Lie Theory: Applications to Problems in Mathematical Finance and Economics

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Abstract

This paper is devoted to show and explain some applications of Lie theory to solve some problems in Economics and Mathematical Finance. So we put forward and discuss mathematical aspects and approaches for several economic problems which have been previously considered in the literature. Besides we also show our advances on this topic, mentioning some open problems for future research.

Keywords and phrases: Mathematical Finance; Multidimensional screening problem; Technical progress; Holotheticity; Lie groups; Lie algebras.

Introduction

It is well-known how to use Lie Theory for solving problems related to other sciences different from Mathematics (like Physics or Engineering). However, most of them are usually experimental or technical; being at present quite more unknown the applications of this theory to other non-experimental, non-technical sciences. In this sense, this paper is devoted to show some applications of Lie Theory to Mathematical Finance and Economics. Let us note that such applications are only known in some very specific financial and economic topics.

Regarding the relation between Lie Theory and some financial and economic problems, we would like to point out the existence of several recent works (all of them within this new century) which consistently represent the application of well-known properties of Lie groups and algebras to several financial and economic problems and concepts. For example, Lo and Hui presented in [13] and [14] (at the beginnings of 2000s) different techniques based on Lie algebras to deal with the valuation of financial derivatives, in particular, *multi-asset derivatives*. They also deal with the PDE with time-dependent coefficients, CEV models and *moving barriers* using Lie Theory (see [11] and [12]). Independently and using Lie algebras, Björk and Landén (2002) studied in [6] some models of *interest-rate*, previously introduced by Björk himself in [4]. Later, Polidoro (2003) studied a financial problem by using nilpotent Lie groups in [17]. Another interesting paper about the application of Lie Theory to Economics is due to Basov (2004), who described some methods based on Lie groups in order to solve the multidimensional *screening problem* in [3]. We also want to recall to Gaspar [8], who studied a general model for the structure of *forward prices* by using the methodology of Lie algebras given by Björk. Finally, Björk [4, 5] studied some applications of Lie algebras to certain economic concepts like *constant volatility* and *constant direction volatility* (i.e. considering that the volatility follows a constant vector field as its "direction"). The interested reader can consult the two previous references for an extensive and complete explanation about the concept of volatility and its different types.

So this paper is devoted first to comment briefly on the basic aspects in some of the works previously cited, reaching an overall view for the current status in the application of Lie Theory to Economics, which constitutes a recent an innovative research. Secondly, we expound our advances on the concept of holotheticity (introduced by Sato in [18]) by endowing it with more mathematical explanations, in a similar way as one of the authors already did in [7]. More concretely, the mathematical groundwork for the concept of holotheticity is explained and checked in order to prove that the definition and some properties given by Sato have a correct mathematical foundation, and some problems relative to notation and formulation are also solved.

1 Lie groups of PDEs applied to multidimensional screening problems

One of the main applications of Lie Theory to Economics is the use of Lie groups of partial differential equations to *multidimensional screening problems*, as Basov showed in [3].

In that paper, he described the use of group-theoretic methods for analyzing boundary problems arising when the Hamiltonian method is applied to solve the relaxed problem for the multidimensional screening problem. According to Basov, this technique provides a useful piece of information about the possible structure of the solutions and sometimes some particular solutions (but not general) can be obtained.

In many industries the relationship between price paid by the customers and quantity purchased is not strictly proportional. Moreover, for nonlinear tariffs, the payment is often determined as a function of several characteristics, which can be differently valued by each customer. Consequently, a one-dimensional characteristic is not enough to capture the typology of customers in general. This leads to a multi-dimensional screening problem.

This problem is described as follows: let us consider a multi-product monopoly producing n goods (or, in a more simplified model, a good with n quality dimensions), being convex its cost function. An arbitrary consumer's preferences over these goods are parameterizable by a m-dimensional vector and the typology of consumers follows a distribution with density function f over a convex open bounded set $\Omega \subset \mathbb{R}^m$. Let also suppose that f is continuously differentiable on Ω and extending by continuity to its closure $\overline{\Omega}$.

