A(k,R)_m$ for every $m \in S$ and $j \geq 0$.

Now, to compute $\mathrm{Tor}^j_A(R,k)_m$, one can tensor the minimal resolution of R by k. This yields $\mathrm{Tor}^j_A(R,k)_m \cong V_j(m)$. In the same way, to compute $\mathrm{Tor}^j_A(k,R)_m$, one can tensor the Koszul complex of $\underline{X}$ by R. Since the image of X_i in R is nothing but $\{n_i\}$, one gets the Koszul complex for $\{\underline{n}\}$ in R and hence, one has $\mathrm{Tor}^j_A(k,R)_m \cong \widetilde{H}_j(\Delta_m)$. Thus the isomorphism $\widetilde{H}_j(\Delta_m) \cong V_j(m)$ as required. □

Corollary 2.2

Given the same assumptions as Theorem 2.1, the depth of the semigroup algebra R is equal to $r-p$ where p is the least integer such that $\widetilde{H}_p(\Delta_m) = 0$ for every $m \in S$. In particular, R is Cohen-Macaulay if and only if $\widetilde{H}_{r-s}(\Delta_m) = 0$ for every $m \in S$, where $s =$ rank $G(S)$. If R is Cohen-Macaulay then the Cohen-Macaulay type τ_R of R is given by

$$\tau_R = \sum_{m \in S} \dim_k \widetilde{H}_{r-s-1}(\Delta_m)\,.$$

Furthermore, R is Gorenstein if and only if R is Cohen-Macaulay and if $\widetilde{H}_{r-s-1}(\Delta_m) \neq 0$ exactly for one m for which

$$\dim_k \widetilde{H}_{r-s-1}(\Delta_m) = 1\,.$$

Proof. The above follows from the Auslander-Buchbaum theorem. The formula for τ_R follows from the fact that $\tau_R = b_{r-s}$ in the Cohen-Macaulay case. □

The following result reflects in terms of combinatorial symmetry the Gorenstein condition on the ring R. This can be seen as a generalization of the well known characterization of Gorensteiness for numerical semigroups due to Kunz [12].

To state the result, notice that Δ_m makes sense for m in $G(S)$. It is clear that for $m \in G(S) - S$, Δ_m is the empty simplicial complex and that therefore $\widetilde{H}_j(\Delta_m) = 0$ for such an m and $j = -1, 0, 1, 2$. Also, notice that Δ_0 is the only complex among the Δ_m's with the property that $\widetilde{H}_{-1}(\Delta_m) \neq 0$ (in fact it is a one dimensional space). Finally, let us set $H_j(\Delta_m) = 0$ for $j \in \mathbb{Z}$, $j < -1$ and $m \in G(S)$

Corollary 2.3

Given the same assumptions a 2.1, assume that the semigroup algebra R is Gorenstein and let $n \in S$ be the element such that $\widetilde{H}_{r-s-1}(\Delta_n) \neq 0$. Then for any pair of elements $m, m' \in G(S)$ with $m + m' = n$ and $j \in \mathbb{Z}$ one has

$$\widetilde{H}_j(\Delta_m) \cong \widetilde{H}_{r-s-j}(\Delta_{m'}).$$

Proof. This follows from 2.1 and the symmetry of the graded resolution in the Gorenstein case. □

Remarks 2.4

(i) If for some $n \in S$ one has the isomorphisms in Corollary 2.3, then R is Gorenstein. In fact, by the symmetry in 2.3 one has $\widetilde{H}_j(\Delta_m) = 0$ for $m \in S$ and $j > r-s$, so R is Cohen-Macaulay. Now, since $\widetilde{H}_{-1}(\Delta_0) \cong k$ and $\widetilde{H}_{-1}(\Delta_m) = 0$ for $m \neq 0$, it follows from the symmetry that $\widetilde{H}_{r-s-1}(\Delta_m) = 0$ for $m \neq n$ and $\widetilde{H}_{r-s-1}(\Delta_m) \cong k$, hence R is Gorenstein.

(ii) If S is a numerical semigroup, i.e. a subsemigroup of $\mathbb{N}$ with $\mathbb{N} - S$ finite, then the isomorphisms in 2.3 are an equivalent condition to the fact that S is a symmetric semigroup, i.e. satisfying the property that if $m, m' \in \mathbb{Z}$ and such that $m + m' = c - 1$, where c is the conductor of S, then either $m \in S$ or $m' \in S$ (see [5] for details). Thus, Corollary 2.3 for numerical semigroups is equivalent to the criteria by Kunz that R is Gorenstein if and only if S is symmetric.

(iii) Further characterizations of Cohen-Macaulayness in combinatorial ways can be found in [15], [17], [4].

3. Computing syzygies from combinatorics

In this section we will put form the isomorphisms in Theorem 2.1 in an explicit way. As a consequence, one can construct minimal systems of homogeneous generators for the successive syzygy modules only by taking the images of the base elements for the homology spaces $\widetilde{H}_j(\Delta_m)$. For $j = 0$ such isomorphisms are not difficult to construct and they were already used in [3] for computing minimal systems of generators for the ideal of the semigroup.

