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COMPUTER ALGEBRA AND ALGEBRAIC ANALYSIS

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RESUMEN. Este artículo describe algunas aplicaciones del Álgebra Computacional al Análisis Algebraico, también conocido como teoría de $\mathcal{D}$-módulos, es decir, el estudio algebraico de sistemas lineales de ecuaciones en derivadas parciales. Mostramos cómo calcular diferentes objetos e invariantes en teoría de $\mathcal{D}$-módulos, utilizando bases de Groebner para anillos de operadores diferenciales lineales.

ABSTRACT. This paper describes some applications of Computer Algebra to Algebraic Analysis also known as $\mathcal{D}$-module theory, i.e. the algebraic study of the systems of linear partial differential equations. One shows how to compute different objects and invariants in $\mathcal{D}$–module theory, by using Groebner bases for rings of linear differential operators.

1. Introduction

The article is intended to provide a short introduction to the use of some Computer Algebra methods in the algebraic study of linear partial differential systems, also known as Algebraic Analysis [24]. Our main tool will be Groebner bases for linear partial differential operators. Some of the algebraic methods developed in this article have been treated by different authors elsewhere. A list of earlier works should include Ch. Riquier [34] and M. Janet [22] both inspired by the works of E. Cartan. Among recent treatments of the topic we can cite the paper [31] and the book [37]. Most of the algorithms presented here have been implemented in the Computer Algebra systems `Macaulay 2`[20], `Risa/Asir`[30] and `Singular`[21].

2. Rings of linear differential operators

For simplicity we are going to mainly consider either the complex numbers $\mathbb{C}$ or the real numbers $\mathbb{R}$ as the base field. Nevertheless, in what follows many results also hold for any base field $\mathbb{K}$ of characteristic zero.

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Let us denote by $\mathbb{K}[x] = \mathbb{K}[x_1, \ldots, x_n]$ the polynomial ring in the variables $x_1, \ldots, x_n$ with coefficients in the field $\mathbb{K}$.

A linear differential operator (LDO), in the variables $x_1, \ldots, x_n$, with polynomial coefficients is a finite sum of the form

$$P(x, \partial) = \sum_{\beta \in \mathbb{N}^n} p_\beta(x) \partial^\beta$$

where each $p_\beta(x)$ is a polynomial in $\mathbb{K}[x]$, $\partial = (\partial_1, \ldots, \partial_n)$ with $\partial_i = \frac{\partial}{\partial x_i}$ and $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$.

The set of such LDOs is denoted by $A_n(\mathbb{K})$ (or simply A_n if no confusion is possible). The set A_n has a natural structure of associative ring (and even of a $\mathbb{K}$–algebra) with unit. The elements in A_n can be added and multiplied in a natural way. Leibniz's rule holds for the multiplication of LDOs: $\partial_i a(x) = a(x)\partial_i + \frac{\partial a(x)}{\partial x_i}$ for any $a(x) \in \mathbb{K}[x]$. The unit of A_n is nothing but the 'constant' operator $1 = x_1^0 \cdots x_n^0 \partial_1^0 \cdots \partial_n^0)$.

The $\mathbb{K}$–algebra A_n is called the *Weyl algebra* of order n with coefficients in the field $\mathbb{K}$. The expressions $P(x, \partial), Q(x, \partial), R(x, \partial), \ldots$ and $P, Q, R, \ldots$ (sometimes with subindexes) will denote LDOs.

The polynomial ring $\mathbb{K}[x]$ has a natural structure of (left) A_n–module, since each operator in A_n acts on each polynomial $f \in \mathbb{K}[x]$ in a natural way (we denote this action by $P(f)$):

$$P(f) = \sum_{\beta \in \mathbb{N}^n} p_\beta(x) \frac{\partial^{\beta_1 + \cdots + \beta_n} f}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}.$$

Definition 2.1. *The* order *of a nonzero operator* $P = \sum_{\beta \in \mathbb{N}^n} p_\beta(x) \partial^\beta$, *denoted by* $\operatorname{ord}(P)$, *is the maximum of the integer numbers* $|\beta| = \beta_1 + \cdots + \beta_n$ *for* $p_\beta(x) \neq 0$ *and the* principal symbol *of* P *is the polynomial*

$$\sigma(P) = \sum_{|\beta| = \operatorname{ord}(P)} p_\beta(x) \xi_1^{\beta_1} \cdots \xi_n^{\beta_n} \in \mathbb{K}[x, \xi]$$

where $\xi = (\xi_1, \ldots, \xi_n)$ *are new variables and* $\mathbb{K}[x, \xi]$ *stands for the polynomial ring in the variables* $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$.

One has $\operatorname{ord}(0) = -\infty$. Sometimes we will write $\sigma(P)(x, \xi)$ to emphasize the fact that $\sigma(P)$ is a polynomial in $\mathbb{K}[x, \xi]$. Notice that $\sigma(P)$ is homogeneous in ξ of degree $\operatorname{ord}(P)$. One has the equality $\sigma(PQ) = \sigma(P)\sigma(Q)$ for $P, Q \in A_n$ and by definition $\sigma(0) = 0$.

Remark 2.2. One can also consider LDOs with coefficients in other rings as

- the ring $\mathcal{O}_{\mathbb{C}^n}(U)$ (resp. $\mathcal{O}_{\mathbb{R}^n}(U)$) of holomorphic (resp. analytic) functions in some open set $U \subset \mathbb{C}^n$ (resp. $U \subset \mathbb{R}^n$).
- the ring of convergent power series $\mathbb{C}\{x\} = \mathbb{C}\{x_1, \ldots, x_n\}$ (or $\mathbb{R}\{x\} = \mathbb{R}\{x_1, \ldots, x_n\}$).
- the ring of formal power series $\mathbb{K}[[x]] = \mathbb{K}[[x_1, \ldots, x_n]]$.

If $\mathcal{R}$ is any of these rings we will denote by $\mathrm{Diff}(\mathcal{R})$ the corresponding ring of LDOs.

One of the goals of the theory of Differential Equations is to study the existence, uniqueness and the properties of the solutions of linear partial differential systems (LPDS)

$$\left\{\begin{array}{ccccc} P_{11}(u_1) & +\cdots+ & P_{1m}(u_m) & = & v_1 \\ \vdots & \vdots & \vdots & & \vdots \\ P_{\ell 1}(u_1) & +\cdots+ & P_{\ell m}(u_m) & = & v_\ell \end{array}\right. \tag{1}$$

where P_{ij} are LDOs, u_j are unknown and v_i are given data. Both u_j and v_j could be functions, distributions, hyperfunctions or more generally elements in any vector space $\mathcal{F}$ endowed with a structure of (left) $\mathrm{Diff}(\mathcal{R})$–module.

Assume System (1) is homogeneous (i.e. $v_1 = \cdots = v_\ell = 0$). If $u = (u_1, \ldots, u_m)$ is a solution[1] of the system then u is also a solution of any equation

$$P_1(u_1) + \cdots + P_m(u_m) = 0$$

with

$$(P_1, \ldots, P_m) = \sum_{i=1}^{\ell} Q_i(P_{i1}, \ldots, P_{im}) \tag{2}$$

for any Q_i in A_n (or more generally $Q_i \in \mathrm{Diff}(\mathcal{R})$ if we are considering any of the rings of Remark 2.2).

For simplicity in what follows, we will assume $\mathrm{Diff}(\mathcal{R}) = A_n = A_n(\mathbb{C})$ unless otherwise stated. The set of all linear combinations

$$(P_1, \ldots, P_m) = \sum_{i=1}^{\ell} Q_i(P_{i1}, \ldots, P_{im})$$

with coefficients Q_i in A_n is the (left) sub-module

$$\sum_{i=1}^{\ell} A_n \underline{P}_i$$

of the free module A_n^m where $\underline{P}_i$ is the vector $(P_{i1}, \ldots, P_{im})$. We also denote by this submodule by $A_n(\underline{P}_1, \ldots, \underline{P}_\ell)$.

B. Malgrange [29], D. Quillen [33] and the Japanese school of M. Sato (e.g. [38] and [24]) have been probably the first to associate to each system of type (1) the (left) quotient A_n–module[2]

$$\frac{A_n^m}{A_n(\underline{P}_1, \ldots, \underline{P}_\ell)} \tag{3}$$

[1]We do not need to precise here the space of the wanted solutions. The result is true for any such space.

