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SOME INEQUALITIES ON TOTALLY REAL SUBMANIFOLDS IN LOCALLY CONFORMAL KAEHLER SPACE FORMS

Alfonso Carriazo, Young Ho Kim∗ and Dae Won Yoon

Abstract. In this article, we establish sharp relations between the sectional curvature and the shape operator and also between the k-Ricci curvature and the shape operator for a totally real submanifold in a locally conformal Kaehler space form of constant holomorphic sectional curvature with arbitrary codimension.

1. Introduction

Nash's Theorem enables us to consider any Riemannian manifold as a submanifold of Euclidean space. This gives us a natural motivation for the study of submanifolds of Riemannian manifolds. In this case, we have intrinsic invariants as well as extrinsic invariants. Among extrinsic invariants, the shape operator and the squared mean curvature are the most important ones. Among the main intrinsic invariants, sectional, Ricci and scalar curvature are the well-known ones. Gauss-Bonnet Theorem, isoperimetric inequality and Chern-Lashof Theorem provide relations between intrinsic invariants and extrinsic invariants for a submanifold in a Euclidean space.

B.-Y. Chen ([1, 2]) established a inequality relating intrinsic quantities and extrinsic ones for submanifolds in a space form with arbitrary codimension. In particular, in [1] he investigated a relation between the sectional curvature and the shape operator for submanifolds in Riemannian space forms. And, in [2] he established a sharp relation between the k-Ricci curvature and the shape operator. On the other hand, for the above mentioned contents K. Matsumoto, I. Mihai and A. Oiaga ([6])

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studied these relations of slant submanifolds in complex space forms, and Y. H. Kim, D. W. Yoon and C. W. Lee ([4]) have recently investigated these relations of slant submanifolds in Sasakian spaces.

In this paper, we study submanifolds of locally conformal Kaehler space forms of constant holomorphic sectional curvature with arbitrary codimension and establish relations between the sectional curvature and the shape operator and also between the k-Ricci curvature and the shape operator for totally real submanifolds in locally conformal Kaehler space forms.

2. Preliminaries

Let $\tilde{M}$ be a Hermitian manifold with almost complex structure J and a Hermitian metric g. A Hermitian manifold $\tilde{M}$ is called a *locally conformal Kaehler manifold* if each point $p \in \tilde{M}$ has an open neighborhood U with a differentiable map $\phi : U \longrightarrow \mathbb{R}$ such that

$$g^* = e^{-2\phi} g|_U \tag{2.1}$$

is Kaehler metric on U (See [3, 7]). On the other hand, the fundamental 2-form w of $\tilde{M}$ is defined by

$$w(X, Y) = g(JX, Y) \tag{2.2}$$

for any tangent vectors X, Y on $\tilde{M}$.

Proposition 2.1 ([3]). *A Hermitian manifold $\tilde{M}$ is a locally conformal Kaehler manifold if and only if there exists a global closed 1-form α satisfying*

$$\begin{aligned} &(\tilde{\nabla}_Z w)(X, Y) \\ &= \beta(Y)g(X, Z) - \beta(X)g(Y, Z) + \alpha(Y)w(X, Z) - \alpha(X)w(Y, Z) \end{aligned} \tag{2.3}$$

for any tangent vectors X, Y, Z on $\tilde{M}$, where $\tilde{\nabla}$ denotes the Levi-Civita connection with respect to g and the 1-form β is given by $\beta(X) = -\alpha(JX)$.

The 1-form α which satisfies is called the *Lee form* and its dual vector field is the Lee vector field. A locally conformal Kaehler manifold having the parallel Lee form is called a generalized Hopf manifold. As a matter of fact, the Hopf manifold diffeomorphic to $S^1 \times S^{2n-1}$ is an example of a locally conformal Kaehler manifold that is not Kaehlerian.

