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Order of growth of distributional irregular entire functions for the differentiation operator

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Abstract

[1] [2] We study the rate of growth of entire functions that are distributionally irregular for the differentiation operator D. More specifically, given $p \in [1, \infty]$ and $b \in (0, a)$, where $a = \frac{1}{2 \max\{2, p\}}$, we prove that there exists a distributionally irregular entire function f for the operator D such that its p-integral mean function $M_p(f, r)$ grows not more rapidly than $e^r r^{-b}$. This completes related known results about the possible rates of growth of such means for D-hypercyclic entire functions. It is also obtained the existence of dense linear submanifolds of $H(\mathbb{C})$ all whose nonzero vectors are D-distributionally irregular and present the same kind of growth.

1 Introduction

In 1988, Beauzamy [5], when trying to describe the erratic dynamics of certain vectors under the action of concrete operators, introduced the following notion.

Definition 1. Let X be a Banach space and $T : X \to X$ be a bounded operator. A vector $x_0 \in X$ is said to be *irregular* for T if $\liminf_{n\to\infty} \|T^n x\| = 0$ and $\limsup_{n\to\infty} \|T^n x\| = \infty$.

Inspired by this definition, and by the notion of distributional chaotic mapping due to Schweizer and Smítal ([25], see also [24]), in [6] it is considered the stronger property contained in Definition 2 below. Recall that, if A is a subset of the set $\mathbb{N}$ of positive integers, then its upper density and its lower density are respectively defined by

$$\overline{\text{dens}}(A) = \limsup_{n\to\infty} \frac{\text{card}(A \cap \{1, 2, \dots, n\})}{n}, \ \underline{\text{dens}}(A) = \liminf_{n\to\infty} \frac{\text{card}(A \cap \{1, 2, \dots, n\})}{n}.$$

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[2]*Key words and phrases.* Differentiation operator, irregular vector, distributionally irregular vector, hypercyclic operator, frequently hypercyclic operator, rate of growth, entire function.

Definition 2. Let X be a Banach space and $T : X \to X$ be a bounded operator. A vector $x_0 \in X$ is said to be *distributionally irregular* for T if there are increasing sequences of positive integers $A = (n_k)_k$ and $B = (m_k)_k$ such that

$$\overline{\mathrm{dens}}(A) = 1 = \overline{\mathrm{dens}}(B), \quad \lim_{k\to\infty} \|T^{n_k} x\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|T^{m_k} x_0\| = \infty.$$

Now, let Y be a Fréchet space, that is, Y is a vector space endowed with an increasing sequence $(\|\cdot\|_k)_{k\in\mathbb{N}}$ of seminorms (called a *fundamental sequence of seminorms*) that defines a metric

$$d(x,y) := \sum_{k=1}^{\infty} \frac{1}{2^k} \min\{1, \|x-y\|_k\} \quad (x, y \in Y),$$

under which Y is complete. By $B(Y)$ we have denoted, as usual, the set of all continuous linear operators $T : Y \to Y$.

The following concepts, that are a generalization of the above ones to Fréchet spaces, were introduced in [12].

Definition 3. Given $T \in B(Y)$ and $x_0 \in Y$, we say that x_0 is an *irregular vector* for T if there are $m \in \mathbb{N}$ and strictly increasing sequences (n_k) and (j_k) of positive integers such that

$$\lim_{k\to\infty} T^{n_k} x_0 = 0 \quad \text{and} \quad \lim_{k\to\infty} \|T^{j_k} x_0\|_m = \infty.$$

Definition 4. Given $T \in B(Y)$ and $x_0 \in Y$, we say that x_0 is a *distributionally irregular vector* for T if there are $m \in \mathbb{N}$ and $A, B \subset \mathbb{N}$ with $\overline{\mathrm{dens}}(A) = 1 = \overline{\mathrm{dens}}(B)$ such that

$$\lim_{\substack{n\to\infty \\ n\in A}} T^n x_0 = 0 \quad \text{and} \quad \lim_{\substack{n\to\infty \\ n\in B}} \|T^n x_0\|_m = \infty.$$

Let us consider the Fréchet space $H(\mathbb{C})$ of entire functions endowed of the family of seminorms $\|f\|_{\infty,\overline{B(0,k)}}$ $(k \in \mathbb{N})$. Here $B(a,r)$ denotes, as usual, the open ball in the complex plane $\mathbb{C}$ with center a and radius $r > 0$; and, for a nonempty set $A \subset \mathbb{C}$, we have set $\|f\|_{\infty,A} := \sup\{|f(z)| : z \in A\}$. Hence the metric $d(f,g) = \displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k} \min\{1, \|f\|_{\infty,\overline{B(0,k)}}\}$ defines the topology of $H(\mathbb{C})$. If $D : f \in H(\mathbb{C}) \mapsto f' \in H(\mathbb{C})$ is the differentiation operator, then the above definitions read as follows.