[2]All the modules and ideals considered here will be left modules and left ideals unless otherwise stated.

This last quotient, that encodes important information about the system, is also called the *differential system* associated with the system[3] (1).

As A_n is left-Noetherian (see Subsection 5) any finitely generated left A_n-module is isomorphic to a quotient of type (3).

When $m = 1$ (i.e. when the system has only one unknown $u = u_1$) then System (1) reduces to (writing $P_{11} = P_1, \ldots, P_{\ell 1} = P_\ell$)

$$\left\{ \begin{array}{lcl} P_1(u) & = & v_1 \\ \vdots & & \vdots \\ P_\ell(u) & = & v_\ell \end{array} \right. \tag{4}$$

and the set of linear combinations $\sum_i Q_i P_i$ with coefficients $Q_i \in A_n$ is a (left) ideal in A_n, denoted by $\sum_{i=1}^{\ell} A_n P_i$ (and also by $A_n(P_1, \ldots, P_\ell)$).

Different systems could have the same associated module, i.e. the corresponding quotient modules could be isomorphic.

Example 2.3. *Let $P(x, \frac{d}{dx})$ be the operator $\left(\frac{d}{dx}\right)^2 + 2x\frac{d}{dx} + 1 \in A_1$ (we write here $x = x_1$) and let us consider the systems*

$$P(u_1) = 0 \tag{5}$$

and

$$\left\{ \begin{array}{lcl} \frac{du_1}{dx} - u_2 & = & 0 \\ u_1 + (\frac{d}{dx} + 2x)u_2 & = & 0 \end{array} \right. \tag{6}$$

The associated quotient modules are isomorphic since

$$\frac{A_1}{A_1 P} \simeq \frac{A_1^2}{N}$$

where $N \subset A_1^2$ is the sub-module generated by $(\frac{d}{dx}, -1)$ and $(1, \frac{d}{dx} + 2x)$. The morphism of A_1–modules sending the class of 1 in the first module to the class of $(1,0)$ in the second one is in fact an isomorphism. This isomorphism encodes the fact that the systems (5) and (6) are equivalent in the sense that the computation of their respective solutions (wherever they lie) are equivalent problems since they can be reduced to each other. A function $u_1 = u_1(x)$ is a solution of Equation (5) if and only if the vector $(u_1, u_2 := \frac{du_1}{dx})$ is a solution of System (6).

Example 2.4. *We also have the isomorphism*

$$\frac{A_2}{A_2(\partial_1^2 + \partial_2^2)} \simeq \frac{A_2^3}{N}$$

where $N \subset A_2^3$ is the sub-module generated by the family $(\partial_1, -1, 0)$, $(\partial_2, 0, -1)$, $(0, \partial_1, \partial_2)$. The following systems

$$(\partial_1^2 + \partial_2^2)(u_1) = 0 \tag{7}$$

[3]This association is also typical in Algebraic Geometry: to a given system of polynomial equations $f_1(x) = 0, \ldots, f_\ell(x) = 0$ one associates the quotient ring $\frac{\mathbb{K}[x]}{\langle f_1, \ldots, f_\ell \rangle}$ where $\langle f_1, \ldots, f_\ell \rangle$ is the ideal in $\mathbb{K}[x]$ generated by the polynomials $f_i(x)$.

and

$$\begin{cases} \partial_1(u_1) - u_2 & = & 0 \\ \partial_2(u_1) - u_3 & = & 0 \\ \partial_1(u_2) + \partial_2(u_3) & = & 0 \end{cases} \tag{8}$$

are equivalent. A suitable function $u_1 = u_1(x_1, x_2)$ *is a solution of Equation (7) if and only if the vector* $(u_1, u_2 := \partial_1(u_1), u_3 := \partial_2(u_1))$ *is a solution of System (8).*

The study of such A_n–modules is the object of the so-called *Algebraic Analysis*[4] or $\mathcal{D}$–module theory.[5]

In the next three Sections we are going to recall the classical definition of characteristic vector of a linear partial differential equation (Section 3), then we will recall the definition and basic properties of Groebner bases for LDOs and we will show how they can be used to compute the characteristic variety of a LPDS (Sections 5 and 4).

3. Classical characteristic vectors

Assume we have just one linear partial differential equation (LPDE)

$$P(x, \partial)(u) = \left(\sum_\beta p_\beta(x) \partial^\beta \right)(u) = v$$

with real-analytic coefficients $p_\beta(x)$ in some open subset $U \subset \mathbb{R}^n$. A vector $\xi_0 \in \mathbb{R}^n$ is called *characteristic* for P at $x_0 \in U$ if $\sigma(P)(x_0, \xi_0) = 0$ and the set of all such ξ_0 is called the *characteristic variety* of the operator P (or of the equation $P(u) = v$) at $x_0 \in U$ and is denoted by $\mathrm{Char}_{x_0}(P)$. Recall that $\sigma(P)$ denotes the principal symbol of the operator P (see Definition 2.1). Notice that here, in contrast to some textbooks, the zero vector could be characteristic.

More generally, the *characteristic variety* of the operator P is by definition the set

$$\mathrm{Char}(P) = \{(x_0, \xi_0) \in U \times \mathbb{R}^n \,|\, \sigma(P)(x_0, \xi_0) = 0\}.$$

For example, if $Q(x, \frac{d}{dx}) = x^2 \frac{d}{dx} + 1$, then its characteristic variety is the union of the two lines $x = 0$ and $\xi = 0$ in the plane $\mathbb{R} \times \mathbb{R}$ with coordinates (x, ξ).

Assume $\mathrm{ord}(P) \geq 1$, then P is said to be *elliptic* at x_0 if P has no nonzero characteristic vectors at x_0 (i.e. $\mathrm{Char}_{x_0}(P) \subset \{0\}$) and it is said to be *elliptic* on U if $\mathrm{Char}(P) \subset U \times \{0\}$.

The *Laplace operator* $\sum_{i=1}^n \partial_i^2$ is elliptic on $\mathbb{R}^n$.

The characteristic variety of the *wave operator* $P = \partial_1^2 - \sum_{i=2}^n \partial_i^2$ is nothing but the hyperquadric defined in $\mathbb{R}^n \times \mathbb{R}^n$ by the equation $\xi_1^2 - \sum_{i=2}^n \xi_i^2 = 0$.

Characteristic vectors are important in the study of singularities of solutions as can be seen in any classical book on Differential Equations. For example, in the case of the equation $Q(u) = 0$ with Q as before, the corresponding singular locus

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[4] The term was introduced by M. Sato; see the introduction of the volume I of [25]. See also [7].

[5] `Mathematics Subject Classification 2010 (MSC2010): 32C38 Sheaves of differential operators and their modules, D-modules [See also 14F10, 16S32, 35A27, 58J15].`

(see Definition 4.3) is just $\{0\}$. In the neighborhood of any point $x_0 \in \mathbb{R} \setminus \{0\}$ one can apply Cauchy's Theorem: in the neighborhood of such a point the space of solutions of the equation $Q(u) = 0$ is generated by the analytic function $\exp(\frac{1}{x})$.

To define the principal symbol and the characteristic vectors for a system (1) of linear differential equations in many variables (even in the case of only one unknown function) is more involved and in general the naive approach of simply considering the principal symbols of the equations turns out to be unsatisfactory (see Example 4.5). We will use *graded ideals* and Groebner bases for LDOs (see Sections 5 and 4) to define and to compute the *characteristic variety* of a general LPDS.

4. Graded ideal, characteristic variety and dimension.

In this Section $A_n = A_n(\mathbb{C})$. Assume $I \subset A_n$ is an ideal (e.g. the ideal generated by operators $P_1, \ldots, P_m$ in the system (4)).

Definition 4.1. *The graded ideal* $\mathrm{gr}(I)$ *associated with* I *is the ideal in* $\mathbb{C}[x, \xi]$ *generated by the set of principal symbols* $\{\sigma(P) \mid P \in I\}$.

Notice that $\mathrm{gr}(I)$ is a homogeneous polynomial ideal with respect to the (ξ)-degree (the degree with respect to the ξ–variables).

If $I = A_nP$ is the principal ideal generated by P then $\mathrm{gr}(I)$ is also principal in $\mathbb{C}[x, \xi]$ and it is in fact generated by $\sigma(P)$.

