INVARIANT SUBSPACES OF PARABOLIC SELF-MAPS IN THE HARDY SPACE

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Abstract. It is shown that the lattice of invariant subspaces of the operator of multiplication by a cyclic element of a Banach algebra consists of the closed ideals of this algebra. As an application, with the help of some elements of the Gelfand Theory of Banach algebras, the lattice of invariant subspaces of composition operators acting on the Hardy space, whose inducing symbol is a parabolic non-automorphism, is found. In particular, each invariant subspace always consists of the closed span of a set of eigenfunctions. As a consequence, such composition operators have no non-trivial reducing subspaces.

1. Introduction

The problem of characterizing the lattices of invariant subspaces of bounded linear operators acting on a separable Hilbert space is probably one of the most interesting and difficult ones in General Operator Theory on Hilbert spaces. While the classes of operators for which their lattices are known are very scarce, to characterize the lattice of a very particular operator can solve the invariant subspace problem. For instance, as shown by Nordgren, Rosenthal and Wintrobe, [13] and [14], this is the case of composition operators C_φ acting on the Hardy space $\mathcal{H}^2$ of the unit disk $\mathbb{D}$ of the complex plane, with φ an automorphism of $\mathbb{D}$ fixing ± 1: solving the invariant subspace problem is equivalent to showing that the minimal invariant subspaces for C_φ are one-dimensional. See also the related work by Mortini [12].

Another instance of what is just said is that while Beurling's Theorem provides a complete description of the invariant subspaces of the shift operator acting on $\mathcal{H}^2$, the lattice of the shift operator acting on the Bergman space is not completely understood, see [1], [2] or [7, Chapters 7 and 8].

In the present work, we will characterize the invariant subspaces of the composition operators C_φ acting on the Hardy space $\mathcal{H}^2$, where φ is a parabolic non-automorphism that takes $\mathbb{D}$ into itself, which has the formula

$$\varphi_a(z) = \frac{(2-a)z+a}{-az+2+a}, \qquad \text{where } \Re a > 0. \tag{1.1}$$

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Since $\varphi_a(\mathbb{D})$ is contained in $\mathbb{D}$, according to Littlewood's Subordination Principle, the composition operator $(C_{\varphi_a}f)(z) = f(\varphi_a(z))$ acts boundedly on $\mathcal{H}^2$, see the book by Cowen and MacCluer [5] for more details.

In connection with linear operator theory on separable (linear) infinitely diemensional Hilbert spaces, one of the fundamental and classical concepts in dynamics is cyclicity, that is, the closure of the linear span of the orbit of a vector under the operator is the whole Hilbert space and which is also a fundamental concept in Operator Theory on Hilbert Spaces as well as in Approximation Theory. Intuitively, a cyclic vector for an operator is one for which the orbit of the vector under the operator exhausts all the infinitely many possible directions in the Hilbert space. Thus cyclic vectors are exactly those with chaotic orbits in the sense of direction, which is the basic pilar where the concept of Hilbert space lays on. Indeed, other stronger and much more recent concepts, like hypercyclicity, in which the closure of the orbits themselves are the whole Hilbert space, have been developed to study stronger forms of cyclicity.

The smallest invariant subspace that contains a given vector is the closure of the linear span of the orbit of an operator. From a point of view of dynamics of an operator, to characterize it lattice of invariant subspaces is equivalent to characterize the closure of the linear span of the orbit of each of the vectors under the given operator. In particular, we characterize the cyclic vectors for C_φ, a rare result in Operator Theory. Indeed, the family of all composition operators induced by parabolic non-automorphism will have common cyclic vectors, Corollaries 1.2. Moreover, each orbit of any vector under all composition operators induced by parabolic non-automrophisms has a common closure, which is an immediate consequence of Theorem 1.1.

To prove our main result, it is essential a result due to Cowen [4], see also [5, Theorem 6.1], in which he found the spectrum of C_{φ_a}. If $\Re a > 0$, the spectrum $\sigma(C_{\varphi_a})$ is the spiral

$$\sigma(C_{\varphi_a}) = \{0\} \cup \{e^{-at} : t \in [0, \infty)\}.$$

Indeed, C_{φ_a} has a well-known family of inner functions as its eigenfunctions,

$$C_{\varphi_a} e_t = e^{-at} e_t, \text{ where } e_t(z) = \exp\left(t\frac{z+1}{z-1}\right) \text{ for each } t \geq 0. \tag{1.2}$$

All invariant subspaces we consider in this work will be closed. Let $\operatorname{Lat} T$ denote the lattice of invariant subspaces of the bounded linear operator T and let $\mathbb{F}[0, \infty)$ denote the set of closed subsets of $[0, \infty)$. As usual, the closed span of the empty set is the trivial subspace consisting of just the zero vector. We will prove

Theorem 1.1. *Let φ be a parabolic non-automorphism that takes the unit disk into itself. Then*

$$\operatorname{Lat} C_\varphi = \{\overline{\operatorname{span}}\, \{e_t : t \in F\} : F \in \mathbb{F}[0, \infty)\}.$$

In particular, any non-trivial invariant subspace of C_φ contains a non-trivial eigenfunction of C_φ. As an immediate corollary of the above theorem, we have

Corollary 1.2. *Composition operators induced by parabolic non-automorphisms that take the unit disk into itself share their lattice and their cyclic vectors.*

Recall that a subspace that is invariant for an operator as well as for its adjoint is called a reducing subspace. Using Theorem 1.1, we will prove

Theorem 1.3. *Let φ be a parabolic non-automorphism that takes the unit disk into itself. Then C_φ has no non-trivial reducing subspace.*

The proof of Theorem 1.1 consists of two steps. First, it is shown that the adjoint operator $C_\varphi^\star$ is similar to the operator of multiplication by a cyclic element in a commutative semisimple regular Banach algebra. Second it is proved that the invariant subspaces of such a multiplication are exactly ideals of the algebra that are characterized by using some elements of the Gelfand Theory.

2. Banach algebras with a cyclic element

Recall that a Banach algebra is a complex Banach space $\mathcal{A}$ equipped with a continuous binary operation $(a, b) \to ab$, which turns $\mathcal{A}$ into a ring over the complex numbers. Since the bilinear map $(a, b) \to ab$ is continuous, there must be a positive constant c such that $\|ab\| \le c\|a\|\,\|b\|$ for each a and b in $\mathcal{A}$. Although it is not required that $c = 1$, this can always be achieved by replacing the initial norm of $\mathcal{A}$ by an equivalent one, see [3] or [8], for instance. Observe also that it is not required that the ring $\mathcal{A}$ has unity. A character on a Banach algebra $\mathcal{A}$ is a linear functional $\varkappa : \mathcal{A} \to \mathbb{C}$ such that $\varkappa(ab) = \varkappa(a)\varkappa(b)$ for each a and b in $\mathcal{A}$. We observe that any character on a Banach algebra is continuous [11, p. 201], that is, it belongs to the dual space $\mathcal{A}^\star$. The spectrum of $\mathcal{A}$ is the set $\Omega(\mathcal{A})$ of non-zero characters of $\mathcal{A}$ equipped with the weak-star topology. It is well-known that the spectrum of any Banach algebra is a Hausdorff locally compact topological space and it is compact whenever $\mathcal{A}$ has unity [11, p. 205].