On a locally conformal Kaehler manifold, a symmetric $(0,2)$-tensor P is defined by

$$P(X,Y) = -(\tilde{\nabla}_X \alpha)Y - \alpha(X)\alpha(Y) + \frac{1}{2}||\alpha||^2 g(X,Y), \tag{2.4}$$

and another skew-symmetric $(0,2)$-tensor $\tilde{P}$ by $\tilde{P}(X,Y) = P(JX,Y)$, where $||\alpha||$ is the norm of α with respect to g.

Let M be an n-dimensional submanifold of an m-dimensional locally conformal Kaehler manifold $\tilde{M}$. Let ∇ be the induced Levi-Civita connection on M. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.5}$$

$$\tilde{\nabla}_X V = -A_V X + D_X V \tag{2.6}$$

for vector fields X, Y tangent to M and a vector field V normal to M, where h denotes the second fundamental form, D the normal connection and A_V the shape operator in the direction of V. The second fundamental form and the shape operator are related by

$$g(h(X,Y),V) = g(A_V X, Y). \tag{2.7}$$

We also use g for the induced Riemannian metric on M as well as the locally conformal Kaehler manifold $\tilde{M}$. Moreover, the mean curvature vector H on M is defined by $H = \frac{1}{n}\mathrm{trace}h$. A submanifold M in $\tilde{M}$ is called *totally geodesic* if the second fundamental form vanishes identically and *totally umbilical* if there is a real number λ such that $h(X,Y) = \lambda g(X,Y)H$ for any tangent vectors X, Y on M.

For an n-dimensional Riemannian manifold M, we denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM, p \in M$. For an orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space T_pM, the scalar curvature τ and the normalized scalar curvature ρ are defined respectively by

$$\tau = \sum_{i<j} K_{ij}, \quad \rho = \frac{2\tau}{n(n-1)},$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i, e_j.

Suppose L is a k-plane section of T_pM and X a unit vector in L. We choose an orthonormal basis $\{e_1, \ldots, e_k\}$ of L such that $e_1 = X$. Define the Ricci curvature Ric_L of L at X by

$$Ric_L(X) = K_{12} + \cdots + K_{1k}. \tag{2.8}$$

We simply call such a curvature a *k-Ricci curvature.* The scalar curvature τ of the k-plane section L is given by

$$\tau(L) = \sum_{1\leq i<j\leq k} K_{ij}. \tag{2.9}$$

For each integer $k, 2 \leq k \leq n$, the Riemannian invariant Θ_k on an n-dimensional Riemannian manifold M is defined by

$$\Theta_k(p) = \frac{1}{k-1} \inf_{L,X} Ric_L(X), \quad p \in M, \tag{2.10}$$

where L runs over all k-plane sections in T_pM and X runs over all unit vectors in L.

Recall that for a submanifold M in a Riemannian manifold, the *relative null space* or the kernel of the second fundamental form of M at a point $p \in M$ is defined by

$$N_p = \{X \in T_pM | h(X, Y) = 0 \quad \text{for all} \quad Y \in T_pM\}. \tag{2.11}$$

3. Totally real submanifolds in locally conformal Kaehler space forms

Let M be an n-dimensional submanifold isometrically immersed in an m-dimensional locally conformal Kaehler manifold $\tilde{M}$. A locally conformal Kaehler manifold $\tilde{M}$ is said to be a *locally conformal Kaehler space form* if the holomorphic sectional curvature of the 2-plane section $\{X, JX\}$ at each point of $\tilde{M}$ is a real constant $\tilde{c}$ along $\tilde{M}$. A locally conformal Kaehler space form will be denoted by $\tilde{M}(\tilde{c})$. Then, the Riemannian curvature tensor $\tilde{R}$ on $\tilde{M}(\tilde{c})$ is given by

(3.1)

$$\begin{aligned}\tilde{R}(X,Y)Z = \frac{\tilde{c}}{4}\{&g(Y,Z)X - g(X,Z)Y + w(Y,Z)JX \\ &- w(X,Z)JY - 2w(X,Y)JZ\} \\ &+ \frac{3}{4}\{g(Y,Z)P_1X - g(X,Z)P_1Y + P(Y,Z)X - P(X,Z)Y\} \\ &- \frac{1}{4}\{w(Y,Z)\tilde{P}_1X - w(X,Z)\tilde{P}_1Y + \tilde{P}(Y,Z)JX \\ &- \tilde{P}(X,Z)JY - 2\tilde{P}(X,Y)JZ - 2w(X,Y)\tilde{P}_1Z\},\end{aligned}$$

where $g(P_1X, Y) = P(X, Y), g(\tilde{P}_1X, Y) = \tilde{P}(X, Y)$.