Definition 5. Given $f \in H(\mathbb{C})$, we say that f is an *irregular function* for D if there are $m \in \mathbb{N}$ and increasing sequences (n_k) and (j_k) of positive integers such that

$$\lim_{k\to\infty} D^{n_k} f = 0 \quad \text{and} \quad \lim_{k\to\infty} \|D^{j_k} f\|_{\infty,\overline{B(0,m)}} = \infty.$$

Definition 6. Given $f \in H(\mathbb{C})$, we say that f is a *distributionally irregular function* for D if there are $m \in \mathbb{N}$ and $A, B \subset \mathbb{N}$ with $\overline{\mathrm{dens}}(A) = \overline{\mathrm{dens}}(B) = 1$ such that

$$\lim_{\substack{n\to\infty \\ n\in A}} D^n f = 0 \quad \text{and} \quad \lim_{\substack{n\to\infty \\ n\in B}} \|D^n f\|_{\infty,\overline{B(0,m)}} = \infty.$$

In [12] is proved that D admits distributionally irregular entire functions. In fact, it is shown in [12] that there are T-distributionally irregular entire functions for any given non-scalar operator T on $H(\mathbb{C})$ that commutes with D.

Another important property related to the dynamics of operators is that of hypercyclicity. If T is an operator defined on a topological vector space X and $x_0 \in X$, then the vector x_0 is called hypercyclic for T provided that the orbit $\{T^n x_0 : n \in \mathbb{N}\}$ is dense in X; and, according to Bayart and Grivaux [2], x_0 is called frequently hypercyclic for T whenever the following stronger property is satisfied: for any prescribed nonempty open set $U \subset X$, $\underline{\text{dens}}(\{n \in \mathbb{N} : U \cap T^n x_0 \neq \varnothing\}) > 0$. The existence of entire functions which are hypercyclic (frequently hypercyclic) for D is known since MacLane [23] (Bayart and Grivaux [2], resp.). For excellent accounts of these topics, we refer the reader to the books [3, 20]. Notice that, if X is a Fréchet space, every T-hypercyclic vector is T-irregular.

Many papers have been devoted to study the *rate of growth* of entire functions that are hypercyclic or frequently hypercyclic for D, see for instance [9, 13, 14, 15, 16, 17, 18, 19, 26]. In this paper, we aim to study the rate of growth of entire functions that are *irregular* or *distributionally irregular* for the differentiation operator. The growth for the "irregular" case will be completely determined, while lower and upper bounds will be provided for the growth of D-distributionally irregular functions. This research is completed by analyzing the distributional irregularity in weighted Banach spaces as well as the existence of large vector subspaces of D-distributionally irregular functions satisfying the mentioned growth conditions.

2 Order of growth of distributional irregular entire functions

For every $r > 0$, every entire function f and $1 \leq p < \infty$ we will consider the integral p-means

$$M_p(f,r) = \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p \, dt\Big)^{1/p}$$

as well as the function $M_\infty(f,r) = \sup_{|z|=r} |f(z)|$.

In 1990, Grosse-Erdmann [19] discovered that there are not D-hypercyclic entire functions f with $M_\infty(f,r)$ growing not more rapidly than $e^r/\sqrt{r}$, while there do exist D-hypercyclic entire functions growing under any prescribed rate speeder than $e^r/\sqrt{r}$. Our first result states that the critical order of growth for hypercyclic and irregular entire functions for the differentiation operator is the same, and that this is independent of the p-mean considered.

Theorem 7. *Let $1 \leq p \leq \infty$. We have:*

(a) *For any function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(r) \to \infty$ as $r \to \infty$ there is a D-irregular entire function f with*

$$M_p(f,r) \leq \varphi(r)\frac{e^r}{\sqrt{r}} \quad \textit{for } r > 0 \textit{ sufficiently large.}$$

(b) *There is no D-irregular entire function f that satisfies*

$$M_p(f,r) \le C\frac{e^r}{\sqrt{r}} \quad \textit{for } r>0,$$

where C is a positive constant.

Proof. Part (a) follows from the corresponding result for D-hypercyclic functions [19] (see also [26]), since all hypercyclic vectors are irregular.