An element a in $\mathcal{A}$ is called cyclic if the subalgebra generated by a is dense in $\mathcal{A}$, in which case, $\mathcal{A}$ is clearly separable and commutative. If $a \in \mathcal{A}$, the operator of multiplication by a acting on $\mathcal{A}$ is

$$M_a x = ax, \qquad x \in \mathcal{A},$$

which is clearly bounded. The proof of Theorem 1.1 will rely heavily on

Proposition 2.1. *Let $\mathcal{A}$ be a Banach algebra. Then the invariant subspaces of multiplication by a cyclic element are exactly the closed ideals of $\mathcal{A}$.*

Proof. First, since $\mathcal{A}$ has a cyclic element, it is commutative. Let a be a cyclic element of $\mathcal{A}$ and let $\mathcal{L}$ be an invariant subspace of M_a. Clearly,

$$\mathcal{M}_{\mathcal{L}} = \{b \in \mathcal{A} : bx \in \mathcal{L} \text{ for all } x \in \mathcal{L}\}$$

is a closed subalgebra of $\mathcal{A}$. Since $\mathcal{L}$ is an invariant subspace of M_a, we find that $a \in \mathcal{M}_{\mathcal{L}}$ and, therefore, $\mathcal{M}_{\mathcal{L}}$ contains the subalgebra generated by a and, being $\mathcal{M}_{\mathcal{L}}$ closed and a cyclic, it follows that $\mathcal{M}_{\mathcal{L}} = \mathcal{A}$. Hence, $\mathcal{L}$ is a left ideal and thus, being $\mathcal{A}$ commutative, an ideal of $\mathcal{A}$. On the other hand, each ideal of $\mathcal{A}$ is invariant with respect to M_a, which finishes the proof. □

Now we turn our attention to the structure of the regular ideals of Banach algebras. An ideal $\mathcal{I}$ of a Banach algebra $\mathcal{A}$ is called regular when the quotient algebra $\mathcal{A}/\mathcal{I}$ has unit. In particular, the kernel of any character is a maximal regular ideal.

Therefore, the mapping $\varkappa \mapsto \ker \varkappa$ defines a one-to-one correspondence between the spectrum of $\mathcal{A}$ and the set of its maximal regular ideals, which is denoted by $\mathfrak{M}$, see [11, p. 202]. Recall also that a complex algebra is called semisimple if the intersection of all maximal regular ideals, called Jacobson's radical, is zero. Thus a commutative Banach algebra $\mathcal{A}$ is semisimple if and only if the elements of $\Omega(\mathcal{A})$ separate points of $\mathcal{A}$, that is, the intersection of kernels of the characters is zero.

Given $x \in \mathcal{A}$ and $\mathcal{M} \in \mathfrak{M}$, we denote $\widehat{x}(\mathcal{M}) = x \bmod \mathcal{M}$ the image of x under the multiplicative linear functional corresponding to $\mathcal{M}$. The mapping $x \mapsto \widehat{x}$ is a homomorphism from $\mathcal{A}$ into $C_0(\mathfrak{M})$ called Gelfand's transform. The Gelfand transform is one-to-one if and only if $\mathcal{A}$ is semisimple [11, p. 207]. The hull $h(\mathcal{I})$ of an ideal $\mathcal{I}$ in $\mathcal{A}$ is the set of all maximal regular ideals $\mathcal{M}$ such that $\mathcal{I}$ is contained in $\mathcal{M}$. Equivalently, $h(\mathcal{I})$ is the set of all $\mathcal{M} \in \mathfrak{M}$ such that $\widehat{x}(\mathcal{M}) = 0$ for all $x \in \mathcal{I}$. The kernel $k(E)$ of a set $E \subset \mathfrak{M}$ is the ideal $\bigcap_{\mathcal{M}\in E} \mathcal{M}$, that is, $k(E)$ is the set of all $x \in \mathcal{A}$ such that $\widehat{x}$ equals zero on E. Recall also that a Banach algebra $\mathcal{A}$ is said to be regular when each point in $\mathcal{A}$ has a neighborhood U such that $k(U)$ is a regular ideal.

For a closed set F in $\mathfrak{M}$ let $J(F,\infty)$ be the union of all ideals $k(U)$, where U is any open set containing F and having compact complement. Since $J(F,\infty)$ is the smallest ideal with hull equal to F, see [15, p. 91], for any closed ideal $\mathcal{I}$ the following holds

$$J(h(\mathcal{I}),\infty) \subset \mathcal{I} \subset k(h(\mathcal{I})).$$

If $\mathcal{A}$ is a semisimple regular algebra, then the closed sets of $\mathfrak{M}$ are exactly the hulls of closed ideals and a closed ideal is an intersection of maximal regular ideals if and only if it is equal to the kernel of its hull. Therefore, see [15, p. 92], we have

Lemma 2.2. *Let $\mathcal{A}$ be a semisimple regular Banach algebra. Then every closed ideal $\mathcal{I}$ of $\mathcal{A}$ is equal to an intersection of maximal regular ideals if and only if $J(h(\mathcal{I}),\infty) = k(h(\mathcal{I}))$.*

Using the definition of $J(h(\mathcal{I}),\infty)$, the equality in the preceding lemma is equivalent to the fact that for each closed ideal $\mathcal{I}$ and each $x \in k(h(\mathcal{I}))$, there exist open sets $U_n \supset h(\mathcal{I})$ with compact complement and $x_n \in h(U_n)$ such that $x_n \to x$. If we define $h(x) = \{\mathcal{M} \in \mathfrak{M} :\ x \in \mathcal{M}\}$, then it is easy to see that $h(x)$ equals to the hull of the ideal generated by x. Thus the equality in Lemma 2.2 is also equivalent to the fact that for each x in $k(h(\mathcal{I}))$, there is a sequence $\{x_n\}$ such that $x_n \to x$ in $\mathcal{A}$ and $\widehat{x}_n$ equals zero in a neighborhood U_n of $h(x)$ with compact complement. Next corollary follows immediately from Proposition 2.1 and Lemma 2.2.

