Michigan Math. J. 58 (2009)

Universal Functions with Prescribed Zeros and Interpolation Properties

LUIS BERNAL-GONZÁLEZ, ANTONIO BONILLA, & MARKUS NIEß

Dedicated to Professor José Méndez on the occasion of his sixtieth birthday

1. Introduction

Roughly speaking, universality means "existence of a dense orbit". Thus, in some sense, universal functions are "uncontrolled". In this paper, we study the existence of functions that are universal with respect to differential operators and that are, at the same time, "controlled" by prescribed interpolation properties, including prescribed zeros and multiplicities. Precise definitions are given in what follows.

We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$, and $\mathbb{N}_0$ the set of positive integers, the set of all integers, the complex plane, and the set $\mathbb{N} \cup \{0\}$, respectively. If $A \subset \mathbb{C}$ then A°, $\bar{A}$, and ∂A will stand (respectively) for the interior, the closure, and the boundary of A in $\mathbb{C}$. We use $\mathbb{C}_\infty$ to denote the extended complex plane. Recall that a *domain* is a nonempty connected open subset of $\mathbb{C}$.

Let $H(\Omega)$ be the linear space of holomorphic functions on a domain Ω. In particular, $H(\mathbb{C})$ is the space of entire functions. Consider the metric

$$d(f,h) := \sum_{j=1}^{\infty} \frac{1}{2^j} \cdot \frac{\|f-h\|_{C_j}}{1+\|f-h\|_{C_j}} \quad (f,h \in H(\Omega)),$$

where

$$\|f-h\|_M := \sup_{z\in M} |f(z)-h(z)|.$$

Here $\{C_j : j \geq 1\}$ is a fixed exhaustive sequence of compact subsets of Ω; that is, $C_j \subset C_{j+1}^\circ$ $(j \geq 1)$ and $\Omega = \bigcup_{j=1}^{\infty} C_j$. It is possible to select $\{C_j : j \geq 1\}$ so that each connected component of $\mathbb{C}_\infty \setminus C_j$ contains some connected component of $\mathbb{C}_\infty \setminus \Omega$; in particular, if Ω is simply connected (i.e., if $\mathbb{C}_\infty \setminus \Omega$ is connected) then we can choose every C_j without "holes".

The aforementioned metric d generates on $H(\Omega)$ the topology of uniform convergence on compact subsets of Ω; see [5]. In the sequel, we will always consider the complete metric space $(H(\Omega), d)$.

Received April 7, 2008. Revision received February 24, 2009.

The first author has been partially supported by the Plan Andaluz de Investigación de la Junta de Andalucía FQM-127 and by MEC Grant MTM2006-13997-C02-01. The second author has been partially supported by MEC and FEDER MTM2008-05891. Both authors have been partially supported by MEC Acción Especial MTM2006-26627-E.

According to Baire's category theorem, every complete metric space X is a Baire space; in other words, the intersection of countably many open dense subsets of X is also dense in X. In a Baire space X, a subset is *residual* when it contains a dense G_δ-subset of X. In particular, this applies to $(H(\Omega), d)$.

Throughout this paper, we will use the following notion of universality.

DEFINITION 1. Let (X, d_X) and (Y, d_Y) be metric spaces and let $\mathcal{L} = (L_j)_{j\in J}$ be a family of continuous mappings $L_j : X \to Y$.

(i) An element $x \in X$ is called $\mathcal{L}$-*universal* if

$$Y = \overline{\{L_j x : j \in J\}}.$$

The set of all such elements $x \in X$ is denoted by $\mathcal{U}(\mathcal{L})$. The family $\mathcal{L}$ is called *universal* if $\mathcal{U}(\mathcal{L}) \neq \emptyset$.

(ii) $\mathcal{L}$ is called *topologically transitive* if it has the following property: For every $x \in X$, $y \in Y$, and $\varepsilon > 0$, there is a $z \in X$ and a $j \in J$ such that

$$d_X(x, z) < \varepsilon \quad \text{and} \quad d_Y(y, L_j z) < \varepsilon.$$

In fact, the last definition can be easily extended to the setting of topological spaces, but such a generality will not be needed here. If $X = Y$ and $L : X \to X$ is a continuous self-map, then L is said to be *universal* (topologically transitive, resp.) if the family $\mathcal{L} = \{L^n : n \geq 1\}$ of its iterates is universal (topologically transitive, resp.). If X, Y are topological vector spaces and the L_j (or L, if we are dealing with self-maps) are linear, then it is customary to say *hypercyclic* instead of *universal*. Readers interested in these concepts are referred to the surveys [10] and [13].

