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ENDPOINT ESTIMATES FOR COMMUTATORS OF SINGULAR INTEGRAL OPERATORS

Dedicated to Professor Guido L. Weiss

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1 Introduction and statements of the results

1.1 The commutator of Coifman, Rochberg and Weiss

Let b be a locally integrable function on $\mathbf{R}^n$ and let T be a Calderón–Zygmund singular integral operator. Consider the commutator operator $[b,T]$ defined for smooth functions f by

$$[b,T]f = b\,T(f) - T(bf).$$

A by now classical result of R. Coifman, R. Rochberg and G. Weiss [1] states that $[b,T]$ is a bounded operator on $L^p(\mathbf{R}^n)$, $1 < p < \infty$, when b is a BMO function. Unlike the classical theory of singular integral operators, the proof of this theorem does not rely on a weak type $(1,1)$ inequality for $[b,T]$. In fact, simple examples show (cf. section 5) that these operators fail to be of weak type $(1,1)$ when $b \in BMO$. The purpose of this paper is to provide an endpoint theory for these operators. Our first theorem is the following:

THEOREM 1.1 *Let T be a Calderón–Zygmund singular integral operator and let b be a function in BMO. Then, there exists a positive constant C such that for each smooth function with compact support f and for all $\lambda > 0$*

$$|\{y \in \mathbf{R}^n : |[b,T]f(y)| > \lambda\}| \leq C\|b\|_{BMO} \int_{\mathbf{R}^n} \frac{|f(y)|}{\lambda}(1 + \log^+(\frac{|f(y)|}{\lambda}))\,dy. \qquad (1)$$

By imposing a condition on b stronger than BMO we can sharpen this estimate getting as close as we want to the L^1 norm. Indeed, we may replace $LlogL$ by the smaller function $L(logL)^\epsilon$, $\epsilon > 0$. To be precise we shall say that a function b belongs to the space $osc_{(expL)^r}$, $1 \leq r < \infty$, if there is a positive constant $c = c_b$ such that

$$\sup_Q \frac{1}{|Q|} \int_Q \exp(|\tfrac{b(y)-b_Q}{c}|^r)\,dy < \infty,$$

where the supremum is taken over all the cubes Q in $\mathbf{R}^n$. The infimum of all these c is denoted by $\|b\|_{osc_{(expL)^r}}$. The case $r = 1$ corresponds to the BMO space by the John–Nirenberg estimate (cf. section 3). Other interesting examples are provided by Trudinger's inequality for Riesz potentials (cf. [11] or [4]). For $0 < \alpha < n$ let $I_\alpha f$ be the Riesz potential or fractional integral of order α. Let f be an arbitary function on $L^{n/\alpha}(\mathbf{R}^n)$. Trudinger's estimate says that $I_\alpha f$ belongs to $osc_{(expL)^{(n/\alpha)'}}$.

Other examples are given by the following fact [6]: let $0 < \epsilon \leq 1$ and suppose that F is a Hölder continuous function of order ϵ, then if $b \in BMO$ $F(b) \in osc_{(expL)^{1/\epsilon}}$. The same method of proof of Theorem 1.1 gives the following.

THEOREM 1.2 *Let T be a Calderón–Zygmund singular integral operator. For $0 < \epsilon < 1$ we let b be a function in $osc_{(expL)^{1/\epsilon}}$. Then, there exists a positive constant C such that for each smooth function with compact support f and for all $\lambda > 0$*

$$|\{y \in \mathbf{R}^n : |[b,T]f(y)| > \lambda\}| \leq C\|b\|_{osc_{(expL)^{1/\epsilon}}} \int_{\mathbf{R}^n} \frac{|f(y)|}{\lambda}(1 + \log^+(\frac{|f(y)|}{\lambda}))^\epsilon \, dy.$$

Also, we are going to consider endpoint estimates related to Hardy type spaces. It is well known that any singular integral operator maps $H^1(\mathbf{R}^n)$ into $L^1(\mathbf{R}^n)$. However, it was observed in [8] that the corresponding result for $[b, H]$ is false when b is a BMO function. Here, we introduce a subspace of $H^1(\mathbf{R}^n)$ from which $[b, H]$ is a bounded operator.

DEFINITION 1.3 *A function a is a b–atom if there is a cube Q for which*

(i) $supp(a) \subset Q$
(ii) $\|a\|_{L^\infty} \leq 1/|Q|$
(iii) $\int_Q a(y)\,dy = 0$
(iv) $\int_Q a(y)\,b(y)dy = 0.$

The space $H^1_b(\mathbf{R}^n)$ consists of the subspace of $L^1(\mathbf{R}^n)$ of functions f which can be written as $f = \sum_j \lambda_j a_j$ where a_j are b-atoms, and λ_j are complex numbers with $\sum_j |\lambda_j| < \infty$.

THEOREM 1.4 *Let T be a Calderón–Zygmund singular integral operator and a let b be a function in BMO. Then,*

$$[b,T] : H^1_b(\mathbf{R}^n) \to L^1(\mathbf{R}^n)$$

Let us give an outline of the proof of Theorem 1.1. Our approach exhibits the close connection existing between commutators and iterations of the Hardy–Littlewood maximal operator. The proof is based on two lemmas, and the first one makes explicit this relationship.

LEMMA 1.5 *Let $\Phi(t) = t(1 + \log^+ t)$. Then, there exists a positive constant C such that for any smooth function with compact support f*

$$\sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} |\{y \in \mathbf{R}^n : |[b,T]f(y)| > t\}| \leq C \, \|b\|_{BMO} \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} |\{y \in \mathbf{R}^n : M^2 f(y) > t\}|,$$

where $M^2 = M \circ M$

The size of M^2 is given by the following.

LEMMA 1.6 *There exists a positive constant C such that for any function f and for all $\lambda > 0$*

$$|\{y \in \mathbf{R}^n : M^2 f(y) > \lambda\}| \leq C \int_{\mathbf{R}^n} \frac{|f(y)|}{\lambda} (1 + \log^+ (\frac{|f(y)|}{\lambda})) \, dy. \qquad (2)$$

The proof of Lemma 1.5 is based on a good–λ type argument as follows. For $\delta > 0$ we denote by L_δ the functional

$$L_\delta(f) = \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} |\{y \in \mathbf{R}^n : M_\delta([b,T]f)(y) > t\}|.$$

Then the operator $[b,T]$ satisfies the following inequality for $0 < \delta < 1$ and $\epsilon > 0$:

$$L_\delta(f) \leq \epsilon \, C \, L_\delta(f) + C(\epsilon) \, \|b\|_{BMO} \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} |\{y \in \mathbf{R}^n : M^2 f(y) > t\}|.$$

By choosing ϵ small enough we get Lemma 1.5 easily.

1.2 Higher order commutators

In section 7 we shall extend all these results to higher order commutators T_b^m, m, $m = 0, 1, 2, \cdots$ as consider for instance in [10] and [3]. These operators are defined by

$$T_b^m f(x) = \int (b(x) - b(y))^m K(x-y) f(y) \, dy.$$

Observe that $T_b^0 = T$, $T_b^m = [b, T_b^{m-1}]$, $m = 1, 2, \cdots$. We shall show the following.