For the monopolist, maximizing its profits is always one of its main goals. An interesting way to achieve this is the suitable choice of a tariff, bearing in mind that a tariff is simply a real function on the set of bundles of goods. Therefore, the tariff allows us to determine a consumer's payment for a particular bundle of goods. Consequently, finding a solution of this problem often involves solving a boundary problem involving a system of nonlinear PDEs.

We must take into consideration that there do not exist any general methods for solving the previous problem. However, when symmetries are present, the problem can be considerably simplified and even solved explicitly. In this sense, Basov applied the theory of Lie groups of PDEs to multidimensional screening problems for the first time in the literature. Besides, Basov obtained particular explicit solutions for some types of these problems using his own methods. For example, he could compute explicit solutions for the monopolist problem. Even when explicit solutions cannot be obtained, the corresponding problem can be simplified and non-explicit solutions are obtained. Next, an example of Basov's method and its application are shown according to the assumptions considered in this section.

For a multi-product monopoly producing n goods, consumers' preferences over the bundles of these goods are parameterized by m-dimensional column vectors and the typology of consumers follows the distribution given by a density function f defined on $\Omega \in \mathbb{R}^m$, which is open, bounded, and convex. Indeed, Basov assumed that:

- $\Omega = \Pi_{i=1}^{m} (a_i, b_i)$.
- f is continuous and strictly positive on a convex open subset of Ω.

When a consumer of type $\alpha \in \Omega$ consumes a bundle $x \in X \subset \mathbb{R}^{+n}$ and pays the tariff t, the utility for him/her is given by:

$$U(\alpha, x, t) = \sum_{i=1}^{m} \alpha_i \, v_i(x) - t(x),$$

where each function v_i (which represents the marginal rate of the superplus for a consumer of type α) is increasing, continuously differentiable and satisfying the Lipschitz condition in $x \in X$. Let us note that the notation $U(\alpha, x, t)$ for the utility of consumers of type α was introduced by Basov himself to define a function simplify the classical notation given in [2, 21]. Hence, given the tariff $t : X \to \mathbb{R}$, the profits of the monopolist can be computed as:

$$\pi = \int [t(x(\alpha)) - c(x(\alpha))] \, f(\alpha) \, d\alpha,$$

where c is the production cost and $x(\alpha)$ is the bundle purchased by every consumer of type α. So the monopolist is interested in determining a tariff t which maximizes the utility on profits:

$$s(\alpha) = \max_{x \in X} U(\alpha, x, t(x)).$$

This is equivalent to solving a system of PDEs, whose solution can be obtained explicitly by Lie Theory.

To obtain the solution of the problem using symmetries, the following steps were considered by Basov:

1. Find an invariance group of the problem.
2. Find all the independent invariants of the group computed in the previous step.
3. Express the problem in terms of invariants of the group.
4. Attempt to solve it.

To do this, it is necessary to introduce some notions related to Lie Theory. Next, we recall the definition of a one-parameter Lie group (which was incorrectly defined by Basov but appropriately applied):

Definition 1.1. *Let us consider an open set $E \subset \mathbb{R}^m$ and a function $F : \mathbb{R}^m \times \mathbb{R} \to E$ smooth at $\alpha \in \mathbb{R}^m$ and analytic at $\tau \in \mathbb{R}$. If the set G of coordinate transformations:*

$$g^\tau : \mathbb{R}^m \to E : \alpha \mapsto F(\alpha, \tau)$$

has a group structure by considering the inner product operation $m : G \times G \to G$ defined by:

$$m(g^{\tau_1}, g^{\tau_2}) = g^{\tau_2} \circ g^{\tau_1} = g^{\tau_1+\tau_2} : \alpha \to F(F(\alpha, \tau_1), \tau_2) = F(\alpha, \tau_1 + \tau_2),$$

the pair (G, m) is called a one-parameter Lie group, where τ is the parameter.