In 1952, MacLane [14] stated that there exist entire functions φ such that the set of derivatives $\{\varphi^{(n)} : n \in \mathbb{N}\}$ is dense in $(H(\mathbb{C}), d)$ or, equivalently, $\varphi \in \mathcal{U}(\mathcal{D})$ for $\mathcal{D} = \{D^n : n \in \mathbb{N}\}$, where D is the differentiation operator on $H(\mathbb{C})$ given by $Df = f'$. In 1994, Herzog [12] posed the following question: Which additional properties of elements of X are compatible with universality? For $\mathcal{U}(\mathcal{L})$ residual and $A \subset X$ a G_δ-subset, he proved that under certain conditions on A and $\mathcal{L}$ (see Section 2) the set $A \cap \mathcal{U}(\mathcal{L})$ is residual in A.

By using his theorem, Herzog derived the existence of $\mathcal{D}$-universal functions having a zero-free qth and $(q+1)$th derivative $(q \in \mathbb{N}_0)$. This result was extended by the first author (see [1] and [2, Thm. 12]) for infinite-order differential operators $\Phi(D) = \sum_{n=0}^{\infty} a_n D^n$, where D is again the differentiation operator (with $D^0 = I$, the identity operator) and $\Phi(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function of *subexponential type*; that is, given $\varepsilon > 0$, there is a positive constant $A = A(\varepsilon)$ with $|\Phi(z)| \leq Ae^{\varepsilon|z|}$ for all $z \in \mathbb{C}$. Recall that an entire function Φ is said to be of *exponential type* if there are positive constants A and B with $|\Phi(z)| \leq Ae^{B|z|}$ for all $z \in \mathbb{C}$. Of course, every entire function of subexponential type is of exponential type. By an *operator* we mean a continuous linear self-map on a topological vector space. It is not difficult to see that if Φ is of subexponential type then $\Phi(D)$ defines an operator on $H(\Omega)$ (and on $H(\mathbb{C})$, assuming only that Φ is of exponential type).

In [15, Thm. 3.3], the third author showed the existence of $\mathcal{D}$-universal functions that solve a given interpolation problem in $\mathbb{C}$. Independently and with a different approach, Costakis and Vlachou [7] arrived at the same conclusion for any simply connected domain. Moreover, in [15, Thm. 2.3] it is proved that there are MacLane-universal entire functions having zeros at prescribed points with prescribed orders. On the other hand, a celebrated result due to Godefroy and Shapiro (see Section 2) asserts the universality on $H(\mathbb{C})$ of every differential operator $\Phi(D)$ as described here that is not a multiple of the identity. Recently, the first author [3] demonstrated the existence of $\Phi(D)$-universal holomorphic functions with given interpolation properties.

Our aim in this paper is to prove the existence of holomorphic functions f on a simply connected domain Ω that *simultaneously* satisfy the following conditions:

(a) f is $\Phi(D)$-universal;
(b) f has zeros (only) at the points of a given subset of Ω, with preassigned orders; and
(c) f assumes prescribed values at prescribed points.

This will be accomplished in Section 3.

The combination of universal Taylor series with the property (b) or (c) has been considered by Costakis [6]. His improvement on Herzog's theorem is also one of our auxiliary results (see Theorem 3).

2. Preliminary Results

This section is devoted to establishing a number of statements that will be needed in the proof of our main result. We begin by presenting the following version, due to Grosse-Erdmann [11], of the well-known Birkhoff transitivity theorem.

THEOREM 2. *Let A be a nonempty G_δ-subset of a complete metric space, let Y be a separable metric space, and let $\mathcal{L} = (L_j)_{j\in J}$ be a family of continuous mappings $L_j : A \to Y$. Then the following assertions are equivalent*:

(i) *there is a dense set of elements of A that are $\mathcal{L}$-universal*;
(ii) *$\mathcal{L}$ is topologically transitive*.

If either condition holds then the set $\mathcal{U}(\mathcal{L})$ is a dense G_δ-set, and so is residual, in A.

Recall that a *Polish space* is a separable complete metric space. Next, we state the Herzog criterion [12] concerning inherited universality—more precisely, the (slightly improved) version due to Costakis [6].

THEOREM 3. *Assume that X is a Polish space and that Y is a separable metric space. Also let d_X, d_Y be the corresponding metrics. Let $L_n : X \to Y$ be a sequence of continuous functions with $\mathcal{U}(\{L_n\})$ residual in X. For any $B \subset X$, let $L_n|_B$ be the restriction of L_n to B. Consider a sequence $\{B_k\}_{k\in\mathbb{N}}$ of Baire spaces that are subsets of X satisfying*

$$A := \bigcap_{k\in\mathbb{N}} B_k \neq \emptyset, \tag{1}$$

$$B_k \cap \mathcal{U}(\{L_n\}) \text{ is residual in } B_k. \tag{2}$$

If, in addition,

$$\lim_{k\to\infty} \sup_{n\in\mathbb{N}} \inf_{h\in A} (d_X(b_k, h) + d_Y(L_n b_k, L_n h)) = 0 \tag{3}$$

holds for every sequence $\{b_k\}_{k\in\mathbb{N}}$ with $b_k \in B_k$, then the set $\mathcal{U}(\{L_n|_A\})$ is residual in A.