Definition 4.2. *The* characteristic variety *of the quotient* A_n*–module* A_n/I *(or of the system defined by* I*) –denoted by* $\mathrm{Char}(A_n/I)$*, is by definition the affine algebraic subvariety of* $\mathbb{C}^{2n}$ *defined by the ideal* $\mathrm{gr}(I) \subset \mathbb{C}[x, \xi]$.

If $I = A_nP$ is a principal ideal then the characteristic variety of A_n/I coincides with the classical characteristic variety of P (see Section 3).

The definition of the characteristic variety $\mathrm{Char}(M)$ of any finitely generated A_n–module M is more involved and uses filtrations on the module M (see e.g. [28, Chapter 11]). The characteristic variety $\mathrm{Char}(M)$ is an affine algebraic subvariety of $\mathbb{C}^{2n}$.

Definition 4.3. *The* singular locus *of a finitely generated* A_n*–module* M *is the Zariski closure of the image of* $\mathrm{Char}(M) \setminus \mathbb{C}^n \times \{0\}$ *under the projection* $\pi : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$*,* $\pi(a, b) = a$.

The notion of singular locus generalizes the one of singular point of an ordinary linear differential equation. One can compute the singular locus of a given system using `Macaulay 2`. See Examples 4.5 and 4.6.

Remark 4.4. In general, if the ideal $I \subset A_n$ is generated by a family $P_1, \ldots, P_m$, then the ideal $\mathrm{gr}(I)$ could be strictly bigger than the ideal generated by the principal symbols $\sigma(P_1), \ldots, \sigma(P_m)$. Such a situation occurs in the following example: Let us consider $P = \partial_1^2 + \partial_1$ and $Q = \partial_1^2 + \partial_2$ in the Weyl algebra $A_2 = A_2(\mathbb{C})$. Let us denote by I the left ideal in A_2 generated by P, Q. It is clear that

$\sigma(P-Q) = \xi_1 - \xi_2 \in \mathrm{gr}(I)$ but $\sigma(P-Q)$ does not belong to the ideal generated by $\sigma(P) = \sigma(Q) = \xi_1^2$ in $\mathbb{C}[x_1, x_2, \xi_1, \xi_2]$. See also Example 4.5 for a more complete example.

Groebner basis theory in A_n can be used to calculate $\mathrm{gr}(I)$. Namely, if $P_1, \ldots, P_\ell$ is a Groebner basis of I[6] then $\sigma(P_1), \ldots, \sigma(P_\ell)$ generate $\mathrm{gr}(I)$ and so these principal symbols define the characteristic variety of A_n/I. See Lemma 5.6 for a more precise statement.

Example 4.5. *If* $I = A_2(P_1, P_2)$ *with* $P_1 = x_1\partial_1 + x_2\partial_2$ *and* $P_2 = x_1\partial_2 + x_2^2\partial_1$ *then* $\mathrm{gr}(I) = \langle \xi_1, \xi_2 \rangle$ *that is strictly bigger than* $\langle \sigma(P_1), \sigma(P_2) \rangle = \langle x_1\xi_1 + x_2\xi_2, x_1\xi_2 + x_2^2\xi_1 \rangle$.

The following `Macaulay 2` *script can be used to compute generators of the graded ideal* $\mathrm{gr}(I)$. *The corresponding* `Macaulay 2` *command is called* `charIdeal`. *We need* `D-modules.m2` *package to this end (see* [20]*). Input lines in* `Macaulay` *are denoted by* `i1, i2, ...` *while the corresponding output lines are* `o1, o2, ...`

The command `R=QQ[x,y]` *defines the ring* `R` *to be the polynomial ring in the variables* `x, y` *and with rational coefficients. The command* `W=makeWA R` *defines the ring* `W` *to be the Weyl algebra of order 2 with coefficients in* `R`.

```
Macaulay 2, version 1.2
with packages: Elimination, IntegralClosure,LLLBases, PrimaryDecomposition,
               ReesAlgebra, SchurRings, TangentCone

i1 : R=QQ[x,y]

i2 : load "D-modules.m2"

i3 : W=makeWA R

i4 : P1=x*dx+y*dy,P2=x*dy+y^2*dx

o4 = (x*dx + y*dy, y^2*dx + x*dy)

i5 : I=ideal(P1,P2)

o5 = ideal (x*dx + y*dy, y^2*dx + x*dy)

i6 : charIdeal I

o6 = ideal (dy, dx)

o6 : Ideal of QQ [x, y, dx, dy]

i7 : J=ideal(dx,dy)

o7 = ideal (dx, dy)

o7 : Ideal of W
```

[6] With respect to a monomial ordering compatible with the order of the differential operators (see Lemma 5.6).

```
i8 : J==I

o8 = true
```

Input line `i4` *defines the operators* `P1, P2` *generating the ideal I (the corresponding definition in* `Macaulay` *is the input line* `i5`*.*

The computation of the input line `i6: charIdeal I` *gives the ideal* `o6: ideal (dy, dx)`*. Notice that as remarked by* `Macaulay` *output* `o6 : Ideal of QQ [x, y, dx, dy]` *the ideal given by* `o6: ideal (dy, dx)` *is in fact an ideal of the ring* `QQ [x, y, dx, dy]` *which is considered to be a commutative polynomial ring while* `W` *is the Weyl algebra of order 2.*

In fact, the last part of the script (from `i7` *to* `o8`*) proves that the ideal I equals the ideal $A_2(\partial_1, \partial_2)$. We are using here $x = x_1$, $y = x_2$.*

In the Weyl algebra `W` *the expressions* `dx, dy` *stand for ∂_1 and ∂_2 while in* `QQ [x, y, dx, dy]` *they stand for ξ_1 and ξ_2 respectively.*

The previous computation can be also made by hand although they are not completely obvious.

If $I = A_2(P_1, P_2)$ as in Example 4.5 we have proven that $\mathrm{gr}(I) = \langle \xi_1, \xi_2 \rangle$ and then the equality $\mathrm{Char}(A_2/I) = \mathbb{C}^2 \times \{(0,0)\}$.

In particular, the singular locus of the differential system A_2/I is the empty set.

Let's see another example using `Macaulay 2`.

Example 4.6. *The following* `Macaulay 2` *script computes $\mathrm{gr}(J)$ for $J = A_2(Q_1, Q_2)$ and $Q_1 = \partial_1^2 - \partial_2$, $Q_2 = x_1\partial_1 + 2x_2\partial_2$.*

```
i2 : R=QQ[x,y]

i3 : W2=makeWA R

i4 : Q1=dx^2-dy,Q2=x*dx+2*y*dy

o4 = (dx^2 - dy, x*dx + 2y*dy)

o4 : Sequence

i5 : J = ideal (Q1,Q2)

o5 = ideal (dx^2 - dy, x*dx + 2y*dy)

o5 : Ideal of W2

i6 : charIdeal J

o6 = ideal (dx^2 , x*dx + 2y*dy)

o6 : Ideal of QQ [x, y, dx, dy]

i7 : singLocus ideal(Q1,Q2)

o7 = ideal(y)
```

The input `J = ideal(Q1, Q2)` *defines the ideal* J *of the Weyl algebra* W *generated by the linear differential operators* `Q1, Q2`*. Then the input* `i6 : charIdeal J` *computes the graded ideal* $\mathrm{gr}(J)$*. Then* $\mathrm{gr}(J)$ *is generated by the polynomials* $\xi_1^2, x_1\xi_1 + 2x_2\xi_2$ *and the characteristic variety* $\mathrm{Char}(A_2/I)$ *is the union of the two planes* $\xi_1 = x_2 = 0$ *and* $\xi_1 = \xi_2 = 0$ *in* $\mathbb{C}^4$*.*

The command `i7: singLocus ideal(Q1,Q2)` *computes the singular locus of the differential system* A_2/J*. This singular locus is the line* $x_2 = 0$ *in the plane* $\mathbb{C}^2$*.*

By definition the *dimension* of a finitely generated nonzero A_n–module M, denoted by $\dim(M)$, is the dimension[7] of its characteristic variety $\dim(\mathrm{Char}(M))$ viewed as an algebraic variety in $\mathbb{C}^{2n}$. The modules A_2/I and A_2/J of Examples 4.5 and 4.6 have both dimension 2 since their characteristic varieties are, in the first case, the plane $\mathbb{C}^2 \times 0$ in $\mathbb{C}^4$ and the union of the planes $\xi_1 = x_2 = 0$ and $\xi_1 = \xi_2 = 0$ (again in $\mathbb{C}^4$) in the second case.