As in the above sections, let S be a f.g.c.f.c.c. semigroup. Fix a commutative field k and a system of generators $n_1, ..., n_r$ of S with $n_i \in S - \{0\}$. Keep all the notations in the above two sections and denote by $L_j = A^{b_j}$, $L'_t = \bigwedge^t A^r$, $L_{t,j}(m)$ the degree m component of $L'_t \otimes L_j$, $\varphi_{t,j} : L_{t,j}(m) \to L_{t,j-1}(m)$, $0 \leq j \leq r-1$, $-1 \leq t \leq r-1$ (resp. $\lambda_{t,j} : L_{t,j} \to L_{t-1,j}, -1 \leq j \leq r-1, 0 \leq t \leq r-1$), the linear map induced by φ_j (resp. λ_t). Notice that, as above $\varphi_{t,j}$ is nothing but the degree m part of the map $Id_{L'_t} \otimes \varphi_j$ (resp. $\lambda_t \otimes Id_{L_j}$). Hence, one has $\lambda_{t,j-1} \circ \varphi_{t,j} = \varphi_{t-1,j} \circ \lambda_{t,j}$,for any j, t with $0 \leq j, t \leq r-1$. Moreover, one has the exact sequences

$$0 \to L_{r-1,j}(m) \overset{\lambda_{r-1,j}}{\to} L_{r-2,j}(m) \to \cdots \to L_{0,j}(m) \overset{\lambda_{0,j}}{\to} L_{-1,j}(m)$$

for $0 \leq j \leq r-1$ and

$$0 \to L_{t,r-1}(m) \stackrel{\varphi_{t,r-1}}{\to} L_{t,r-2}(m) \to \cdots \to L_{t,0}(m) \stackrel{\varphi_{t,0}}{\to} L_{t,-1}(m)\,.$$

Now, consider the vector subspace $T_{t,j}(m)$ of $L_{t,j}(m)$ given by

$$\begin{aligned} T_{t,j}(m) = &\ \ker(\lambda_{t,j}) \cap \ker(\varphi_{t,j}) \ \text{ if } \ j \geq 0 \ \text{ and } \ t \geq 0\,, \\ T_{t,-1}(m) = &\ \ker(\lambda_{t,-1}) \cap \mathrm{Im}(\varphi_{t,0}) \ \text{ for } t \geq 0\,, \\ T_{-1,j}(m) = &\ \mathrm{Im}(\lambda_{0,j}) \cap \ker(\varphi_{-1,j}) \ \text{ for } j \geq 0\,. \end{aligned}$$

Notice that one has $\ker(\lambda_{t,j}) = \mathrm{Im}(\lambda_{t+1,j})$ and $\ker(\varphi_{t,j}) = \mathrm{Im}(\varphi_{t,j+1})$ for $j \geq 0$ and $t \geq 0$.

Lemma 3.1

(i) For $t \geq 0$, $T_{t,-1}(m)$ is canonically isomorphic to the zycle space $\widetilde{Z}_t(\Delta_m)$.
(ii) For $j \geq 0$, one has $T_{-1,j}(m) = (N_j)_m \subset \left(A^{b_j}\right)_m = L_{-1,j}(m)$.

Proof. Since $\varphi_{t,0}$ is surjective, one has $T_{t,-1}(m) = \ker(\lambda_{t,-1})$. Now, $T_{t,-1}(m)$ is canonically isomorphic to the order t chain vector space and $\lambda_{t,-1}$ corresponds to the boundary of the reduced homology for Δ_m. Thus, one has $T_{t,-1}(m) = \widetilde{Z}_t(\Delta_m)$ which shows (i).

Since $\ker(\varphi_{-1,j}) = (N_j)_m$, to prove (ii) it is enough to shows that $(N_j)_m$ is included in $\mathrm{Im}(\lambda_{0,j}) = (PA^{b_j})_m$. Let $\underline{a} = (a^1, ..., a^{b_j}) \in (N_j)_m$. Then $\underline{a}$ is a syzygy of the S-graded module N_{j-1} relative to a minimal system of homogeneous generators, so one should have $a^l \in P$ for each l. This shows $\underline{a} \in (PA^{b_j})_m$ as required. □

In the sequel we will use the following two basic correspondences

$$\begin{aligned} \sigma_{t,j} &= (\lambda_{t,j+1}) \circ (\varphi_{t,j+1})^{-1} \ \text{ for } \ j \geq -1,\ t \geq 0\,, \\ \gamma_{t,j} &= (\varphi_{t+1,j}) \circ (\lambda_{t+1,j})^{-1} \ \text{ for } \ j \geq 0,\ t \geq -1\,. \end{aligned}$$

Since $\varphi_{t,j+1}$ (resp. $\lambda_{t,j+1}$) is not necessarily an injective map, the correspondence $\sigma_{t,j}$ (resp. $\gamma_{t,j}$) is seen as a multivalued function from $\mathrm{Im}(\varphi_{t,j+1})$ (resp. $\mathrm{Im}(\lambda_{t+1,j})$) to $L_{t-1,j+1}(m)$ (resp. $L_{t+1,j-1}(m)$).

Lemma 3.2

(i) The correspondence $\sigma_{t,j}$ takes $T_{t,j}(m)$ to $T_{t-1,j+1}(m)$.
(ii) The correspondence $\gamma_{t,j}$ takes $T_{t,j}(m)$ to $T_{t+1,j-1}(m)$.