A submanifold M isometrically immersed in $\tilde{M}(\tilde{c})$ is called *totally real* if the almost complex structure J of $\tilde{M}(\tilde{c})$ carries each tangent space of

M into its corresponding normal space. On the other hand, for a totally real submanifold M on $\widetilde{M}(\tilde{c})$ we have $w(X, Y) = 0$ for vector fields X, Y tangent to M. We denote by R the Riemannian curvature tensor field of M. Then, the equation of Gauss on M is given by

(3.2)

$$\begin{aligned} g(R(X,Y)Z,W) &= \frac{\tilde{c}}{4}\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\} \\ &+ \frac{3}{4}\{g(X,W)P(Y,Z) - g(Y,W)P(X,Z) + P(X,W)g(Y,Z) \\ &- P(Y,W)g(X,Z)\} + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)), \end{aligned}$$

for any vector fields X, Y, Z, W tangent to M.

Let $\{e_1, \ldots, e_n\}$ be any orthonormal basis in T_pM. Then it is easily seen that the scalar curvature τ of M at p is obtained by

$$2\tau(p) = n^2||H||^2 - ||h||^2 + \frac{1}{4}n(n-1)(\tilde{c} + 6\sigma), \tag{3.3}$$

where $||H||^2$ and $||h||^2$ are the squared mean curvature and the squared norm of the second fundamental form respectively, and we have put $\sigma = \frac{1}{n}\sum_{i=1}^{n} P(e_i, e_i)$.

4. Sectional curvature and shape operator

B.-Y. Chen ([1]) established a relation between the sectional curvature and the shape operator for submanifolds in real space forms. Also, K. Matsumoto, I. Mihai and A. Oiaga ([6]) and Y. H. Kim, D. W. Yoon and C. W. Lee ([4]) have recently investigated these relations for slant submanifolds into complex space forms and Sasakian spaces, respectively. We prove a similar inequality for an n-dimensional totally real submanifold M into an m-dimensional locally conformal Kaehler space form $\widetilde{M}(\tilde{c})$ of constant holomorphic sectional curvature $\tilde{c}$.

Lemma 4.1. *Let $x : M \longrightarrow \widetilde{M}(c)$ be an isometric immersion of an n-dimensional totally real submanifold with normalized scalar curvature ρ into an m-dimensional locally conformal Kaehler space form $\widetilde{M}(\tilde{c})$ of constant holomorphic sectional curvature $\tilde{c}$. Then, we have*

$$||H||^2 \geq \rho - \frac{1}{4}(\tilde{c} + 6\sigma), \tag{4.1}$$

equality holding at a point $p \in M$ if and only if p is a totally umbilical point.

Proof. Let p be a point of M. We choose an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ for the tangent space T_pM and $\{e_{n+1}, \dots, e_{2m}\}$ for the normal space $T_p^{\perp}M$ at p such that the normal vector e_{n+1} is in the direction of the mean curvature vector and $e_1, e_2, \dots, e_n$ diagonalize the shape operator A_{n+1}. Then we have

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_n \end{pmatrix}, \tag{4.2}$$

$$A_r = (h_{ij}^r), \qquad \sum_{i=1}^{n} h_{ii}^r = 0, \quad 1 \le i, j \le n; n+2 \le r \le 2m.$$

From the equation of Gauss (3.2)

$$n^2||H||^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - \frac{1}{4}n(n-1)(\tilde{c} + 6\sigma). \tag{4.3}$$

On the other hand,

$$\sum_{i<j} (a_i - a_j)^2 = (n-1) \sum_{i=1}^{n} a_i^2 - 2 \sum_{i<j} a_i a_j. \tag{4.4}$$