Corollary 2.3. *Let $\mathcal{A}$ be a semisimple regular commutative Banach algebra such that a is a cyclic element of $\mathcal{A}$. Then*

$$\operatorname{Lat} M_a = \left\{ \bigcap_{\varkappa\in F} \ker \varkappa :\ F \text{ is closed in } \Omega(\mathcal{A}) \right\}.$$

if and only if for each $x \in \mathcal{A}$, there exists a sequence $\{x_n\}$ tending to x in $\mathcal{A}$ and $\widehat{x}_n$ vanishes on a neighborhood U_n of $h(x)$ with compact complement.

3. An isomorphism from $\mathcal{H}^2$ onto the Sobolev space $W^{1,2}[0,\infty)$

The Sobolev space $W^{1,2}[0,\infty)$ consists of those functions f in $L^2[0,\infty)$ absolutely continuous on each bounded subinterval of $[0,\infty)$ and whose derivative belong to $L^2[0,\infty)$. It is well-known and easy to check that the space $W^{1,2}[0,\infty)$ becomes a Hilbert space endowed with the inner product

$$\langle f,g\rangle_{1,2}=\frac{1}{2}\int_0^\infty (f(t)\overline{g(t)}+f'(t)\overline{g'(t)})\,dt.$$

The corresponding norm will be denoted by $\|\cdot\|_{1,2}$. Similarly, we can define $W^{1,2}(\mathbb{R})$.

We will show up an isomorphism, which is closely related to the eigenfunctions of C_φ, between the Hardy space $\mathcal{H}^2$ and the Sobolev space $W^{1,2}[0,\infty)$ that will be crucial to prove Theorem 1.1. The inner functions $e_t(z)=\exp(t(z+1)/(z-1))$, with $t\geq 0$, allow us to consider a complex valued function for each f in $\mathcal{H}^2$ defined by

$$(\Phi f)(t)=\langle f,e_t\rangle_{\mathcal{H}^2},\qquad t\geq 0.$$

The key point to prove that Φ is an isomorphism from $\mathcal{H}^2$ onto $W^{1,2}[0,\infty)$ is to consider the operator Ψ that for each f in $L^2(\mathbb{T})$, where $\mathbb{T}$ denotes the unit circle, defined as

$$(\Psi f)(t)=\langle f,e_t\rangle_{L^2(\mathbb{T})},\qquad t\in\mathbb{R}.$$

Let $W_0^{1,2}[0,\infty)$ denote the subspace of functions in $W^{1,2}(\mathbb{R})$ that vanish on $(-\infty,0]$. The space $W_0^{1,2}(-\infty,0]$ is defined similarly. Finally, let Π denote the upper half-plane of the complex plane. The Hardy space of the upper half-plane $\mathcal{H}^2(\Pi)$ consists of those functions f analytic on Π for which the norm

$$\|f\|^2_{\mathcal{H}^2(\Pi)}=\sup_{y>0}\int_{-\infty}^\infty |f(x+iy)|^2\,dx$$

is finite, see [16, p. 372]. We will still maintain the symbol $\mathcal{H}^2$ for the Hardy space of the unit disk. We have

Theorem 3.1. *The operator Ψ is an isometric isomorphism from $L^2(\mathbb{T})$ onto $W^{1,2}(\mathbb{R})$. In addition, $\Psi(z\mathcal{H}^2)=W_0^{1,2}[0,\infty)$ and $\Psi(\bar{z}\overline{\mathcal{H}}^2)=W_0^{1,2}(-\infty,0]$.*

Proof. For each f in $L^2(\mathbb{T})$, we have

$$(\Psi f)(t)=\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\exp\left(t\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)\,d\theta,\qquad t\in\mathbb{R}.$$

The change of variables $x=i(1+e^{i\theta})/(1-e^{i\theta})$ yields

$$(\Psi f)(t)=\frac{1}{\pi}\int_{-\infty}^\infty f\left(\frac{x-i}{x+i}\right)\frac{e^{-itx}}{1+x^2}dx,\qquad t\in\mathbb{R}. \tag{3.1}$$

Therefore, $\Psi=\mathcal{F}MT$, where $\mathcal{F}$ denotes the Fourier transform,

$$(Mg)(y)=\frac{1}{\sqrt{\pi}}\frac{g(y)}{\sqrt{1+y^2}}\quad\text{and}\quad (Tf)(x)=\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{1+x^2}}f\left(\frac{x-i}{x+i}\right).$$

The obvious change of variables shows that T is an isometric isomorphism from $L^2(\mathbb{T})$ onto $L^2(\mathbb{R})$. In addition, using the properties of the Fourier transform, one easily checks, with the help of Plancherel's Theorem, that $\mathcal{F}M$ is an isometric isomorphism from $L^2(\mathbb{R})$ onto $W^{1,2}(\mathbb{R})$, which proves the first statement of the proposition.

Now, let f be in $z\mathcal{H}^2$, that is, $f(z) = zg(z)$ with g in $\mathcal{H}^2$. Using (3.1), we obtain

$$(\Psi f)(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} g\left(\frac{x-i}{x+i}\right)\frac{e^{-itx}}{(x+i)^2}\,dx, \qquad \text{for each } t \in \mathbb{R}.$$

Since the map

$$h \to \frac{1}{\sqrt{\pi}(x+i)}h\left(\frac{x-i}{x+i}\right)$$

is an isometric isomorphism from $\mathcal{H}^2$ onto $\mathcal{H}^2(\Pi)$, see [9, p. 106], and multiplication by $(w+i)^{-1}$ is bounded on $\mathcal{H}^2(\Pi)$, we find that Ψf is the Fourier transform of a function of $\mathcal{H}^2(\Pi)$. Thus, the Paley-Wiener Theorem, see [16, p. 372], shows that Ψf, which is continuous, must vanish on $(-\infty, 0]$ and, therefore, $\Psi(z\mathcal{H}^2) \subset W_0^{1,2}[0,\infty)$. Similarly, $\Psi(\bar{z}\overline{\mathcal{H}}^2) \subset W_0^{1,2}(-\infty, 0]$. The fact that $\Psi(z\mathcal{H}^2) = W_0^{1,2}[0,\infty)$ and $\Psi(\bar{z}\overline{\mathcal{H}}^2) = W_0^{1,2}(-\infty,0]$ follows immediately from the orthogonal decomposition $W^{1,2}(\mathbb{R}) = W_0^{1,2}(-\infty,0] \oplus [e^{-|t|}] \oplus W_0^{1,2}[0,\infty)$, which in turns follows, being Ψ an isometric isomorphism, from the orthogonal decomposition $L^2(\mathbb{T}) = \bar{z}\overline{\mathcal{H}}^2 \oplus [1] \oplus z\mathcal{H}^2$ and the fact that $\Psi 1 = e^{-|t|}$, where $[f]$ denotes the one-dimensional linear space spanned by the vector f. The proof is complete. □