THEOREM 1.7 *Let $m = 0, 1, 2, \cdots$, and let b be a function in BMO. Then, there exists a positive constant C such that for each $\lambda > 0$*

$$|\{y \in \mathbf{R}^n : |T_b^m f(y)| > \lambda\}| \leq C \, \|b\|_{BMO}^m \int_{\mathbf{R}^n} \frac{|f(y)|}{\lambda}(1 + \log^+(\frac{|f(y)|}{\lambda}))^m \, dy.$$

The proof of this theorem uses an induction argument from the case $m = 1$. It would be interesting to give a direct proof of this result which would avoid the induction argument.

The analog of $H_b^1(\mathbf{R}^n)$ space that we need is the following.

DEFINITION 1.8 *A function a is a b–atom of order m if there is a cube Q for which*

(i) $supp(a) \subset Q$
(ii) $\|a\|_{L^\infty} \leq 1/|Q|$
(iii) $\int a(y)\, b(y)^m dy = 0, \; m = 0, 1, 2, \cdots$.

The space $H_{b,m}^1(\mathbf{R}^n)$ consists of the subspace of $L^1(\mathbf{R}^n)$ of functions f which can be written as $f = \sum_j \lambda_j a_j$ where a_j are b-atoms of order m, and λ_j are complex numbers with $\sum_j |\lambda_j| < \infty$.

THEOREM 1.9 *Let T be a Calderón–Zygmund singular integral operator and a let b be a function in BMO. Then,*

$$T_b^m : H_{b,m}^1(\mathbf{R}^n) \to L^1(\mathbf{R}^n)$$

with constant less or equal than a multiple of $\|b\|_{BMO}^m$.

2 Some preliminaries and notation

In this section we shall collect some known results or variants of them and set some notation. The basic tool we are going to use is a modification of the sharp maximal operator $M^\#$ of C. Fefferman and E. Stein: for $\delta > 0$ we define the δ–sharp maximal operator $M_\delta^\#$ as

$$M_\delta^\#(f) = M^\#(|f|^\delta)^{1/\delta}.$$

Recall that $M^{\#}$ is defined by

$$M^{\#}f(x) = \sup_{x\in Q} \inf_{c} \frac{1}{|Q|}\int_Q |f(y)-c|\,dy \approx \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)-f_Q|\,dy$$

The idea of relating commutators with the sharp maximal operator is due to J. O. Strömberg (cf. [5]). The basic estimate is contained in the following lemma of C. Fefferman and E. Stein. Recall that $M_\delta f = M(f^\delta)^{1/\delta}$

LEMMA 2.1 *Let $M = M^d$ be the dyadic Hardy–Littlewood maximal operator. Then, there exists a positive dimensional constant C for which the following good–λ inequality holds*

$$\Big|\{y\in\mathbf{R}^n : Mf(y)>\lambda, M^{\#}f(y)\le\lambda\epsilon\}\Big| \le C\,\epsilon\Big|\{y\in\mathbf{R}^n : Mf(y)>\tfrac{\lambda}{2}\}\Big| \qquad (3)$$

for all $\lambda,\epsilon>0$.

As a consequence we have the following estimate:

Let $\varphi:(0,\infty)\to(0,\infty)$ be a doubling function. Then, there exists a positive constant C such that

$$\sup_{\lambda>0}\varphi(\lambda)|\{y\in\mathbf{R}^n : M_\delta f(y)>\lambda\}| \le C\sup_{\lambda>0}\varphi(\lambda)\Big|\{y\in\mathbf{R}^n : M_\delta^{\#}f(y)>\lambda\}\Big|$$

for all functions f such that the left side is finite. Then, there exists a positive constant C such that

$$\sup_{\lambda>0}\varphi(\lambda)|\{y\in\mathbf{R}^n : M_\delta f(y)>\lambda\}| \le C\sup_{\lambda>0}\varphi(\lambda)\Big|\{y\in\mathbf{R}^n : M_\delta^{\#}f(y)>\lambda\}\Big|$$

for all functions f such that the left side is finite.

A simple proof of (3) can be found in [7] p. 42.

Throughout the paper we will assume that the singular integral operators we are working with

$$Tf(x) = v.p.\int_{\mathbf{R}^n} K(x-y)f(y)\,dy$$

are regular (following [2] p. 204), which means that the kernel satisfies

(i) $|K(x)|\le c/|x|^n \qquad x\neq 0$

(ii) $|K(x-y)-K(x)|\le |y|/|x|^{n+1} \qquad |x|>2|y|>0.$

All the results still hold for the more general class of Calderón–Zygmund operators [7].

Most the notation we use is standard. Q will always denote a cube with sides parallel to the axes. λQ, $\lambda > 0$ denotes the cube Q dilated by λ. For a locally integrable function f, f_Q denotes the average $f_Q = \frac{1}{|Q|}\int_Q f(y)\,dy$. Also $B = B(x,r)$ will denote a ball centered at x with radius r and corresponding notation stands for λB and f_B.

As usual, a function $A : [0,\infty) \to [0,\infty)$ is a Young function if it is continuous, convex and increasing satisfying $A(0) = 0$ and $A(t) \to \infty$ as $t \to \infty$. We define the A–average of a function f over a cube Q by means of the following Luxemburg norm

$$\|f\|_{A,Q} = \inf\{\lambda > 0 : \frac{1}{|Q|}\int_Q A\left(\frac{|f(y)|}{\lambda}\right) dy \le 1\}. \tag{4}$$

The following generalized Hölder's inequality holds:

$$\frac{1}{|Q|}\int_Q |f(y)g(y)|\,dy \le \|f\|_{A,Q}\|g\|_{\bar{A},Q}, \tag{5}$$

where $\bar{A}$ is the complementary Young function associated to A.

The main example that we are going to be using is $A(t) = t(1+\log^+ t)^m$, $m = 1,2,3,\cdots$, with maximal function denoted by $M_{L(logL)^m}$. The complementary Young fuction is given by $\bar{A}(t) \approx e^{t^{1/m}}$ with corresponding maximal function denoted by $M_{(expL)^{1/m}}$.

3 Pointwise estimates

In this section we shall prove the following pointwise inequality for commutators which is the key estimate for the proof of Lemma 1.5.

LEMMA 3.1 *Let $b \in BMO$ and let $0 < \delta < \epsilon < 1$. Then, there exists a positive constant $C = C_{\delta,\epsilon}$ such that,*

$$M_\delta^{\#}([b,T]f)(x) \le C\|b\|_{BMO}(M_\epsilon(Tf)(x) + M^2 f(x))$$

for all smooth functions f.