Therefore, from the above equation we have

$$n^2||H||^2 = (\sum_{i=1}^{n} a_i)^2 = \sum_{i=1}^{n} a_i^2 + 2 \sum_{i<j} a_i a_j \le n \sum_{i=1}^{n} a_i^2. \tag{4.5}$$

Combining (4.3) and (4.5)

$$n(n-1)||H||^2 \ge 2\tau - \frac{1}{4}n(n-1)(\tilde{c} + 6\sigma) + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 \tag{4.6}$$

which implies inequality (4.1). If the equality sign of (4.1) holds at a point $p \in M$ then from (4.4) and (4.6) we get $A_r = 0$ $(r = n+2, \dots, 2m)$ and $a_1 = \dots = a_n$. Consequently, p is a totally umbilical point. The converse is trivial. □

THEOREM 4.2. *Let $x : M \longrightarrow \tilde{M}(\tilde{c})$ be an isometric immersion of an n-dimensional totally real submanifold M into an m-dimensional locally conformal Kaehler space form $\tilde{M}(\tilde{c})$ of constant holomorphic sectional curvature $\tilde{c}$. If there exist a point $p \in M$ and a number $c > \frac{1}{4}(\tilde{c} + 6\sigma)$*

such that inf $K(p) = K \geq c$ at p. Then the shape operator at the mean curvature vector satisfies

$$A_H > \frac{n-1}{n}\left\{c - \frac{1}{4}(\tilde{c} + 6\sigma)\right\} I_n \quad \text{at} \quad p, \tag{4.7}$$

where I_n is the identity map of T_pM.

Proof. Assume that M is a totally real submanifold in $\tilde{M}(\tilde{c})$. Let $p \in M$ and a number $c > \frac{1}{4}(\tilde{c} + 6\sigma)$ such that $K \geq c$ at p. Choose an orthonormal basis $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2m}\}$ at p such that e_{n+1} is parallel to the mean curvature vector H and $e_1, \ldots, e_n$ diagonalize the shape operator A_{n+1}. Then we have the relation (4.2). We have put $u_{ij} = u_{ji} = a_i a_j$. From the Gauss's equation we get

$$u_{ij} \geq c - \frac{1}{4}(\tilde{c} + 6\sigma) + \sum_{r=n+2}^{2m} (h^r_{ij})^2 - \sum_{r=n+2}^{2m} h^r_{ii} h^r_{jj}, \quad 1 \leq i \neq j \leq n. \tag{4.8}$$

We need the following lemmas in order to complete the proof of the theorem. □

Lemma 4.3. *The following statements hold.*

(1) *For any fixed $i \in \{1, \ldots, n\}$, we have $\sum_{j \neq i} u_{ij} \geq (n-1)\{c - \frac{1}{4}(\tilde{c} + 6\sigma)\}$.*

(2) *$u_{ij} \neq 0$ for $i \neq j$.*

(3) *For distinct i, j, k, we have $a_i^2 = u_{ij} u_{ik} u_{jk}^{-1}$.*

Proof. From (4.2) and (4.8), we get

$$\begin{aligned}\sum_{j \neq i}^{n} u_{ij} &\geq (n-1)\{c - \frac{1}{4}(\tilde{c} + 6\sigma)\} + \sum_{r=n+2}^{2m} \{\sum_{j \neq i} (h^r_{ij})^2 - h^r_{ii} \sum_{j \neq i} h^r_{jj}\} \\ &= (n-1)\{c - \frac{1}{4}(\tilde{c} + 6\sigma)\} + \sum_{r=n+2}^{2m} \sum_{i,j} (h^r_{ij})^2,\end{aligned}$$

which yields statement (1). For statement (2), assume that $u_{ij} = 0$ for $i \neq j$, then $a_i = 0$ or $a_j = 0$. $a_i = 0$ implies that $u_{it} = 0$ for any $i \neq t$. Hence $\sum_{i \neq t} u_{it} = 0$ which contradicts statement (1). Statement (3) follows from $u_{ij} u_{ik} = a_i^2 a_j a_k = a_i^2 u_{jk}$. □

We put $S_k = \{B \subset \{1, \ldots, n\} : |B| = k\}$. For any $B \in S_k$ we denote by $\bar{B} = \{1, \ldots, n\} \backslash B$.