As for (b), assume that f is an entire function satisfying $M_p(f,r) \le C\frac{e^r}{\sqrt{r}}$ $(r>0)$ for some $C>0$. Fix $m \in \mathbb{N}$ and a radius $R>m$. From Cauchy's integral formula for derivatives of holomorphic functions, we get for all $n \in \mathbb{N}$ that

$$D^n f(z) = \frac{n!}{2\pi i}\oint_{|\xi|=R} \frac{f(\xi)}{(\xi - z)^{n+1}}\, d\xi \quad (z \in \overline{B(0,r)}).$$

¿From here and the triangle inequality one derives that

$$\|D^n f\|_{\infty,\overline{B(0,m)}} \le \frac{n!\,R}{(R-m)^{n+1}} M_1(f,R).$$

Since $M_1(f,R) \le M_p(f,R) \le C\,e^R/\sqrt{R}$, we find that $\|D^n f\|_{\infty,\overline{B(0,m)}} \le \frac{C\,n!\,e^R\,R^{1/2}}{(R-m)^{n+1}}$ for all $n \in \mathbb{N}$. Letting $R=n$ we get

$$\|D^n f\|_{\infty,\overline{B(0,m)}} \le \frac{C\,n!\,e^n}{n^{n+\frac{1}{2}}(1-\frac{m}{n})^{n+1}}$$

for all $n>m$. Now Stirling's formula $n! \sim \sqrt{2\pi n}\, e^{-n}\, n^n$ $(n \to \infty)$ and the fact $(1-\frac{m}{n})^{n+1} \to e^{-m}$ $(n \to \infty)$ imply that the sequence $\{\|D^n f\|_{\infty,\overline{B(0,m)}}\}_{n\ge 1}$ is bounded, so that f cannot be irregular. □

The following auxiliary result, which can be found in [13, Lemma 2.2], will be needed in the sequel.

Lemma 8. *Let $0<\alpha\le 2$ and $\beta \in \mathbb{R}$. Then there is a constant $C>0$ such that*

$$\sum_{n=0}^{\infty} \frac{r^{\alpha n}}{(n+1)^{\beta}(n!)^{\alpha}} \le C r^{\frac{1-\alpha-2\beta}{2}} e^{\alpha r} \quad \textit{for all } r>0.$$

Let $2 \le p \le \infty$, by the Hausdorff–Young inequality (see, for example, [21]) we obtain that

$$M_p\Big(\sum a_n \frac{z^n}{n!}, r\Big) \le \Big(\sum |a_n|^q \frac{r^{qn}}{(n!)^q}\Big)^{1/q},$$

where $q := \frac{p}{p-1}$ is the conjugate exponent of p.

Our second result provides the construction of a D-distributionally irregular entire function having, in some sense, prescribed control on their Taylor coefficients.

Theorem 9. *Assume that $\{\omega_n\}_{n\ge 1} \subset [0,+\infty)$ is a sequence with $\lim_{n\to\infty}\omega_n = +\infty$. Then there exists an entire function f satisfying the following properties:*

(a) $|f^n(0)| \leq \omega_n$ *for all* $n \geq 1$, *and*

(b) f *is D-distributionally irregular in* $H(\mathbb{C})$.

Proof. Firstly, the radius of convergence of the series $\sum_{n=1}^{\infty} n^{1+\frac{n}{2}} \frac{z^n}{n!}$ is

$$R = \lim_{n\to\infty} \frac{n^{1+\frac{n}{2}}/n!}{(n+1)^{1+\frac{n+1}{2}}/(n+1)!} = +\infty.$$

Therefore it converges at every $z \in \mathbb{C}$. In particular, we have

$$\lim_{k\to\infty} \sum_{n=k}^{\infty} n^{1+\frac{n}{2}} \frac{R^n}{n!} = 0 \text{ for any } R > 0. \tag{1}$$

Let us introduce some notation. Define $\widetilde{\omega_n} := \min\{\omega_n, n\}$ $(n \geq 1)$. Since $\frac{\widetilde{\omega_n}}{n!} \leq \frac{1}{(n-1)!}$ for all $n \in \mathbb{N}$, we have obtain that, for any selection of the set $S \subset \mathbb{N}$, the expression $\sum_{n\in S} \widetilde{\omega_n} \frac{z^n}{n!}$ defines an entire function.

The core of the proof is the search for an entire function f having large derivatives $D^j f$ for many indexes j and, simultaneously, having small derivatives $D^j f$ for other (many) indexes j.