Proof. If $t \geq 0$, $j \geq -1$ one has $T_{t,j}(m) = \ker(\lambda_{t,j}) \cap \mathrm{Im}(\varphi_{t,j+1})$ and $T_{t-1,j+1}(m) = \mathrm{Im}(\lambda_{t,j+1}) \cap \ker(\varphi_{t-1,j+1})$. Thus, in particular, $\sigma_{t,j}$ is defined on $T_{t,j}(m)$. On the other hand, since each element of $T_{t,j}(m)$ is in $\ker(\lambda_{t,j})$, the image by $\lambda_{t,j+1}$ of any inverse image in $L_{t,j+1}(m)$ of such an element belongs to $\mathrm{Im}(\lambda_{t,j+1}) \cap \ker(\varphi_{t-1,j+1}) = T_{t-1,j+1}(m)$. This shows (i).

In the same way, if $j \geq 0$, $t \geq -1$ one has $T_{t,j}(m) = \mathrm{Im}(\lambda_{t+1,j}) \cap \ker(\varphi_{t,j})$ and $T_{t+1,j-1}(m) = \ker(\lambda_{t+1,j-1}) \cap \mathrm{Im}(\varphi_{t+1,j})$. Again, in particular $\gamma_{t,j}$ is defined on $T_{t,j}(m)$. On the other hand, since each element in $T_{t,j}(m)$ is in $\ker(\varphi_{t,j})$, the image by $\varphi_{t+1,j}$ of any inverse image of such an element in $L_{t+1,j}(m)$ belongs to $\ker(\lambda_{t+1,j-1}) \cap \mathrm{Im}(\varphi_{t+1,j})$. This shows (ii). □

We now come to the main construction of the section. If one fixes $j \geq 0$ and $m \in S$, then Lemmas 3.1 and 3.2 show that one has the following two correspondences

$$\sigma_j := (\sigma_{0,j-1}) \circ (\sigma_{1,j-2}) \circ \cdots \circ (\sigma_{j,-1}) : \widetilde{Z}_j(\Delta_m) \to (N_j)_m \,,$$
$$\gamma_j := (\gamma_{j-1,0}) \circ (\gamma_{j-2,1}) \circ \cdots \circ (\gamma_{-1,j}) : (N_j)_m \to \widetilde{Z}_j(\Delta_m) \,.$$

By composing σ_j (resp. γ_j) with the quotient maps $(N_j)_m \to (N_j)_m/(PN_j)_m = V_j(m)$ (resp. $\widetilde{Z}_j(\Delta_m) \to \widetilde{H}_j(\Delta_m)$) one gets two new correspondences

$$\overline{\sigma_j} : \widetilde{Z}_j(\Delta_m) \to V_j(m),$$
$$\overline{\gamma_j} : (N_j)_m \to \widetilde{H}_j(\Delta_m) \,.$$

A priori, the correspondences $\overline{\sigma_j}$ and $\overline{\gamma_j}$ are multivalued functions. The next theorem shows how they are, in fact, linear univalued functions inducing the isomorphisms in Theorem 2.1.

Theorem 3.3

Given the same assumptions and notations as above, for any $m \in S$ and $j \geq 0$ one has

(1) The correspondences $\overline{\sigma_j}$ and $\overline{\gamma_j}$ are well defined k-linear maps. The map $\overline{\sigma_j}$ takes boundaries for the reduced homology to zero. The map $\overline{\gamma_j}$ takes elements in $(PN_j)_m$ to zero. $\overline{\sigma_j}$ takes boundaries for the reduced homology to 0. $\overline{\gamma_j}$ takes elements in $(PN_j)_m$ to 0.
(2) The k-linear maps $\widetilde{H}_j(\Delta_m) \to V_j(m)$ and $V_j(m) \to \widetilde{H}_j(\Delta_m)$ induced, respectively, by $\overline{\sigma_j}$ and $\overline{\gamma_j}$ are inverse to one another.

Proof. Assume $t \geq 1$. Since $\lambda_{t-1,j+2} \circ \lambda_{t,j+2} = 0$, it is clear that the images of any element in $T_{t,j}(m)$ by the correspondence $\sigma_{t-1,j+1} \circ \sigma_{t,j}$ do not depend on the concrete choice of the inverse image by $\varphi_{t,j+1}$ made for such an element. By recurrence, for $j \geq 0$ the images by σ_j of a cycle $c \in \widetilde{Z}_j(\Delta_m)$ only depend on the choice of the inverse image by $\varphi_{0,j}$ of any of the elements in the set $(\sigma_{1,j-2}) \circ (\sigma_{2,j-3}) \circ \cdots \circ (\sigma_{j,-1})(c) \subset T_{0,j-1}(m)$. The difference of two such choices is in $\ker(\varphi_{0,j}) = \oplus_{i=1}^{r}(N_j)_{m-n_i}$. Hence, the image by $\lambda_{0,j}$ of this difference belongs to $(PN_j)_m$. This shows that the set $\overline{\sigma_j}(c)$ consists of only one element in $V_j(m)$. The linearity of $\overline{\sigma_j}$ follows from that of λ's and φ's taking into account that the choices of inverse images by the φ's can be done linearly.