We also present the following result about the "internal control" property of differential operators (see [4]). We omit its easy proof, which is based on the Cauchy integral formula for derivatives.

THEOREM 4. *Let $\Omega \subset \mathbb{C}$ be a domain and let Φ be an entire function of subexponential type. Assume that K, L are compact sets in $\mathbb{C}$ with $L \subset K^\circ$. Then there exists a constant $C = C(K, L) \in (0, +\infty)$ such that*

$$\|\Phi(D) f\|_L \leq C \|f\|_K \quad \text{for all } f \in H(\Omega).$$

The previously mentioned theorem of Godefroy and Shapiro states that if Φ is an entire function of exponential type then the differential operator $\Phi(D)$ is universal on $H(\mathbb{C})$ [8, Sec. 5]. By restricting the class of operators, we may extend the result to all domains without holes. Specifically, we have the following assertion, which can be found in [2, Thm. 8].

THEOREM 5. *Let Φ be a nonconstant entire function of subexponential type and consider the operator $S = \Phi(D)\colon H(\Omega) \to H(\Omega)$, where Ω is a simply connected domain of $\mathbb{C}$. Then S is universal. In fact, $\mathcal{U}(\mathcal{S})$ is residual in $(H(\Omega), d)$ for $\mathcal{S} = \{S^n : n \in \mathbb{N}\}$.*

The final lemma in this section combines interpolation and approximation. It is a kind of Hermite interpolation using exponential functions instead of polynomials. The result improves [3, Lemma 2.1]. By e_a ($a \in \mathbb{C}$) we denote the function $e_a(z) := \exp(az)$, and span X_0 will stand for the linear span of a subset X_0 of a vector space.

LEMMA 6. *Assume that L, K are compact subsets of a domain $\Omega \subset \mathbb{C}$ with $L \subset K^\circ$, that $a_1, \ldots, a_n$ are different points in L, that m is a natural number, and that G is a nonempty open subset of $\mathbb{C}$. Then there exist a positive constant $M = M(L, K, a_1, \ldots, a_n, G, m)$ and a finite set of functions*

$$\{\alpha_{j,k} : j = 1, \ldots, n;\ k = 0, \ldots, m-1\} \subset \operatorname{span}\{e_a : a \in G\},$$

depending only on G, m, and the points $a_1, \ldots, a_n$, that satisfy the following property. For each pair of functions $f, h \in H(\Omega)$, the function φ defined by

$$\varphi(z) = h(z) + \sum_{j=1}^{n} \sum_{k=0}^{m-1} (f^{(k)}(a_j) - h^{(k)}(a_j))\alpha_{j,k}(z)$$

satisfies:

(a) $\varphi \in H(\Omega)$;
(b) $\varphi^{(\sigma)}(a_j) = f^{(\sigma)}(a_j)$ $(j = 1, \dots, n;\ \sigma = 0, \dots, m-1)$;
(c) $\|\varphi - f\|_L \le M\|h - f\|_K$.

Proof. We can assume that $G \neq \mathbb{C}$, so we choose a point $c \in G$ and select a positive number d satisfying

$$d < \frac{1}{m(n+1)} \inf\{|z - c| : z \in \mathbb{C} \setminus G\} \tag{4}$$

and

$$d < \min_{\substack{j,l \in \{1,\dots,n\} \\ j \neq l}} \frac{1}{|a_j - a_l|}. \tag{5}$$

We define

$$\Pi_j(z) := \prod_{\substack{l=1 \\ l \neq j}}^{n} (e_d(z - a_l) - 1)^m \quad (j = 1, \dots, n),$$

$$\beta_{j,k}(z) := e_c(z - a_j) \frac{(e_d(z - a_j) - 1)^k}{k!\, d^k} \frac{\Pi_j(z)}{\Pi_j(a_j)} \qquad (j = 1, \dots, n;\ k = 0, \dots, m-1).$$

From (5), it follows that

$$0 < d|a_j - a_l| < 1 < 2\pi$$

for all $j, l \in \{1, \dots, n\}$ with $j \neq l$, so $\Pi_j(a_j) \neq 0$ for all $j \in \{1, \dots, n\}$. Also, an easy calculation shows that

$$\beta_{j,k}^{(\sigma)}(a_t) = \begin{cases} 1 & \text{if } t = j \text{ and } \sigma = k, \\ 0 & \text{if } t \neq j \text{ or } \sigma < k, \end{cases}$$

where $\sigma \in \{0, 1, \dots, m-1\}$ is always assumed. We have no information about the values of $\beta_{j,k}^{(\sigma)}(a_t)$ for $t = j$ and $\sigma > k$. Hence, for each $j \in \{1, \dots, n\}$ we set

$$\alpha_{j,m-1}(z) := \beta_{j,m-1}(z) \quad \text{and}$$

$$\alpha_{j,k}(z) := \beta_{j,k}(z) - \sum_{\nu=k+1}^{m-1} \beta_{j,k}^{(\nu)}(a_j)\alpha_{j,\nu}(z) \quad (k = 0, 1, \dots, m-2),$$

where the last expression makes sense only if $m \ge 2$. By induction we obtain

$$\alpha_{j,k}^{(\sigma)}(a_t) = \begin{cases} 1 & \text{if } t = j \text{ and } \sigma = k, \\ 0 & \text{if } t \neq j \text{ or } \sigma \neq k. \end{cases}$$