Thanks to (1), we can select $\alpha_1 \in \mathbb{N}$ such that

$$\sum_{n=\alpha_1}^{\infty} n^{1+\frac{n}{2}} \frac{1}{n!} < 1.$$

Now, we choose $\beta_1 \in \mathbb{N}$ with $\beta_1 > 2\alpha_1^2$. ¿From (1), again, we obtain an $\alpha_2 \in \mathbb{N}$ with $\alpha_2 > \beta_1^2$ such that

$$\sum_{n=\alpha_2}^{\infty} n^{1+\frac{n}{2}} \frac{2^n}{n!} < \frac{1}{2}.$$

Choose $\beta_2 \in \mathbb{N}$ with $\beta_2 > 2\alpha_2^2$. Proceeding in this way, we can recursively obtain two sequences $\{\alpha\}_{n\geq 1}$ and $\{\beta\}_{n\geq 1}$ of natural numbers satisfying

$$\alpha_1 < 2\alpha_1^2 < \beta_1 < \beta_1^2 < \cdots < \alpha_n < 2\alpha_n^2 < \beta_n < \beta_n^2 < \alpha_{n+1} < \cdots$$

and

$$\sum_{n=\alpha_N}^{\infty} n^{1+\frac{n}{2}} \frac{N^n}{n!} < \frac{1}{N} \text{ for all } N \in \mathbb{N}. \tag{2}$$

Now, define the sets

$$A := \bigcup_{n=1}^{\infty} \{\alpha_n, \alpha_n + 1, \ldots, \alpha_n^2\} \text{ and } B := \bigcup_{n=1}^{\infty} \{\beta_n, \beta_n + 1, \ldots, \beta_n^2\}.$$

Notice that

$$\begin{aligned}\overline{\text{dens}}\,(A) &= \limsup_{n\to\infty} \frac{\text{card}(A\cap\{1,\dots,n\})}{n} \\ &\geq \limsup_{n\to\infty} \frac{\text{card}(A\cap\{1,\dots,\alpha_n^2\})}{\alpha_n^2} \\ &= \limsup_{n\to\infty} \frac{\text{card}(\{\alpha_n,\alpha_n+1,\dots,\alpha_n^2\})}{\alpha_n^2} \\ &= \lim_{n\to\infty} \frac{\alpha_n^2-\alpha_n+1}{\alpha_n^2} = 1.\end{aligned}$$

Hence $\overline{\text{dens}}\,(A)=1$ and, analogously, $\overline{\text{dens}}\,(B)=1$.

Next, we set, for $n\in\mathbb{N}$:

$$\omega_n^* := \begin{cases} \widetilde{\omega_n} & \text{if } n\in B \\ 0 & \text{otherwise}\end{cases}$$

and define the entire function

$$f(z) = \sum_{n=1}^{\infty} \omega_n^* \frac{z^n}{n!} = \sum_{n\in B} \widetilde{\omega_n} \frac{z^n}{n!}.$$

Clearly, $|f^{(n)}(0)| = \omega_n^* \leq \widetilde{\omega_n} \leq \omega_n$ for all $n\geq 1$. Therefore, our only task is to prove that f is distributionally irregular for D in $H(\mathbb{C})$.

With this aim, fix any $m\in\mathbb{N}$. If $n\in B$ then $\|D^n f\|_{\infty,\overline{B(0,m)}} \geq |f^{(n)}(0)| = \widetilde{\omega_n} = \min\{\omega_n,n\}$, which entails that $\lim_{\substack{n\to\infty\\ n\in B}} \|D^n f\|_{\infty,\overline{B(0,m)}} = \infty$.

Finally, in order to show that $\lim_{\substack{n\to\infty\\ n\in A}} D^n f = 0$ it is enough to prove that $\lim_{\substack{n\to\infty\\ n\in A}} \|f^{(n)}\|_{\infty,\overline{B(0,m)}} = 0$ for every $m\in\mathbb{N}$. So, fix $m\in\mathbb{N}$ as well as an $\varepsilon>0$. Choose $N_0\in\mathbb{N}$ with $N_0\geq m$ and $1/N_0<\varepsilon$. Denote $j_0 := \alpha_{N_0}$. For every $j\in A$ with $j\geq j_0$ there is (a unique) $N\geq N_0$ such that $\alpha_N\leq j\leq \alpha_N^2$. Then we obtain

$$\begin{aligned} f^{(j)}(z) &= \sum_{n=j}^{\infty} \omega_n^*\, n(n-1)(n-2)\cdots(n-j+1)\,\frac{z^n}{n!} \\ &= \sum_{n=2\alpha_N^2}^{\infty} \omega_n^*\, n(n-1)(n-2)\cdots(n-j+1)\,\frac{z^n}{n!},\end{aligned}$$

