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SURVEY PAPER

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Some questions in metric fixed point theory, by A. W. Kirk, revisited

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Abstract In this survey, we comment on the current status of several questions in Metric Fixed Point Theory which were raised by W. A. Kirk in 1995.

Mathematics Subject Classification 47H09 · 47H10 · 46B20

الملخص

نعلق في هذا الاستعراض على الوضع الحالي لعدة أسئلة في نظرية القيمة الثابتة المترية والتي أثيرت من قبل و. أ. كيرك عام 1995 م.

1 Introduction

Although the most important result in Metric Fixed Point Theory is the Contractive Mapping Principle, given by S. Banach in 1922, it was in 1965 that the discovery of a fundamental fixed point theorem for the class of non-expansive mappings provided the foundation for much of the subsequent theory. The central result in [3] asserts that if X is a Hilbert space and C is a convex closed bounded subset of X, then every non-expansive mapping $T : C \to C$ has a fixed point. A similar result was proved the same year for uniformly convex Banach spaces [4] and for reflexive Banach spaces with normal structure [22]. From then, many results have appeared proving the existence of a fixed point for non-expansive mappings defined on many classes of Banach spaces and many questions concerning this subject have been raised. The monographs [17,24] provide detailed information on this subject. In 1995, W. A. Kirk gave a series of lectures in the III International Conference in Fixed Point Theory and Applications, held in Seville [23], listing and discussing some open questions concerning the existence of fixed points for non-expansive mappings. Since many of them had long remained unanswered, the common belief was that most of these questions would remain open forever. Despite that many questions remaining unanswered, we want to show that there has been great progress in responding to these questions during the past 17 years, opening several new fruitful research fields.

We will use the following notation. Assume that X is a Banach space and C a subset of X. We say that C satisfies the fixed point property (FPP) if every non-expansive self-mapping defined on C has a fixed point. We say that X satisfies the FPP if every convex closed bounded subset of X does so, and that X satisfies the weak fixed point property (w-FPP) if every convex weakly compact subset of X satisfies the FPP. Replacing weakly compact sets by weakly-star compact sets yields the weak-star compact fixed point property (w^*-FPP).

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2 Reflexivity and the fixed point property

Probably, the most interesting (and difficult) problem in Metric Fixed Theory is to determine the relationship between the FPP and the reflexivity. The following question in [23] was perhaps the most fundamental and well-known open problem in the theory:

QUESTION I: *X reflexive* $\Longleftrightarrow$ *X has the FPP?*

Some basic facts can suggest that this is the case. For instance, Dowling and Lennard [12] proved in 1997: *A subspace of* L^1 *satisfies the FPP* $\Longleftrightarrow$ *it is reflexive.*

Furthermore for the classic non-reflexive spaces, c_0, ℓ_1, ℓ_∞, it is well known that there are some convex closed bounded subsets which fail to satisfy the FPP. In fact, another question appearing in [23] and very related to the previous one is the following:

QUESTION IV: *Can either* ℓ_1 *or* c_0 *be renormed so that the resulting space has the FPP?*

In the case of ℓ_1, there is a fact which suggests the contrary. Indeed, James' Distortion Theorem states the following:

If X is a Banach space isomorphic to ℓ_1*, then there exists a null sequence* (ϵ_n) *and a sequence* (x_n) *in X such that*

$$(1-\epsilon_k)\sum_{n=k}^{\infty}|t_n| \le \left\|\sum_{n=k}^{\infty} t_n x_n\right\| \le \sum_{n=k}^{\infty}|t_n|$$

for every sequence $(t_n) \in \ell_1$.

In addition, Dowling et al. [14] proved in 2002:

Let X be a Banach space which contains an asymptotically isometric ℓ_1 *-basis, i.e., a sequence* (x_n) *such that there exists a null sequence* (ϵ_n) *satisfying*

$$\sum_{n=1}^{\infty}(1-\epsilon_n)|t_n| \le \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \le \sum_{n=1}^{\infty}|t_n|.$$

Then, X fails to satisfy the FPP.