Proof Let $B = B(x,r)$ be an arbitrary ball. Since $0 < \delta < 1$ implies $||\alpha|^\delta - |\beta|^\delta| \le |\alpha - \beta|^\delta$ for $\alpha, \beta \in \mathbf{R}$ it is enough to show for some complex constant $c = c_B$ that there exists $C = C_\delta > 0$ such that,

$$\left(\frac{1}{|B|} \int_B |[b,T]f(y) - c|^\delta \, dy \right)^{1/\delta} \le C \, Mf(x). \tag{6}$$

Let $f = f_1 + f_2$, where $f_1 = f \, \chi_{2B}$. For arbitrary constant a we can write

$$[b,T]f = (b-a)Tf - T((b-a)f_1) - T((b-a)f_2).$$

If we pick $c = (T((b-a)f_2))_B$, $a = b_{2B}$ we can estimate the left hand side of (6) by a multiple of

$$\left(\frac{1}{|B|} \int_B |(b(y) - b_{2B})Tf(y)|^\delta \, dy \right)^{1/\delta} + \left(\frac{1}{|B|} \int_B |T((b - b_{2B})f_1)(y)|^\delta \, dy \right)^{1/\delta} +$$

$$\left(\frac{1}{|B|} \int_B |T((b - b_{2B})f_2) - (T((b - b_{2B})f_2))_B|^\delta \, dy \right)^{1/\delta} = I + II + III.$$

To estimate I we use Hölder's inequality with exponents r and r' where $1 < r < \frac{\epsilon}{\delta}$:

$$I \le \left(\frac{1}{|B|} \int_B |b(y) - b_{2B})|^{\delta r'} \, dy \right)^{1/\delta r'} \left(\frac{1}{|B|} \int_B |Tf(y)|^{\delta r} \, dy \right)^{1/\delta r} \le$$

$$\le C \, \|b\|_{BMO} M_{\delta r}(Tf)(x) \le C \, \|b\|_{BMO} M_\epsilon(Tf)(x).$$

Since $T : L^1(\mathbf{R}^n) \to L^{1,\infty}(\mathbf{R}^n)$ and $0 < \delta < 1$, Kolmogorov's inequality (cf. [2] p. 485 for instance) yields

$$II \le \frac{C}{|B|} \int_B |(b(y) - b_{2B})f_1(y)| \, dy = \frac{C}{|2B|} \int_{2B} |(b(y) - b_{2B})f(y)| \, dy \le$$

$$\le C \, \|b - b_{2B}\|_{expL,2B} \, \|f\|_{LlogL,2B},$$

by the generalized Hölder's inequality (5). We claim the following: there is a positive constant C such that for all ball B

$$\|b - b_B\|_{expL,B} \le C \, \|b\|_{BMO}. \tag{7}$$

Indeed, this is equivalent to

$$\frac{1}{|B|}\int_B \exp(\frac{|b(y)-b_B|}{C\,\|b\|_{BMO}})\,dy \le C_0,$$

which is the fundamental estimate of F. John and L. Nirenberg (see [2] p. 166). Then,

$$II \le C\,\|b\|_{BMO}\,M_{LlogL}f(x).$$

To take care of III we observe that Jensen's inequality together with Fubini's theorem yield,

$$III \le \frac{1}{|B|}\int_B |T((b-b_{2B})f_2)(y) - (T((b-b_{2B})f_2))_B|\,dy \le$$

$$\le \frac{1}{|B|^2}\int_B\int_B\int_{\mathbf{R}^n-2B} |k(y-w)-k(z-w)||(b(w)-b_{2B})f(w)|dw\,dz\,dy \le$$

$$\le \frac{1}{|B|^2}\int_B\int_B\sum_{j=1}^{\infty}\int_{2^jr\le|w-x|<2^{j+1}r}\frac{|y-z|}{|x-w|^{n+1}}|b(w)-b_{2B}||f(w)|dw\,dz\,dy \le$$

$$\le C\sum_{j=1}^{\infty}\frac{2^{-j}}{(2^{j+1}r)^n}\int_{2^{j+1}B}|b(w)-b_{2B}||f(w)|\,dw \le$$

$$\le C\sum_{j=1}^{\infty}\frac{2^{-j}}{(2^{j+1}r)^n}\int_{2^{j+1}B}|b(w)-b_{2^{j+1}B}||f(w)|\,dw +$$

$$+\,C\sum_{j=1}^{\infty}2^{-j}|b_{2^{j+1}B}-b_{2B}|\frac{1}{(2^{j+1}r)^n}\int_{2^{j+1}B}|f(w)|\,dw \le$$

$$\le C\sum_{j=1}^{\infty}2^{-j}\|b-b_{2^{j+1}B}\|_{expL,2^{j+1}B}\,\|f\|_{LlogL,2^{j+1}B} + C\,\|b\|_{BMO}\sum_{j=1}^{\infty}\frac{j}{2^j}Mf(x) \le$$

$$\le C\,\|b\|_{BMO}M_{LlogL}f(x) + C\,\|b\|_{BMO}\,Mf(x) \le$$

$$\le C\,\|b\|_{BMO}M_{LlogL}f(x),$$

where we have used that $|b_{2^{j+1}B}-b_{2B}| \le 2j\,\|b\|_{BMO}$, inequality (7) and $Mf(x) \le M_{LlogL}f(x)$ by the generalized Jensen's inequality. Finally, since $M^2f \approx M_{LlogL}$ (cf. (21) below) we are done. □

4 Proof of the basic Lemmata

In this section we give the proofs of both lemmata 1.5 and 1.6.

Proof of Lemma 1.5

Let f be smooth with compact support. We have to prove that

$$\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}|\{y\in\mathbf{R}^n:|[b,T]f(y)|>t\}|\le C\,\|b\|_{BMO}\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}|\{y\in\mathbf{R}^n:M^2f(y)>t\}|, \tag{8}$$

with a constant C independent of f. Instead of working with the functional

$$\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}|\{y\in\mathbf{R}^n:|[b,T]f(y)|>t\}|,$$

we shall consider the following larger functional by the Lebesgue differentiation theorem: for $\delta>0$, $L_{\Phi,\delta}(f)=L_\delta(f)$ is defined by

$$L_\delta(f)=\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}|\{y\in\mathbf{R}^n:M_\delta([b,T]f)(y)>t\}|.$$

We claim that the operator $[b,T]$ satisfies the following inequality for arbitrary $0<\delta<1$, $\epsilon>0$:

$$L_\delta(f)\le\epsilon\,C\,L_\delta(f)+C(\epsilon)\,\|b\|_{BMO}\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}|\{y\in\mathbf{R}^n:M^2f(y)>t\}| \tag{9}$$

To prove this estimate we use Lemma 2.1: for $t>0$ and $\delta>0$

$$\begin{aligned}&|\{y\in\mathbf{R}^n:M_\delta([b,T]f)(y)>t\}|= \qquad (10)\\&=\Big|\{y\in\mathbf{R}^n:M(|[b,T]f|^\delta)(y)>t^\delta\}\Big|\le\\&\le c_n\epsilon\Big|\{y\in\mathbf{R}^n:M(|[b,T]f|^\delta)(y)>\tfrac{t^\delta}{2}\}\Big|+\\&+\Big|\{y\in\mathbf{R}^n:M^{\#}(([b,T]f)^\delta)(y)>\epsilon t^\delta\}\Big|=I+II.\end{aligned}$$