Corollary 3.2. *The operator Φ defines an isomorphism from $\mathcal{H}^2$ onto $W^{1,2}[0,\infty)$. Indeed, $\|\Phi f\|_{1,2}^2 = \|f\|_{\mathcal{H}^2}^2 - |f(0)|^2/2$.*

Proof. Upon applying Theorem 3.1, Φ and Ψ coincide on $z\mathcal{H}^2$ and, therefore, Φ defines an isometric isomorphism from $z\mathcal{H}^2$ onto $W_0^{1,2}[0,\infty)$. Since $e^{-|t|}$ is orthogonal to $W_0^{1,2}[0,\infty)$, so is $e^{-t}\chi_{[0,\infty)}$ and, therefore, $W^{1,2}[0,\infty) = [e^{-t}\chi_{[0,\infty)}] \oplus W_0^{1,2}[0,\infty) = (\Phi 1) \oplus \Phi(z\mathcal{H}^2) = \Phi(\mathcal{H}^2)$, which proves that Φ is an isomorphism. The formula for the norm is trivial. The proof is complete. □

Now, we shall see that the adjoint of composition operators induced by parabolic non-automorphism can be seen as a multiplication operator on $W^{1,2}[0,\infty)$.

Proposition 3.3. *Let φ_a, with $\Re a \geq 0$, be as in* (1.1)*. Then the adjoint of C_{φ_a} acting on $\mathcal{H}^2$ is similar under Φ to the multiplication operator M_ψ, where $\psi(t) = e^{-\bar{a}t}$, acting on $W^{1,2}[0,\infty)$.*

Proof. Using the eigenvalue equation (1.2), for each $f \in \mathcal{H}^2$, we have

$$(\Phi C_{\varphi_a}^\star f)(t) = \langle C_{\varphi_a}^\star f, e_t\rangle_{\mathcal{H}^2} = \langle f, C_{\varphi_a} e_t\rangle_{\mathcal{H}^2} = e^{-\bar{a}t}\langle f, e_t\rangle_{\mathcal{H}^2} = e^{-\bar{a}t}(\Phi f)(t),$$

for each $t \geq 0$. Thus $M_\psi = \Phi C_{\varphi_a}^\star \Phi^{-1}$. The result is proved. □

Proposition 3.4. *The operator M_ψ, where $\psi(t) = e^{-\bar{a}t}$ and $\Re a > 0$, acting on $W^{1,2}[0,\infty)$ is cyclic with cyclic vector ψ.*

Proof. Let $k_\alpha(z) = (1-\bar{\alpha}z)^{-1}$, where $\alpha = (a-1)/(a+1)$, be the reproducing kernel at $\alpha \in \mathbb{D}$ in the Hardy space $\mathcal{H}^2$. Since $\Phi k_\alpha = \psi$, by Proposition 3.3, it is enough to show k_α is cyclic for $C^\star_{\varphi_a}$. Suppose that f in $\mathcal{H}^2$ is orthogonal to the orbit of k_α under $C^\star_{\varphi_a}$. Then, for each $n \geq 0$, we have

$$0 = \langle C^{\star n}_{\varphi_a} k_\alpha, f\rangle_{\mathcal{H}^2} = \langle k_\alpha, C^n_{\varphi_a} f\rangle_{\mathcal{H}^2} = \langle k_\alpha, C_{\varphi_{na}} f\rangle_{\mathcal{H}^2} = \langle k_\alpha, f\circ\varphi_{na}\rangle_{\mathcal{H}^2} = f(\varphi_{na}(\bar\alpha)).$$

Since $\{\varphi_{na}(\alpha)\}$ is not a Blaschke sequence, we find that f is the null function and the result follows. □

An interesting consequence of Corollary 3.2 is a summability theorem for the Laguerre polynomials. Set $u_n(z) = z^n$. Then $\widetilde{u}_n(t) = (\Phi u_n)(t) = L_n^{(-1)}(2t)e^{-t}\chi_{[0,\infty)}$, where $L_n^{(-1)}(t)$ is the Laguerre polynomial of degree n and of index -1. Indeed, since $\widetilde{u}_n = \langle z^n, e_t(z)\rangle_{\mathcal{H}^2}$ is the n-th coefficient of the Taylor series of $e_t(z)$, by definition of the Laguerre polynomials see [17, p. 97], we have

$$e_t(z) = e^{-t}\exp\left(-\frac{2tz}{1-z}\right) = \sum_{n=0}^{\infty} e^{-t} L_n^{(-1)}(2t) z^n. \tag{3.2}$$

Therefore, the following follows immediately

Corollary 3.5. *Let $\{a_n\}_{n\geq 0}$ be a sequence of complex numbers. Then the series $\widetilde{f}(t) = \sum_{n=0}^\infty a_n L_n^{(-1)}(2t)e^{-t}\chi_{[0,\infty)}$ converges in $W^{1,2}[0,\infty)$ if and only if $\{a_n\}$ is in the sequence space ℓ^2. Indeed, $\|\widetilde{f}\|^2_{1,2} = -|a_0|^2/2 + \|\{a_n\}_{n\geq 1}\|_2^2$.*

Remark. In [6, Chaps. IV and V], it is also considered the isomorphism Φ. However, the norm on the space $\Phi(\mathcal{H}^2)$ is defined as $\|\Phi(f)\| = \|f\|_{\mathcal{H}^2}$, without identifying $\Phi(\mathcal{H}^2)$ with $W^{1,2}[0,\infty)$, and, consequently, more difficult to handle.

4. The Sobolev space $W^{1,2}[0,\infty)$ as a Banach algebra

In this section, we will show that $W^{1,2}[0,\infty)$ is a semisimple regular Banach algebra with respect to the pointwise multiplication. First, we need to state some basic properties of $W^{1,2}[0,\infty)$.

Proposition 4.1. *Each f in $W^{1,2}[0,\infty)$ satisfies $\|f\|_\infty \leq \sqrt{2}\|f\|_{1,2}$ and vanishes at ∞. In particular, each f in $W^{1,2}[0,\infty)$ is uniformly continuous and norm convergence in $W^{1,2}[0,\infty)$ implies uniform convergence.*

Proof. By Corollary 3.5, we can write $f(t) = \sum_{n=0}^\infty a_n L_n^{(-1)}(2t)e^{-t}$, where $\{a_n\}$ is in ℓ^2. The Cauchy-Schwarz inequality and Corollary 3.5, for each $t \geq 0$, yields

$$|f(t)| = \left|\sum_{n=0}^\infty a_n L_n^{(-1)}(2t)e^{-t}\right| \leq \|f\|_{1,2}\left(2e^{-2t} + \sum_{n=1}^\infty (L_n^{(-1)}(2t))^2 e^{-2t}\right)^{1/2}.$$

Since $\|e_t\|_{\mathcal{H}^2} = 1$, using (3.2), one easily checks that the quantity into the brackets above equals to $1 + e^{-2t} \leq 2$ and, therefore, $\|f\|_\infty \leq \sqrt{2}\|f\|_{1,2}$.