However, in [13], the authors gave an example of a renorming of ℓ_1, which does not contain any asymptotically isometric ℓ_1-basis, nominally the following:

$$|||(a_n)||| = \sup_{k\in N} \gamma_k \sum_{n=k}^{\infty}|a_n|$$

for all $x=(a_n)\in \ell_1$ where $\gamma_n = 8^n/(1+8^n)$.

This result could provide an expectation of obtaining a renorming of ℓ_1 with the FPP and, indeed, in 2008, Lin [27] has proved that ℓ_1 endowed with this norm does satisfy the FPP. From then, a series of papers have appeared [16,18–20,28], showing either several other renormings of ℓ_1 with the FPP or some renormings of some other non-reflexive spaces with the same property. For instance, in [18], the authors obtain a renorming of the space $\bigoplus_1 \sum_n \ell_1^n$ which satisfies the FPP. It must be noted that this space is not isomorphic to ℓ_1. Thus, Lin's result answers positively the part of Question IV concerning ℓ_1 and negatively the reverse arrow in Question I. It is worth noting that the part of Question IV concerning c_0 remains unanswered.

On the other hand, there are some spaces which cannot be renormed to satisfy the FPP. For instance, by using a refinement of James' Distortion Theorem, Dowling et al. [13] proved in 2002 that every renorming of ℓ_∞ fails to satisfy the FPP. Thus, a very important problem in this theory is now to determine the class of all Banach spaces which can be renormed to satisfy the FPP. In particular, we can recall another question in [23]:

QUESTION VI: *Can any reflexive Banach space be renormed so that the resulting space has the FPP?*

Day et al. [6] proved in 1971 that every separable Banach space has a UCED renorming. Since this property implies normal structure, we have:

Every separable reflexive space can be renormed to satisfy the FPP.

However, as proved by Kutzarova and Troyanski [25] in 1982, there are some reflexive spaces without any equivalent norm which is UCED. Thus, the above result cannot be extended to the class of all reflexive spaces. On the other hand, some relevant renorming results have been proved for nonseparable spaces. For instance, Amir and Lindenstrauss [2] proved in 1968 that every WCG Banach space has an equivalent norm, which is strictly convex, and Troyanski [38] proved in 1970 that every WCG Banach space has an equivalent norm, which is locally uniformly convex.

In 2007, a renorming with the FPP was considered for more general spaces, nominally for spaces which are continuously embedded in $c_0(\Gamma)$ for some set Γ, i.e., there exists a bounded one–one linear operator J from X into $c_0(\Gamma)$. This property is satisfied by subspaces of a space with Markushevich basis (separable spaces, reflexive spaces, WCG spaces, dual of separable spaces as ℓ_∞, etc). Nominally, the following renorming result was proved in [8]:

Assume that X is a Banach space such that there exists a bounded one–one linear operator from X into $c_0(\Gamma)$. Then, X has an equivalent norm such that every non-expansive mapping T for the new norm, defined from a weakly compact convex subset C of X into C, has a fixed point.

In fact, this norm is defined by

$$|x|^2 = \|x\|^2 + \|Jx\|_\infty^2$$

and it is, in general, different from the strict convex norm used by Amir and Lindenstrauss and the locally uniformly convex norm used by Troyanski for weakly compactly generated spaces. Note that if, for instance, X is c_0 and J is the identity, the above norm is just the supremum norm which does not satisfy any convexity property.

3 Stability of the FPP

The result in [27] also proves that the FPP is not preserved under isomorphisms and so negatively answers the following question in [23]:

QUESTION III: *If $(X, \|\cdot\|)$ has the FPP, and if $\|\cdot\|_1$ is a norm on X which is equivalent to $\|\cdot\|$, then does $(X, \|\cdot\|_1)$ satisfy the FPP?*

But this non-preservation of the FPP under isomorphisms leads to another classic problem in Metric Fixed Point Theory which is usually called **the stability problem**; nominally:

Assume that X is a Banach space satisfying the FPP. Does there exist a positive number $d > 1$ such that if Y is another Banach space, which is isomorphic to X and the Banach-Mazur distance $d(X; Y) < d$, then Y satisfies the FPP?