Observe that each function $\beta_{j,k}$, and so each function $\alpha_{j,k}$, is a finite linear combination of functions of the form e_{c+sd} with $0 \le s < m(n+1)$. But each point $c + sd$ is in G because of (4). Hence, the functions $\alpha_{j,k}$ are in $\operatorname{span}\{e_a : a \in G\}$.

Let L, K be compact subsets as in the statement. By using Theorem 4 (with $\Phi(z) = z^k$) or simply the Cauchy estimates, we obtain

$$\|g^{(k)}\|_L \leq M_k \|g\|_K \quad (k \in \mathbb{N}_0,\ g \in H(\Omega))$$

for some positive constant M_k that is independent of g. Now we set

$$M := 1 + \sum_{j=1}^{n} \sum_{k=0}^{m-1} M_k \|\alpha_{j,k}\|_L.$$

Finally, if we fix holomorphic functions f, h on Ω and define the corresponding function φ as in the statement, then properties (a), (b), and (c) are obvious. □

3. Main Result

Suppose that Ω is a domain in $\mathbb{C}$, and let $w = \{w_k\}_{k\in\mathbb{N}}$, $\gamma = \{\gamma_k\}_{k\in\mathbb{N}}$, $m = \{m_k\}_{k\in\mathbb{N}}$, and $\beta = \{\beta_k\}_{k\in\mathbb{N}}$ be sequences satisfying: $w \subset \Omega$, $\gamma \subset \Omega$, $\beta \subset \mathbb{C} \setminus \{0\}$, $m \subset \mathbb{N}$, $w \cap \gamma = \emptyset$; the points $w_k, k \in \mathbb{N}$ (as well as the points $\gamma_k, k \in \mathbb{N}$) are pairwise distinct; and neither w nor γ have accumulation points in Ω. To each such set of sequences we can associate the set $A = A(w, m; \gamma, \beta)$ defined by

$$\begin{aligned} A := \{f \in H(\Omega) : f(w_k) = 0 \text{ of order } m_k,\ f(\gamma_k) = \beta_k\ (k \in \mathbb{N}), \\ \text{and } f(z) \neq 0 \text{ if } z \in \Omega \setminus w\}. \end{aligned} \tag{6}$$

In other words, A is the set of holomorphic functions in Ω with prescribed zeros and interpolation conditions (corresponding to w, m, γ, β). First, we note in the following proposition that A possesses good topological properties.

PROPOSITION 7. *The set A defined in* (6) *is a nonempty G_δ-subset of $H(\Omega)$; in addition, it is a Baire space when endowed with the compact-open topology inherited from $H(\Omega)$.*

Proof. By the Weierstraß factorization theorem (see e.g. [18, Thm. 15.9]), we know that there exists a function $h \in H(\Omega)$ such that

$$h(w_k) = 0 \text{ of order } m_k\ (k \in \mathbb{N}) \quad \text{and} \quad h(z) \neq 0 \text{ if } z \neq w_k\ (k \in \mathbb{N}).$$

Each γ_k is different from all the w_k, so $\beta_k / h(\gamma_k)$ is well-defined. For each $k \in \mathbb{N}$, let α_k be a fixed logarithm of $\beta_k / h(\gamma_k)$. By [18, Thm. 15.13] there exists a function $g \in H(\Omega)$ with $g(\gamma_k) = \alpha_k$. Then the function

$$f(z) := e^{g(z)} \cdot h(z)$$

is an element of A, hence $A \neq \emptyset$.

Now consider the exhaustive sequence $\{C_n : n \geq 1\}$ given in Section 1. Setting

$$M_n := \{z \in C_n : |z - w_k| \geq 1/n \text{ for all } k \in \mathbb{N}\},$$

we obtain that A is the intersection of the open sets

$$\begin{aligned} A_n := \{f \in H(\Omega) : |f^{(\nu)}(w_k)| < \tfrac{1}{n} \text{ for } 0 \leq \nu < m_k,\ f^{(m_k)}(w_k) \neq 0, \\ |f(\gamma_k) - \beta_k| < \tfrac{1}{n} \text{ for } 1 \leq k \leq n,\ \min_{z\in M_n} |f(z)| > 0\}, \end{aligned}$$

where $n \geq 1$. So, A is a G_δ-subset of $(H(\Omega), d)$. Finally, since A is a G_δ-subset in a complete metric space, Alexandroff's theorem (see [17]) guarantees that the topological space A is completely metrizable. Hence, it is a Baire space. □

Now, we suppose that Ω is a simply connected domain. Before establishing the promised result on interpolation in its full strength, we present the following "discrete" version of it, which will be used in the proof of Theorem 9.