Remark 1.2. In Definition 1.1, the function F has two inputs, a vector $\alpha \in \mathbb{R}^m$ and a real number τ_1. The unique output of F is a vector $F(\alpha, \tau_1)$ belonging to the open set $E \subset \mathbb{R}^m$. Consequently, $F(\alpha, \tau_1)$ is a vector in $\mathbb{R}^m$ and hence $(F(\alpha, \tau_1), \tau_2)$ belongs to the domain of the function F, for all $\tau_2 \in \mathbb{R}$.

For screening problems, the arisen PDEs are of first and second-order and its expression is the following:

$$\Phi(\alpha, u, \nabla(u), D^2u) = 0,$$

where Φ is a continuously differentiable function; $\alpha \in \mathbb{R}^m$; $u : \mathbb{R}^m \to \mathbb{R}$ is twice continuously differentiable; $\nabla(u)$ is the gradient of u and $D^2(u)$ is the symmetric Hessian matrix.

For such a PDE, let us consider the following transformations of the independent and dependent variables:

$$\begin{cases} \overline{\alpha}_i = F_i(\alpha, u, \tau), \\ \overline{u} = G(\alpha, u, \tau), \end{cases}$$

where functions F_i and G are infinitely differentiable at α and u and analytic at $\tau \in \mathbb{R}$, with $F_i(\alpha, u, 0) = \alpha_i$ and $G(\alpha, u, 0) = u$. The set of transformations previously defined also has a structure of one-parameter Lie group when defining the product of two transformations as the inner product operation defined in Definition 1.1.

Definition 1.3. *A* symmetry group *of the equation* $\Phi(\alpha, u, \nabla(u), D^2u) = 0$ *is any subgroup of the Lie group of transformations leaving invariant the equation.*

Therefore, solving a PDE is necessary to solve the problem presented in this section. This can be done by computing and using symmetry groups of such a PDE.

2 Non-linear PDEs in Mathematical Finance

Another application of PDEs and Lie Theory to Economics is related to Mathematical Finance. This section is devoted to explaining Polidoro's work [17] about this application. He studied a non-linear PDE arisen from solving a problem in Mathematical Finance. To solve such a PDE, he used some properties of both Lie commutator and Lie groups.

More concretely, Polidoro studied a non-linear degenerate Cauchy problem arising from Mathematical Finance, proving the existence of a locally strong solution. Besides, the regularity of this solution was also studied in the paper by considering the framework of sub-elliptic operators on nilpotent Lie groups. Moreover, he gave sufficient conditions for the existence of global solutions.

Let us consider the following PDE with three variables $z = (x, y, t) \in \mathbb{R}^3$:

$$Lu \equiv \partial_{xx}u + u\,\partial_y u - \partial_t u = f, \qquad (1)$$

that first appeared in a financial problem studied by Antonelli, Barucci and Mancino [1]. Polidoro proved that there exists a unique utility function for the previos cited financial problem. Moreover, this solution is also the unique viscosity solution of the Cauchy problem corresponding to the given PDE with a suitable initial datum, namely:

$$u(x,y,0) = g(x,y). \qquad (2)$$

Besides, he proved that the solution of the Cauchy problem (1)-(2) is defined in a short time interval $[0,T)$ satisfying the following condition:

$$\mid u(x,y,t) - u(\xi,\eta,\tau) \mid \leq C(\mid x-\xi \mid + \mid y-\eta \mid + \mid t-\tau \mid)^{\frac{1}{2}}. \qquad (3)$$

for all $(x,y,t),(\xi,\eta,\tau) \in \mathbb{R}^2 \times [0,T)$ and assuming that f and g are uniformly Lipschitz continuous functions. Indeed, Polidoro was interested in the interior regularity of the solution. First, we must note that both Standard Regularity Theory for distributional solutions and regularity results for viscosity solutions cannot be applied to the problem studied by Polidoro. On the contrary, the operator L can be hyperbolic in some cases. However, under some suitable hypotheses (namely, u is a classical solution of $Lu \equiv f$ with $f \in \mathcal{C}^\infty$ and $\partial_x u > 0$), the solution was proved to be smooth by Polidoro. To obtain such results, he applied the framework of Lie groups analysis.