The above argument applied to $\overline{\gamma_j}$ (changing the roles of φ's and λ's) shows that for any class in $V_j(m)$, its image by $\overline{\gamma_j}$ only depends on the choice of an inverse image by $\lambda_{j,0}$ of any element in the image set by $(\gamma_{j-2,1}) \circ \cdots \circ (\gamma_{-1,j})$ of the class. Again, the difference of two choices is in $\ker(\lambda_{j,0}) = \mathrm{Im}(\lambda_{j+1,0})$, and, hence, the image by $\varphi_{0,j}$ of the difference is a boundary for the reduced homology. One concludes that $\overline{\gamma_j}$ is well defined and k-linear as above.

Now, if c is a boundary (i.e., if $c = \lambda_{j+1,-1}(c')$ with $c' \in T_{j+1,-1}(m)$) then, since $\varphi_{0,j+1}$ is surjective ($\lambda_{j,0} \circ \lambda_{j+1,0} = 0$) it follows that $0 \in (\sigma_{j,-1})(c)$. By the definition of $\overline{\sigma_j}$ one has $\overline{\sigma_j}(c) = 0$. In the same way, if an element $\underline{b} \in (N_j)_m$ is in $(PN_j)_m$, then it has an inverse image by $\lambda_{0,j}$ which is in $\ker(\varphi_{j,0})$, so, as above, one has $0 \in (\gamma_{-1,j})(\underline{b})$ and hence $\overline{\gamma_j}(\underline{b}) = 0$. This shows (1). (2) follows from (1) and the definitions of $\overline{\sigma_j}$ and $\overline{\gamma_j}$. □

Corollary 3.4

Given the same assumptions as 3.3, one can successively construct minimal systems of homogeneous generators for the syzygy modules N_j (i.e., the minimal resolution of R) in terms of the reduced homology of the simplicial complexes Δ_m.

Proof. Take $j \geq 0$. By recurrence, assume that one has already constructed minimal systems of homogeneous generators for $N_0 = I, N_1, \ldots, N_{j-1}$. It follows that, in practice, for each homogeneous syzygy $\underline{b} \in N_i$, $0 \leq i \leq j-1$, one can find an inverse image of $\underline{b}$ by φ_{i+1}. Thus, taking into account that the λ's are explicit maps, one has that for each $m \in S$ an each $c \in \widetilde{H}_j(\Delta_m)$ one can find an element in $\sigma_j(c) \subset (N_j)_m$.

To find a minimal system of generators for N_j, it is enough to do the following. For any m over the set of elements in S with $\widetilde{H}_j(\Delta_m) \neq 0$:

1. Take a set $B_j(m)$ of cycles of $\widetilde{Z}_j(\Delta_m)$ inducing a basis in the homology.
2. Construct a syzygy in $\sigma_j(c)$ for c ranging over $B_j(m)$. □

Remark 3.5. The proof of 3.4 shows how, in practice, it is possible to take inverse images by the Φ's. If one knows how to find inverse images by the λ's, then one could also have images by the γ_j's.

Remark 3.6. 3.4 shows the existence of an algorithmic method, based on combinatorics, to write down the minimal resolution of R. Let us detail the first step and the recurrence one of such method.

1.- For each m with $\widetilde{H}_0(\Delta_m) \neq 0$, i.e., with non connected Δ_m, a set $B_0(m)$ can be constructed by picking, for each component v of Δ_m, a point (i.e. a 0-dimensional face) P_v in it and, then, taking as $B_0(m)$ a set of differences $P_v - P_{v'}$ where the involved pairs (v, v') are the edges of a tree for the set of connected components.

Now, for a cycle $c = \sum_P \lambda_P P$ in $\widetilde{Z}_0(\Delta_m)$ (i.e. $\sum_P \lambda_P = 0$), one has an element F in $\sigma_0(c) \subset I_m$ by setting $G = \sum_P \lambda_P X_P M_P$, where M_P is any monomial of degree $m - n_P$. In particular, for a cycle of type $P - P'$ the polynomial $F \in I_m$ is a binomial and so, the minimal system of generators of I associated with the above choice of $B_0(m)$'s is a set of binomials.

2.- By recurrence, assume that one already has minimal systems of homogeneous generators $\{G_1^{(i)}, \ldots, G_{b_{i+1}}^{(i)}\}$ for the modules N_i, $0 \leq i \leq j-1$. Again, to construct such a system for N_j it is enough to compute images by σ_j of the cycles in $B_j(m)$, for those m with $\widetilde{H}_j(\Delta_m) \neq 0$. Now, for a cycle $c = \sum_F \lambda_F \cdot F \in \widetilde{Z}_j(\Delta_m)$, one can compute a syzygy in $\sigma_j(c)$ in j steps as follows. First, for any F' with $\dim F' = j-1$, consider $b_{F'} \in I_{m-n_{F'}}$ given by

$$b_{F'} = \sum_F \varepsilon_{F'F} \, \frac{X_F}{X_{F'}} \, M_F$$

where M_F is a monomial of degree $m - n_F$ for any $F \in \Delta_m$. Now, take $\underline{a}_{F'} = (a_{F'}^l) \in (A^{b_1})_{m-n_{F'}}$ with $b_{F'} = \sum a_{F'}^l . G_l^{(0)}$, and for each F'' with $\dim F'' = j-2$ consider $\underline{b}_{F''} \in (N_1)_{m-n_{F''}}$ given by

$$\underline{b}_{F''} = \sum_{F'} \varepsilon_{F''F'} \, \frac{X_{F''}}{X_{F'}} \, \underline{a}_{F'}$$

Again construct $\underline{a}_{F''} \in (A^{b_2})_{m-n_{F''}}$ by the same method and continue in the same way. In the j-th step, for the dimension -1 one gets an element $\underline{b}$ in $(N_j)_m$ given by

$$\underline{b} = \sum_P X_P \, \underline{a}_P$$

where $\underline{a}_P \in (A_j^{b_j})_{m-n_P}$ corresponds to the face dimension 0.