LEMMA 8. *Let L be a compact subset of Ω, let $w_1, \ldots, w_{n_1}$ and $\gamma_1, \ldots, \gamma_{n_2}$ be pairwise distinct points in L°, and let $m_1, \ldots, m_{n_1} \in \mathbb{N}$ and $\beta_1, \ldots, \beta_{n_2} \in \mathbb{C} \setminus \{0\}$, where $n_1, n_2 \in \mathbb{N}$. We define*

$$\begin{aligned} B := \{f \in H(\Omega) : {} & f(w_k) = 0 \text{ of order } m_k \text{ if } k = 1, \ldots, n_1, \\ & f(\gamma_k) = \beta_k \text{ if } k = 1, \ldots, n_2, \\ & f(z) \neq 0 \text{ if } z \in L \setminus \{w_k : k = 1, \ldots, n_1\}\}. \end{aligned}$$

Endow B with the compact-open topology inherited from $H(\Omega)$. Let Φ be a nonconstant entire function of subexponential type, and let $S = \Phi(D)$ and $\mathcal{S} = \{S^n : n \in \mathbb{N}\}$. Then $\mathcal{U}(\mathcal{S}) \cap B$ is a dense G_δ-subset of B.

Proof. In a similar manner as for the set A in Proposition 7, we deduce that B is also a G_δ-subset of $H(\Omega)$. According to Theorem 2, it suffices to show that $\mathcal{S}$ is topologically transitive. We therefore fix $f \in B$, $g \in H(\Omega)$, a compact set $K \subset \Omega$, and a number $\varepsilon > 0$. We have to show the existence of some $\varphi \in B$ and some $N \in \mathbb{N}$ with

$$\sup_{z \in K} |\varphi(z) - f(z)| < \varepsilon \tag{7}$$

and

$$\sup_{z \in K} |S^N \varphi(z) - g(z)| < \varepsilon. \tag{8}$$

Since $f \not\equiv 0$ and Ω is simply connected, we can find a compact set $L_1 \subset \Omega$ satisfying the following properties:

- $\mathbb{C} \setminus L_1$ is connected;
- $K \cup L \subset L_1^\circ$;
- ∂L_1 is a regular Jordan curve; and
- f is zero-free on ∂L_1.

Thus

$$d := \min_{z \in \partial L_1} |f(z)| > 0.$$

If f has further zeros on L_1° (apart from $w_1, \ldots, w_{n_1}$), we denote them by $\zeta_1, \ldots, \zeta_{n_3}$. Denote by $r_1, \ldots, r_{n_3}$ their respective orders.

By hypothesis, the entire function Φ is nonconstant, so the open set

$$G := \{z \in \mathbb{C} : |\Phi(z)| < 1\}$$

is nonempty. Choose a compact subset $L_2 \subset \Omega$ with $L_2^\circ \supset L_1$, and let

$$M = M(L_1, L_2, w_1, \dots, w_{n_1}, \gamma_1, \dots, \gamma_{n_2}, \zeta_1, \dots, \zeta_{n_3}, G, \max\{m_1, \dots, m_{n_1}, r_1, \dots, r_{n_3}\})$$

be the positive constant provided by Lemma 6. According to Theorem 5, there exists an $\mathcal{S}$-universal function $h \in H(\Omega)$ with

$$\|h - f\|_{L_2} < \frac{1}{M} \min\{d, \varepsilon\}.$$

By Lemma 6 one can find a function $\varphi = h + \alpha \in H(\Omega)$, with $\alpha \in \operatorname{span}\{e_a : a \in G\}$, that satisfies

$$\begin{aligned} \varphi^{(\sigma)}(w_k) &= f^{(\sigma)}(w_k) = 0 \quad (\sigma = 0, \dots, m_k - 1;\ k = 1, \dots, n_1), \\ \varphi^{(\sigma)}(\zeta_k) &= f^{(\sigma)}(\zeta_k) = 0 \quad (\sigma = 0, \dots, r_k - 1;\ k = 1, \dots, n_3), \\ \varphi(\gamma_k) &= f(\gamma_k) = \beta_k \quad (k = 1, \dots, n_2), \end{aligned}$$

and

$$\|\varphi - f\|_{L_1} \le M \|h - f\|_{L_2} < \min\{d, \varepsilon\}. \tag{9}$$

Condition (7) follows directly from (9) and from the fact $K \subset L_1$. By (9) we have

$$|\varphi(z) - f(z)| < d \le |f(z)| \quad \text{for all } z \in \partial L_1,$$

so we conclude from Rouché's theorem [18, Chap. 10] that the functions f and φ have the same number of zeros, counting multiplicities, in L_1°. Owing to the interpolation properties of φ (observe that the order of the zero w_k, ζ_k for φ is at least m_k, r_k, respectively), it follows that $\varphi \in B$.