To estimate II we apply Lemma 3.1 with $\epsilon=r\delta$, $1<r<1/\delta$:

$$\begin{aligned}II&=\Big|\{y\in\mathbf{R}^n:M^{\#}_\delta([b,T]f)(y)>\epsilon^{1/\delta}t\}\Big|\le\\&\le\Big|\{y\in\mathbf{R}^n:M_{\delta r}(Tf)(y)+M^2f(y)>\tfrac{\epsilon^{1/\delta}\,t}{C\|b\|_{BMO}}\}\Big|\le\end{aligned}$$

$$\leq \left|\{y \in \mathbf{R}^n : M_{\delta r}(Tf)(y) > \tfrac{\epsilon^{1/\delta}\, t}{2C\|b\|_{BMO}}\right| + \left|\{y \in \mathbf{R}^n : M^2 f(y) > \tfrac{\epsilon^{1/\delta}\, t}{2\,C\,\|b\|_{BMO}}\right|.$$

Let $a = \frac{\epsilon^{1/\delta}}{2\,C\,\|b\|_{BMO}}$. Then, dividing (10) by $\Phi(\frac{1}{t})$ and using that Φ is doubling we have

$$\frac{1}{\Phi(\frac{1}{t})}|\{y \in \mathbf{R}^n : M_\delta([b,T]f)(y) > t\}| \leq$$

$$\leq \frac{C\,\epsilon}{\Phi(\frac{1}{t})}\left|\{y \in \mathbf{R}^n : M_\delta([b,T]f)(y) > \tfrac{t}{2^{1/\delta}}\}\right| + \frac{1}{\Phi(\frac{1}{t})}\,|\{y \in \mathbf{R}^n : M_{\delta r}(Tf)(y) > a\,t| +$$

$$+\frac{1}{\Phi(\frac{1}{t})}|\{y \in \mathbf{R}^n : M^2 f(y) > a\,t| \leq$$

$$\leq \frac{C\,\epsilon}{\Phi(\frac{2^{1/\delta}}{t})}\left|\{y \in \mathbf{R}^n : M_\delta([b,T]f)(y) > \tfrac{t}{2^{1/\delta}}\}\right| + \frac{C\,\|b\|_{BMO}}{\Phi(\frac{1}{a\,t})}\,|\{y \in \mathbf{R}^n : M_{\delta r}(Tf)(y) > a\,t| +$$

$$+\frac{C\,\|b\|_{BMO}}{\Phi(\frac{1}{a\,t})}|\{y \in \mathbf{R}^n : M^2 f(y) > a\,t| \leq$$

$$\leq C\,\epsilon\,L_\delta(f) + C\,\|b\|_{BMO}\,\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}|\{y \in \mathbf{R}^n : M_{\delta r}(Tf)(y) > t\}| +$$

$$+C\,\|b\|_{BMO}\,\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}|\{y \in \mathbf{R}^n : M^2 f(y) > t\}|.$$

Now, since $0 < r\,\delta < 1$, we can use the estimate

$$M_\alpha^{\#}(Tf)(y) \leq C\,Mf(y), \tag{11}$$

which holds for all $0 < \alpha < 1$, together with part a) of Lemma 2.1 to obtain

$$L_\delta(f) \leq$$

$$\leq C\,\epsilon\,L_\delta(f) + C\,\|b\|_{BMO}\,\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}\left|\{y \in \mathbf{R}^n : M_{\delta r}^{\#}(Tf)f(y) > t\}\right| +$$

$$+C\,\|b\|_{BMO}\,\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}|\{y \in \mathbf{R}^n : M^2 f(y) > t\}| \leq$$

$$\leq C\,\epsilon\,L_\delta(f) + C\,\|b\|_{BMO}\,\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}|\{y \in \mathbf{R}^n : Mf(y) > t\}| +$$

$$+C\,\|b\|_{BMO}\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}|\{y\in\mathbf{R}^n: M^2f(y)>t\}|\le$$

$$\le C\,\epsilon\,L_\delta(f)+C\,\|b\|_{BMO}\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}|\{y\in\mathbf{R}^n: M^2f(y)>t\}|.$$

To finish the proof of the lemma we need to show that $L_\delta(f)$ is finite so that we can choose $\epsilon<\frac{1}{C}$ to conclude that

$$L_\delta(f)\le C\,\|b\|_{BMO}\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}|\{y\in\mathbf{R}^n: M^2f(y)>t\}|. \tag{12}$$

It is clear that this implies (8).

For each $m=1,2,3,\cdots$ we let $b_m=\inf\{b,m\}$. Since $\|b_m\|_{BMO}\le c\,\|b\|_{BMO}$ with c independent of m we shall prove that $L_{\Phi,\delta,b}$ is finite with b replaced by b_m. Therefore, (12) will hold with constant C independent of m. Since $b_m\to b$ as $m\to\infty$, we shall let $m\to\infty$ to conclude the proof of inequality (12).

Now, since f is bounded and has compact support, assume that $suppf\subset B(0,R)$ for some $R>0$. Recalling that $b=b_m$ and that $\|b_m\|_{L^\infty}\le m$, we have for $|x|>2\,R$

$$|[b,T]f(x)|\le C\int_{B(0,R)}\frac{|b(x)-b(y)|}{|x-y|^n}|f(y)|\,dy\le$$

$$\le\frac{2Cm}{|x|^n}\int_{B(0,c|x|)}|f(y)|\,dy\le Cm\,Mf(x).$$

Using this and that $0<\delta<1$, we have for $t>0$

$$\frac{1}{\Phi(\frac{1}{t})}|\{x\in\mathbf{R}^n: M_\delta([b,T]f)(x)>t\}|\le$$

$$\le\frac{1}{\Phi(\frac{1}{t})}\Big|\{x\in\mathbf{R}^n: M(\chi_{B(0,2R)}\,[b,T]f)(x)>t/2\}\Big|+$$

$$+\frac{1}{\Phi(\frac{1}{t})}\Big|\{x\in\mathbf{R}^n: M(\chi_{\mathbf{R}^n\setminus B(0,2R)}\,[b,T]f)(x)>t/2\}\Big|\le$$

$$\le\frac{1}{\Phi(\frac{1}{t})}\frac{1}{t}\int_{B(0,2R)}|[b,T]f(x)|\,dx+\frac{1}{\Phi(\frac{1}{t})}|\{x\in\mathbf{R}^n: M^2f(x)>c_m\,t\}|\le$$

$$\le C\,|B(0,2R)|\left(\frac{1}{|B(0,2R)|}\int_{B(0,2R)}|[b,T]f(y)|^2\,dy\right)^{1/2}+\frac{C}{\Phi(\frac{1}{t})}\int_{\mathbf{R}^n}\Phi(\frac{|c_m\,f(y)|}{t})\,dy\le$$

$$\leq C\,|B(0,2R)|\|b\|_{BMO}\left(\frac{1}{|B(0,R)|}\int_{B(0,R)}|f(y)|^2\,dy\right)^{1/2}+C\int_{B(0,R)}\Phi(f(y))\,dy,$$

using that M is of weak type $(1,1)$ and the analog for M^2 given in Lemma 1.6. Since f is smooth with compact support last expression is finite, and we are done. □