because $\omega_n^*=0$ if $n\notin B$. From (2), the triangle inequality and the fact $\alpha_N^2 > 2\,\alpha_N$ we get for every $z\in\overline{B(0,m)}$ that

$$\begin{aligned} |f^{(j)}(z)| &\leq \sum_{n=2\alpha_N^2}^{\infty} \omega_n^*\, n(n-1)(n-2)\cdots(n-j+1)\,\frac{m^n}{n!} \\ &\leq \sum_{n=2\alpha_N^2}^{\infty} \omega_n^*\, n^j\,\frac{m^n}{n!} \leq \sum_{n=2\alpha_N^2}^{\infty} n\cdot n^{\frac{n}{2}}\,\frac{N^n}{n!} \\ &\leq \sum_{n=\alpha_N}^{\infty} n^{1+\frac{n}{2}}\,\frac{N^n}{n!} < \frac{1}{N} \leq \frac{1}{N_0} < \varepsilon.\end{aligned}$$

Therefore $\|f^{(j)}\|_{\infty,\overline{B(0,m)}} < \varepsilon$ whenever $j \in A$ and $j \geq j_0$. In other words, $\lim_{\substack{j\to\infty \\ j\in A}} \|f^{(j)}\|_{\infty,\overline{B(0,m)}} = 0$, and we are done. □

Remarks 10. 1. The idea of making large gaps in the sequence of Taylor coefficients, given in the previous proof, is inspired by the construction of scrambled sets for weighted backward shifts on Köthe sequences spaces due to Wu [27] and Wu *et al.* [28].

2. Observe that the function f constructed in the proof of Theorem 9 is D-distributionally irregular in a sense stronger than the one given in Definition 6, because the set B satisfying $\lim_{\substack{n\to\infty \\ n\in B}} \|D^n f\|_{\infty,\overline{B(0,m)}} = \infty$ holds *for any* $m \in \mathbb{N}$.

3. Observe that the sequences $\{\alpha_n\}_{n\geq 1}$ and $\{\beta_n\}_{n\geq 1}$ (hence the sets A and B) are *independent* of the sequence $\{\omega_n\}_{n\geq 1}$. This fact will be exploited in the next section.

Now, we are ready to present the main result of this section.

Theorem 11. *Let* $1 \leq p \leq \infty$, *and set* $a = \dfrac{1}{2\max\{2,p\}}$. *Then, for every* $\varepsilon > 0$, *there exists a distributionally irregular entire function* f *for the differentiation operator acting on* $H(\mathbb{C})$ *such that*

$$M_p(f,r) \leq C\frac{e^r}{r^{a-\varepsilon}} \quad (r > 0)$$

for some constant $C > 0$.

Proof. Since

$$M_p(f,r) \leq M_2(f,r) \quad \text{for } 1 \leq p < 2,$$

we need only prove the result for $p \geq 2$.

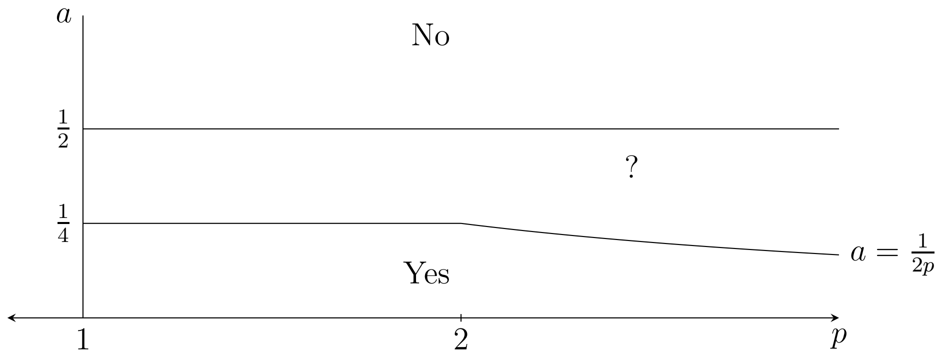
Then fix $p \geq 2$ as well as an $\varepsilon > 0$ and denote, as usual, by q the conjugate exponent of p. Take $\omega_n := n^\varepsilon$ $(n \geq 1)$. Of course, we have $\omega_n \to +\infty$. By Theorem 9 there exists a D-distributionally irregular entire function f satisfying ($f(0) = 0$ and) $|f^{(n)}(0)| \leq n^\varepsilon$ for all $n \geq 1$.