A fundamental result due to I.N. Bernstein ([3], [4]) says that if $M \neq 0$ then $\dim(M) \geq n$.

If $M = A_n/I$ (and more generally if M is a quotient of a free A_n–module) the dimension of M can be computed using Groebner basis in A_n. To this end we first notice that $\dim(A_n/I) = \dim \mathrm{Char}(A_n/I)$ is nothing but the Krull dimension of the quotient ring $\mathbb{C}[x,\xi]/\mathrm{gr}(I)$ (see e.g. [27, Chap. 8]). We first compute, using Groebner basis algorithm, a system of generators of $\mathrm{gr}(I)$ –assuming that a system of generators of I is given– and then, applying again Groebner basis computation, this time in the polynomial ring $\mathbb{C}[x,\xi]$, we compute the Krull dimension of $\mathbb{C}[x,\xi]/\mathrm{gr}(I)$[8].

Computer Algebra systems `Macaulay 2` [20] and `Risa/Asir` [30] support command computing the dimension of a differential system with coefficients in A_n. `Singular` [21] supports a command deciding is a A_n–module is holonomic (see Definition 4.7).

Definition 4.7. *A finitely generated* A_n*–module* M *is said to be* holonomic *(or a* holonomic system*) if either* $M = (0)$ *or* M *is nonzero and* $\dim(M) = n$*.*

Holonomic systems generalize the classical notion of maximally overdetermined systems (see [23]). The previous examples A_2/I and A_2/J are holonomic.

Remark 4.8. If $K = A_nP$ is the principal ideal generated by $P \in A_n$ and the quotient $M = A_n/K$ is non zero then M is holonomic if and only if $n = 1$. In fact $\mathrm{gr}(K)$ is just generated by the principal symbol $\sigma(P) \in \mathbb{C}[x,\xi]$ and the characteristic variety $\mathrm{Char}(M)$ is the hypersurface defined by the polynomial $\sigma(P)(x,\xi)$ in $\mathbb{C}^{2n}$. So $\dim(M) = 2n - 1$ and $\dim(M) = n$ if and only if $n = 1$.

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[7] We are considering here the Krull dimension (see e.g. [27, Chap. 8]).

[8] Actually only a single Groebner basis of I is needed if the monomial ordering is suitably chosen.

Let $I \subset A_n$ be an ideal. We define, following [37], the *holonomic rank* of the ideal I as

$$\mathrm{rank}(I) = \dim_{\mathbb{C}(x)} \frac{\mathbb{C}(x)[\xi]}{\mathbb{C}(x)[\xi]\mathrm{gr}(I)}$$

where $\mathbb{C}(x)$ is the field of rational functions and $\mathrm{gr}(I) \subset \mathbb{C}[x, \xi]$ is the graded ideal associated with I.

It is easy to see that if A_n/I is holonomic then $\mathrm{rank}(I) < +\infty$ and that the converse is not true (see e.g. [37, Prop. 1.4.9]). See Remark 7.3 for a result relating the holonomic rank with the number of independent holomorphic solutions of the system.

5. Groebner bases for rings of differential operators

The definition and construction of Groebner bases for polynomial rings [8, 9] can be adapted to the case of rings of linear differential operators [6, 12], see also [37] for the Weyl algebra.

Definition 5.1. *Let $r > 0$ be an integer number. A well ordering $\prec$ on $\mathbb{N}^r$ is said to be a* monomial order *if it is compatible with the sum. That is: $\alpha \prec \beta$ implies $\alpha + \gamma \prec \beta + \gamma$ for all $\gamma \in \mathbb{N}^r$.*

Remark 5.2. For any monomial order $\prec$ on $\mathbb{N}^r$ one has $0 = (0, \ldots, 0) \prec \alpha$ for all $\alpha \in \mathbb{N}^r$. Moreover, for $\alpha, \beta \in \mathbb{N}^r$ such that $\alpha_i \leq \beta_i$ for all i one has $\alpha \prec \beta$. In other words, any monomial order refines the componentwise order on $\mathbb{N}^r$.

We usually translate any order $\prec$ on $\mathbb{N}^r$ to an order –also denoted by $\prec$– on the set of monomial $\{x^\alpha \,|\, \alpha \in \mathbb{N}^r\}$ just by writing $x^\alpha \prec x^\beta$ if and only if $\alpha \prec \beta$.

Let $P = P(x, \partial) = \sum_{\beta \in \mathbb{N}^n} p_\beta(x)\partial^\beta$ be a differential operator in A_n. The operator P can be rewritten as

$$P = \sum_{\alpha\beta} p_{\alpha\beta} x^\alpha \partial^\beta$$

just by writing the polynomial $p_\beta(x)$ as $p_\beta(x) = \sum_\alpha p_{\alpha\beta} x^\alpha$, with $p_{\alpha\beta} \in \mathbb{C}$.

We will denote by $\mathcal{N}(P)$ the *Newton diagram* of P. One has by definition $\mathcal{N}(P) = \{(\alpha, \beta) \in \mathbb{N}^{2n} | p_{\alpha\beta} \neq 0\}$.

Definition 5.3. *Let us fix a monomial order $\prec$ on $\mathbb{N}^{2n}$. We call* privileged exponent *with respect to $\prec$ of a nonzero operator P –and we denote it by* $\exp_\prec(P)$*– the maximum $(\alpha, \beta) \in \mathbb{N}^{2n}$ such that $p_{\alpha\beta} \neq 0$. We will write simply* $\exp(P)$ *if no confusion is possible.*

The equality $\exp(PQ) = \exp(P) + \exp(Q)$ is satisfied for all nonzero $P, Q \in A_n$. The notion of privileged exponent of a differential operator generalizes the one of privileged exponent of a power series, due to H. Hironaka. It was introduced in Lejeune and Teissier [26] (see also Aroca et al.[2]).

If I is a nonzero ideal in A_n, we denote (as in the polynomial case) by $E_\prec(I)$ (or simply $E(I)$) the set of privileged exponents of the nonzero elements in I. Since $E(I) + \mathbb{N}^{2n} = E(I)$ there exists a finite subset $G \subset I$ such that $E(I)$ is generated by $\{\exp(P)\,|\,P \in G\}$ (this is a consequence of Dickson's Lemma; see e.g. [19, p. 12]).

Definition 5.4. *Let $I \subset A_n$ be a nonzero ideal. A finite subset $\{P_1,\dots,P_r\} \subset I$ such that $E_\prec(I)$ is generated by $\{\exp_\prec(P_i)\,|\,i=1,\dots,m\}$, is called a Groebner basis of I with respect to the fixed monomial order $\prec$.*

Remark 5.5. If the nonzero ideal $I \subset A_n$ is principal and generated by an operator $P \in A_n$ then $E(I)$ is the hyper-quadrant generated by $\exp(P)$ in $\mathbb{N}^{2n}$: one has $E(I) = \exp(P) + \mathbb{N}^{2n}$. Moreover, $\{P\}$ is a Groebner basis of I (with respect to any monomial order $\prec$ in $\mathbb{N}^{2n}$).

Lemma 5.6. *Assume the monomial ordering $\prec$ is compatible with the order of the differential operators.*[9] *Let I be an ideal in A_n and $\mathcal{G} = \{P_1,\dots,P_m\}$ be a subset in I. Then if $\mathcal{G}$ is a Groebner basis of I (with respect to $\prec$) then the set $\{\sigma(P_1),\dots,\sigma(P_m)\}$ is a Groebner basis of the graded ideal* $\mathrm{gr}(I)$ *(with respect to $\prec$.)*

Proof. Notice that $\mathrm{gr}(I)$ is an ideal in the polynomial ring $\mathbb{C}[x,\xi]$ (see Definition 4.1). The statement is a consequence of the equality $\exp_\prec(P) = \exp_\prec(\sigma(P))$ for all $P \in A_n$ which implies the equality $\mathrm{Exp}_\prec(I) = \mathrm{Exp}_\prec(\mathrm{gr}(I))$. □

Theorem 5.7 (Division in A_n)**.** *Let us fix $\prec$ a monomial order in $\mathbb{N}^{2n}$. Let $(P_1,\dots,P_m)$ be an m–tuple of nonzero elements of A_n Then, for any P in A_n, there exists a $(m+1)$–tuple $(Q_1,\dots,Q_m,R)$ of elements in A_n, such that:*

1. $P = Q_1P_1 + \cdots + Q_mP_m + R$.
2. $\exp_\prec(P) = \max\{\exp_\prec(Q_1P_1),\dots,\exp_\prec(Q_mP_m),\exp_\prec(R)\}$.
3. $\mathcal{N}(R) \cap \left(\bigcup_{i=1}^m (\exp_\prec(P_i) + \mathbb{N}^{2n})\right) = \emptyset$.