In fact, the operator defined in (1) is a second-order operator and the matrix of coefficients for second-order derivatives is semi-definite positive. Therefore, the solution of the equation $Lu \equiv f \in \mathcal{C}^\infty$ is smooth in the directions in which the matrix is non-degenerate, but not necessarily in other directions. As an example of this problem, Polidoro considered the Kohn-Laplace operator in $\mathbb{R}^3$, defined as follows:

$$L \equiv (\partial_x + 2y\partial_t)^2 + (\partial_y - 2x\partial_t)^2.$$

This operator is defined as the addition of the directional derivatives $X = \partial_x + 2y\partial_t$ and $Y = \partial_y - 2x\partial_t$. So every solution of $Lu = 0$ is smooth in both directions. However, these solutions are also smooth in the direction of the commutator of both derivatives: $[X,Y] = -4\partial_t$.

The Kohn-Laplace operator is the simplest meaningful example studied by Hörmander [10]. In this way, Hörmander considered a set $\{X_0,\ldots,X_p\}$ of directional derivatives (i.e. smooth vector fields on Ω) defined as:

$$X_j = \sum_{i=1}^{n} a_{ij}(x)\,\partial_{xi}, \quad \forall j = 0,\ldots,p$$

where $a_{ij} \in \mathcal{C}^\infty(\Omega)$ for some open set $\Omega \in \mathbb{R}^n$. If $f \in \mathcal{C}^\infty(\Omega)$, Hörmander proved the following result: if u is a solution on Ω of the equation

$$\sum_{i,j} X_i\, X_j\, u + X_0\, u = f, \qquad (4)$$

and $\mathbb{R}^n$ is the linear span of the vector fields $X_0, \ldots, X_p$ and their respective commutators, then $u \in \mathcal{C}^\infty(\Omega)$.

This result has been the starting point in the research of regularity properties for the operators involved in (4), as well as their relationship with some Lie group structures on $\mathbb{R}^n$. For the non-linear equation (1), Polidoro applied the previously cited linear theory to study the corresponding regularity problem. In this way, Polidoro considered the *linearized* operator $L_u = \partial_x^2 + u\,\partial_y - \partial_t$, where u is used as a coefficient. He checked that the smoothness of the coefficients a_{ij} was an essential hypothesis which cannot be removed. Obviously, this assumption could not be done for u as a coefficient, because the goal of his paper was precisely proved the smoothness of the solution u.

Let us note that the reader can also consider the Stochastic Calculus of Variations, developed by Malliavin [16] in 1976. It was originally introduced to study the smoothness of the densities of solutions of stochastic differential equations, obtaining a probabilistic proof of a condition for a partial differential operator to be hypo-elliptic. Malliavin's works involve the use of Lie brackets of vector fields for diffusion matrices satisfying a kind of Hörmander's condition.

Taking into consideration the nonexistence of a general theory for operators with non-smooth coefficients, Polidoro had to represent the operator L following the structure in (4):

$$Lu = X^2 u + Y u,$$

by considering $X = \partial_x$ and $Y = u\partial_y - \partial_t$. In this way, for any $\varsigma = (\xi, \eta, \tau) \in \Omega$ he took the operator approximation as:

$$L_\varsigma u = X^2 u + Y_\varsigma u,$$

where $Y_\varsigma = (u(\varsigma) + (x - \xi)\partial_x u(\varsigma))\,\partial_y - \partial_t$.