Finally, making use of the Hausdorff-Young inequality and Lemma 8 (with $\alpha = q$ and $\beta = -\varepsilon q$), we have that

$$M_p(f,r) \leq \left(\sum_{n=1}^{\infty} |f^{(n)}(0)|\,(\frac{r^n}{n!})^q\right)^{\frac{1}{q}} \leq \left(\sum_{n=1}^{\infty} n^{\varepsilon q}(\frac{r^n}{n!})^q\right)^{\frac{1}{q}} \leq C\,\frac{e^r}{r^{\frac{1}{2p}-\varepsilon}} = C\,\frac{e^r}{r^{a-\varepsilon}}$$

for all $r > 0$ and some positive constant C, which proves the theorem. □

The following figure shows our present knowledge of possible or impossible rates $\frac{e^r}{r^a}$ for distributionally irregular entire function for differentiation operator.



Problem 12. For each $p \in [1, \infty]$, give the critical order of growth between $\frac{e^r}{\sqrt{r}}$ and $\frac{e^r}{r^{a-\varepsilon}}$ (with $a = \frac{1}{2\max\{2,p\}}$) of a distributionally irregular entire function for the differentiation operator.

3 Dense linear manifolds of distributionally irregular functions

The study of lineability, that is, the search of linear (or, in general, algebraic) structures within nonlinear sets has become a trend in the last two decades, see e.g. the survey [11]. In particular, if X is a topological vector space (over $\mathbb{R}$ or $\mathbb{C}$) and S is a subset of X, then S is called *dense-lineable* in X provided that there exists a dense vector subspace M of X such that $M \subset S \cup \{0\}$. If, in addition, M can be found with $\dim(M) = \dim(X)$, then S is said to be *maximal dense-lineable* in X. In this section, we consider the lineability of the family of D-distributionally irregular entire functions having growth restrictions.

To this end, we will need the next lemma, whose content can be found in [11] (see also [1, 7, 8, 10]). Following [1], if $\mathcal{A}$, $\mathcal{B}$ are subsets of a vector space, then we say that $\mathcal{A}$ is *stronger than* $\mathcal{B}$ whenever $\mathcal{A} + \mathcal{B} \subset \mathcal{A}$.

Lemma 13. *Assume that X is a metrizable topological vector space. Let $\mathcal{A} \subset X$. Suppose that there exists a dense vector subspace $\mathcal{B} \subset X$ such that $\mathcal{A}$ is stronger than $\mathcal{B}$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$. Suppose also that $\mathcal{A} \cup \{0\}$ contains a vector subspace whose dimension equals* $\dim(X)$*. Then $\mathcal{A}$ is maximal dense-lineable.*

Denote by $\mathcal{C}$ the class of functions satisfying simultaneously all properties and growth restrictions considered in the previous section. More precisely, we denote

$$\begin{aligned}\mathcal{C} := \{f \in H(\mathbb{C}) : \; & f \text{ is } D\text{-distributionally irregular in } H(\mathbb{C}) \text{ and} \\ & \sup_{r>0} r^{\frac{1}{2\max\{2,p\}} - \varepsilon} e^{-r} M_p(f, r) < \infty \\ & \text{for all } \varepsilon > 0 \text{ and all } p \in [1, \infty]\}.\end{aligned}$$

The following theorem reveals a rich linear structure inside this (seemingly small) class.

Theorem 14. *The set $\mathcal{C}$ is maximal dense-lineable in $H(\mathbb{C})$.*

Proof. We apply Theorem 9 to the sequence $\omega_n := (\log(n+1))^t$, where $t > 0$. By the construction given in the proof of Theorem 9 (whose notation we keep here) and Remark 10.3, there are subsets A and B of $\mathbb{N}$ with maximal upper density satisfying that, for each $t > 0$, the entire function

$$f_t(z) := \sum_{n \in B} \min\{n, (\log(n+1))^t\} \, \frac{z^n}{n!}$$

is D-distributionally irregular in $H(\mathbb{C})$. In fact, on one hand, we have that $|f_t^{(n)}(0)| = \min\{n, (\log(n+1))^t\}$ for all $n \in B$ and, on the other hand, the proof of Theorem 9 allows to obtain that $\lim_{\substack{n\to\infty \\ n\in A}} \|D^n f_t\|_{\infty, \overline{B(0,m)}} = 0$ for all $m \in \mathbb{N}$ and all $t > 0$.