Proof of Lemma 1.6

The proof of the estimate is standard once we observe that

$$M^2 f(x) \leq C\, M_{LlogL} f(x) \tag{13}$$

for each $x \in \mathbf{R}^n$ (in fact they are equivalent). Assuming this for the moment we see that it is enough to prove

$$|\{y \in \mathbf{R}^n : M_{LlogL} f(y) > \lambda\}| \leq C\int_{\mathbf{R}^n}\frac{f(y)}{\lambda}(1+\log^+(\frac{f(y)}{\lambda}))\,dy.$$

Now, let K be an arbitrary compact set contained in $\{y \in \mathbf{R}^n : M_{LlogL} f(y) > \lambda\}$. By an standard covering lemma, it is possible to choose cubes $Q_1, \cdots, Q_m$ with pairwise disjoint interiors such that $K \subset \cup_{i=1}^m 3Q_i$, and with $\|f\|_{LlogL,Q_i} > \lambda$, $i = 1, \cdots, m$. This implies

$$|Q_i| < \int_{Q_i}\frac{f(y)}{\lambda}(1+\log^+(\frac{f(y)}{\lambda}))\,dy,$$

which clearly gives the estimate.

To prove (13) let $x \in \mathbf{R}^n$ and fix a cube $x \in Q$. Let $f = f_1 + f_2$, where $f_1 = f\,\chi_{3Q}$. Then

$$\frac{1}{|Q|}\int_Q Mf(y)\,dy \leq \frac{1}{|Q|}\int_Q Mf_1(y)\,dy + \frac{1}{|Q|}\int_Q Mf_2(y)\,dy = I + II.$$

Now, II is comparable to $\inf_{z\in Q} Mf(z)$ (see [2] p. 160 for instance) and hence $II \leq C\,Mf(x)$. To estimate I we claim that

$$\frac{1}{|Q|}\int_Q Mf(y)\,dy \leq C\,\|f\|_{LlogL,Q} \tag{14}$$

for all f such that $supp f \subset Q$. By homogeneity we can take f with $\|f\|_{LlogL,Q} = 1$ which implies

$$\frac{1}{|Q|}\int_Q f(y)(1+\log^+(f(y)))\,dy \leq 1.$$

Hence, it is enough to prove

$$\frac{1}{|Q|}\int_Q Mf(y)\,dy \leq C\,(1+\frac{1}{|Q|}\int_Q f(y)\,\log^+(f(y))\,dy)$$

for all f with $suppf \subset Q$. But this is a well known local estimate (see [2] p. 147 for instance).

Finally, using (14) with Q replaced by $3Q$ we have

$$I + II \leq \frac{C}{|3Q|}\int_{3Q} Mf_1(y)\,dy + C\,Mf(x) \leq$$

$$\leq C\,\|f\|_{LlogL,3Q} + C\,M_{LlogL}f(x) \leq C\,M_{LlogL}f(x).$$

This completes the proof of the lemma. □

5 The LlogL estimate and a counterexample

In this section we shall give the proof of Theorem 1.1, namely

$$|\{y \in \mathbf{R}^n : |[b,T]f(y)| > \lambda\}| \leq C\|b\|_{BMO}\int_{\mathbf{R}^n}\frac{|f(y)|}{\lambda}(1+\log^+(\frac{|f(y)|}{\lambda}))\,dy$$

when T is a Calderón–Zygmund singular integral operator and b is a BMO function. We first show, however, that $[b,T]$ fails to be of weak type $(1,1)$. Consider the Hilbert transform

$$Hf(x) = pv\int_{\mathbf{R}}\frac{f(y)}{x-y}\,dy,$$

and consider the BMO function $b(x) = \log|1+x|$. Choose $f \approx \delta$, the point mass at the origin such that $\int_{\mathbf{R}}|f(y)|\,dy < \infty$.

Now, for $\lambda > 0$

$$\lambda\,|\{x \in \mathbf{R} : |[b,T]f(x)| > \lambda\}| = \lambda\left|\{x \in \mathbf{R} : |\tfrac{\log|1+x|}{x}| > \lambda\}\right| \geq$$

$$\lambda\left|\{x > e : \tfrac{\log x}{x} > \lambda\}\right| = \lambda\,(\varphi^{-1}(\lambda) - e),$$

where φ is the decreasing function $\varphi : (e,\infty) \to (0,e^{-1})$, given by $\varphi(x) = \frac{\log x}{x}$. To conclude observe that the right hand side of the estimate is unbounded as $\to 0$:

$$\lim_{\lambda\to 0}\lambda\varphi^{-1}(\lambda) = \lim_{\lambda\to\infty}\varphi(\lambda)\lambda = \infty.$$

Proof of Theorem 1.1

To prove that for some positive constant C the inequality

$$|\{y \in \mathbf{R}^n : |[b,T]f(y)| > \lambda\}| \leq C\|b\|_{BMO} \int_{\mathbf{R}^n} \frac{|f(y)|}{\lambda}(1+\log^+(\frac{|f(y)|}{\lambda}))\,dy, \qquad (15)$$

holds for each f and for all $\lambda > 0$, it is sufficient to consider by homogeneity the case $\lambda = 1$, namely

$$|\{y \in \mathbf{R}^n : |[b,T]f(y)| > 1\}| \leq C\|b\|_{BMO} \int_{\mathbf{R}^n} |f(y)|(1+\log^+|f(y)|)\,dy. \qquad (16)$$

Also, by a density argument we may also assume that f is smooth with compact support.

By Lemma 1.5 we have

$$\sup_{t>0} \frac{1}{\Phi(\frac{1}{t})}|\{y \in \mathbf{R}^n : |[b,T]f(y)| > t\}| \leq C\,\|b\|_{BMO} \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})}|\{y \in \mathbf{R}^n : M^2 f(y) > t\}|.$$

Also, by Lemma 1.6 we have for all $t > 0$

$$|\{y \in \mathbf{R}^n : M^2 f(y) > t\}| \leq C \int_{\mathbf{R}^n} \Phi(\frac{|f(y)|}{t})\,dy \leq C \int_{\mathbf{R}^n} \Phi(|f(y)|)\Phi(\frac{1}{t})\,dy,$$

since Φ is submultiplicative. Hence,

$$|\{y \in \mathbf{R}^n : |[b,T]f(y)| > 1\}| \leq \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})}|\{y \in \mathbf{R}^n : |[b,T]f(y)| > t\}| \leq$$

$$\leq C\,\|b\|_{BMO} \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})}|\{y \in \mathbf{R}^n : M^2 f(y) > t\}| \leq$$

$$\leq C\|b\|_{BMO} \int_{\mathbf{R}^n} \Phi(|f(y)|)\,dy = C\|b\|_{BMO} \int_{\mathbf{R}^n} |f(y)|(1+\log^+|f(y)|)\,dy,$$

which yields the desired estimate. $\square$

6 The H^1_b estimate

Proof of Theorem 1.4

Let b be a function in BMO. We need to prove that there exists a constant C such that for each function f in $H^1_b(\mathbf{R}^n)$

$$\int_{\mathbf{R}^n} |[b,T]f(y)|\, dy \le C\, \|b\|_{BMO} \|f\|_{H^1_b(\mathbf{R}^n)}. \tag{17}$$