Remark 5.8. Theorem 5.7 is analogous to the division theorem for polynomials in the polynomial ring $\mathbb{C}[x]$ (see e.g. [19, p. 9] or [1, Th. 1.5.9]). We call here Division (or Division Theorem) in A_n what is sometimes called *weak* Division in A_n. The proof of Theorem 5.7 can be read in [12, 13] and also in [37].

Remark 5.9. The linear differential operator Q_i in the theorem is called a i–th *quotient* and R is called a *remainder* of the division of P by $(P_1,\dots,P_m)$.

Let us write $\mathcal{F} = \{P_1,\dots,P_m\}$. If $P = Q_1P_1 + \cdots + Q_mP_m + R$ as in Theorem 5.7 we say that P reduces to R modulo $\mathcal{F}$.

Proof. (Theorem 5.7) By linearity it is enough to prove the result for the monomials $x^\alpha\partial^\beta \in A_n$. We will use induction on (α,β). If $x^\alpha\partial^\beta = 1$ (i.e. if $\alpha = \beta = (0,\dots,0)$), then either $\exp(P_i) \neq 0 \in \mathbb{N}^{2n}$ for all i and in this case it is enough to write $1 = \sum_{i=1}^m 0P_i + 1$ (and 1 satisfies the third condition in the statement of the theorem) or there exists an integer j such that $\exp(P_j) = 0 \in \mathbb{N}^{2n}$. In this case P_j is a nonzero constant because 0 is the first element in $\mathbb{N}^{2n}$ with

[9] A monomial order $\prec$ on $\mathbb{N}^{2n}$ is said to be compatible with the order of the differential operators if for any $(\alpha,\beta),(\gamma,\delta)$ with $|\beta| < |\delta|$ one has $(\alpha,\beta) \prec (\gamma,\delta)$.

respect to the well ordering $\prec$. We write

$$1 = \sum_{i \neq j} 0 \cdot P_i + (1/P_j)P_j + 0.$$

This proves the existence at the first step of the induction.

Assume that the result is proved for any (α', β') strictly smaller than (with respect to $\prec$) some $(\alpha, \beta) \neq 0 \in \mathbb{N}^{2n}$.

If

$$(\alpha, \beta) \notin \bigcup_{i=1}^{m} (\exp_{\prec}(P_i) + \mathbb{N}^{2n})$$

then we write

$$x^\alpha \partial^\beta = \sum_{i=1}^{m} 0 \cdot P_i + x^\alpha \partial^\beta$$

and this expression satisfies the theorem.

If

$$(\alpha, \beta) \in \bigcup_{i=1}^{m} (\exp_{\prec}(P_i) + \mathbb{N}^{2n})$$

then there exist $j = 1, \ldots, m$ and $(\gamma, \delta) \in \mathbb{N}^{2n}$ such that $(\alpha, \beta) = (\gamma, \delta) + \exp(P_j)$. We can write

$$x^\alpha \partial^\beta = \frac{1}{c_j} x^\gamma \partial^\delta P_j + G_j$$

where c_j is the coefficient of the privileged monomial of P_j and all the monomials in G_j are strictly smaller (with respect to $\prec$) than (α, β). By the induction hypothesis there exists $(Q'_1, \ldots, Q'_m, R')$ satisfying the conditions of the theorem for $P = G_j$. In particular we have:

$$x^\alpha \partial^\beta = \sum_{i \neq j} Q'_i P_i + \left(\frac{1}{c_j} x^\gamma \partial^\delta + Q'_j \right) P_j + R'.$$

This proves the result for (α, β). Thus, the result is proved for any $P \in A_n$. □

Corollary 5.10. *Let I be a nonzero ideal of A_n and let $\mathcal{G} := \{P_1, \ldots, P_m\} \subset I$. The following conditions are equivalent:*

1. *$\mathcal{G}$ is a Groebner basis of I (with respect to a fixed monomial order in $\mathbb{N}^{2n}$).*
2. *For any P in A_n, we have: $P \in I$ if and only if P reduces to 0 modulo $\mathcal{G}$.*

Corollary 5.11. *Let I be a nonzero (left) ideal of A_n and let $P_1, \ldots, P_m$ be a Groebner basis of I. Then $P_1, \ldots, P_m$ is a system of generators of I. In particular the ring A_n is (left) noetherian.*

Division Theorem 5.7 and Groebner bases can be also considered, in a straightforward way, for right ideals (or more generally for right sub-modules of a free module A_n^m). In particular, A_n is a right-Noetherian ring and so actually a Noetherian ring.

The Division Theorem and the theory of Groebner basis can be also extended for sub-modules of free modules A_n^m for any integer number $m \geq 1$ [12, 13].

Buchberger's algorithm for polynomials (see [9]) can be adapted to the Weyl algebra A_n [12], see also [37]. We do not reproduce here the generalization of Buchberger's algorithm to the Weyl algebra (the reader can consult previous references). Considering as input a monomial order $\prec$ in $\mathbb{N}^{2n}$ and a finite set $\mathcal{F} = \{P_1, \ldots, P_m\}$ of differential operators, one can algorithmically compute a Groebner basis, with respect to $\prec$, of the ideal $I \subset A_n$ generated by $\mathcal{F}$. So, one can also compute a finite set of generators of the subset $E(I) \subset \mathbb{N}^{2n}$.

Remark 5.12. Similarly to the commutative polynomial case, Groebner bases in A_n are used to compute, in an explicit way, some invariants in A_n-module theory. Most of the algorithms in this subject appears in Oaku and Takayama[31]. In particular, Groebner bases in A_n are used:

a) to compute a generating system of $Syz_{A_n}(P_1, \ldots, P_m)$, the A_n–module of syzygies of a given family $P_1, \ldots, P_m$ in a free module A_n^r ($r \geq 1$).
b) to solve the membership problem (i.e. to decide if a given vector $P \in A_n^r$ belongs to the sub-module generated by the vectors $P_1, \ldots, P_m$) and to decide if two sub-modules of A_n^r are equal.
c) to compute the graded ideal associated with a (left) ideal I in A_n (see Definition 4.1 and Lemma 5.6) and to compute the dimension of a quotient module A_n/I.
d) to decide if a finitely presented A_n–module is holonomic (i.e. to decide if its characteristic variety has dimension n. See Definition 4.7).
e) to construct a *finite free resolution* of a given finitely presented A_n–module.
f) to decide if a *finite complex of free* A_n–modules is exact.

Many computer algebra systems can handle this kind of computations. Among the most used should be mentioned `Macaulay` [20], `Risa/Asir` [30] and `Singular` [21].

Remark 5.13 (Division theorem and Groebner bases in $\mathcal{D}$ and $\widehat{\mathcal{D}}$). A Division Theorem (analogous to Theorem 5.7) can be proved for elements in $\mathcal{D}$ or in $\widehat{\mathcal{D}}$ (see Briançon and Maisonobe [6] and Castro[12]). Recall that $\mathcal{D}$ (resp. $\widehat{\mathcal{D}}$) stands for the ring of linear differential operators with coefficients in the ring $\mathbb{C}\{x\}$ (resp. $\mathbb{C}[[x]]$) of convergent (resp. formal) power series.

This is not straightforward from the Weyl algebra case because Definition 5.3 of *privileged exponent* for an element in A_n doesn't work for general operators in $\mathcal{D}$ or in $\widehat{\mathcal{D}}$.

Nevertheless, Groebner bases also exist for left (or right) ideals in $\mathcal{D}$ (and in $\widehat{\mathcal{D}}$) and the analogous of Corollaries 5.10 and 5.11 also hold in $\mathcal{D}$ and $\widehat{\mathcal{D}}$. This proves in particular that $\mathcal{D}$ and $\widehat{\mathcal{D}}$ are Noetherian rings. We will not give here the details and we refer the interested reader to the references above.

6. The solution spaces of $P(u) = v$

Let us consider a single LPDE

$$P(u) = P(x, \partial)(u) = 0$$

and suppose we want to compute its solutions in some function space $\mathcal{F}$ where A_n acts naturally. The space $\mathcal{F}$ should be then a (left) A_n–module.