LEMMA 4.4. *For a fixed k, $1 \leq k \leq [\frac{n}{2}]$, and each $B \in S_k$, we have*

$$\sum_{j\in B}\sum_{t\in\bar{B}} u_{jt} \geq (n-k)k\{c - \frac{1}{4}(\tilde{c}+6\sigma)\}.$$

Proof. Without loss of generality, we may assume $B = \{1,\ldots,k\}$. From (4.8) we find

$$\begin{aligned}
&\sum_{j\in B}\sum_{t\in\bar{B}} u_{jt} \\
&\geq (n-k)k\{c - \frac{1}{4}(\tilde{c}+6\sigma)\} + \sum_{r=n+2}^{2m}\sum_{j=1}^{k}\sum_{t=k+1}^{n}\{(h^r_{jt})^2 - h^r_{jj}h^r_{tt}\} \\
&= (n-k)k\{c - \frac{1}{4}(\tilde{c}+6\sigma)\} + \sum_{r=n+2}^{2m}\{\sum_{j=1}^{k}\sum_{t=k+1}^{n}(h^r_{jt})^2 + \sum_{j=1}^{k}(h^r_{jj})^2\},
\end{aligned}$$

which implies the lemma. □

LEMMA 4.5. *For any $1 \leq i \neq j \leq n$, we have $u_{ij} > 0$.*

Proof. Assume $u_{1n} < 0$. Then, by statement (3) of Lemma 4.3, we get $u_{1i}u_{in} < 0$ for $1 < i < n$. Without loss of generality, we may assume

$$\begin{cases} u_{12},\ldots,u_{1l},u_{(l+1)n},\ldots,u_{(n-1)n} > 0, \\ u_{1(l+1)},\ldots,u_{1n},u_{2n},\ldots,u_{ln} < 0, \end{cases} \tag{4.9}$$

for some $[\frac{n+1}{2}] \leq l \leq n-1$.

Let $l = n-1$, then $u_{1n} + u_{2n} + \cdots + u_{(n-1)n} < 0$ which contradicts to statement (1) of Lemma 4.3. Thus, $l < n-1$. From statement (3) of Lemma 4.3, we get

$$a_n^2 = \frac{u_{in}u_{tn}}{u_{it}} > 0, \tag{4.10}$$

where $2 \leq i \leq l$ and $l+1 \leq t \leq n-1$. By (4.9) and (4.10) we have $u_{it} < 0$ which implies

$$\sum_{i=1}^{l}\sum_{t=l+1}^{n} u_{it} = \sum_{i=2}^{l}\sum_{t=l+1}^{n-1} u_{it} + \sum_{i=1}^{l} u_{in} + \sum_{t=l+1}^{n} u_{1t} < 0.$$

This contradicts Lemma 4.4. □

Now, we return to the proof of Theorem 4.2. From Lemma 4.5, it follows that $a_1,\ldots,a_n$ are of the same sign. Therefore, the shape operator A_H is positive-definite. Assume $a_j > 0$ for all $j \in \{1, \dots, n\}$. Then from statement (1) of Lemma 4.3, we obtain

$$\begin{aligned} na_i||H|| - a_i^2 &= a_i(a_1 + \cdots + a_n) - a_i^2 \\ &= a_i \sum_{i \neq j} a_j = \sum_{i \neq j} a_i a_j = \sum_{i \neq j} u_{ij} \\ &\geq (n-1)\{c - \frac{1}{4}(\tilde{c} + 6\sigma)\}, \end{aligned}$$

which implies (4.7). This completes the proof of the theorem. □

5. k-Ricci curvature and shape operator

In this section, we establish a relation between the k-Ricci curvature and the shape operator for an n-dimensional totally real submanifold M into an m-dimensional locally conformal Kaehler space form $\tilde{M}(\tilde{c})$ of constant holomorphic sectional curvature $\tilde{c}$.