By an standard argument, it is enough to show that there is a constant C such that

$$\int_{\mathbf{R}^n} |[b,T]a(y)|\, dy \le C\|b\|_{BMO}$$

for each b–atom a. To prove this, suppose that $supp(a) \subset B$ for some ball B. Then

$$\int_{\mathbf{R}^n} |[b,T]a(y)|\, dy = \int_{2B} |[b,T]a(y)|\, dy + \int_{\mathbf{R}^n \setminus 2B} |[b,T]a(y)|\, dy = I + II.$$

The estimate for I follows from the boundedness of $[b,T]$ on $L^2(\mathbf{R}^n)$ ([1])

$$I \le C\,|B| \left(\frac{1}{|2B|} \int_{2B} |[b,T]a(y)|^2\, dy \right)^{1/2} \le C\, \|b\|_{BMO}\, |B| \left(\frac{1}{|B|} \int_B |a(y)|^2\, dy \right)^{1/2} \le$$

$$\le C\, \|b\|_{BMO} |B|\, \|a\|_{L^\infty(\mathbf{R}^n)} \le C\, \|b\|_{BMO},$$

by the definition of b–atom.

Now, to majorize II we split $[b,T]$ in the following way $[b,T]a = (b - b_B)Ta - T((b - b_B)a)$, then

$$II \le \int_{\mathbf{R}^n - 2B} |(b(x) - b_B)Ta(x)|\, dx + \int_{\mathbf{R}^n - 2B} |T((b - b_B)a)(x)|\, dx = III + IV.$$

To estimate III we only use the cancellation of a $\int_B a(y)\, dy = 0$, so that if x_B denotes the center of B and r the radius of B

$$III \le \int_B |a(y)| \int_{\mathbf{R}^n - 2B} |k(x-y) - k(x - x_B)||b(x) - b_B| dx\, dy =$$

$$= \int_B |a(y)| \sum_{j=1}^{\infty} \int_{2^j r \le |x - x_B| < 2^{j+1} r} \frac{|y - x_B|}{|x - x_B|^{n+1}} |b(x) - b_B| dx\, dy \le$$

$$\leq \int_B |a(y)|\,dy \sum_{j=1}^{\infty} \frac{2^{-j}}{|2^{j+1}B|} \int_{2^{j+1}B} |b(x) - b_B|\,dx \leq$$

$$\leq C \sum_{j=1}^{\infty} 2^{-j} \left[\frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |b(x) - b_{2^{j+1}B}|\,dx + |b_{2^{j+1}B} - b_B| \right]$$

$$\leq C \sum_{j=1}^{\infty} 2^{-j} [\|b\|_{BMO} + (j+1)\|b\|_{BMO}] = C\,\|b\|_{BMO},$$

since $|b_{2^{j+1}B} - b_B| \leq 2(j+1)\,\|b\|_{BMO}$.

Now, by the definition of a

$$\int_B (b(y) - b_B)a(y)\,dy = \int_B a(y)\,b(y)dy - b_B \int_B a(y)\,dy = 0,$$

and hence we estimate IV using Lemma 3.3 from [2] p. 413:

$$IV \leq C \int_B |b(x) - b_B||a(x)|\,dx \leq C \frac{1}{|B|} \int_B |b(x) - b_B|\,dx \leq C\,\|b\|_{BMO}.$$

This concludes the proof. □

7 Commutator of order m

In this section we extend the results of the previous sections to higher order commutators.

Recall that for $m = 0, 1, 2, \cdots$ the commutator of order m is defined by

$$T_b^m f(x) = \int (b(x) - b(y))^m K(x - y) f(y)\,dy.$$

Observe that $T_b^0 = T$, $T_b^1 = [b, T]$ and that $T_b^m = [b, T_b^{m-1}]$.

Proof of Theorem 1.7

As in the proof of Theorem 1.1, it is sufficient to consider the case $\lambda = 1$

$$|\{y \in \mathbf{R}^n : |T_b^m f(y)| > 1\}| \leq C \int_{\mathbf{R}^n} |f(y)|(1 + \log^+ |f(y)|)^m\,dy, \tag{18}$$

for each f smooth with compact support.

We need the following version of Lemma 3.1:

LEMMA 7.1 *For each $b \in BMO$, $0 < \delta < \epsilon < 1$, there exists $C = C_{\delta,\epsilon} > 0$ such that,*

$$M_\delta^{\#}(T_b^m f)(x) \leq C\|b\|_{BMO} \sum_{j=0}^{m-1} M_\epsilon(T_b^j)(x) + \|b\|_{BMO}^m \, M^{m+1} f(x) \tag{19}$$

for all smooth functions f, and where $M_\epsilon(f) = M(|f|^\epsilon)^{1/\epsilon}$. Let $\Phi_m(t) = t(1 + \log^+ t)^m$. Then it follows from (19) that there exists a constant C such that for each smooth function f

$$\sup_{t>0} \frac{1}{\Phi_m(\frac{1}{t})} |\{y \in \mathbf{R}^n : |T_b^m f(y)| > t\}| \leq C \, \|b\|_{BMO} \sup_{t>0} \frac{1}{\Phi_m(\frac{1}{t})} |\{y \in \mathbf{R}^n : M^{m+1} f(y) > t\}| \tag{20}$$

Let us postpone the proof of the lemma for the moment and continue with the proof of (18). We need the following analog of Lemma 1.6:

$$|\{y \in \mathbf{R}^n : M^{m+1} f(y) > t\}| \, \leq C \int_{\mathbf{R}^n} \frac{|f(y)|}{t} (1 + \log^+(\frac{|f(y)|}{t}))^m \, dy$$

for all $t > 0$. The proof is the same as for the case $m = 1$ using that

$$M^{m+1} f(x) \approx M_{L(logL)^m} f(x) \tag{21}$$

for each $x \in \mathbf{R}^n$ and $m = 0, 1, 2, \cdots$. We shall omit it. Therefore, using the notation $\Phi_m(t) = t(1 + \log^+ t)^m$ we have

$$|\{y \in \mathbf{R}^n : M^{m+1} f(y) > t\}| \, \leq C \int_{\mathbf{R}^n} \Phi_m(\frac{|f(y)|}{t}) \, dy \leq C \int_{\mathbf{R}^n} \Phi_m(|f(y)|) \Phi_m(\frac{1}{t}) \, dy,$$

since Φ_m is submultiplicative. Hence,

$$\sup_{t>0} \frac{1}{\Phi_m(\frac{1}{t})} |\{y \in \mathbf{R}^n : M^{m+1} f(y) > t\}| \leq C \int_{\mathbf{R}^n} \Phi_m(|f(y)|) \, dy.$$

This together with (20) yields the desired estimate since

$$|\{y \in \mathbf{R}^n : |T_b^m f(y)| > 1\}| \, \leq \sup_{t>0} \frac{1}{\Phi_m(\frac{1}{t})} |\{y \in \mathbf{R}^n : |T_b^m f(y)| > t\}|.$$