Typical examples of such spaces are function spaces (continuous functions, real analytic or holomorphic functions, polynomial functions ...), spaces of multivalued functions and spaces of distributions among others.

A central question in the theory of Differential Equations is to compute the solution set

$$\mathrm{Sol}(P;\mathcal{F}) = \{u \in \mathcal{F} \text{ such that } P(u) = 0\}.$$

Actually, $\mathrm{Sol}(P;\mathcal{F})$ is a vector space as it is nothing but the kernel $\ker(P(\,))$ of the morphism

$$P(\,) : \mathcal{F} \to \mathcal{F}$$

defined by the action of P on $\mathcal{F}$. Notice that as A_n is a non commutative ring the map $P(\,)$ is only $\mathbb{C}$–linear.

Lemma 6.1. *Let us denote $M = A_n/A_nP$. The solution vector space* $\mathrm{Sol}(P,\mathcal{F})$ *is isomorphic to* $\mathrm{Hom}_{A_n}(M,\mathcal{F})$ *the vector space of A_n-morphisms from M to $\mathcal{F}$.*

Proof. Each solution $u \in \mathrm{Sol}(P;\mathcal{F})$ determines the morphism (of A_n–modules)

$$\phi_u : M \to \mathcal{F}$$

defined by $\phi_u(\overline{Q}) = Q(u)$ for $Q \in A_n$, where $\overline{Q}$ stands for the class of Q modulo the ideal A_nP. On the other hand, each A_n–module morphism

$$\phi : M \to \mathcal{F}$$

(i.e. each $\phi \in \mathrm{Hom}_{A_n}(M,\mathcal{F})$) determines the solution

$$u_\phi = \phi(\overline{1})$$

since $P(\phi(\overline{1})) = \phi(P \cdot \overline{1}) = \phi(\overline{0}) = 0$.

The map sending $u \in \mathrm{Sol}(P;\mathcal{F})$ to $\phi_u \in \mathrm{Hom}_{A_n}(M,\mathcal{F})$ is an isomorphism of vector spaces whose inverse is just the map sending $\phi \in \mathrm{Hom}_{A_n}(M,\mathcal{F})$ to $u_\phi \in \mathrm{Sol}(P;\mathcal{F})$. □

Let us return to the case of the complete equation $P(u) = v$ where v is in $\mathcal{F}$. The obstruction to solve this equation is given by the vector space $\mathcal{F}/P(\mathcal{F}) = \mathrm{coker}(P(\,))$ that is the cokernel of the map $P(\,) : \mathcal{F} \to \mathcal{F}$. That is, for a fixed $v \in \mathcal{F}$, the equation $P(u) = v$ has a solution u in $\mathcal{F}$ if and only if $v \in P(\mathcal{F})$ or equivalently if and only if the class of v in the quotient space $\mathcal{F}/P(\mathcal{F})$ is zero.

More concretely, the complete equation has a solution u for each v if and only if $\mathcal{F} = P(\mathcal{F})$ (or equivalently if and only if $\mathcal{F}/P(\mathcal{F}) = \mathrm{coker}(P(\,)) = (0)$).

We will see that $\mathrm{coker}(P(\,))$ is naturally isomorphic, as vector space, to the first extension group $\mathrm{Ext}^1_{A_n}(M,\mathcal{F})$ of M by $\mathcal{F}$ (in this case it is actually a vector space).

First of all, let us consider the natural exact sequence of modules and morphisms

$$0 \to A_n \xrightarrow{\phi_P} A_n \xrightarrow{\pi} M = \frac{A_n}{A_nP} \to 0. \tag{9}$$

where the morphism ϕ_P is defined by $\phi_P(Q) = QP$ for $Q \in A_n$ and π is the natural projection. Then by truncating the previous one we consider the complex (of A_n-modules)

$$0 \to A_n \xrightarrow{\phi_P} A_n \to 0. \tag{10}$$

We then apply to this complex the functor $\mathrm{Hom}_{A_n}(-, \mathcal{F})$ and we get the complex of vector spaces

$$0 \to \mathrm{Hom}_{A_n}(A_n, \mathcal{F}) \xrightarrow{(\phi_P)^*} \mathrm{Hom}_{A_n}(A_n, \mathcal{F}) \to 0 \tag{11}$$

where $(\phi_P)^*(\eta) = \eta \circ \phi_P$ for $\eta \in \mathrm{Hom}_{A_n}(A_n, \mathcal{F})$.

The vector space $\mathrm{Hom}_{A_n}(A_n, \mathcal{F})$ has a natural structure of A_n–module which is in fact isomorphic to $\mathcal{F}$. This is a general fact in ring theory: to each morphism $\eta \in \mathrm{Hom}_{A_n}(A_n, \mathcal{F})$ we associate $\eta(1) \in \mathcal{F}$ and this correspondence is in fact an isomorphism whose inverse is the map sending an element $u \in \mathcal{F}$ to the morphism

$$\eta_u : A_n \longrightarrow \mathcal{F}$$

defined by $\eta_u(Q) = Q(u)$. Under this isomorphism the last complex can be read as

$$0 \to \mathcal{F} \xrightarrow{P(\,)} \mathcal{F} \to 0$$

Then we have natural isomorphisms of vector spaces $\ker(P(\,)) \simeq \mathrm{Hom}_{A_n}(M, \mathcal{F}) = \mathrm{Ext}^0_{A_n}(M, \mathcal{F})$ (which we have described before, see Lemma 6.1) and $\mathcal{F}/P(\mathcal{F}) = \mathrm{coker}(P(\,)) \simeq \mathrm{Ext}^1_{A_n}(M, \mathcal{F})$.

Definition 6.2. *The vector spaces* $\mathrm{Ext}^i_{A_n}(M, \mathcal{F})$ *for* $i = 0, 1$ *are called the* solutions spaces *of the equation* $P(u) = v$ *(or more precisely of the* A_n*–module* $M = A_n/A_nP$*) in* $\mathcal{F}$.

7. The solution spaces of a differential system

In order to generalize the notion of *solutions spaces* for a general System (1) we have to consider first the A_n-module (or differential system) $M = A_n^m/A_n(\underline{P_1}, \ldots, \underline{P_\ell})$ associated with the system.

First of all, similarly to the construction done in Section 6 one can describe an isomorphism between the solution space $\mathrm{Sol}(\mathcal{S}_h; \mathcal{F})$ and $\mathrm{Hom}_{A_n}(M, \mathcal{F})$ where $\mathcal{S}_h$ is the homogeneous system associated with System (1).

This isomorphism associates to each solution $u = (u_1, \ldots, u_m) \in \mathrm{Sol}(\mathcal{S}_h; \mathcal{F})$ the morphism $\phi_u \in \mathrm{Hom}_{A_n}(M, \mathcal{F})$ defined by $\phi_u(\overline{Q}) = Q(u)$. In particular, if I is an ideal in A_n, the solution space $\mathrm{Sol}(I; \mathcal{F})$ is isomorphic to $\mathrm{Hom}_{A_n}(A_n/I, \mathcal{F})$.

A somehow analogous situation can be found in Algebraic Geometry. Assume the system $\mathcal{S} = \{f_1(x) = 0, \ldots, f_\ell(x) = 0\}$ of complex polynomial equations (in n variables) has only finitely many solutions (that is the set $\mathcal{V}(\mathcal{S}) = \{a \in \mathbb{C}^n | f_1(a) = \cdots = f_\ell(a) = 0\}$ is finite). There exists a natural bijection from $\mathcal{V}(\mathcal{S})$ to $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[x]/\langle\mathcal{S}\rangle, \mathbb{C})$ defined by attaching to each solution $\underline{a} \in \mathcal{V}(\mathcal{S})$ the corresponding evaluation homomorphism $(\overline{g(x)} \mapsto g(\underline{a}))$.

Let M be an A_n–module. Inspired by the situation described in Section 6 we can give the following

Definition 7.1. *The* solutions spaces *of the A_n–module M with values in $\mathcal{F}$ are the vector spaces* $\mathrm{Ext}^i_{A_n}(M,\mathcal{F})$ *for* $i = 0,\dots,n$.