Remark 3.7. To apply the algorithm in 3.6 one needs to find finite sets $C_t \subset S$, $t \geq 0$, such that $\widetilde{H}_t(\Delta_m) = 0$ if $m \notin C_t$ and, for each $m \in C_t$ a basis for the vector space $\widetilde{H}_t(\Delta_m)$. A general method to find these sets and vector bases is given in [4]. For the cases of affine and projective curves, this method is applied ([4]) to compute the sets C_t in terms of the semigroup generators. Vector bases for $\widetilde{H}_t(\Delta_m)$ can be also computed in terms of some finite graphs. This computation obviously depends on the characteristic of the field.

In the next section we show how the method in [4] can be also used to compute finite sets C_t and homology vector bases for monomial affine and simplicial projective surfaces. Thus, the algorithm in 3.6 is able to compute the syzygies for such surfaces.

4. Syzygies of monomial surfaces

We take an affine monomial surface to be an affine surface of type $k[S]$ where S is a subsemigroup of $\mathbb{N}^2$ containing at least two elements of type $e_1 = (d_1, 0)$, $e_2 = (0, d_2)$ with $d_1 > 0$, $d_2 > 0$. Obviously S is a f.g.c.f.c.c. semigroup. We take a simplicial projective monomial surface to be a projective surface of type $\mathrm{Proj}(k[S])$ where S is a subsemigroup of $\mathbb{N}^3$ containing the elements $e_1 = (d, 0, 0)$, $e_2 = (0, d, 0)$, $e_3 = (0, 0, d)$, $d > 0$, and generated by elements in the hyperplane of $\mathbb{Z}^3$ given by $x + y + z = d$. Again S is a f.g.c.f.c.c. semigroup.

Let us consider a system of generators $e_1 = (d_1, 0)$, $e_2 = (0, d_2)$, $a_1 = (a_{11}, a_{12})$,..., $a_s = (a_{s1}, a_{s2})$ for the semigroup of an affine monomial surface, or a system of generators $e_1 = (d, 0, 0)$, $e_2 = (0, d, 0)$, $e_3 = (0, 0, d)$, $a_1 = (a_{11}, a_{12}, a_{13})$,..., $a_s = (a_{s1}, a_{s2}, a_{s3})$ with $a_{i1} + a_{i2} + a_{i3} = d$, $i = 1, 2, \ldots, s$, for the semigroup of a simplicial projective surface. Write $A := \{a_1, \ldots, a_s\}$ and $E := \{e_1, e_2\}$ in the affine case and $E := \{e_1, e_2, e_3\}$ in the projective one.

For each subset $I \subset A$ (resp. $J \subset E$) denote by a_I (resp. e_J) the element $a_I := \sum_{i \in I} a_i$ (resp. $e_J := \sum_{j \in J} e_j$). For each $m \in S$ consider the simplicial subcomplex T_m of $\mathcal{P}(E)$ given by

$$T_m := \{J \subset E \mid m - e_J \in S\}.$$

For $j \geq -1$, let $D(j)$ be the subset of S given by

$$D(j) := \{m \in S \mid \widetilde{H}_j(T_m) \neq 0\}.$$

Notice that $D(-1) = \{m \in S \mid m - e \notin S \;\; \forall e \in E\}$, and that $D(j) = \emptyset$ for $j \geq 1$ in the affine case, and for $j \geq 2$ in the projective case. Thus, one only has three significative sets $D(-1)$, $D(0)$, $D(1)$.

Now, for every $t \geq -1$ set

$$C_t := \{m \in S \mid m = d + a_I \text{ with } d \in D(j) \text{ for some } j \text{ and } \mathrm{card}(I) = t - j\}\,.$$

For $m \in S$ also consider the simplicial subcomplexes of $\mathcal{P}(\Lambda)$ given by

$$K_m := \{F \in \Delta_m \mid F \cap E \neq \emptyset \text{ or } F \subset A \text{ and } m - a_F \in S - D(-1)\}\,,$$

$$\overline{K}_m := K_m \cup \{I \cup J \mid I \subset A, J \subset E,\ \ m - a_I - e_J \notin S,\ m - a_I - e \in S\ \forall e \in J\}\,.$$

Two of the main results proved in [4] are summarized in the following proposition.

Proposition 4.1

(1) The simplicial complexes $\overline{K}_m$ are acyclic.
(2) The sets C_t are finite.