L_ς is a linear Hörmander operator and a good approximation of L because, under Condition (3), it holds:

$$\mid Lu(z) - L_\varsigma u(z) \mid = \mid u(z) - u(\varsigma) - (x - \xi)\,\partial_x u(\varsigma) \mid \cdot \mid \partial_y u(z) \mid \leq C \mid z - \varsigma \mid .$$

Under these conditions, results about regularity could be obtained using representations of the solution u in terms of the fundamental solution of L_ς. Consequently, the regularity problem had to be studied using Lie groups analysis. Besides, it was necessary to consider and apply the Lie commutator of X and Y (i.e. $[X, Y] = \partial_x u \cdot \partial_y$) and take into consideration that Hörmander condition is satisfied if $\partial_x u(z) \neq 0$, for every z.

With this example, we have shown another application of Lie Theory for dealing with and solving questions related to problems in Finance. In fact, there are several papers giving explicit or concrete examples about how to apply the Lie group analysis for solving problems in Mathematical Finance, like [14, 15, 20]. Besides, the reader can also consult Gazizov and Ibragimov [9] to consult how Lie group analysis can be applied in general to problems in Finance involving PDEs.

3 Technical Progress and Lie groups

A praiseworthy application of Lie Theory to Economics is the corresponding to study changes in a technology over the time. These changes can be differentiated in two types: those due to a *technical progress* or those produced by *scale effects* (both notions will be discussed in some detail later). This distinction between the two types of effects leads to Solow-Stigler Controversy, which will be also explained later. At this point, we want to highlight that Sato [18, 19] introduced the concept of *holotheticity* for a production function in the 1980s, solving the previously cited controversy using Lie Theory. The solution given by Sato will be explained next.

Any given technology is expressed according to two variables: the capital K and the labor L. In this way, such a technology can be represented by a production function $Y = f(K, L)$, where K and L are two real vectors (whose dimensions can be, in general, different).

Definition 3.1. *A production function $Y = f(K, L)$ is called neoclassical if the following two conditions hold:*

1. *Y is homogeneous of degree 1 (constant returns to scale).*

2. *Y is smoothly diminishes returns to individual factors (law of diminishing returns).*

The considered technology varies as time goes by due to modifications in the capital or improvements in research. These changes affect its production function. In Economics, these changes over the technology are represented and explained by the notion of *technical change* and the most restrictive one of *technical progress*.

Definition 3.2. *A technical change in a technology is any change in the production function altering the relationship between inputs (i.e. capital and labor) and outputs (i.e. productions). When the output increases for any given input (with respect to the one before the change), the technical change is called a technical progress.*

By considering the technical progress, the production function can be written as $\overline{Y} = \overline{f}(K, L, t)$, where t is the time parameter of the technical progress and $\overline{Y}$ is the production for the capital K and the labor L after applying the technical progress.

Definition 3.3. *The functions ϕ and ψ of technical progress for capital K and labor L are those which allow us to express the variations of both capital and labor depending on the factors K, L and the parameter t:*

$$T_t : \overline{K} = \phi(K, L, t), \quad \overline{L} = \psi(K, L, t).$$

The variables $\overline{K}$ and $\overline{L}$ are called effective capital and effective labor, respectively.

As we have already commented, one of the reasons why Lie Theory is used in Economics is the interest in distinguishing between the impact of a given technical progress and the scale effects in a technology. This problem is called Solow-Stigler controversy and it was solved by Sato using the concept of *holotheticity* for a given production function.