Recall that $\mathrm{Hom}_{A_n}(M,\mathcal{F}) = \mathrm{Ext}^0_{A_n}(M,\mathcal{F})$ and that the space $\mathrm{Ext}^i_{A_n}(M,\mathcal{F})$ for $i \geq 1$ can be described by using the right derived functors of the functor $\mathrm{Hom}_{A_n}(-,\mathcal{F})$. Moreover, by definition $\mathrm{Ext}^i_{A_n}(M,\mathcal{F})$ can be calculated as the i-th cohomology group of the complex $\mathrm{Hom}_{A_n}(\mathcal{L}_\bullet,\mathcal{F})$ where $\mathcal{L}_\bullet$ is a *free resolution* of M.

As a consequence of Kashiwara's constructibility theorem [23] we have the following

Theorem 7.2. *Assume the A_n–module is holonomic then the solution $\mathbb{C}$–vector spaces* $\mathrm{Ext}^i_{A_n}(M,\mathbb{C}\{x\})$ *and* $\mathrm{Ext}^i_{A_n}(M,\mathbb{C}[x])$ *have finite dimension for* $i = 0,\dots,n$.

The holonomicity condition on M is of course necessary: in dimension 2, we have $\mathrm{Ext}^0_{A_2}(\frac{A_2}{A_2\partial_1},\mathbb{C}\{x_1,x_2\}) = \mathbb{C}\{x_2\}$ and this is an infinite dimensional vector space.

For general systems as (1) and general function spaces $\mathcal{F}$ there is no algorithm to compute the solution spaces $\mathrm{Ext}^i_{A_n}(M,\mathcal{F})$.

Nevertheless, if $M = A_n/I$ is holonomic (see Definition 4.7) there are algorithms computing a basis of $\mathrm{Ext}^i_{A_n}(M,\mathbb{C}[x])$ for all i, ([32], [42]). Moreover, in [40] an algorithm computing a basis of $\mathrm{Ext}^i_{A_n}(M,\mathbb{C}[[x]])$ ($i = 0,\dots,n$) is described.

Remark 7.3. As a consequence of Cauchy Theorem (see e.g. [37, Th. 1.4.19]) we have

$$\dim_{\mathbb{C}} \mathrm{Sol}(I;\mathcal{O}_{\mathbb{C}^n}(U)) = \dim_{\mathbb{C}} Ext^0_{A_n}(A_n/I,\mathcal{O}_{\mathbb{C}^n}(U)) = \mathrm{rank}(I)$$

where the system A_n/I is holonomic and $\mathcal{O}_{\mathbb{C}^n}(U)$ stands for the space of holomorphic functions on an open set $U \subset \mathbb{C}^n \setminus Z$ where Z is the singular locus of A_n/I (see Definition 4.3).

All the algorithms mentioned above use Groebner basis computations in the Weyl algebra A_n. A key ingredient of the algorithms is the effective computation of a free resolution of the given A_n–module M (see Remark 5.12).

8. Operators annihilating a rational function

Let us consider a nonzero polynomial $f = f(x)$ in $\mathbb{C}[x]$. We are going to explain how to use some tools in Computer Algebra in order to explicitly compute the annihilating ideal, in the Weyl algebra A_n, of the rational function $\frac{1}{f}$, that is

$$Ann(\frac{1}{f}) = \{P \in A_n | P(\frac{1}{f}) = 0\}.$$

We first treat the elementary case when $f = x_1$. It is clear that the operators $P_1 = x_1\partial_1 + 1, P_2 = \partial_2,\dots,P_n = \partial_n$ annihilate $\frac{1}{f}$.

We will prove that $Ann(\frac{1}{x_1}) = A_n(P_1,\dots,P_n)$. Assume $P \in A_n$ is such that $P(1/x_1) = 0$. We write

$$P = Q(x,\partial_1) + S(x,\partial)$$

where $Q = Q(x, \partial_1) = \sum_{j=0}^{d} a_j(x)\partial_1^j$ (for some integer $d \geq 0$ and $a_j(x) \in \mathbb{C}[x]$) and $S = S(x, \partial)$ belongs to the ideal $A_n(P_2, \ldots, P_n)$. We will prove that Q belongs to the ideal $A_n P_1$; which proves that $P \in A_n(P_1, \ldots, P_n)$.

We divide the operator Q by $P_1 = x_1\partial_1 + 1$ (this is very particular case of the division theorem 5.7). We write

$$Q = Q_1 P_1 + R$$

where Q_1 depends only on x and ∂_1 and the remainder $R = R_0 + R_1$ has the following form (see the statement of Division Theorem 5.7):

$$R_0 = R_0(x) = \sum_{j=0}^{e_0} b_j(x')x_1^j, \qquad R_1 = \sum_{\ell=1}^{e_1} c_\ell(x')\partial_1^\ell$$

for some integers $e_0 \geq 0, e_1 \geq 1$ and polynomials $b_j(x'), c_\ell(x')$ in the polynomial ring $\mathbb{C}[x'] := \mathbb{C}[x_2, \ldots, x_n]$.

Assume R_1 is nonzero and $c_{e_1}(x') \neq 0$. Since $Q(1(x_1) = 0$ we also have

$$R(1/x_1) = \frac{R_0(x)}{x_1} + R_1(1/x_1) = 0.$$

The pole of the rational function $R_1(1/x_1)$ at $x_1 = 0$ has order $e_1 + 1$ and thus it can not be cancelled with $R_0(x)/x_1$. This yields a contradiction and so R_1 must be zero. In this case $R_0(x)/x_1 = 0$ implies $R_0 = 0$. Then $R = R_0 + R_1 = 0$ and $Q = Q_1 P_1$.

It is obvious that the previous procedure can not be applied for general rational function of the form $1/f$ with $f \in \mathbb{C}[x]$.

T. Oaku and N. Takayama [31] described an algorithm for computing a finite system of generators of the annihilating ideal $Ann(1/f)$. The algorithm uses Groebner basis and elimination theory in A_n.

Due to the high complexity of Groebner basis algorithm[10] it is difficult in practice to compute $Ann(1/f)$. This annihilating ideal can be approximated by the intermediate $Ann^{(k)}(1/f)$ which is by definition the (left) ideal in A_n generated by the operators in $Ann(1/f)$ of order less than or equal to k, for each integer $k \geq 1$. One has the following chain of ideals in A_n

$$Ann^{(1)}\left(\frac{1}{f}\right) \subseteq Ann^{(2)}\left(\frac{1}{f}\right) \subseteq \cdots \subseteq Ann^{(k)}\left(\frac{1}{f}\right) \subseteq \cdots \subseteq Ann\left(\frac{1}{f}\right).$$

Since the ring A_n is (left) Noetherian there exists an integer k such that $Ann^{(k)}(1/f) = Ann(1/f)$.

The case of the ideal $Ann^{(1)}(1/f)$ deserves the following explanation. An operator P of order 1 has the following form:

$$P = \sum_{i=1}^{n} a_i(x)\partial_i + a_0(x)$$

for some $a_j(x) \in \mathbb{C}[x]$ for $j = 0, 1, \ldots, n$.

[10]This complexity equals the one in commutative polynomial rings.

Assume that $P(1/f) = 0$. Then we have the equality

$$\sum_i a_i(x) \frac{\partial}{\partial x_i}\left(\frac{1}{f}\right) + \frac{a_0(x)}{f} = \sum_i -\frac{a_i(x)}{f^2}\frac{\partial f}{\partial x_i} + \frac{a_0(x)}{f} = 0.$$

Previous equality determines (up to sign) the syzygy $(a_1(x), \ldots, a_n(x), -a_0(x))$ of the polynomials $(\partial_1(f), \ldots, \partial_n(f), f)$ where $\partial_i(f)$ stands for $\frac{\partial f}{\partial x_i}$ for $i = 1, \ldots, n$. The set of all the polynomial syzygies of $(\partial_1(f), \ldots, \partial_n(f), f)$ is denoted by

$$Syz(\partial_1(f), \ldots, \partial_n(f), f).$$

This set is in fact a $\mathbb{C}[x]$–module and, by using commutative Groebner basis techniques, one can compute one of its finite generating systems (see e.g. [1]).

Moreover, if $P = \sum_{i=1}^n a_i(x)\partial_i + a_0(x)$ is an operator of order 1 annihilating $1/f$ then the vector field $\sum_{i=1}^n a_i(x)\partial_i$ is *logarithmic* (see [36]) with respect to f as we have the equality

$$\sum_{i=1}^n a_i(x)\partial_i(f) = -a_0(x)f.$$

Reciprocally, for any logarithmic vector field (also called logarithmic derivation) $\delta = \sum_{i=1}^n a_i(x)\partial_i$ with respect to f the operator $\delta + \frac{\delta(f)}{f}$ annihilates $1/f$ and it is of order 1. So, the ideal $Ann^{(1)}(1/f)$ is closely related to the logarithmic derivations associated with (or with respect to) f.