THEOREM 5.1. *Let $x : M \longrightarrow \tilde{M}(\tilde{c})$ be an isometric immersion of an n-dimensional totally real submanifold M into an m-dimensional locally conformal Kaehler space form $\tilde{M}(\tilde{c})$ of constant holomorphic sectional curvature $\tilde{c}$. Then, for any integer k $2 \leq k \leq n$, and any point $p \in M$, we have*

(1) *If $\Theta_k(p) \neq \frac{1}{4}(\tilde{c} + 6\sigma)$, then shape operator at the mean curvature satisfies*

$$A_H > \frac{n-1}{n}\left\{\Theta_k(p) - \frac{1}{4}(\tilde{c} + 6\sigma)\right\} I_n \quad \text{at} \quad p, \tag{5.1}$$

where I_n denotes the identity map of T_pM.

(2) *If $\Theta_k(p) = \frac{1}{4}(\tilde{c} + 6\sigma)$, then $A_H \geq 0$ at p.*

(3) *A unit vector $X \in T_pM$ satisfies*

$$A_H X = \frac{n-1}{n}\left\{\Theta_k(p) - \frac{1}{4}(\tilde{c} + 6\sigma)\right\} X \tag{5.2}$$

if and only if $\Theta_k(p) = \frac{1}{4}(\tilde{c} + 6\sigma)$ and $X \in N_p$.

(4) *$A_H = \frac{n-1}{n}\{\Theta_k(p) - \frac{1}{4}(\tilde{c} + 6\sigma)\}I_n$ at p if and only if p is a totally geodesic point.*

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of T_pM. Denote by $L_{i_1 \cdots i_k}$ the k-plane section spanned by $e_{i_1}, \dots, e_{i_k}$. It follows from (2.8) and (2.9) that

$$\tau(L_{i_1\cdots i_k}) = \frac{1}{2}\sum_{i\in\{i_1,\ldots,i_k\}} Ric_{L_{i_1\cdots i_k}}(e_i), \tag{5.3}$$

$$\tau(p) = \frac{1}{\binom{n-2}{k-2}}\sum_{1\le i_1<\cdots<i_k\le n} \tau(L_{i_1\cdots i_k}). \tag{5.4}$$

Combining (2.10), (5.3) and (5.4), we obtain

$$\tau(p) \ge \frac{n(n-1)}{2}\Theta_k(p). \tag{5.5}$$

We choose an orthonormal basis $\{e_1,\ldots,e_n,e_{n+1},\ldots,e_{2m}\}$ at p such that e_{n+1} is parallel to the mean curvature vector $H(p)$ and $e_1,\ldots,e_n$ diagonalize the shape operator A_{n+1}. Then we have the relation (4.2). Furthermore (4.1) can be rewritten as the form

$$||H||^2 \ge \frac{2\tau}{n(n-1)} - \frac{1}{4}(\tilde{c}+6\sigma), \tag{5.6}$$

which implies

$$||H||^2 \ge \Theta_k(p) - \frac{1}{4}(\tilde{c}+6\sigma). \tag{5.7}$$

This show that $H(p) = 0$ may occurs only when $\Theta_k(p) \le \frac{1}{4}(\tilde{c}+6\sigma)$. Consequently, if $H(p) = 0$, statements (1) and (2) hold automatically. Therefore, without loss of generality, we may assume $H(p) \neq 0$. From the Gauss's equation we get

(5.8)

$$a_i a_j = K_{ij} - \frac{1}{4}(\tilde{c}+6\sigma) + \sum_{r=n+2}^{2m}(h_{ij}^r)^2 - \sum_{r=n+2}^{2m} h_{ii}^r h_{jj}^r, \quad 1\le i\neq j\le n.$$