It remains to show the existence of an $N \in \mathbb{N}$ such that (8) is fulfilled. By definition, $|\Phi(a)| < 1$ for every $a \in G$. Since $\Phi(D)e_c = \Phi(c)e_c$ for all $c \in \mathbb{C}$, we have $\Phi(D)^n e_c = \Phi(c)^n e_c$ for every $n \in \mathbb{N}$. Hence $\Phi(D)^n e_a \to 0$ $(n \to \infty)$ compactly on $\mathbb{C}$ whenever $a \in G$ and, as a result,

$$\Phi(D)^n \alpha \to 0 \ (n \to \infty) \ \text{ compactly on } \mathbb{C} \text{ for every } \alpha \in \operatorname{span}\{e_a : a \in G\}.$$

In particular, there is an $N_1 \in \mathbb{N}$ such that

$$\|S^n \alpha\|_K < \varepsilon/2 \quad (n \ge N_1).$$

Because h is $\mathcal{S}$-universal, there is an $N \ge N_1$ with $\|S^N h - g\|_K < \varepsilon/2$. Finally, since $\varphi = h + \alpha$ with $\alpha \in \operatorname{span}\{e_a : a \in G\}$, the linearity of S together with the triangle inequality drives us to (8), as required. □

We are now ready to state our main result.

THEOREM 9. *Assume that $\Omega \subset \mathbb{C}$ is a simply connected domain and that Φ is a nonconstant entire function of subexponential type. Let $S = \Phi(D)$ and $\mathcal{S} = \{S^n : n \in \mathbb{N}\}$. Suppose that $w = \{w_k\}$, $\gamma = \{\gamma_k\}$, $m = \{m_k\}$, and $\beta = \{\beta_k\}$ are sequences as in the beginning of this section, and suppose the set $A = A(w, m; \gamma, \beta)$ is defined as in* (6). *Then the set $A \cap \mathcal{U}(\mathcal{S})$ is residual in A. In particular, in A there is a dense G_δ-subset all of whose functions are $\Phi(D)$-universal.*

Proof. The essential tool for the proof of this theorem will be Theorem 3. Therefore, let $X := (H(\Omega), d) =: Y$. Recall that $(H(\Omega), d)$ is a Polish space, so it is a separable metric space as well. Let $L_n = S^n$ $(n \in \mathbb{N})$. From Theorem 5 we know that $\mathcal{U}(\mathcal{S})$ is residual in $(H(\Omega), d)$.

Without loss of generality, we can assume that the exhaustive sequence $\{C_k : k \in \mathbb{N}\}$ of compact sets defining the metric d of $H(\Omega)$ (see the Introduction) satisfies that there are two strictly increasing sequences $\{j_1(k)\}_1^\infty, \{j_2(k)\}_1^\infty \subset \mathbb{N}$ such that $C_k \cap w = \{w_1, \dots, w_{j_1(k)}\} = C_k^\circ \cap w$ and $C_k \cap \gamma = \{\gamma_1, \dots, \gamma_{j_2(k)}\} = C_k^\circ \cap \gamma$. We define

$$\begin{aligned} B_k := \{f \in H(\Omega) : f(w_j) = 0 \text{ of order } m_j \text{ if } j = 1, \dots, j_1(k), \\ f(\gamma_j) = \beta_j \text{ if } j = 1, \dots, j_2(k), \\ f(z) \neq 0 \text{ if } z \in C_k \backslash \{w_j : j = 1, \dots, j_1(k)\}\}. \end{aligned}$$

Obviously, the intersection of the sets B_k is exactly the set A, which is nonempty by Proposition 7. An argument similar to the one given in the proof of that proposition shows that each B_k is a Baire space. Now, Lemma 8 shows the correctness of condition (2) in Theorem 3.

In order to apply Theorem 3, it remains to prove that for every sequence $\{b_k\}_{k \in \mathbb{N}}$ with $b_k \in B_k$ we have

$$\lim_{k \to \infty} \sup_{n \in \mathbb{N}} \inf_{\varphi \in A} (d(b_k, \varphi) + d(S^n b_k, S^n \varphi)) = 0. \tag{10}$$

Let $k \in \mathbb{N}$ and $f \in B_k$ be fixed. Hence, the zeros of f in C_k are exactly given by the points w_j (of orders m_j) with $j \in \{1, \dots, j_1(k)\}$. Choose a Jordan subdomain $U_k \subset \Omega$ such that $U_k \supset C_k$ and

$$w \cap (U_k \setminus C_k) = \emptyset = \gamma \cap (U_k \setminus C_k),$$

which is possible because neither w nor γ have accumulation points in Ω.