Using reduced relative homology for the pair $(\overline{K}_m, K_m)$, it follows from (1) that one has an isomorphism $\widetilde{H}_t(K_m) \cong \widetilde{H}_{t+1}(\overline{K}_m, K_m)$. Moreover, direct and inverse images by this isomorphism can be computed explicitly as shown in [4]. Combining the above isomorphisms with the reduced relative homology for the pair (Δ_m, K_m) one gets the following long exact sequence in homology

$$(*)\quad \cdots \to \widetilde{H}_{t+1}(\Delta_m, K_m) \overset{\mu_{t+1}}{\to} \widetilde{H}_{t+1}(\overline{K}_m, K_m) \overset{j_t}{\to} \widetilde{H}_t(\Delta_m) \overset{\rho_t}{\to} \widetilde{H}_t(\Delta_m, K_m) \to \cdots$$

where the images of elements by the maps $\mu.$, $j.$, $\rho.$, can be computed in practice. One concludes that, for each m, a basis for the homology $\widetilde{H}_t(\Delta_m)$ can be computed (in terms of linear algebra) from the knowledge of bases for the relative homologies $\widetilde{H}_j(\Delta_m, K_m)$ and $\widetilde{H}_j(\overline{K}_m, K_m)$ for $j = t, t+1$.

Now, take $m \notin C_t$. Since $m - a_F \notin D(-1)$ for every $F \subset A$, with card $F = t+1$, it follows that $\widetilde{H}_t(\Delta_m, K_m) = 0$. Moreover, the homology $\widetilde{H}_{t+1}(\overline{K}_m, K_m)$ can be described by using the filtration of $K_m = M_m^{(-1)} \subset M_m^{(0)} \subset \cdots \subset M_m^{(s)} = \overline{K}_m$, where

$$M_m^{(i)} = K_m \cup \big\{I \cup J \in \overline{K}_m \mid I \subset A,\ J \subset E \text{ and card } I \leq i\big\}\,.$$

Since, from definitions, one has (see again [4]),

$$\widetilde{H}_{t+1}\big(M_m^{(i)}, M_m^{(i-1)}\big) \cong \bigoplus_{\substack{I \subset A \\ \mathrm{card}\ I = i}} \widetilde{H}_{t-i}(T_{m-a_I})\,,$$

if follows that $m \notin C_t$, also implies $\widetilde{H}_{t+1}(\overline{K}_m, K_m) = 0$.

Thus, one leads to the following two results.

Proposition 4.2 ([4])

With above assumptions one has
(3) $\widetilde{H}_t(\Delta_m) = 0$, if $m \notin C_t$.
(4) For $m \in C_t$ a basis for the homology $\widetilde{H}_t(\Delta_m)$ can be obtained from the exact sequence $(*)$, if one knows bases for $\widetilde{H}_j(\Delta_m, K_m)$, $\widetilde{H}_j(\overline{K}_m, K_m)$ for $j = t, t+1$.

The chain complex for both pairs (Δ_m, K_m) and $(\overline{K}_m, K_m)$ has, in practice, much less faces than that of Δ_m. In [4] is shown how the homologies $\widetilde{H}_j(\Delta_m, K_m)$, $\widetilde{H}_j(\overline{K}_m, K_m)$ can be computed from the data in certain graphs.

Remark 4.3. From results (1)-(4) in the propositions, it follows that to describe the syzygies of monomial surfaces that what one needs is to compute the sets $D(j)$, $j = -1, 0, 1$, and, from them, the sets C_t. Then, for any $m \in C_t$, one calculates bases for the relative homologies for the pairs (Δ_m, K_m), $(\overline{K}_m, K_m)$ and hence, by using $(*)$, the homology $\widetilde{H}_t(\Delta_m)$. Notice that the computation of bases for those vector spaces depends on the characteristic of the field k, as the involved linear system of equations to derive the homologies has integer coefficients. Finally, by Remark 3.6 one can obtain the syzygies.

Next, we will give the explicit description of the sets C_t for monomial surfaces.

First, consider the case of affine monomial surface. Set $Q := D(-1)$ and $D := D(0)$. Fix a class c in the group $\overline{G} := G(S)/(d_1\mathbb{Z} \oplus d_2\mathbb{Z})$ given by a representative (α, β) with $0 \leq \alpha < d_1$, $0 \leq \beta < d_2$. Denote by Q_c (resp. D_c) the set of elements of Q (resp. D) which are in the class c. Notice that one has $Q = \cup_{c\in\overline{G}} Q_c$, $D = \cup_{c\in\overline{G}} D_c$.

Lemma 4.4

$Q_c \neq \emptyset$ and card $Q_c =$ card $D_c + 1$, *for every $c \in \overline{G}$.*

Proof. Let S_c be the set of elements of S in the class c. Then, $S_c + \mathbb{N}e_1 + \mathbb{N}e_2 = S_c$ where $\mathbb{N}$ is the set of non negative integers. It follows that S_c is a ladder whose vertices (i.e. the elements in the minimal set V_c such that $V_c + \mathbb{N}e_1 + \mathbb{N}e_2 = S_c$) are the elements of Q_c. Between two vertices with consecutive abscise there is only one element of D_c (the point for which each coordinate is the maximum of the corresponding coordinates of both vertices). This proves that card $Q_c =$card $D_c + 1$. To see that $Q_c \neq \emptyset$, notice that for every i, $1 \leq i \leq s$, there exists an integer N_i such that $N_i a_i \in \mathbb{N}d_1 \oplus \mathbb{N}d_2$. Thus if the class c is represented by the element $m = \sum_{i=1}^{s} l_i a_i + t_1 e_1 + t_2 e_2 \in G(S)$ where l_i, t_1, $t_2 \in \mathbb{Z}$, then c is also represented by $m' = m + Me_1 + Me_2$ where $M \in \mathbb{N}$ is chosen to be sufficiently large in order to have $m' = \sum_{i=1}^{s} l'_i a_i + t'_1 e_1 + t'_2 e_2$ with $l'_i \geq 0$, $t'_1 \geq 0$, $t'_2 \geq 0$. Thus, $m' \in S$ and it is in the class c, so $Q_c \neq \emptyset$. □