The first results in this direction were obtained by Bynum [5] who proved, for instance, that if $d(X, \ell_p) < 2^{1/p}$ for $1 < p < \infty$, then X satisfies the FPP. These, stability bounds were later improved. For instance, it is known that X satisfies the FPP if $d(X, \ell_p) < \left(1 + 2^{\frac{1}{p-1}}\right)^{\frac{p-1}{p}}$ [7] or $d(X, \ell_p) < c_p$ where c_p is the maximum of the function $2\frac{(1+t)^p+t^p}{1+2t^p}$ [21] or $d(X, \ell_2) < \sqrt{\frac{5+\sqrt{17}}{2}}$ [31]. It can be interesting to compute the maximum value for d satisfying the above condition, that is, the sharpest stability bound of the space (with respect to the FPP). However, nothing is known about how sharp the above bounds are.

On the other hand, in [19] it is shown that if you endow $X = \ell_1$ with the norm defined by Lin and considered in Sect. 2, there is no positive number d greater than 1 such that $d(X, \ell_1) < d$ implies the FPP. In fact, as proved in [10], it is not possible to obtain an equivalent norm in ℓ_1 with a stability bound $d > 1$. Indeed, having again in mind James' Distortion Theorem, it is clear that for every renorming X of ℓ_1 and every $\epsilon > 1$, X contains a subspace Y which is almost isometric to ℓ_1 in the sense that the Banach–Mazur distance between Y and ℓ_1 is less than ϵ. The following result is proved in [10]:

Let $(X, \|\cdot\|)$ be a Banach space and Y a subspace of X. Assume that for some $\epsilon > 0$ there is a norm p on Y such that $(1-\epsilon)\|x\| \le p(x) \le \|x\|$ for every $x \in Y$. Then, there exists a norm q on X such that $q(x) = p(x)$ for every $x \in Y$ and $(1-\epsilon)\|x\| \le q(x) \le \|x\|$ for every $x \in X$.

Hence, the ℓ_1-norm on Y can be lifted to a norm on X which is less than ϵ-separated from the norm of X. Thus, there is a renorming of X, which is as close to the norm of X as required and failing the FPP. Of course, the same situation is true for c_0, because James' Theorem also holds for this space. Thus, we do not know if c_0 can be renormed to satisfy the FPP, but we at least know that it cannot be renormed to satisfy the FPP with a stability bound $d > 1$. Looking at all these facts, the following question is now very natural:

QUESTION: *Does a sharpest stability bound for the FPP in a Banach space exist, which is different from either 1 or* ∞?

Note that if this were not the case, the FPP would be preserved by isomorphisms for all spaces satisfying the FPP with a non-trivial stability bound (Hilbert spaces, uniformly convex spaces, etc). But we only know that the sharpest stability bound is equal to 1 for spaces satisfying James' Theorem and ∞ for finite-dimensional spaces.

A similar study can be achieved for the w-FPP.

QUESTION: *Does a sharpest stability bound for the w-FPP in a Banach space belonging to* $(1, \infty)$ exist?

For instance, we know that for Schur spaces, the sharpest stability bound for the w-FPP is ∞. On the other hand, it is well known that certain spaces failing the w-FPP can be renormed to satisfy it (as commented in Section 2 concerning UCED renormings of separable spaces and having in mind that $L^1([0, 1])$ fails the w-FPP [1]). However, no stability bounds greater than 1 can be obtained by using these renormings. The following example, inspired by the one appearing in [19], illustrates this fact:

Example 1 We know that ℓ_∞ fails to satisfy the w-FPP (because it contains isometrically any separable space as $L_1([0, 1])$). However, it is known [6,41] that it can be renormed to be UCED. The same method can be applied to every Banach space such that there exists a separating sequence in its dual space. Indeed, we consider the equivalent norm in ℓ_∞ defined by

$$|||x|||^2 = \|x\|^2 + \sum_{n=1}^{\infty} \frac{\xi_n^2}{2^n}$$

where $x = (\xi_n)$ and $\|\cdot\|$ is the supremum norm. It is easy to check that this equivalent norm is UCED and so the space X with this norm has normal structure. However, we claim that there is no stability bound, greater than 1, for the w-FPP in $(\ell_\infty, |||\cdot|||)$. Otherwise, there exists $a > 1$ such that $d(X, (\ell_\infty, |||\cdot|||)) < a$ implies that X satisfies the w-FPP. In particular, $(\ell_\infty, |\cdot|)$ satisfies the w-FPP where

$$|x|^2 = \|x\|^2 + \sum_{n=1}^{k} \frac{\xi_n^2}{2^n}$$

whenever $2^k/(2^k - 1) < a^2$ because $|x| \le |||x|||$ and

$$\frac{|x|^2}{|||x|||^2} = 1 - \frac{\sum_{n=k+1}^{\infty} \frac{\xi_n^2}{2^n}}{\|x\|^2 + \sum_{n=1}^{\infty} \frac{\xi_n^2}{2^n}} \ge 1 - \frac{\sum_{n=k+1}^{\infty} \frac{\|x\|^2}{2^n}}{\|x\|^2 + \sum_{n=1}^{\infty} \frac{\|x\|^2}{2^n}} \ge 1 - \sum_{n=k+1}^{\infty} \frac{1}{2^n} = \frac{2^k - 1}{2^k},$$

which implies

$$\sqrt{\frac{2^k}{2^k - 1}}\,|||x||| \le |x| \le |||x|||$$

and so $d((\ell_\infty, |\cdot|), (\ell_\infty, |||\cdot|||) < a$. Hence, the subspace $E = \{(\xi_n) \in \ell_\infty : \xi_n = 0 \text{ if } n \le k\}$ of $(\ell_\infty, |\cdot|)$ satisfies the w-FPP. But this subspace is isometric to $(\ell_\infty, \|\cdot\|)$ which is a contradiction.

To give some answers to the above question, we need to recall the notions of distorted norms, defined in [29] and inspired in James' Distortion Theorem:

If $(X, \|\cdot\|)$ *is a Banach space, an equivalent norm* $|||\cdot|||$ *on* X *is said to be a distorted norm if the space* $(X, |||\cdot|||)$ *does not contain almost isometric copies of* $(X, \|\cdot\|)$, *i.e., there exists* $c > 1$ *such that for every subspace* Y *of* $(X, |||\cdot|||)$ *and every isomorphism* $T : Y \to (X, \|\cdot\|)$ *we have* $\|T\|\|T^{-1}\| > c$. *We will say that* X *is non-distortable if no equivalent norm is distorted.*

In this sense, James' Theorem states that c_0 and ℓ_1 are non-distortable, and Partington [32] has proved that ℓ_∞ is non-distortable. We will denote by $\mathcal{P}(X)$ the metric space formed by all equivalent norms defined on X with the topology of the uniform convergence on the unit ball. With this notation, it is easy to check that the previous example is just a particular case of the following theorem [10]:

Let X *be a Banach space that contains isomorphically a non-distortable space* Y *which fails the w-FPP. Then, the subset of* $\mathcal{P}(X)$ *failing the w-FPP is dense in* $\mathcal{P}(X)$. *In particular, this assertion holds for any Banach space containing* ℓ_∞.

In the opposite direction, it can be interesting to find a Banach space failing the w-FPP such that it can be renormed to satisfy this property and for which there exists $d > 1$ such that the w-FPP is shared by any other isomorphic space Y, which is less than d separated away from X. The usual renormings which we have already commented cannot give a solution to this problem, because these renormings apply to ℓ_∞ and the above result assures that the renormings failing the w-FPP are dense in $\mathcal{P}(\ell_\infty)$.