Definition 3.4. *Let f and T be a production function and a technical progress, respectively, where T is defined by the functions $(\phi,\ \psi)$ of the technical progress. The function f is to be said holothetic under the technical progress T if the overall effect of the technical progress T over f can be represented by a strictly monotonic function F. This condition can be expressed by the following chain of inequalities:*

$$\begin{aligned}\overline{Y} &= \overline{f}(K, L, t) = f(\overline{K}, \overline{L}) = f[\phi(K, L, t),\ \psi(K, L, t)] \\ &= g(f(K, L), t) = F(Y, t) = F_{(t)}(Y).\end{aligned}$$

Remark 3.5. Let us note that Sato [19] used the notation $F_{(t)}(Y)$ instead of $F(Y,t)$, which is more usual and natural. The reason is the following: he wanted to emphasize the production as the most important input in the function, considering the time as a secondary input.

Next, we explain the *Lie-type* technical progress, which is needed to study the holotheticity of a given production function f. To affirm that a technical progress is of Lie-type, the conditions of a one-parameter Lie group have to hold. Such conditions are as follows:

- (GL1) The result of applying successively two transformations:

$$T_{t_1} = \begin{cases} \overline{K} = \phi(K, L, t_1); \\ \overline{L} = \psi(K, L, t_1) \end{cases} \quad \text{and} \quad T_{t_2} = \begin{cases} \overline{\overline{K}} = \phi(\overline{K}, \overline{L}, t_2); \\ \overline{\overline{L}} = \psi(\overline{K}, \overline{L}, t_2). \end{cases}$$

  is the same as the obtained applying the transformation:

$$T_{t_1+t_2} = \begin{cases} \overline{\overline{K}} = \phi(K, L, t_1 + t_2); \\ \overline{\overline{L}} = \psi(\overline{K}, \overline{L}, t_1 + t_2). \end{cases}$$

- (GL2) The transformation obtained for the value $-t$ of the parameter in the technical progress matches with the transformation inverse to the one obtained for the value t:

$$T_t^{-1} : K = \phi(\overline{K}, \overline{L}, -t), \quad L = \psi(\overline{K}, \overline{L}, -t).$$

- (GL3) The transformation obtained for the value $t_0 = 0$ is exactly the identity transformation:

$$T_0 = \begin{cases} \overline{K} = \phi(K, L, t_0) = \phi(K, L, 0) = K; \\ \overline{L} = \psi(K, L, t_0) = \psi(K, L, 0) = L. \end{cases}$$

Definition 3.6. *If a technical progress T has the three properties of a one-parameter Lie group, T is called a Lie-type technical progress.*

Let us see now an economic interpretation for the three previous properties. If t is the year in which the technical progress takes place and $\overline{K}$ and $\overline{L}$ are the capital and the labor for the year t, respectively, Property GL1 can be interpreted as follows: if the variations of capital and labor are known in the first year (i.e. $t = 1$), the respective variations can be obtained for any year by considering $t = \sum_{i=1}^{n} 1$, where n is the year involved. Property GL2 implies to know both capital and labor in the initial time starting from the functions of technical progress, because it is only needed to use the parameter $-t$ corresponding to the known time. Finally, Property GL3 can be interpreted and enunciated as follows: the initial capital equal the effective ones when no technical progresses happen.

When considering Lie-type technical progresses, its most useful property is their infinitesimal transformation. After some computations, a unique expression can be obtained to relate the derivatives of the production function depending on time with the Lie operator associated with this infinitesimal transformation:

$$U = \xi(K, L)\,\frac{\partial}{\partial K} + \eta(K, L)\,\frac{\partial}{\partial L},$$

where U is the Lie operator associated with the infinitesimal transformation of the technical progress: $\xi(K, L) = (\frac{\partial \phi}{\partial t})_{t=0}$ and $\eta(K, L) = (\frac{\partial \psi}{\partial t})_{t=0}$.

There exists a way of interpreting the operator U so obtained. It allows us to determine a measure of the technical progress given.

Definition 3.7. *Given a production function $Y = f(K, L)$ and a Lie-type technical progress T, the first-order measure for the impact of the technical progress over the identity transformation is defined by the following derivative:*

$$M(T) = \left(\frac{\partial \overline{Y}}{\partial t}\right)_{t=0}.$$

The function $M(T)$ of the technical progress T can be denominated, shortly, by the measure of the technical progress.