Now, let S_1 be the semigroup generated by the integers $a_{11}, \ldots, a_{s1}, d_1$. Notice that above generators need not be different and that some of the a_{i1} can be equal to 0.

Fix $n \in S_1$ with $n \equiv \alpha (\text{mod } d_1)$. Consider the integer $L_c(n)$ given by the least integer $l \in \mathbb{N}$ with $l \equiv \beta (\text{mod } d_2)$ and such that there exist $i_1, \ldots, i_p$ with $1 \leq i_t \leq s$

for $t = 1, \ldots, p$ and $k \geq 0$ such that $l = a_{i_1 2} + \cdots + a_{i_p 2}$ and $n = a_{i_1 1} + \cdots + a_{i_p 1} + kd_1$. If such l does not exist, set $L_c(n) = \infty$. Also, set $L_c(n) = \infty$, if $n \notin S_1$, or if $n \in S_1$ and $n \not\equiv \alpha (\text{mod } d_1)$.

Lemma 4.5

With assumptions and notations as above, take $n \in S_1$, $n \equiv \alpha (\text{mod } d_1)$. Then one has $(n, l) \in Q_c$ for some integer l if and only if one has $L_c(n - d_1) > L_c(n)$ and, in that case, one has $l = L_c(n)$. The elements of D_c are exactly those of type $(n, L_c(n - d_1))$ where $(n, L_c(n)) \in Q_c$ and $L_c(n - d_1) < \infty$.

Proof. If $(n, l) \in Q_c$ then one has $(n, l) - e_1 \notin S$, $(n, l) - e_2 \notin S$, so, by definitions of $L_c(n)$ and $L_c(n - d_1)$, one should have $l = L_c(n)$ and $L_c(n - d_1) > L_c(n)$. Conversely, if $L_c(n - d_1) > L_c(n)$ and $n \in S_1$, then $(n, L_c(n)) \in Q_c$, again by the definition of $L_c(n)$ and $L_c(n - d_1)$.

On the other hand, it is clear that elements of D_c should be those of type $(n, L_c(n - d_1))$ where $(n, L_c(n)) \in Q_c$ and $L_c(n - d_1) < \infty$. $\square$

Lemma 4.5 shows how the sets Q and D can be computed in arithmetical terms from the semigroup generator system. Thus, by using the exact sequences $(*)$ and Remark 4.3 one gets the following result.

Theorem 4.6

The syzygies of an affine monomial surface can be determined from the the knowledge of the semigroup generators and the characteristic of the field.

Now, let us consider the projective case. As above set $Q := D(-1)$. Consider the subsemigroup S_{12} of $\mathbb{N}^2$ generated by the elements

$$\Lambda_{12} := \left\{(a_{11}, a_{12}), \ldots, (a_{s1}, a_{s2}), (d, 0), (0, d)\right\}.$$

Notice that S_{12} is nothing but the projection of S on the plane corresponding to the two first coordinates. The semigroup S_{12} defines an affine monomial surface, so by the computation of the set $Q = Q_{12}$ for S_{12} in Lemma 4.5, and taking into account the proof of Lemma 4.4, the elements of S_{12} in a class modulo $d\mathbb{Z} \oplus d\mathbb{Z}$ can be determined.

For each $(a, b) \in S_{12}$ denote by $l(a, b)$ the minimum integer l such that $(a, b) = n_1 + \cdots + n_l$ with $n_i \in \Lambda_{12}$ for $i = 1, \ldots, l$. Set $l(a, b) = \infty$ if $(a, b) \notin S_{12}$. Next,we will characterize those (a, b) for which there exists some integer t such that $(a, b, t) \in Q$ (resp. $(a, b, t) \in D(0)$) (resp. $(a, b, t) \in D(1)$).

For this, notice that one has $m \in Q$ if and only if $T_m = \{\emptyset\}$, and $m \in D(1)$ if and only if $T_m = T$ where

$$T = \big\{\emptyset, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}\big\}\,.$$

If $m \in D(0)$, then T_m should be one of the following seven complexes (the only possible non connected complexes with three vertices).

$U_1 = \{\emptyset, \{e_1\}, \{e_2\}, \{e_3\}\}$
$U_2 = \{\emptyset, \{e_1\}, \{e_2\}\}$
$U_3 = \{\emptyset, \{e_1\}, \{e_3\}\}$
$U_4 = \{\emptyset, \{e_2\}, \{e_3\}\}$
$U_5 = \{\emptyset, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_2\}\}$
$U_6 = \{\emptyset, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_3\}$
$U_7 = \{\emptyset, \{e_1\}, \{e_2\}, \{e_3\}, \{e_2, e_3\}\}\,.$

Let us denote by $D(0)_j$ the subset of $D(0)$ consisting of those $m \in S$ such that $T_m = U_j$. Let $\overline{Q}$, $\overline{D(1)}$, $\overline{D(0)}_j$ be the respective projections of Q, $D(1)$, $D(0)_j$ on the (1,2)-plane.