However, we can follow a very different approach as in [10]. Assume that $(X, \|\cdot\|)$ is a Banach space with a Schauder basis $\{e_n\}$. Consider the set $\mathcal{G}$ formed by all nondecreasing bounded sequences of nonnegative integers $g = \{p(n)\}$. As in [29] (Lemma 5.3), for any $a \in (-1, 0)$ we can consider the equivalent norm on X defined by $\|x\|_a = \sup\{\|g(x)\| : g \in \mathcal{G}\}$ where $g(x) =: \sum_{n=1}^{\infty} a^{p(n)} t_n e_n$ for $g = \{p(n)\}$ and $x = \sum_{n=1}^{\infty} t_n e_n$. The following result was proved in [10]:

Let Y be a Banach space isomorphic to X such that the Banach–Mazur distance between Y and $(X, \|\cdot\|_a)$ is less than $\frac{2}{\sqrt{3+28(1-a^8)}}$. Then, Y satisfies the w-FPP.

As a consequence, we obtain the following.

Let X be a separable Banach space. Then, there exist $d > 1$ and an equivalent norm $|\cdot|$ on X, such that Y satisfies the w-FPP whenever Y is a Banach space isomorphic to X and the Banach-Mazur distance between Y and $(X, |\cdot|)$ is less than d.

Since $C([0, 1])$ and $L_1([0, 1])$ fail the w-FPP, from the above result, we obtain two examples of Banach spaces for which the sharpest stability bound for the w-FPP belongs to $(1, \infty)$.

The similar problem for the w^*-FPP becomes much more simple. Indeed, it is well known that ℓ_1 satisfies the w^*-FPP with stability bound equal to 2. This is the sharpest stability bound because there is a renorming of ℓ_1 such that this space with the resultant norm fails to satisfy the w^*-FPP and the Banach–Mazur distance between the spaces produced endowing ℓ_1 with the standard and the resultant norm is 2. (See [11,26] for details.)

4 Quasi-normal structure and Kannan mappings

Questions VII and VIII in [23] concern the notion of quasi-normal structure. Recall the definition of normal structure:

A Banach space X is said to have normal structure if each of its non-empty bounded convex subset D of X which contains more than one point contains a non-diametral point, i.e., a point x_0 such that $\sup\{\|x_0 - y\| y \in D\} <$ *diam* (D).

A slightly weaker notion is the following:

A subset K of a Banach space X is said to have quasi-normal structure if each bounded closed convex subset H of K for which diam $(K) > 0$ contains a point u such that $\|u - v\| <$ diam (H) for every $v \in H$.

Despite the similarity of the definitions, both notions occur in very different settings. In fact, while normal structure is very important to study the existence of fixed point for non-expansive mappings, quasi-normal structure is very related to the existence of fixed points of Kannan mappings (this is a subject that has often appeared in the fixed point literature in the last 20 years).

Let M be a metric space. A mapping $T : M \to M$ is said to be a Kannan mapping if

$$d(Tx, Ty) \le \frac{1}{2}(d(x, Tx) + d(y, Ty))$$

for every $x, y \in M$.

In [40], the following result is proved:

Let K be a convex weakly compact subset of a Banach space X. Then, every Kannan mapping $T : K \to K$ has a (unique) fixed point if and only if K has quasi-normal structure.

In [39] it is proved that X has quasi-normal structure if either X is separable or X is strictly convex. In particular, by [2] every Banach space which can be embedded in $c_0(\Gamma)$ (and so every reflexive Banach space) can be renormed to have quasi-normal structure. Due to this fact, the following question seems very natural

QUESTION: *Assume that X is a Banach space which can be embedded in $c_0(\Gamma)$. Can X be renormed to have normal structure?*

The answer to this question is already known [13]. When Γ is uncountable, any renorming of $c_0(\Gamma)$ contains an asymptotically isometrical basis of c_0 and so it fails to have a normal structure. However, the question remains unanswered if X is reflexive. This is, in fact, **QUESTION VII** in [23].