To show that f vanishes at ∞, for each positive integer m, we observe that

$$|f(t)| \leq \left|\sum_{n=0}^m a_n L_n^{(-1)}(2t)e^{-t}\right| + \left|\sum_{n=m+1}^\infty a_n L_n^{(-1)}(2t)e^{-t}\right|.$$

The second term in the right-hand side above is bounded by $\sqrt{2}\|\{a_n\}_{n\geq m+1}\|_2$ and, thus, we can take large enough m so that this term be small enough for each

$t \geq 0$. For this m and large enough t, the first term in the right-hand side above is clearly as small as desired. The proof is complete. □

As a consequence of Proposition 4.1, we find that $W^{1,2}[0,\infty)$ is a Banach algebra.

Proposition 4.2. *The space* $W^{1,2}[0,\infty)$ *with the pointwise multiplication is a Banach algebra without unity.*

Proof. Let f and g be in $W^{1,2}[0,\infty)$. Upon applying Proposition 4.1, we see that

$$\|fg\|_2 \leq \|f\|_2\|g\|_\infty \leq 2\|f\|_{1,2}\|g\|_{1,2}$$

and

$$\|(fg)'\|_2 = \|f'g + fg'\|_2 \leq \|f'\|_2\|g\|_\infty + \|g'\|_2\|f\|_\infty \leq 4\|f\|_{1,2}\|g\|_{1,2},$$

which show that the statement holds. □

We will need a special dense subspace of $W^{1,2}[0,\infty)$. Let $\mathcal{C}_c^\infty[0,\infty)$ denote the space of infinitely differentiable complex functions on $[0,\infty)$ that have compact support. The content of the next proposition is known, we include a proof for the sake of completeness.

Proposition 4.3. *The space* $\mathcal{C}_c^\infty[0,\infty)$ *is dense in* $W^{1,2}[0,\infty)$.

Proof. Suppose that f in $W^{1,2}[0,\infty)$ satisfies

$$\int_0^\infty f(t)\overline{g(t)}\,dt + \int_0^\infty f'(t)\overline{g'(t)}\,dt = 0, \qquad \text{for each } g \in C_c^\infty[0,\infty).$$

Since g has compact support, integrating by parts and putting everything under the same integral sign, we find that

$$\int_0^\infty \left[f'(x) - \left(\int_0^x f(t)\,dt\right)\right]\overline{g'(x)}\,dx = 0, \qquad \text{for each } g \in C_c^\infty[0,\infty).$$

Observe that since g' has compact support, the second integral above is always over a finite interval. Let $a > 0$ be fixed. Since the set of functions g' with g in $\mathcal{C}_c^\infty[0,a)$ is dense in $L^2[0,a]$, we have

$$f'(x) - \int_0^x f(t)\,dt = 0, \qquad \text{for each } 0 \leq x \leq a.$$

Therefore, it follows that $f(x) = c_1e^x + c_2e^{-x}$, for $0 \leq x \leq a$, where c_i, $i = 1,2$, is constant. Since a was arbitrary, it follows that $f(x) = c_1e^x + c_2e^{-x}$ for $0 \leq x < \infty$. But $c_1 = 0$ because f is in $W^{1,2}[0,\infty)$ and $c_2 = 0$ because $f'(0) = 0$. Thus f is the zero function and the result follows. □

For each $t \geq 0$, let δ_t denote the reproducing kernel at t, that is, $f(t) = \langle f, \delta_t\rangle_{1,2} = \langle \Phi^{-1}f, e_t\rangle_{\mathcal{H}^2}$ for each $f \in W^{1,2}[0,\infty)$ and where Φ is the transform defined in Section 3. Recall that the spectrum $\Omega = \Omega(W^{1,2}[0,\infty))$ is the space of characters endowed with the weak-star topology that, since $W^{1,2}[0,\infty)$ is a Hilbert space, coincides with the weak topology.

Proposition 4.4. *The spectrum of the Banach algebra $W^{1,2}[0,\infty)$ is*

$$\Omega(W^{1,2}[0,\infty)) = \{\delta_t : t \geq 0\}.$$

Furthermore, the mapping that to each t assigns δ_t is a homeomorphism from $[0,\infty)$ onto $\Omega(W^{1,2}[0,\infty))$

Proof. Clearly, for each $t \geq 0$, the functional δ_t is a character on $W^{1,2}[0,\infty)$, that is, δ_t is in $\Omega = \Omega(W^{1,2}[0,\infty))$. To prove that each character on $W^{1,2}[0,\infty)$ is one of the δ_t's, we begin by considering the Banach algebra $\mathcal{C}^1[0,1]$, with pointwise multiplication, endowed with the norm $\|f\| = \max\{\|f\|_\infty, \|f'\|_\infty\}$. Consider also its Banach subalgebra $\mathcal{A}_0 = \{f \in \mathcal{C}^1[0,1] : f(1) = 0\}$. Then, it is easy to check that $(Tf)(x) = f(x/(1+x))$ defines a bounded operator from $\mathcal{A}_0$ into $W^{1,2}[0,\infty)$, which is also an algebra homomorphism. Now, if $\varkappa$ is a character of $W^{1,2}[0,\infty)$, then it is easy to see that the functional $\widetilde{\varkappa}$ on $\mathcal{C}^1[0,1]$ defined by $\widetilde{\varkappa}(f) = \varkappa(T(f - f(1))) + f(1)$ is also a character. Since the characters of $\mathcal{C}^1[0,1]$ are exactly the point evaluations $f \to f(s)$, with $0 \leq s \leq 1$, see [11, p. 204], there is $0 \leq s \leq 1$ such that $\widetilde{\varkappa}(f) = f(s)$ for each f in $\mathcal{C}^1[0,1]$. If $s = 1$, it follows immediately that $\varkappa(Tf) = 0$ for each f in $\mathcal{A}_0$. Hence $\varkappa$ vanishes on the range of T, which is dense because it contains $\mathcal{C}_c^\infty[0,\infty)$, see Proposition 4.3, and, therefore, $\varkappa$ is the zero functional. If $s \neq 1$, then set $t = s/(1-s) \geq 0$ and observe that $\varkappa(Tf) = (Tf)(t)$ for each $f \in \mathcal{A}_0$. Hence $\varkappa$ and δ_t coincide on a dense set, which implies that $\varkappa = \delta_t$. Thus we have shown that $\Omega = \{\delta_t : t \geq 0\}$.