Lemma 4.7

The projections $Q \to \overline{Q}$, $D(1) \to \overline{D(1)}$, $D(0)_j \to \overline{D(0)}_j$ for $j = 1, \ldots, 7$ are bijective maps.

Proof. The maps are obviously surjective. To prove that they are injective we will use the following fact: each one of the simplicial complexes $\Delta = \{\emptyset\}, T, U_1, \ldots, U_7$ has associated a face $J \in \mathcal{P}(E)$ such that $e_3 \in J$, $J \notin \Delta$ and $J - \{e_3\} \in \Delta$. Now, let Δ be any one of the above nine simplicial complexes and let H be the set Q, $D(1)$ or $D(0)_j$ which correspond to the complex Δ. Take $(a,b) \in \overline{H}$. Denote by t the least integer such that, if $m = (a,b,t)$, then $T_m = \Delta$ (i.e. $m \in H$). Elements of S in the fiber of (a,b) by the projection on the (1,2)-plane are of type $m' = (a,b,t+\lambda d)$ with $\lambda \in \mathbb{Z}$. If $\lambda < 0$, then $T_{m'} \neq \Delta$ by the minimality of t. If $\lambda > 0$ then $T_{m'} \neq \Delta$ as one has $J \notin T_m$ and $J \in T_{m'}$, J being a face associated to Δ with the property indicated in the fact at the beginning of the proof. This shows that the projection $H \to \overline{H}$ is injective as required. □

Lemma 4.7 shows that in order to compute the sets Q, $D(1)$, $D(0)_j$, it is sufficient to compute the finite sets $\overline{Q}$, $\overline{D(1)}$, $\overline{D(0)}_j$ and, for each element (a,b) on each one of those sets, the value of t such that the semigroup element (a,b,t) realizes the corresponding simplicial complex $\{\emptyset\}$, T, U_j.

Lemma 4.8

With assumptions and notations as above, for $(a, b) \in S_{12}$ one has:

(i) $(a, b) \in \overline{Q}$ *if and only if* $l(a, b) \leq l(a-d, b)$ *and* $l(a, b) \leq l(a, b-d)$.

(ii) $(a, b) \in \overline{D(1)}$ *if and only if* $l(a-d, b-d) \geq l(a-d, b)$, $l(a-d, b-d) \geq l(a, b-d)$, *and* $l(a-d, b-d) < \infty$.

(iii) $(a, b) \in \overline{D(0)_1}$ *if and only if* $l(a, b) \leq l(a-d, b) = l(a, b-d) \leq l(a-d, b-d)$ *and* $l(a-d, b) < \infty$.

(iv) $(a, b) \in \overline{D(0)}_2$ *if and only if* $l(a, b) > l(a, b-d) = l(a-d, b) \leq l(a-d, b-d)$.

(v) $(a, b) \in \overline{D(0)}_3$ *if and only if* $l(a, b) \leq l(a-d, b) < l(a, b-d)$.

(vi) $(a, b) \in \overline{D(0)}_4$ *if and only if* $l(a, b) \leq l(a, b-d) < l(a-d, b)$.

(vii) $(a, b) \in \overline{D(0)}_5$ *if and only if* $l(a, b) \leq l(a-d, b) = l(a, b-d)$ *and* $l(a-d, b-d) < l(a-d, b)$.

(viii) $(a, b) \in \overline{D(0)}_6$ *if and only if* $l(a, b) \leq l(a, b-d) \leq l(a-d, b-d)$ *and* $l(a-d, b) < l(a, b-d)$.

(ix) $(a, b) \in \overline{D(0)}_7$ *if and only if* $l(a, b) \leq l(a-d, b) \leq l(a-d, b-d)$ *and* $l(a, b-d) < l(a-d, b)$.

The value of t such that $(a, b, t) \in Q, D(1), D(0)_j$ respectively in (i)-(ix) is given by $ld-a-b$ where l is given by (i) $l(a, b)$, (ii) $l(a-d, b-d)+2$, (iii) $l(a-d, b)+1$, (iv) $l(a, b)$, (v) $l(a-d, b)+1$, (vi) $l(a, b-d)+1$, (vii) $l(a-d, b)+1$, (viii) $l(a, b-d)+1$, (ix) $l(a-d, b)+1$.

Proof. Let (a, b, t) satisfy $a+b+t = ld$, then $(a, b, t) \in S$ if and only if $l \geq l(a, b)$. Using the above fact, (i)-(ix) follow by inspection case by case using the definition of $l(a, b)$ and the same kind of arguments than in the proof of Lemma 4.5. □

Lemma 4.8 allows to compute the sets Q, $D(1)$, $D(0)$ in arithmetic terms from the generator system of the semigroup. Again, by using $(*)$ and Remark 4.3, one gets the following result.

Theorem 4.9

The syzygies of a simplicial projective monomial surface can be determined from the knowledge of the semigroup generators and the characteristic of the field.

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