Observe first that the functions f_t $(t > 0)$ are linearly independent. Indeed, assume that $c_1, \dots, c_s$ are complex numbers (with $s \geq 2$ and $c_s \neq 0$) and that $0 < t_1 < \cdots < t_s$. If the linear combination

$$F := \sum_{k=1}^{s} c_k f_{t_k} \tag{3}$$

is identically zero then, after derivation, we get

$$|F^{(n)}(0)| = \Big| \sum_{k=1}^{s} c_k f_t^{(n)}(0) \Big| = \Big| \sum_{k=1}^{s} c_k f_t^{(n)}(0) \Big| = \Big| \sum_{k=1}^{s} c_k \, \min\{n, (\log(n+1))^{t_k}\} \Big| \ (n \in B).$$

Since $(\log(n+1))^{t_k} \leq (\log(n+1))^{t_s}$ $(n \in \mathbb{N};\, k = 1, \dots, s)$ and $\frac{n}{(\log(n+1))^{t_s}} \to +\infty$ as $n \to \infty$, one derives that $|F^{(n)}(0)| = |\sum_{k=1}^{s} c_k \log(n+1))^{t_k}|$ for $n \in B$ large enough. Hence $|F^{(n)}(0)| \to +\infty$ as $n \to \infty$ $(n \in B)$, which is absurd. Then F is not identically 0, which shows the linear independence of $\{f_t : t > 0\}$.

Define $M := \operatorname{span}\{f_t : t > 0\}$. According to the previous paragraph, M is a vector subspace of $H(\mathbb{C})$ with $\dim(M) = \mathfrak{c} = \dim(H(\mathbb{C}))$, where $\mathfrak{c}$ denotes the cardinality of the continuum. Fix any $F \in M \setminus \{0\}$. Then F has the form (3), with the c_k's and the t_k's as above. Therefore, for given $m \in \mathbb{N}$, we have $\lim_{\substack{n\to\infty \\ n\in B}} \|D^n F\|_{\infty, \overline{B(0,m)}} \geq \lim_{\substack{n\to\infty \\ n\in B}} |F^{(n)}(0)| = +\infty$. In addition, by the triangle inequality,

$$\lim_{\substack{n\to\infty \\ n\in A}} \|D^n F\|_{\infty, \overline{B(0,m)}} \leq \sum_{k=1}^{s} |c_k| \lim_{\substack{n\to\infty \\ n\in A}} \|D^n f_{t_k}\|_{\infty, \overline{B(0,m)}} = 0.$$

This shows that F is D-distributionally irregular. Now, given $\varepsilon > 0$, there is a constant $K = K(\varepsilon, c_1, \dots, c_s, t_1, \dots, t_s) \in (0, +\infty)$ such that $|F^{(n)}(0)| \leq K\, n^{\varepsilon}$ for all $n \geq 1$. The approach of the final part of the proof of Theorem 11 yields the existence of positive constants $C = C_{\varepsilon,p}$ $(p \in [1, \infty])$ satisfying

$$M_p(F, r) \leq C_{\varepsilon,p} \, \frac{e^r}{r^{\frac{1}{2\max\{2,p\}} - \varepsilon}} \quad \text{for all } r > 0.$$

In other words, $F \in \mathcal{C}$. Thus, our class $\mathcal{C}$ contains, except for 0, a vector space having maximal dimension.

Finally, let $X := H(\mathbb{C})$, $\mathcal{A} := \mathcal{C}$ and $\mathcal{B} := \{\text{complex polynomials}\}$. Recall that $\mathcal{B}$ is a dense vector subspace of $H(\mathbb{C})$. Moreover, given $P \in \mathcal{B}$, one has $P^{(n)} = 0$ for n large enough. It follows, trivially, that P is not D-distributionally irregular but $P + f$ is D-distributionally irregular if f is. In addition, it is an easy exercise to prove that, for every $b \in \mathbb{R}$ and every $p \in [1, \infty]$, the set $\{f \in H(\mathbb{C}) : \sup_{r>0} r^b e^{-r} M_p(f, r) < \infty\}$ is a vector space containing the polynomials. Consequently, $\mathcal{A} \cap \mathcal{B} = \varnothing$ and $\mathcal{A} + \mathcal{B} \subset \mathcal{A}$. An application of Lemma 13 yields the maximal dense-lineability of $\mathcal{C}$. $\square$

4 Weighted Banach spaces of entire functions

In this brief section, we establish the existence of distributionally irregular vectors for the differentiation operator acting on certain weighted Banach spaces of entire functions. As a sub-product, large linear manifolds consisting of such vectors will be obtained again.