Next, since each f in $W^{1,2}[0,\infty)$ is continuous, so is the mapping $t \to \delta_t$ from $[0,\infty)$ onto Ω. Since $\|\delta_t\|_{1,2} \leq \|\Phi^{-1}\|\|e_t\|_{\mathcal{H}^2} = \|\Phi^{-1}\|$, we find that Ω is norm bounded on the dual space. Since the weak topology of a separable Hilbert space is metrizable on bounded sets, we may conclude that Ω is metrizable. Thus, to prove that $t \to \delta_t$ is a homeomorphism, it suffices to show that $t_n \to t_0$ whenever $\delta_{t_n} \to \delta_{t_0}$. Suppose that this is not the case, then there is $\varepsilon > 0$ such that $|t_n - t_0| > \varepsilon$ for each positive integer n. Consider the $W^{1,2}[0,\infty)$-function defined for $t \geq 0$ by

$$f(t) = \begin{cases} \varepsilon - |t_0 - s|, & \text{if } |t_0 - s| \leq \varepsilon; \\ 0, & \text{otherwise.} \end{cases}$$

Since $\delta_{t_n}(f) = 0$ and $\delta_{t_0}(f) = \varepsilon$, we find that δ_{t_n} cannot converge to δ_{t_0}. Therefore, the mapping $t \to \delta_t$ is a homeomorphism. The result is proved. □

Proposition 4.5. *The Banach algebra $W^{1,2}[0,\infty)$ is semisimple and regular and the mapping $F \to \bigcap_{t\in F} \ker \delta_t$ is one-to-one from $\mathbb{F}[0,\infty)$ onto the set of closed ideals of $W^{1,2}[0,\infty)$.*

Proof. Since the characters δ_t's separate points, $W^{1,2}[0,\infty)$ is semisimple. To prove that $W^{1,2}[0,\infty)$ is also regular, consider a maximal regular ideal $\mathcal{M}_0$ corresponding to the reproducing kernel δ_{t_0}. Suppose that $t_0 \in [0,b) \subset [0,\infty)$, with $0 < b < \infty$ and let U be the image of $[0,b)$ under the homeomorphism furnished by Proposition 4.4. Then U is an open neighborhood of δ_{t_0} and $k(U) = \{f \in W^{1,2}[0,\infty) : f \equiv 0 \text{ on } [0,b)\}$ is a regular ideal. Indeed, $W^{1,2}[0,\infty)/k(U) = W^{1,2}[0,b)$ that clearly has a unit.

It remains to show that the hypotheses of Lemma 2.2 are fulfilled. Indeed, the Gelfand transform of a function in $W^{1,2}[0,\infty)$ vanishes on a set in Ω if and only if the function vanishes on its preimage under the homeomorphism furnished by Proposition 4.4. Clearly, for each f in $W^{1,2}[0,\infty)$ there is a sequence $\{f_n\}$ in $\mathcal{C}_c^\infty[0,\infty)$ converging to f and such that the zero set of each f_n contains an open neighborhood U_n of the zero set of f. Then, by Lemma 2.2, each closed ideal of $W^{1,2}[0,\infty)$ is of the form $\bigcap_{t\in F}\ker\delta_t$ for some F in $\mathbb{F}[0,\infty)$, so the mapping $F\to\bigcap_{t\in F}\ker\delta_t$ is onto and since $\bigcap_{t\in F}\ker\delta_t\neq\bigcap_{t\in G}\ker\delta_t$ whenever $F\neq G$, it is also one-to-one. The result is proved. □

Now, we have all the tools at hand to prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 3.4, the symbol ψ is a cyclic element of the semisimple regular Banach algebra $W^{1,2}[0,\infty)$. Thus, using Corollary 2.3 we obtain that $F\to\bigcap_{t\in F}\ker\delta_t$ a one-to-one correspondence from the set of closed subsets of $\Omega(\mathcal{A})$ and $\operatorname{Lat} M_\psi$. By Proposition 4.4, we see that the map $F\to I_F=\{f\in W^{1,2}[0,\infty): f \text{ vanishes on } F\}$ is one-to-one from $\mathbb{F}[0,\infty)$ onto $\operatorname{Lat} M_\psi$. Since $M_\psi=\Phi C_\varphi^\star\Phi^{-1}$, it follows that the map $F\to J_F=\{f\in\mathcal{H}^2:\langle f,e_t\rangle_{\mathcal{H}^2}=0 \text{ for } t\in F\}$ is one-to-one from $\mathbb{F}[0,\infty)$ onto $\operatorname{Lat} C_\varphi^\star$. Since $\operatorname{Lat} C_\varphi$ consists of the orthogonal complements of $\operatorname{Lat} C_\varphi^\star$, we find that the map $F\to J_F^\perp$ is one-to-one from $\mathbb{F}[0,\infty)$ onto $\operatorname{Lat} C_\varphi$. It remains to notice that $J_F^\perp=\overline{\operatorname{span}}\{e_t:t\in F\}$ for each F in $\mathbb{F}[0,\infty)$. The proof is complete. □

Now, the proof of Theorem 1.3 follows easily.

Proof of Theorem 1.3. Let F be in $\mathbb{F}[0,\infty)$ for which $N_F=\overline{\operatorname{span}}\{e_t:t\in F\}$ is non-trivial. We must show that its orthogonal complement $N_F^\perp$ is not invariant under C_φ. We need the following formula, which is easily checked

$$\langle e_t,e_s\rangle=e^{-|t-s|},\qquad \text{for each } t,s\geq 0. \tag{4.1}$$

First assume that 0 is not in F. Set $t_0=\min F$. One easily checks that $f_{t_0}=1-e^{-t_0}e_{t_0}$ is orthogonal to e_t for each $t\geq t_0$, which means that f_{t_0} is in $N_F^\perp$. If $N_F^\perp$ is invariant under C_φ, then $f_{t_0}-C_\varphi f_{t_0}$ is in $N_F^\perp$. But $f_{t_0}-C_\varphi f_{t_0}=e^{-t_0}(1-e^{-at_0})e_{t_0}$ is also in N_F, which means that $f_{t_0}-C_\varphi f_{t_0}=0$. Hence, $f_{t_0}\equiv 1$, a contradiction.