By (5.8) we have

$$\begin{aligned} &a_1(a_{i_2}+\cdots+a_{i_k}) \\ &= Ric_{L_{1i_2\ldots i_k}}(e_1) - \frac{1}{4}(k-1)(\tilde{c}+6\sigma) \\ &\quad + \sum_{r=n+2}^{2m}\sum_{j=2}^{k}(h_{1i_j}^r)^2 - \sum_{r=n+2}^{2m}\sum_{j=2}^{k} h_{11}^r h_{i_j i_j}^r, \quad 1<i_2<\cdots<i_k, \end{aligned} \tag{5.9}$$

from this

$$(5.10)\quad \begin{aligned} a_1(a_2+\cdots+a_n) = & \frac{1}{\binom{n-2}{k-2}} \sum_{2\le i_2<\cdots<i_k\le n} Ric_{L_{1i_2\ldots i_k}}(e_1) \\ & -\frac{1}{4}(n-1)(\tilde{c}+6\sigma)+\sum_{r=n+2}^{2m}\sum_{j=1}^{n}(h^r_{1j})^2. \end{aligned}$$

From (2.10) and (5.10) we have

$$(5.11)\quad a_1(a_2+\cdots+a_n)\ge (n-1)\{\Theta_k(p)-\frac{1}{4}(\tilde{c}+6\sigma)\}.$$

Then

$$(5.12)\quad \begin{aligned} a_1(a_1+\cdots+a_n) &= a_1^2+a_1(a_2+\cdots+a_n) \\ &\ge a_1^2+(n-1)\{\Theta_k(p)-\frac{1}{4}(\tilde{c}+6\sigma)\}. \end{aligned}$$

Similar inequalities hold when the index 1 were replaced by $j\in\{2,\ldots,n\}$. Hence, we have

$$a_j(a_1+\cdots+a_n)\ge a_j^2+(n-1)\{\Theta_k(p)-\frac{1}{4}(\tilde{c}+6\sigma)\},\quad j\in\{1,\ldots,n\},$$

which yields

$$A_H\ge \frac{n-1}{n}\{\Theta_k(p)-\frac{1}{4}(\tilde{c}+6\sigma)\}I_n.$$

The equation does not hold because in our case $H(p)\ne 0$. The statement (2) is obvious.

(3) Let X be a unit vector in T_pM satisfying (5.2). By (5.10) and (5.12) one has $a_1=0$ and $h^r_{1j}=0$, for all $j\in\{1,\ldots,n\}, r\in\{n+2,\ldots,2m\}$, respectively. The above conditions imply $\Theta_k(p)=\frac{1}{4}(\tilde{c}+6\sigma)$ and $X\in N_p$. The converse is clear.

(4) The equality (5.2) holds for any $X\in T_pM$ if and only if $N_p=T_pM$, i.e., p is a totally geodesic point. This completes the proof of the theorem. □

6. Ricci curvature and squared mean curvature

In this section, we establish a relation between the Ricci curvature and the squared mean curvature for an n-dimensional totally real submanifold M into an m-dimensional locally conformal Kaehler space form $\tilde{M}(\tilde{c})$ of constant holomorphic sectional curvature $\tilde{c}$.

THEOREM 6.1. *Let $x : M \longrightarrow \tilde{M}(\tilde{c})$ be an isometric immersion of an n-dimensional totally real submanifold M into an m-dimensional locally conformal Kaehler space form $\tilde{M}(\tilde{c})$ of constant holomorphic sectional curvature $\tilde{c}$. Then,*

(1) *For each unit vector $X \in T_pM$, we have*

$$\mathrm{Ric(X)} \leq \frac{1}{4}\{n^2||H||^2 + (n-1)(\tilde{c}+6\sigma)\}. \tag{6.1}$$

(2) *If $H(p) = 0$, then a unit tangent vector X at p satisfies the equality case of (6.1) if and only if $X \in N_p$.*

(3) *The equality case of (6.1) holds identically for all unit tangent vectors at p if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Proof. Let $X \in T_pM$ be a unit tangent vector at p. We choose an orthonormal basis $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2m}\}$ in $T_p\tilde{M}$ such that $e_1, \ldots, e_n$ are tangent to M at p with $e_1 = X$. Then, from (3.3) we get