□

Proof of Lemma 7.1 Following [3], we expand $b(x) - b(y) = (b(x) - \lambda) - (b(y) - \lambda)$, where λ is an arbitrary constant as follows:

$$T_b^m f(x) = \int (b(x) - b(y))^m K(x-y) f(y)\, dy =$$

$$= \sum_{j=0}^{m} C_{j,m}\, (b(x) - \lambda)^j \int (b(y) - \lambda)^{m-j} K(x-y) f(y)\, dy =$$

$$= \sum_{j=1}^{m} C_{j,m}\, (b(x) - \lambda)^j \int (b(y) - \lambda)^{m-j} K(x-y) f(y)\, dy + T((b-\lambda)^m f)(x) =$$

$$= \sum_{j=1}^{m} C_{j,m}\, (b(x)-\lambda)^j \int (b(y) - b(x) + b(x) - \lambda)^{m-j} K(x-y) f(y)\, dy + T((b-\lambda)^m f)(x) =$$

$$= \sum_{j=1}^{m} \sum_{h=0}^{m-j} C_{j,m,h}\, (b(x)-\lambda)^{h+j} \int (b(x) - b(y))^{m-j-h} K(x-y) f(y)\, dy + T((b-\lambda)^m f)(x) =$$

$$= \sum_{\alpha=0}^{m-1} C_{\alpha,m}\, (b(x) - \lambda)^{m-\alpha} T_b^{\alpha} f(x) + T((b-\lambda)^m f)(x).$$

For arbitrary ball B centered at x and with radius r, we split f in the usual way: $f = f_1 + f_2$ where $f_1 = f\,\chi_{2B}$. Then, with $\lambda = b_{2B}$ and $c = (T((b - b_{2B})^m f_2))_B$ we have

$$\left(\frac{1}{|B|} \int_B |T^m f(y) - c|^{\delta}\, dy \right)^{1/\delta} =$$

$$\left(\frac{1}{|B|} \int_B | \textstyle\sum_{\alpha=0}^{m-1} C_{\alpha,m}\, (b(y) - \lambda)^{m-\alpha} T^{\alpha} f(y) + T((b-\lambda)^m f(y) - c|^{\delta}\, dy \right)^{1/\delta} \leq$$

$$\leq C \sum_{\alpha=0}^{m-1} \left(\frac{1}{|B|} \int_B (b(y) - \lambda)^{(m-\alpha)\delta} |T^{\alpha} f(y)|^{\delta}\, dy \right)^{1/\delta} + \left(\frac{1}{|B|} \int_B |T((b - b_{2B})^m f_1)(y)|^{\delta}\, dy \right)^{1/\delta} +$$

$$+ \left(\frac{1}{|B|} \int_B |T((b - b_{2B})^m f_2) - (T((b - b_{2B})^m f_2))_B|^{\delta}\, dy \right)^{1/\delta} = I + II + III.$$

To estimate I we use Hölder's inequality with exponents q and q' where $1 < q < \frac{\epsilon}{\delta}$:

$$I \leq \sum_{\alpha=0}^{m-1} \left(\frac{1}{|B|} \int_B |b(y) - b_{2B})|^{(m-\alpha)\delta q'} \, dy \right)^{1/\delta q'} \left(\frac{1}{|B|} \int_B |T^\alpha f(y)|^{\delta q} \, dy \right)^{1/\delta q} \leq$$

$$\leq C \sum_{\alpha=0}^{m-1} \|b\|_{BMO} M_{\delta q}(T^\alpha f)(x) \leq C \sum_{\alpha=0}^{m-1} \|b\|_{BMO} M_\epsilon(T^\alpha f)(x).$$

Again, since $T : L^1(\mathbf{R}^n) \to L^{1,\infty}(\mathbf{R}^n)$ and $0 < \delta < 1$, Kolmogorov's inequality yields

$$II \leq \frac{C}{|B|} \int_B |(b(y) - b_{2B})^m f_1(y)| \, dy = \frac{C}{|2B|} \int_{2B} |b(y) - b_{2B}|^m |f(y)| \, dy \leq$$

$$\leq C \, \|(b - b_{2B})^m\|_{(expL)^{1/m}, 2B} \, \|f\|_{L(logL)^m, 2B} = C \, \|b - b_{2B}\|^m_{expL, 2B} \, \|f\|_{L(logL)^m, 2B}.$$

Then,

$$II \leq C \, \|b\|^m_{BMO} \, M_{L(logL)^m} f(x) \approx \|b\|^m_{BMO} \, M^{m+1} f(x).$$

The last term III is estimated as follows

$$III \leq \frac{1}{|B|} \int_B |T((b - b_{2B})^m f_2)(y) - (T(((b - b_{2B})^m f_2)_B| \, dy \leq$$

$$\leq \frac{1}{|B|^2} \int_B \int_B \int_{\mathbf{R}^n - 2B} |k(y - w) - k(z - w)||b(w) - b_{2B}|^m |f(w)| dw \, dz \, dy \leq$$

$$\leq \frac{1}{|B|^2} \int_B \int_B \sum_{j=1}^{\infty} \int_{2^j r \leq |w - x| < 2^{j+1} r} \frac{|y - z|}{|x - w|^{n+1}} |b(w) - b_{2B}|^m |f(w)| dw \, dz \, dy \leq$$

$$\leq C \sum_{j=1}^{\infty} \frac{2^{-j}}{(2^{j+1} r)^n} \int_{2^{j+1} B} |b(w) - b_{2B}|^m |f(w)| \, dw \leq$$

$$\leq C \sum_{j=1}^{\infty} \frac{2^{-j}}{(2^{j+1} r)^n} \int_{2^{j+1} B} |b(w) - b_{2^{j+1} B}|^m |f(w)| \, dw +$$

$$+ C \sum_{j=1}^{\infty} 2^{-j} |b_{2^{j+1} B} - b_{2B}|^m \frac{1}{(2^{j+1} r)^n} \int_{2^{j+1} B} |f(w)| \, dw \leq$$

$$\leq C \sum_{j=1}^{\infty} 2^{-j} \|b - b_{2^{j+1} B}\|^m_{expL, 2^{j+1} B} \, \|f\|_{L(logL)^m, 2^{j+1} B} + C \, \|b\|^m_{BMO} \, Mf(x) \sum_{j=1}^{\infty} 2^{-j} j^m \leq$$

$$\leq C\,\|b\|_{BMO}^m M_{L(logL)^m} f(x) \sum_{j=1}^{\infty} 2^{-j} + C\,\|b\|_{BMO}^m\, Mf(x) \leq$$

$$\leq C\,\|b\|_{BMO}^m\, M_{L(logL)^m} f(x) \approx \|b\|_{BMO}^m\, M^{m+1} f(x).$$

The proof of (20) can be given by iterating the pointwise estimate (19) m times starting from the basic estimate of Fefferman–Stein Lemma 2.1 (notice that each time Φ_j is doubling); when the case $m = 1$ is reached we use Theorem 1.1. Since we iterate m times we obtain $\|b\|_{BMO}^m$ as a constant. □