Assume now that 0 is in F. Let $s>0$ be fixed and consider the operator M_{e_s} of multiplication by e_s. We have

$$M_{e_s}(N_F)=e_s\overline{\operatorname{span}}\{e_t:\ t\in F\}=\overline{\operatorname{span}}\{e_{s+t}:\ t\in F\}=N_{s+F}. \tag{4.2}$$

Clearly, M_{e_s} is a Hilbert space isometry preserving inner products. Therefore,

$$M_{e_s}\left(N_F^\perp\right)=\left(M_{e_s}(N_F)\right)^\perp. \tag{4.3}$$

Proceeding by contradiction, assume that $N_F^\perp$ is also invariant under C_φ. Then

$$M_{e_s}(C_\varphi(N_F^\perp))\subseteq M_{e_s}(N_F^\perp).$$

Since, for f in $\mathcal{H}^2$, we have $C_\varphi(M_{e_s}f)=C_\varphi(e_sf)=e^{-as}e_sC_\varphi f=e^{-as}M_{e_s}(C_\varphi f)$, from the above display, it follows that $C_\varphi(M_{e_s}(N_F^\perp))$ is included in $M_{e_s}(N_F^\perp)$.

Therefore, from (4.2) and (4.3), we immediately see that $C_\varphi(N_{s+F}^\perp) \subseteq N_{s+F}^\perp$, which is a contradiction because 0 is not in $s+F$. The proof is complete. □

5. The automorphism case

When $\Re a = 0$ in formula (1.1), then φ is a parabolic automorphism of $\mathbb{D}$ and still satisfies the eigenfunction equation (1.2) with the same eigenfunctions. But, instead of a spiral, the spectrum is the unit circle. Now, the lattice becomes much more complicated. The reason for this is that the eigenspaces are infinite dimensional. If we fix t_0 with $0 \le t_0 < 2\pi/|a|$, then it is clear that

$$\ker(C_{\varphi_a} - e^{-at_0}I) = \overline{\operatorname{span}}\,\{e_{t_0+2\pi n/|a|} : n = 0, 1, \ldots\}.$$

We have

Proposition 5.1. *Let $a \neq 0$ with $\Re a = 0$ and $\lambda = e^{-at_0}$, where $0 \le t_0 < 2\pi/|a|$. Then ℓ^2 is isomorphic to $\ker(C_{\varphi_a} - \lambda I)$ under the operator that to each sequence $\{a_n\}$ assigns the function $f = \sum_{n=0}^\infty a_n e_{t_0+2\pi n/|a|}$.*

Proof. Suppose that $\Im a > 0$. If $\Im a < 0$, the proof runs analogously. Since the operators C_{φ_a}, with $\Im a > 0$, are similar to each other, we may assume that $a = i/(2\pi)$. Since multiplication by e^{-it_0} is an isometric isomorphism, we may also assume that $t_0 = 0$. Using (4.1), one immediately checks that $f_n = e_n - e^{-1}e_{n+1}$, $n \ge 0$, are pairwise orthogonal and since $e_0 = \sum_{k=0}^\infty e^{-k} f_k$, they form a complete orthogonal system of $\ker(C_{\varphi_{i/(2\pi)}} - I)$. Thus, since $\|f_n\|_{\mathcal{H}^2}^2 = 1 - e^{-2}$, we need only prove that the operator T defined by $Tf_n = e_n = \sum_{k=0}^\infty e^{-k} f_{n+k}$ is bounded with bounded inverse. But $T = (I - e^{-1}S)^{-1}$, where S is defined by $Sf_n = f_{n+1}$. Clearly, $I - e^{-1}S$ is bounded and has bounded inverse because $\|e^{-1}S\| < 1$. The result is proved. □

The following proposition shows that there are a lot of invariant subspaces which are not spanned by eigenfunctions.

Proposition 5.2. *Let φ_a be a parabolic automorphism of the unit disk. Then C_{φ_a} has a non-trivial infinite-dimensional invariant subspace with at most the eigenfunction* 1.

Proof. By Corollary 3.2 and Proposition 3.3, it is enough to prove that there is an invariant subspace $\mathcal{M}$ for M_ψ such that its orthogonal complement $\mathcal{M}^\perp$ has no eigenfunction for the adjoint $M_\psi^\star$ but the eigenfunction δ_0.

As in the proof of Proposition 5.1, it suffices to consider the case $a = i/2\pi$. We take f_0 in $W^{1,2}[0,\infty)$ such that $f_0(t) \neq 0$ for each $t > 0$ and

$$\int_0^{1/2} \ln|f_0(t)|\,dt = -\infty. \tag{5.1}$$

We also take f_1 in $W^{1,2}[0,\infty)$ such that $f_1(t) > 0$ for each $t > 1$ and vanishing on $[0,1]$. For each $n \ge 2$ set $x_n = n - 2 + 2^{-n+1}$ and take f_n in $W^{1,2}[0,\infty)$ such that $f_n(t) \neq 0$ for $t \in (x_n, x_{n+1})$ and $f_n(t) = 0$ otherwise. The required subspace is

$$\mathcal{M} = \overline{\operatorname{span}}\,\{M_\psi^k f_n : k \in \mathbb{Z} \text{ and } n = 0, 1, 2, \ldots\}.$$

Clearly, $\mathcal{M}$ is invariant under M_ψ.

Now assume that an eigenfunction $h = \sum_{j=0}^{\infty} a_j \delta_{t_0+j}$ of $M_\psi^\star$ with $0 \le t_0 < 1$ and $\{a_j\}$ in ℓ^2 is orthogonal to $\mathcal{M}$.

If $0 < t_0 < 1$, then $t_0 + j = x_n$, $j \ge 0$ and $n \ge 2$, holds for at most just one $n \ge 2$. If $t_0+j \ne x_n$ for every $j \ge 1$, then t_0+j, for each $j \ge 1$, belongs to a unique (x_n, x_{n+1}). It follows that $0 = \langle f_n, h\rangle = \bar{a}_j f_n(t_0+j)$ for each $j \ge 1$, which implies that $a_j = 0$ for each $j \ge 1$. Then $h = a_0\delta_{t_0}$, but we have $0 = \langle f_0, h\rangle = \bar{a}_0 f_0(t_0)$, thus $a_0 = 0$ and h is the zero function. If there is n such that $t_0 + k = x_n$, then again $a_j = 0$ for every j different from k. In addition, since h is orthogonal to f_0 and f_1, we have $\bar{a}_0 f_0(t_0) + \bar{a}_k f_0(t_0+k) = 0$ and $\bar{a}_0 f_1(t_0) + \bar{a}_k f_1(t_0+k) = 0$. Since f_1 vanishes only on $[0,1]$, then $a_k = 0$ and thus $a_0 = 0$ and h is the zero function again.

If $t_0 = 0$, then $j \ne x_n$ for each $j \ge 0$ and each $n \ge 2$. In this case $j \ge 1$ belongs at most one interval (x_n, x_{n+1}). Hence, as in the previous case, $a_j = 0$ for each $j \ge 1$. Then $h = a_0\delta_0$. But since $f_n(0) = 0$ for each $n \ge 0$, we find that δ_0 belongs to the orthogonal of $\mathcal{M}$. Thus h need not be the zero function.