(6.2)

$$\begin{aligned}
n^2||H||^2 &= 2\tau + ||h||^2 - \frac{1}{4}n(n-1)(\tilde{c}+6\sigma) \\
&= 2\tau + \sum_{r=n+1}^{2m}[(h_{11}^r)^2 + (h_{22}^r + \cdots + h_{nn}^r)^2 + 2\sum_{1\leq i<j\leq n}(h_{ij}^r)^2] \\
&\quad - 2\sum_{r=n+1}^{2m}\sum_{2\leq i<j\leq n}h_{ii}^r h_{jj}^r - \frac{1}{4}n(n-1)(\tilde{c}+6\sigma) \\
&= 2\tau + \frac{1}{2}\sum_{r=n+1}^{2m}\Big[(h_{11}^r + h_{22}^r + \cdots + h_{nn}^r)^2 \\
&\quad + (h_{11}^r - h_{22}^r - \cdots - h_{nn}^r)^2\Big] \\
&\quad + 2\sum_{r=n+1}^{2m}\sum_{1\leq i<j\leq n}(h_{ij}^r)^2 - 2\sum_{r=n+1}^{2m}\sum_{2\leq i<j\leq n}h_{ii}^r h_{jj}^r \\
&\quad - \frac{1}{4}n(n-1)(\tilde{c}+6\sigma).
\end{aligned}$$

It follows that

(6.3)

$$\frac{1}{2}n^2||H||^2 \geq 2\tau - \frac{1}{4}n(n-1)(\tilde{c}+6\sigma) - 2\sum_{r=n+1}^{2m}\sum_{2\leq i<j\leq n}[h_{ii}^r h_{jj}^r - (h_{ij}^r)^2].$$

From the equation of Gauss, we have

$$\sum_{2\le i<j\le n} K_{ij} = \sum_{r=n+1}^{2m} \sum_{2\le i<j\le n} [h^r_{ii}h^r_{jj} - (h^r_{ij})^2] + \frac{(n-1)(n-2)\tilde{c}}{8} + \frac{3}{4}(n-1)(n-2)\sigma. \tag{6.4}$$

Substituting (6.4) in (6.3), we get

$$\frac{1}{2}n^2||H||^2 \ge 2\mathrm{Ric(X)} - \frac{1}{2}(n-1)(\tilde{c}+6\sigma),$$

or equivalently (6.1).

(2) Assume $H(P) = 0$. Equality holds in (6.1) if and only if

$$\begin{cases} h^r_{12} = \cdots = h^r_{1n} = 0, \\ h^r_{11} = h^r_{22} + \cdots + h^r_{nn}, \quad r \in \{n+1, \dots, 2m\}. \end{cases}$$

Then $h^r_{1j} = 0$ for all $j \in \{1, \dots, n\}, r \in \{n+1, \dots, 2m\}$, that is, $X \in N_p$.

(3) Then equality case of (6.1) holds for all unit tangent vectors at p if and only if

$$\begin{cases} h^r_{ij} = 0, \quad i \ne j, \quad r \in \{n+1, \dots, 2m\}, \\ h^r_{11} + \cdots + h^r_{nn} - 2h^r_{ii} = 0, \quad i \in \{1, \dots, n\}, \quad r \in \{n+1, \dots, 2m\}. \end{cases}$$

We distinguish two cases:

(a) $n \ne 2$, then p is a totally geodesic point;

(b) $n = 2$, it follows that p is a totally umbilical point.

The converse is trivial. □

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Alfonso Carriazo
Department of Geometry and Topology
Faculty of Mathematics
University of Seville
Apartado de Correos 1160
41080-Sevilla, Spain
E-mail: carriazo@us.es

Young Ho Kim
Department of Mathematics
College of Natural Sciences
Kyungpook National University
Taegu 702-701, Korea
E-mail: yhkim@knu.ac.kr

Dae Won Yoon
Department of Mathematics Education and RINS
Gyeongsang National University
Chinju 660-701, Korea
E-mail: dwyoon@gsnu.ac.kr