Let $g \in A$. Since f/g is holomorphic and zero-free in U_k, there exists a function ϕ holomorphic in U_k such that

$$\frac{f(z)}{g(z)} = e^{\phi(z)} \quad (z \in U_k). \tag{11}$$

For $z = \gamma_j$ with $j \leq j_2(k)$, the left-hand side of (11) is 1 and so $\phi(\gamma_j) = \mu_j \cdot 2\pi i$ for some $\mu_j \in \mathbb{Z}$ if $j \leq j_2(k)$. Next, let $h \in H(\Omega)$ with zeros exactly at the points γ_j with $j > j_2(k)$ and with $h(\gamma_j) = 1$ if $j \leq j_2(k)$. That such a function exists can be proved by an argument similar to the one used in the proof of Proposition 7. Furthermore, we set

$$c_1 := \|h\|_{\bar{V}_k}, \quad c_2 := \|g\|_{\bar{V}_k}, \quad c_3 := \exp(\|\phi\|_{\bar{V}_k} + 1),$$

where V_k is again a fixed Jordan domain, this time satisfying $C_k \subset V_k \subset \bar{V}_k \subset U_k$.

Suppose $\varepsilon > 0$. By Walsh's theorem on simultaneous approximation and interpolation [19], there exists a polynomial p satisfying

$$\left\| \frac{\phi}{h} - p \right\|_{\bar{V}_k} < \min\left\{ \frac{\varepsilon}{c_1 \cdot c_2 \cdot c_3}, 1, \frac{1}{c_1} \right\}$$

and, moreover,

$$p(\gamma_j) = \frac{\phi(\gamma_j)}{h(\gamma_j)} = \phi(\gamma_j) \quad (j = 1, \ldots, j_2(k)). \tag{12}$$

We conclude that

$$\|\phi - ph\|_{\bar{V}_k} < \min\left\{1, \frac{\varepsilon}{c_2 \cdot c_3}\right\}.$$

Together with the elementary inequality $|e^y - e^w| \leq e^{\max\{|y|,|w|\}}|y - w|$, we obtain the next estimate:

$$\begin{aligned}\left\|\frac{f}{g} - e^{ph}\right\|_{\bar{V}_k} &= \|e^{\phi} - e^{ph}\|_{\bar{V}_k} \\ &\leq \exp(\max\{\|\phi\|_{\bar{V}_k}, \|ph\|_{\bar{V}_k}\})\|\phi - ph\|_{\bar{V}_k} < c_3 \cdot \frac{\varepsilon}{c_2 \cdot c_3} = \frac{\varepsilon}{c_2}.\end{aligned}$$

Thus,

$$\|f - ge^{ph}\|_{\bar{V}_k} < \varepsilon.$$

Moreover, by (12) and the properties of h, the function

$$\varphi(z) := g(z)e^{p(z)h(z)} \quad (z \in \Omega)$$

satisfies $\varphi \in A$. Hence, we obtain the existence of a sequence $\{\varphi_s\} \subset A$ fulfilling

$$\|f - \varphi_s\|_{\bar{V}_k} \to 0 \quad (s \to \infty). \tag{13}$$

Of course, this implies

$$\|f - \varphi_s\|_{C_k} \to 0 \quad (s \to \infty). \tag{14}$$

Given (13) and the estimate in Theorem 4, it now follows that, for every $n \in \mathbb{N}$,

$$\|S^n f - S^n \varphi_s\|_{C_k} \to 0 \quad (s \to \infty). \tag{15}$$

According to our definition of the metric d, we obtain from (14) and (15) that, for each $k \in \mathbb{N}$, each $f \in B_k$ and each $n \in \mathbb{N}$,

$$\begin{aligned}&\inf_{\varphi \in A}(d(f, \varphi) + d(S^n f, S^n \varphi)) \\ &\leq \inf_{\varphi \in A}\left[\|f - \varphi\|_{C_k} + \sum_{j=k+1}^{\infty} \frac{1}{2^j} + \|S^n f - S^n \varphi\|_{C_k} + \sum_{j=k+1}^{\infty} \frac{1}{2^j}\right] = 2\sum_{j=k+1}^{\infty} \frac{1}{2^j}.\end{aligned}$$

Therefore,

$$\sup_{n \in \mathbb{N}} \inf_{\varphi \in A}(d(f, \varphi) + d(S^n f, S^n \varphi)) \leq \frac{1}{2^{k-1}} \quad (f \in B_k, k \in \mathbb{N}).$$

Consequently (10) holds for every sequence $\{b_k\}$ with $b_k \in B_k$. Altogether, this completes the proof. □

FINAL REMARKS. 1. For the case $\Omega = \mathbb{C}$, the result cannot be extended to all entire functions Φ of exponential type. Indeed, let us consider for $\tau > 0$ the function $\Phi(z) = e^{\tau z}$. Then, for every entire function f,

$$\Phi(D)f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n!}\tau^n = f(z+\tau).$$

This means that a $\Phi(D)$-universal function is universal with respect to translates. But in [16] it is pointed out that, as a consequence of Hurwitz's theorem, this always causes additional zeros for f that are not necessarily covered by the set A.