For a given nonzero $f \in \mathbb{C}[x]$ we denote by $\omega(f)$ the smallest k such that $Ann^{(k)}(1/f) = Ann(1/f)$ and in this case we say that $Ann(1/f)$ is generated by operators of order less that or equal to k. Very few is known about the behavior of the function $\omega(f)$ when f varies in $\mathbb{C}[x]$. For any quasi-homogeneous polynomial $f \in \mathbb{C}[x, y]$ it is proven in [43] (using results of [10]) that $\omega(f) = 1$.

In the following `Macaulay 2` scripts we will compute $Ann(1/f)$ for some examples.

First of all we will treat the case $f = x^2 + y^2 + z^2$ (we use here x, y, z instead of x_1, x_2, x_3). As f is homogeneous of order 2 the have the equality $\chi(f) = 2f$ where $\chi = x\partial_x + y\partial_y + z\partial_z$ is the Euler operator. Then $\chi + 2$ annihilates $1/f$. It is also obvious that the operators $P = x\partial_y - y\partial_x, Q = x\partial_z - z\partial_x, R = y\partial_z - z\partial_y$ also annihilate $1/f$. But it is not completely easy to prove that, in this case, $Ann(1/f)$ is generated by $\chi + 2, P, Q, R$. We will do that by using the package `D-modules.m2` in `Macaulay 2`.

```
Macaulay 2, version 1.2
with packages: Elimination, IntegralClosure, LLLBases, PrimaryDecomposition,
               ReesAlgebra, SchurRings, TangentCone

i1 : load "D-modules.m2";

i2 : R=QQ[x,y,z];

i3 : W=makeWA R;

i4 : X=x*dx+y*dy+z*dz, P=x*dy-y*dx, Q=x*dz-z*dx, R=y*dz-z*dy;
```

```
i5 : f=x^2+y^2+z^2, g=x+1-x

       2    2    2
o5 = (x  + y  + z , 1)

o5 : Sequence

i6 : I=RatAnn(g,f)

o6 = ideal (z*dy - y*dz, z*dx - x*dz, y*dx - x*dy, x*dx + y*dy +
z*dz + 2)

o6 : Ideal of W

i7 : J=ideal(X+2,P,Q,R);

o7 : Ideal of W

i8 : J==I

o8 = true
```

Remark 8.1. *Comments on the previous script.*
Command `i6 : I=RatAnn(g,f)` calculates the annihilating ideal of `1/f` in the Weyl algebra of order three and associates its value to the name `I`. Notice that by definition `g=x+1-x=1`. This is a trick just to force `Macaulay 2` to consider `1` as an element in the Weyl algebra (or more precisely `of class W`) (the expression `g=1` considers `1` to be `of class ZZ`).

Command `i7 : J=ideal(X+2,P,Q,R);` associates to the name `J` the ideal generated by the four operators `X+2,P,Q,R` (i.e. $\{\chi+2, P, Q, R\}$).

Finally the command `i8 : J==I` checks if both ideals `J` and `I` are equal. Since the answer is `true` that proves that the annihilating ideal of $1/f$ is generated by the four operators defined above.

Previous script shows in particular the equality (as ideals in the Weyl algebra of order 3) $Ann(1/f) = Ann^{(1)}(1/f)$ for $f = x^2 + y^2 + z^2$.

Let us continue our previous `Macaulay 2` session as described in the following script.

```
i9 : f=x^3+y^3+z^3;

i10 : P=x^2*dy-y^2*dx, Q=x^2*dz-z^2*dx, R=y^2*dz-z^2*dy

          2      2       2      2       2      2
o10 = (- y dx + x dy, - z dx + x dz, - z dy + y dz)

o10 : Sequence

i11 : I=RatAnn(g,f)

                                        2      2     2      2     2      2
o11 = ideal (x*dx + y*dy + z*dz + 3, z dy - y dz, z dx - x dz, y dx - x dy,
```

```
      --------------------------------------------------------------------------

            2          2          2
      y*z*dx  + x*z*dy  + x*y*dz )

o11 : Ideal of W

i12 : J=ideal (X+3,P,Q,R);

o12 : Ideal of W

i13 : J==I

o13 = false

i14 : P1=y*z*dx^2+x*z*dy^2+x*y*dz^2

            2          2          2
o14 = y*z*dx  + x*z*dy  + x*y*dz

o14 : W

i15 : P1\%J

            2          2          2
o15 = y*z*dx  + x*z*dy  + x*y*dz

o15 : W
```

Remark 8.2. *Comments on the previous script.*
Command `i9 :` defines f to be the polynomial $x^3+y^3+z^3$ which is a homogeneous polynomial of order 3. So the operator $\chi+3$ annihilates $1/f$. Also the operators P, Q, R defined by command `i10 :` annihilate $1/f$. Command `i12 :` defines J as the ideal generated by $\chi+3, P, Q, R$.

Command `i11 :` defines I as the annihilating ideal of $1/f$. Command `i13 :` checks the equality of ideals I and J. The answer `false` means that both ideals are not equal. Moreover, output `o12 :` tells us that the ideal I equals the ideal generated by J and the operator $P_1 = yz\partial_x^2+xz\partial_y^2+xy\partial_z^2$ (defined using command `i14 :`). Finally, command `i14 :` `P1%J` shows us that the reduction of P_1 modulo the ideal J is not zero giving a different proof of the inequality $J \neq I$ (here `P1%J` stands for the reduction of the division of P_1 by the ideal J).

Moreover, the following script proves that J is in fact the ideal $Ann^{(1)}(1/f)$. Previous discussion tells us that $Ann(1/f)$ is not generated by operators of order 1 for $f = x^3+y^3+z^3$.

The following script computes a system of generators of the syzygy module $Syz(\partial_x(f), \partial_y(f), \partial_z(f), -f)$ for $f = x^3+y^3+z^3$. Each column of the matrix given in output `o16 :` represents a syzygy vector. The four syzygy vectors yields (up to sign) the coefficients of the corresponding generators $\chi+3, P, R, Q$ of the ideal J.

```
i16 : kernel matrix({{diff(x,f),diff(y,f),diff(z,f),-f}})
```

```
o16 = image {2} | x y2  0   z2  |
            {2} | y -x2 z2  0   |
            {2} | z 0   -y2 -x2 |
            {3} | 3 0   0   0   |

                                4
o16 : W-module, submodule of W
```

Ideals $Ann(1/f)$ and $Ann^{(k)}(1/f)$ are related to the comparison between the meromorphic de Rham cohomology and the logarithmic de Rham cohomology with respect to the hypersurface define by $f = 0$ in $\mathbb{C}^n$ (see e.g. [10], [11], [16], [43], [15], [41]).

Conclusions

We have described some applications of Computer Algebra methods to the algebraic study of systems of linear partial differential equations. Using Groebner basis theory for linear differential operators we have described how to calculate the characteristic variety of such a system as well as its dimension (which gives an algorithmic procedure to decide whether the system is holonomic). Algorithms by Oaku and Takayama [32] and by Tsai and Walther [42] compute the solutions spaces $\mathrm{Ext}_{A_n}(A_n/I, \mathbb{C}[x])$ for $i = 0, \ldots, n$ if A_n/I is holonomic.

One has also an algorithm for computing the annihilating ideal $Ann(1/f)$ of a rational function $1/f$ where f is a polynomial in $\mathbb{C}[x]$. By computation of some syzygy module in $\mathbb{C}[x]$ one can also compute the first approximation $Ann^{(1)}(1/f)$ of the previous annihilating ideal.

The use of Groebner basis theory in $\mathcal{D}$–module theory is motivated by somehow analogous situations in Commutative Algebra and Algebraic Geometry.

Bibliographical notes

The content of Section 2 can be found in any book on $\mathcal{D}$–module theory (e.g. [18], [5]). Most of the material of Sections 3, 4 and 5 appears in Castro[12, 13], Briançon and Maisonobe[6], Saito et al.[37], Castro and Granger[14] and Castro[17]. The presentation of the content of Sections 6 and 7 follows Castro[17]. Finally, the content of Section 8 is inspired by Ucha[43] and Castro and Ucha[15].

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