A *weight* v on $\mathbb{C}$ will be is a strictly positive continuous function on $\mathbb{C}$ which is radial, i.e. $v(z) = v(|z|)$ $(z \in \mathbb{C})$, such that $v(r)$ is non-increasing on $[0, \infty)$ and satisfies $\lim_{r\to\infty} r^m v(r) = 0$ for each $m \in \mathbb{N}$.

We define, for $1 \leq p \leq \infty$ and a weight function v, the following spaces as in [22]:

$$B_{p,\infty} = B_{p,\infty}(\mathbb{C}, v) := \{f \in H(\mathbb{C}) : \sup_{r>0} v(r) M_p(f, r) < \infty\}$$

and

$$B_{p,0} = B_{p,0}(\mathbb{C}, v) := \{f \in H(\mathbb{C}) : \lim_{r\to\infty} v(r) M_p(f, r) = 0\}.$$

These spaces are Banach spaces with the norm

$$\|f\|_{p,v} = \|f\|_{p,\infty,v} := \sup_{r>0} v(r) M_p(f, r).$$

According to [22, Theorem 2.1], the polynomials are contained in $B_{p,0}$ for all $1 \leq p \leq \infty$ and form a dense subset in it. In particular, each space $B_{p,0}$ is separable.

It is worth noting that, if X is a Banach space, then an operator $T \in B(X)$ happens to be distributionally chaotic in the sense of Schweizer and J. Smítal [25] if and only if T has a distributionally irregular vector [12, Theorem 12].

Theorem 15. *Let v be a weight function such that $\lim_{r\to\infty} v(r) \frac{e^r}{r^{\frac{1}{2p}}} = 0$ for some $1 \leq p \leq \infty$. If the differentiation operator $D : B_{p,0} \to B_{p,0}$ is continuous, then there is a dense vector subspace of $B_{p,0}$ all of whose nonzero functions are distributionally irregular for this operator.*

Proof. By [15, Theorem 2.3], if v is a weight function such that $\lim_{r\to\infty} v(r) \frac{e^r}{r^{\frac{1}{2p}}} = 0$ for some $1 \leq p \leq \infty$ and $D : B_{p,0} \to B_{p,0}$ is continuous, then D is frequently hypercyclic. Moreover, the set X_0 of the polynomials is a dense subset in $B_{p,0}$ such that $D^n f$ tends to 0 in $B_{p,0}$ for all $f \in X_0$.

On the other hand, Bayart and Ruzsa have recently proved [4, Corollary 15] that if X is a Banach space and $T \in B(X)$ is a frequently hypercyclic operator such that T^n tends pointwise to 0 on a dense subset of X then T is distributionally chaotic. Hence D is distributionally chaotic on $B_{p,0}$. And since, once again, there exists a dense set X_0 such that $D^n f$ tends pointwise to 0 for all $f \in X_0$, then by [12, Theorem 15] D admits a dense vector subspace consisting (except for zero) of distributionally irregular functions in $B_{p,0}$. □

Remark 16. For any $b \in \mathbb{R}$, let be the weight $v(r) := r^b e^{-r}$ for $r \geq b$, which is non-increasing and satisfies that $\sup_{r>0} \frac{v(r)}{v(r+1)} < \infty$. Then, according to [15, Proposition 2.1], the operator $D : B_{p,0} \to B_{p,0}$ is continuous. As a consequence of Theorem 15 we obtain that for every $p \in [1, 2)$ and every $\varepsilon > 0$ there exists an entire function f such that

$$M_p(f, r) \leq C \frac{e^r}{r^{\frac{1}{2p} - \varepsilon}} \quad (r > 0)$$

for some constant $C > 0$ as well as two increasing sequences of integers $A = (n_k)_k$ and $B = (m_k)_k$ such that $\overline{\text{dens}}(A) = \overline{\text{dens}}(B) = 1$, satisfying that $\lim_{k\to\infty} D^{n_k} f = 0$ in $B_{p,0}$ (hence $D^{n_k} f \to 0$ in the topology of $H(\mathbb{C})$, because convergence in $B_{p,0}$ implies uniform convergence on compacta) and $\lim_{k\to\infty} \|D^{m_k} f\|_{p,v} = \infty$. But we do not know whether there exists $m \in \mathbb{N}$ such that $\lim_{k\to\infty} \|D^{m_k} f\|_{\infty,\overline{B(0,m)}} = \infty$; that is, we do not know whether such an f is D-irregular in $H(\mathbb{C})$, so we have not obtained an improvement of Theorem 11.

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