Proposition 3.8. *Under the conditions of the previous definition, the measure of the technical progress matches with the Lie operator applied to the production function f; that is:*

$$M(T) = U(f) = \xi(K, L)\,\frac{\partial f}{\partial K} + \eta(K, L)\,\frac{\partial f}{\partial L}.$$

The Lie operator is also used to obtain a characterization of holothetic production functions, as is shown in the following:

Theorem 3.9. *A production function f is holothetic under a Lie-type technical progress T if and only if the measure of the technical progress is a function of the production function itself; that is: $U(f) = G(f)$.*

However, the previous theorem does not assure the existence of a production function which is holothetic under a given Lie-type technical progress. Indeed, its existence and uniqueness was proved by Sato. The result due to Sato is stated in the following theorem without giving the proof:

Theorem 3.10. *There exists a unique technology holothetic under a fixed and given Lie-type technical progress. Moreover, for such a technology, the effects of the technical progress are completely turned on scale effects.*

Finally, Sato took advantage of all these results to answer Solow-Stigler Controversy by means of the following theorem:

Theorem 3.11. *The effects of a Lie-type technical progress T and the scale effects of a given production function f are independently identifiable if and only if f is not holothetic under the technical progress T.*

4 Open problems and other considerations

It is not difficult to understand that there exists a lot of possibilities for applying Lie Theory to several economic subjects, as many already considered as even for determining. Next, we indicate some of them:

- As it was shown in the previous section, one of the applications of Lie Theory to Economics is to use the properties of Lie groups to define Lie-type technical progresses. This type of technical progresses allows us to study separately scale effects and those coming from the technical progress. To do it, the concept of holotheticity (introduced by Sato) has to be applied.

  Analogously, it would be interesting to consider another possible applications of Lie groups to Economics: the relation between technical progress and economic invariance.

  Another interesting economic concept studied starting from Lie Theory is the Substitution Marginal Rate (SMR), which can be dealt with using a Lie-group approach can be found in [7]). Lie Theory can be also applied to study the compatibility between the technical progress and the inner structure of a given technology. Indeed, this compatibility problem can be dealt with using particular conditions of the corresponding Lie brackets with respect to the representation of the SMR

Lie group. The holotheticity of a production function under a given technical progress would be equivalent to express the technical progress as a linear combination of both the SMR representation and another Lie-type technical progress (compatible with the inner structure of the given technology). This involves the possibility of analyzing when the conditions over the goods are compatibles with the ones required to the different types of existing Lie algebras.

- Lie groups are also used to study the neutrality of a given technical progress by means of their invariance under some Lie transformations groups (this is denominated G-neutrality). Besides, Lie groups also allow us to deal with the holotheticity of implicit technologies, although Lie-type technical progresses under which a given implicit technology is holothetic cannot be determined. Let us recall that this problem does not happen for explicit technologies, because technical progresses are determined by those technologies.

- Apart from that, some results on the properties of differential equations under continuous Lie groups made possible that Emmy Noether (1918) enunciated the denominated Noether's Theorem. This theorem could be interpreted, under economic applications, as a setting of conservative laws for dynamical systems, which would depend on the economic model involved.

- It would be also possible to use Lie Theory to take advantage in Index Number Theory. At present, the tests of index numbers can be translated to conditions of infinitesimal transformations of Lie groups, which can be considered as Lie group actions on the index number. Obviously, this index number must satisfy the invariance properties of group transformations to be useful for measuring prices and quantities.

So, in conclusion, let us note the usefulness of Lie Theory in the study of some economic problems still open. To study these applications, our next efforts will be devoted.

Acknowledgment

The authors would like to thank the referee for his/her valuable suggestions and comments which helped me to improve the quality of this paper.

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