In fact, having in mind that every reflexive space is WCG and so it can be continuously embedded in $c_0(\Gamma)$, the above question is strongly connected with the following from [23]:

QUESTION V: *Does every reflexive space X have an equivalent norm relative to which it has normal structure?*

Since a reflexive space cannot contain a copy of c_0, the above consideration cannot be applied for a reflexive space and so the problem remains open.

Next question in [23] is the following:

QUESTION VIII: *If X is a reflexive Banach space and has the FPP, then does it have quasi-normal structure?*

By using certain results in [37] we can prove that is not, in general, the case. Indeed, consider the space $\ell_2(\Gamma)$ where Γ is uncountable. Of course, $\ell_2(\Gamma)$, as any Hilbert space, is strictly convex and so it has quasi-normal structure. But when we consider the equivalent norm

$$\|x\|_{\sqrt{2}} = \max\left\{\frac{\|x\|_2}{\sqrt{2}}, \|x\|_\infty\right\}$$

we know that ℓ_2 with the resultant norm satisfies the FPP (note that the Banach–Mazur distance between ℓ_2 with the Euclidean norm and ℓ_2 with the norm $\|\cdot\|_{\sqrt{2}}$ is less than the stability bound mentioned in Sect. 3). However, $(\ell_2(\Gamma), \|x\|_{\sqrt{2}})$ does not have quasi-normal structure. Indeed, let C be the closure of the convex hull of $\{e_\gamma : \gamma \in \Gamma\}$ in $(\ell_2(\Gamma), \|x\|_{\sqrt{2}})$. It is clear that diam $C = 1$ and for every $x \in \ell_2(\Gamma)$ we know that $\#\text{supp}x \leq \aleph_0$. Thus, $\|x - e_\gamma\|_{\sqrt{2}} \geq 1$ if $\gamma \notin \text{supp}x$.

We see that there is no connection between the w-FPP for Kannan maps and the w-FPP for non-expansive maps. Indeed, we know that $L_1([0, 1])$ fails to satisfy the w-FPP for non-expansive mappings, but satisfies the w-FPP for Kannan maps because it is separable. The space $(\ell_2(\Gamma), \|x\|_{\sqrt{2}})$ is an example of the converse situation.

5 The FPP and unbounded sets

Section 5 in the Kirk's paper is entitled "The FPP and unbounded sets". Despite that many papers have appeared proving the existence of fixed points for non-expansive mappings defined on *bounded* convex closed subsets of certain classes of Banach spaces, only few have considered the existence of fixed point for *unbounded* sets. The problem is completely solved only in the case of Hilbert spaces, because Ray [33] has proved that every *unbounded* convex subset of a Hilbert space fails to satisfy the FPP. The proof is strongly based upon the structure of Hilbert spaces, nominally, on the existence of an orthogonal basis with some specific properties. A much more simple proof of the same result has been given by Sine [36], but his proof is still based upon a property that, for dimension greater than 2, is a characteristic of Hilbert spaces, namely that every convex closed subset of a Hilbert space is a non-expansive retraction of the whole space. Due to this strong dependence on some characteristic properties of Hilbert spaces, one could think that the failure of the FPP for every unbounded convex set could be a characterization of Hilbert spaces. This is the motivation of the following question in [22]:

QUESTION XIV: *Suppose that X is a Banach space which has the property that a closed convex subset C of X has the FPP only if C is bounded. Is X a Hilbert space?*

Recently, it has been proved [9] that there are Banach spaces other than Hilbert spaces which satisfy that convex unbounded sets fail to satisfy the FPP. Nominally, it is proved in [9] that any unbounded convex set in c_0 fails to satisfy this property. The method in the proof had to be completely different from that in Hilbert spaces. In fact, the main tool used in the proof is the failure of the FPP for some **bounded** convex closed subsets of c_0. The failure of the FPP for certain bounded convex closed subsets of c_0 is well known (see, for instance, [30]). In fact, Dowling et al. [15] proved in 2004 the following complete characterization of the FPP for bounded convex closed subsets of c_0:

Let C be a bounded convex closed set in c_0. Then, C satisfies the FPP if, and only if, C is weakly compact.