Finally, we see that there are infinitely many functions other than δ_0 in $\mathcal{M}^\perp$. In fact, $\mathcal{M}^\perp$ is infinite-dimensional since $\mathcal{M}$ cannot span all functions in $W^{1,2}[0,1/2]$. Indeed, f_0 is the only function non vanishing on $[0,1/2]$, by (5.1), Szegö's Theorem, see [9], implies that $\{e^{kat} f_0(t)\}_{k\in\mathbb{Z}}$ does not span $L^2[0,1/2]$ and therefore neither $W^{1,2}[0,1/2]$. The result is proved. $\square$

6. Final remarks

In this final section, we isolate conditions on an operator on a Banach space that are sufficient to characterize its invariant subspaces. Let T be a bounded linear operator acting on a Banach space $\mathcal{B}$. Let Ω be a set and let $\alpha : \Omega \to \mathbb{C}$, $\phi : \Omega \to \mathcal{B}^*$ be maps such that

(i) The functionals in $\phi(\Omega)$ separate points of $\mathcal{B}$, that is, $\bigcap\limits_{\omega\in\Omega} \ker \phi(\omega) = \{0\}$.

(ii) For each $\omega \in \Omega$, we have $T^*\phi(\omega) = \alpha(\omega)\phi(\omega)$.

Clearly, (i) means that the transform $x \mapsto \widehat{x}$ defined from $\mathcal{B}$ into the complex-valued functions on Ω as

$$\widehat{x}(\omega) = \langle x, \phi(\omega)\rangle$$

is a one-to-one linear operator. It is also elementary to check that (ii) is then equivalent to

$$\widehat{Tx} = \alpha\widehat{x}, \qquad \text{for} \quad x \in \mathcal{B}. \tag{6.1}$$

In addition, suppose that

(iii) The space $\widehat{\mathcal{B}} = \{\widehat{x} : x \in \mathcal{B}\}$ is a complex algebra with respect to the pointwise multiplication.

(iv) There exists $a \in \mathcal{B}$ such that $\alpha = \widehat{a}$; then (6.1) reads as

$$\widehat{Tx} = \widehat{a}\widehat{x} \ \text{ for each } \ x \in \mathcal{B}. \tag{6.2}$$

Proposition 6.1. *Let T be a bounded linear operator acting on a Banach space $\mathcal{B}$. Then the following are equivalent*

(a) *The operator T is similar to an operator of multiplication by an element on a semisimple commutative Banach algebra.*
(b) *There exist a set Ω, $\alpha : \Omega \to \mathbb{C}$, $\phi : \Omega \to \mathcal{B}^*$ and $a \in \mathcal{B}$ such that* (i) *through* (iv) *are satisfied.*

Proof. Suppose T is similar to an operator of multiplication by an element on a semisimple commutative Banach algebra. Upon transferring the multiplication to $\mathcal{B}$ by the similarity operator, we may assume that there is a multiplication $(x, y) \mapsto xy$ on $\mathcal{B}$ turning $\mathcal{B}$ into a semisimple commutative Banach algebra and $a \in \mathcal{B}$ such that $Tx = ax$ for each $x \in \mathcal{B}$. Let Ω be the spectrum of $\mathcal{B}$ and let $\phi : \Omega \to \mathcal{B}^*$ be the identity embedding. Then the map $x \mapsto \widehat{x}$ becomes the Gelfand transform and (iii) is trivially satisfied, that is, $\widehat{\mathcal{B}}$ is a subalgebra of $C(\Omega)$. Since $\mathcal{B}$ is semisimple, we also have (i). On the other hand,

$$\langle Tx, \phi(\omega)\rangle = \omega(ax) = \omega(a)\omega(x) = \omega(a)\langle x, \phi(\omega)\rangle,$$

which implies (ii) with $\alpha(\omega) = \widehat{a}(\omega)$. Thus, (iv) is also satisfied.

Conversely, properties (i) and (iii) allow us to define the multiplication of each $x, y \in \mathcal{B}$ by taking xy to be the unique element in $\mathcal{B}$ such that $\widehat{xy} = \widehat{x}\widehat{y}$. This multiplication turns $\mathcal{B}$ into a commutative complex algebra.

Let σ be the weakest topology on $\mathcal{B}$, with respect to which all functionals $\phi(\omega)$, with $\omega \in \Omega$, are continuous. Clearly, σ is weaker than the initial topology and, according to (i), is also Hausdorff. Moreover, for each $x \in \mathcal{B}$, the linear operator $y \mapsto xy$ is σ to σ continuous and, therefore, has closed graph in the topological square of $(\mathcal{B}, \sigma)$. Since σ is weaker than the initial topology, we see that this operator has closed graph in $\mathcal{B} \times \mathcal{B}$. The Closed Graph Theorem implies that for each $x \in \mathcal{B}$ the operator $y \mapsto xy = yx$ is bounded, which means that the multiplication $(x, y) \mapsto xy$ is separately continuous. According to the Uniform Boundedness Principle, this multiplication is continuous and, therefore, turns $\mathcal{B}$ into a commutative Banach algebra. On the other hand, from the definition of multiplication on $\mathcal{B}$, it follows that $\phi(\omega)$ are characters on $\mathcal{B}$ and hence, according to (i), the characters on $\mathcal{B}$ separate points. Thus, $\mathcal{B}$ is semisimple. Finally, from (6.2), it also follows that $Tx = ax$ for each $x \in \mathcal{B}$. The result is proved. ☐

Proposition 6.1, Proposition 2.1 and Corollary 2.3 immediately imply.

Corollary 6.2. *Let T be a bounded linear operator acting on a Banach space $\mathcal{B}$. Suppose also that there exist a set Ω, $\alpha : \Omega \to \mathbb{C}$, $\phi : \Omega \to \mathcal{B}^*$ and $a \in \mathcal{B}$ such that conditions* (i) *through* (iv) *are satisfied. Let also $a_n \in \mathcal{B}$ be such that $\widehat{a}_n = \widehat{a}^n$ for each positive integer n. If the span of $\{a_n : n = 1, 2, \dots\}$ is dense in $\mathcal{B}$, then the invariant subspaces of T are exactly the ideals of the algebra $\mathcal{B}$ with the multiplication satisfying $\widehat{xy} = \widehat{x}\widehat{y}$ for all $x, y \in \mathcal{B}$.*

Remark. Operators satisfying (i) through (iii) also admit an 'algebraic' characterization. Namely, a bounded linear operator T acting on a Banach space $\mathcal{B}$ is similar to an operator commuting with all multiplication operators on a semisimple commutative Banach algebra if and only if there exist a set Ω, $\alpha : \Omega \to \mathbb{C}$ and $\phi : \Omega \to \mathcal{B}^*$ such that (i) through (iii) are satisfied.

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