2. With essentially the same proof, a little more can be shown. Namely, the sequence $\Phi(D)^n\colon A \to H(\Omega)$ $(n \in \mathbb{N})$ is not only universal—or, equivalently, topologically transitive—but also topologically mixing. Recall that a sequence of continuous mappings $L_n\colon X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is said to be *topologically mixing* provided that, for every $x \in X$, $y \in Y$, and $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for every $n \geq N$ there exists a $z = z(n) \in X$ for which $d_X(x,z) < \varepsilon$ and $d_Y(y, L_n z) < \varepsilon$. It is easy to see that (L_n) is topologically mixing if and only if (L_{n_k}) is topologically transitive for any subsequence $\{n_1 < n_2 < \cdots\} \subset \mathbb{N}$ (see e.g. [9]). Then the proofs of Lemma 8 and Theorem 9 also work for every subsequence $(\Phi(D)^{n_k})$ simply by taking into account that Godefroy–Shapiro's theorem also holds for subsequences.

3. Finally, we note that if $\{\Phi_k : k \in \mathbb{N}\}$ is a sequence of nonconstant entire functions of subexponential type with $S_k = \Phi_k(D)$ $(k \in \mathbb{N})$, then there exists a function $\varphi \in A$ that is universal with respect to every family $\mathcal{S}_k = \{S_k^n : n \in \mathbb{N}\}$. This is a trivial consequence of the fact that, in every Baire space, the countable intersection of residual sets is residual.

ACKNOWLEDGMENT. We thank the referee for useful comments and remarks.

References

[1] L. Bernal-González, *On universal entire functions with zero-free derivatives,* Arch. Math. (Basel) 68 (1997), 145–150.

[2] ———, *Hypercyclic sequences of differential and antidifferential operators,* J. Approx. Theory 96 (1999), 323–337.

[3] ———, *Interpolation by hypercyclic functions for differential operators,* J. Approx. Theory 157 (2009), 134–143.

[4] L. Bernal-González, A. Bonilla, M. C. Calderón, and J. A. Prado-Bassas, *Maximal cluster sets of L-analytic functions on arbitrary curves,* Constr. Approx. 25 (2007), 211–219.

[5] J. B. Conway, *Functions of one complex variable,* Grad. Texts in Math. 11, Springer-Verlag, New York, 1973.

[6] G. Costakis, *Zeros and interpolation by universal Taylor series on simply connected domains,* Math. Proc. Cambridge Philos. Soc. 139 (2005), 149–159.

[7] G. Costakis and V. Vlachou, *Interpolation by universal-hypercyclic functions,* preprint.

[8] G. Godefroy and J. H. Shapiro, *Operators with dense, invariant, cyclic vector manifolds,* J. Funct. Anal. 98 (1991), 229–269.

[9] S. Grivaux, *Hypercyclic operators, mixing operators, and the bounded steps problem,* J. Operator Theory 54 (2005), 147–168.

[10] K.-G. Grosse-Erdmann, *Universal families and hypercyclic operators,* Bull. Amer. Math. Soc. (N.S.) 36 (1999), 345–381.
[11] ———, *Construction vs. Baire category in universality,* preprint.
[12] G. Herzog, *On zero-free universal entire functions,* Arch. Math. (Basel) 63 (1994), 329–332.
[13] S. Kolyada and L. Snoha, *Some aspects of topological transitivity—A survey,* Iteration theory (Opava, 1994), Grazer Math. Ber., 334, pp. 3–35, Karl-Franzens-Univ. Graz, Graz, 1997.
[14] G. R. MacLane, *Sequences of derivatives and normal families,* J. Anal. Math. 2 (1952), 72–87.
[15] M. Nieß, *MacLane functions with prescribed zeros and interpolation properties,* Analysis (Munich) 27 (2007), 323–332.
[16] ———, *Generic approach to multiply universal functions,* Complex Var. Elliptic Equ. 53 (2008), 819–831.
[17] J. C. Oxtoby, *Measure and category,* Grad. Texts in Math., 2, Springer-Verlag, New York, 1980.
[18] W. Rudin, *Real and complex analysis,* McGraw-Hill, New York, 1987.
[19] J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain,* Amer. Math. Soc. Colloq. Publ., 20, Amer. Math. Soc., Providence, RI, 1965.

L. Bernal-González
Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad de Sevilla
41080 Sevilla
Spain

lbernal@us.es

A. Bonilla
Departamento de Análisis Matemático
Universidad de La Laguna
38271 La Laguna (Tenerife)
Spain

abonilla@ull.es

M. Nieß
Katholische Universität Eichstätt-Ingolstadt
Mathematisch-Geographische Fakultät
D-85071 Eichstätt
Germany

markus.niess@ku-eichstaett.de