Their proof is based on the following fact, which is a consequence of Eberlein-Šmulian and Alouglu Theorems:

Every bounded convex closed set in c_0, which is not weakly compact, contains a sequence which is $\sigma(\ell_\infty, \ell_1)$ -convergent to a point $u \in \ell_\infty \setminus c_0$.

The second main tool, needed in the above result, is taken from the theory concerning the approximate fixed point property (AFPP). Recall that

A subset C of a Banach space X is said to satisfy the AFPP (for non-expansive mappings) if $\inf\{\|x - Tx\| : x \in C\} = 0$ *for every non-expansive mapping* $T : C \to C$.

In contrast to the rare knowledge about the existence of fixed point for non-expansive mappings in unbounded sets, the problem of the existence of approximate fixed points is completely solved in this setting. Indeed, Reich proved the following characterization of the sets satisfying the AFPP in a reflexive space:

Let C be a convex closed subset of a reflexive Banach space X. Then, C satisfies the AFPP if and only if C is linearly bounded, that is, $C \cap r$ *is bounded for every line r in X.*

Shafrir [35] defined the notion of directionally boundedness (which is equivalent to linearly boundedness in reflexive spaces) and extended the above result to arbitrary Banach spaces. In [35], the notion of directionally bounded set is stated for any hyperbolic metric space. Since we are only interested in the case of a Banach space, we can consider the following equivalent definition (see [35] Theorems 2.4 and 3.2 for the equivalence):

A convex subset C of a Banach space X is directionally bounded if for every sequence (x_n) *in C such that* $\|x_n\| \to \infty$ *and every* $f \in X^*$, $\|f\| = 1$ *one has*

$$\limsup_{n\to\infty} f\left(\frac{x_n}{\|x_n\|}\right) < 1.$$

By using this notion I. Shafrir gave the following complete characterization of the convex closed subsets of an arbitrary Banach space with the AFPP:

Let C be a convex closed subset of a Banach space X. Then, C satisfies the AFPP if, and only if, C is directionally bounded.

In [9], the behavior of the divergent sequences defined on unbounded convex directionally bounded set in c_0 is studied. Note that such a sequence satisfies that

$$\lambda(k) = \limsup_{n\to\infty} \frac{x_n(k)}{\|x_n\|} < 1$$

because the mapping $x \to x(k)$, where $x = (x(k))_k$, is a normalized functional in c_0^*. Thus, $\lambda = (\lambda(k))_k$ lies in the unit ball of ℓ_∞, but we actually have the following:

Let C be a directionally bounded convex set in c_0 *and* (x_n) *a sequence in C such that* $\|x_n\| \to \infty$. *Then,* $\|\lambda\| > |\lambda(k)|$ *for every* $k \in N$. *In particular,* $\lambda \notin c_0$.

By using this lemma, it can be proved that every unbounded convex directionally bounded set in c_0 contains a sequence which is $\sigma(\ell_\infty, \ell_1)$-convergent to a point $u \in \ell_\infty \setminus c_0$ and as a consequence of this fact, the proof in [15] can be adapted to the unbounded case to prove that every unbounded convex closed set in c_0 fails to satisfy the FPP. Hence, concerning Question XIV, we know that the failure of the FPP for every unbounded convex closed set is not a characteristic of Hilbert spaces.

Remark 1 Another question which has long remained open in Fixed Point Theory and which is very related to this previous one, is the following:

QUESTION XV: *Does there exist an unbounded convex subset of a Banach space which satisfies the FPP for non-expansive self-mappings?*

The above considerations show that this is not the case in Hilbert spaces and in c_0, but it is completely unknown what happens for any other Banach space. The existence of approximated fixed points for non-expansive mappings in any directionally bounded set of a Banach space and unbounded directionally bounded convex set in any Banach space [35] can help us to guess that such a set could exist.

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