Proof of Theorem 1.9

Let b be a function in BMO. We need to prove that there exists a constant C such that for each function f in $H_b^m(\mathbf{R}^n)$

$$\int_{\mathbf{R}^n} |T_b^m f(x)|\, dx \leq C\,\|b\|_{BMO}^m \|f\|_{H^1_{b,m}(\mathbf{R}^n)}. \tag{22}$$

By an standard argument, it is enough to show that there is a costant C such that

$$\int_{\mathbf{R}^n} |T_b^m a(x)|\, dx \leq C\|b\|_{BMO}^m$$

for each b–atom a of order m. To prove this, suppose that $supp(a) \subset B$ for some ball B. Then

$$\int_{\mathbf{R}^n} |T_b^m a(x)|\, dx = \int_{2B} |T_b^m a(x)|\, dx + \int_{\mathbf{R}^n\setminus 2B} |T_b^m a(x)|\, dx = I + II.$$

The estimate for I follows from the boundedness of T_b^m on $L^2(\mathbf{R}^n)$

$$I\; \leq C\,|B| \left(\frac{1}{|2B|}\int_{2B} |T_b^m a(x)|^2\, dx\right)^{1/2} \leq C\,\|b\|_{BMO}^m\,|B| \left(\frac{1}{|B|}\int_B |a(x)|^2\, dx\right)^{1/2} \leq$$

$$\leq C\,\|b\|_{BMO}^m |B|\;\|a\|_{L^\infty(\mathbf{R}^n)} \leq C\,\|b\|_{BMO}^m$$

by the definition of b–atom of order m.

Now, to majorize II we can write using that $\int a(y)\, b(y)^m dy = 0$, for each $m = 0, 1, 2, \cdots$

$$T_b^m a(x) = \int (b(x) - b(y))^m K(x-y) a(y)\, dy =$$

$$= \int (b(x) - b(y))^m [K(x-y) - K(x-x_B)] a(y)\, dy,$$

where x_B denotes the center of B. Then, if r denotes the radius of B we have

$$\int_{\mathbf{R}^n - 2B} |T_b^m a(x)|\, dx \leq \int_B |a(y)| \int_{\mathbf{R}^n - 2B} |b(x) - b(y)|^m |k(x-y) - k(x-x_B)| dx\, dy \leq$$

$$\leq \sum_{j=1}^{\infty} \int_B |a(y)| \int_{2^j r \leq |x - x_B| < 2^{j+1} r} \frac{|y - x_B|}{|x - x_B|^{n+1}} |b(x) - b(y)|^m \, dx\, dy \leq$$

$$C \sum_{j=1}^{\infty} \int_B |a(y)|\, dy \frac{2^{-j}}{|2^{j+1}B|} \int_{2^{j+1}B} |b(x) - b(y)|^m \, dx\, dy \leq$$

$$\leq C \sum_{j=1}^{\infty} 2^{-j} \int_B |a(y)| \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} (|b(x) - b_B|^m + |b_B - b(y)|^m)\, dx\, dy \leq$$

$$\leq C \sum_{j=1}^{\infty} 2^{-j} \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |b(x) - b_B|^m \, dx + \sum_{j=1}^{\infty} 2^{-j} \int_B |a(y)| |b_B - b(y)|^m \, dy \leq$$

$$\leq C \sum_{j=1}^{\infty} 2^{-j} \left[\frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |b(x) - b_{2^{j+1}B}|^m \, dx + |b_{2^{j+1}B} - b_B|^m \right] +$$

$$+ C \sum_{j=1}^{\infty} 2^{-j} \frac{1}{|B|} \int_B |b_B - b(y)|^m \, dy \leq C\, \|b\|_{BMO}^m.$$

□

8 Remarks on weighted inequalities

In this section we sketch how to extend the main result of the paper to the weighted case. We recall some definitions. A weight w is in the class A_1 if there is a positve constant C such that $Mw(x) \leq Cw(x)$, *a.e.* $x \in \mathbf{R}^n$. We denote by $[w]_{A_1}$ the infimum of all these C's. A weight w is in the class A_∞ if there are positve constants C, ϵ such that

$$\frac{w(E)}{w(Q)} \leq c\, (\frac{|E|}{|Q|})^{\epsilon},$$

for all cubes Q and all measurable sets $E \subset Q$. We denote by $[w]_{A_\infty}$ the infimum of all these C's. Recall that it is well known that $A_1 \subset A_\infty$.

THEOREM 8.1 *Let $m = 0, 1, 2, \cdots$, and let b be a function in BMO. Suppose that w is an A_1 weight. Then, there exists a positive constant C such that for each smooth function with compact support f and for all $\lambda > 0$*

$$w(\{y \in \mathbf{R}^n : |T_b^m f(y)| > \lambda\}) \leq C\|b\|_{BMO}^m [w]_{A_1} \int_{\mathbf{R}^n} \frac{|f(y)|}{\lambda}(1+\log^+(\frac{|f(y)|}{\lambda}))^m w(y)dy. \tag{23}$$

The proof of this result goes along the same line as that of the unweighted case once we give the weighted version of the basic lemmata 1.5 and 1.6.

Recall that we use the notation $\Phi_m(t) = t(1 + \log^+ t)^m$. Then, we have

LEMMA 8.2 *Let w be an A_∞ weight. Then, there exists a positive constant C such that for each smooth function f*

$$\sup_{t>0} \frac{1}{\Phi_m(\frac{1}{t})} w(\{y \in \mathbf{R}^n : |T_b^m f(y)| > t\}) \leq$$

$$C \|b\|_{BMO}^m [w]_{A_\infty} \sup_{t>0} \frac{1}{\Phi_m(\frac{1}{t})} w(\{y \in \mathbf{R}^n : M^{m+1} f(y) > t\})$$

The proof of this is based on the pointwise estimate (19) and the weighted version of the basic estimate of the Fefferman– Stein Lemma 2.1: Let w be an A_∞ weight, then

$$w(\{y \in \mathbf{R}^n : Mf(y) > \lambda, M^{\#}f(y) \leq \lambda\epsilon\}) \leq c_n [w]_{A_\infty} \epsilon w(\{y \in \mathbf{R}^n : Mf(y) > \frac{\lambda}{2}\}), \tag{24}$$

for all $\lambda, \epsilon > 0$. As a consequence we have the following estimates for $0 < \delta$ and $\varphi : (0, \infty) \to (0, \infty)$ doubling. Then, there exists a positive constant C such that

$$\sup_{\lambda>0} \varphi(\lambda) w(\{y \in \mathbf{R}^n : M_\delta f(y) > \lambda\}) \leq C[w]_{A_\infty} \sup_{\lambda>0} \varphi(\lambda) w(\{y \in \mathbf{R}^n : M_\delta^{\#} f(y) > \lambda\})$$

LEMMA 8.3 *There exists a positive constant C such that for any weight w and all $\lambda > 0$*

$$w(\{y \in \mathbf{R}^n : M^{m+1} f(y) > \lambda\} \leq C \int_{\mathbf{R}^n} \frac{|f(y)|}{\lambda}(1 + \log^+(\frac{|f(y)|}{\lambda}))^m Mw(y)dy$$

for every locally integrable function f

The proof of this result is standard and we shall omit it.

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