The sub-supersolution method for Kirchhoff systems: applications

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ABSTRACT. In this paper we prove that the sub-supersolution method works for general Kirchhoff systems. We apply the cited method to prove the existence of positive solutions for some specific models.

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1 Introduction

In this note we study the existence of solutions of a nonlinear Kirchhoff system

$$\begin{cases} -M_1(\|u_1\|^2)\Delta u_1 = f_1(x,u_1,u_2) & \text{in } \Omega, \\ -M_2(\|u_2\|^2)\Delta u_2 = f_2(x,u_1,u_2) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a regular and bounded domain,

$$\|u\|^2 := \int_\Omega |\nabla u|^2 dx, \qquad \text{for } u \in H_0^1(\Omega),$$

M_i, $i = 1, 2$ are continuous functions verifying

(M) $\qquad M_i : \mathbb{R}_+ \mapsto \mathbb{R}_+$ and $\exists m_0 > 0$ such that $M_i(t) \geq m_0 > 0\ \forall t \in \mathbb{R}_+$,

and $f_i \in C(\overline{\Omega} \times \mathbb{R}^2)$. We assume (M) along the paper.

Basically, in our knowledge, similar systems to (1.1) have been analyzed in several papers. In [8], [3], [4], [6], [10] and references therein, variational methods have been applied to prove existence and multiplicity of positive solutions for systems as (1.1). In [1] and [2] the sub-supersolution method has been used to prove the existence of solution with M_i increasing and bounded from above and below for positive constants, that is, there exist positive constants $0 < m_i \leq m_i^\infty < \infty$ such that

$$0 < m_i \leq M_i(t) \leq m_i^\infty < \infty \quad i = 1, 2, \quad \forall t \geq 0.$$

However, in both papers the authors use a comparison principle (see for instance Lemma 2.1 in [1]) which seems not to be correct, see [5].

In this paper, we prove that the sub-supersolution method works for system (1.1), when the sub-supersolution is defined in an appropriate way, see Theorem 3.3. Indeed, in this case, the definition of sub-supersolution depends on the monotony of the non-linear reaction term (in a similar way to the local problems, see for instance [9]) and on the functions M_i. In order to prove this result, we transform our Kirchhoff system (1.1) into another with general non-local term depending only on the unknown variable u_i but not the $\|u_i\|^2$. So, as consequence, we establish a very general sub-supersolution method for for a large class of systems with nonlinear and non-local terms (see Theorem 2.2).

The paper is organized as follows. In Section 2 we show that the sub-supersolution method works for general non-local systems. In Section 3, under very general conditions on M_i, we transform our system (1.1) into a non-local systems, and apply the method of Section 2. Section 4 is devoted to apply our method for different particular systems.

2 The sub-super method for non-local systems

First of all we show that the sub-supersolution method works well for non-local systems of the following type

$$\begin{cases} -\Delta u_1 = g_1(x, u_1, u_2, B_1(u_1), B_2(u_2), C_1(u_1, u_2)) & \text{in } \Omega, \\ -\Delta u_2 = g_2(x, u_1, u_2, B_1(u_1), B_2(u_2), C_2(u_1, u_2)) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $g_i : \Omega \times \mathbb{R}^5 \mapsto \mathbb{R}$ is a continuous function, $B_i : L^\infty(\Omega) \mapsto \mathbb{R}$, $C_i : (L^\infty(\Omega))^2 \mapsto \mathbb{R}$ are continuous operators. Given $w \leq z$ a.e. in Ω, we denote by

$$[w, z] := \{u : w(x) \leq u(x) \leq z(x) \quad \text{a.e. } x \in \Omega\}.$$

Definition 2.1. *We say that the pair* $(\underline{u}_1, \overline{u}_1)$, $(\underline{u}_2, \overline{u}_2)$, *with* $\underline{u}_i, \overline{u}_i \in H^1(\Omega) \cap L^\infty(\Omega)$, *is a pair of sub-supersolution of (2.1) if*

1. $\underline{u}_i \leq \overline{u}_i$ *in* Ω *and* $\underline{u}_i \leq 0 \leq \overline{u}_i$ *on* $\partial\Omega$ *for* $i = 1, 2$,

2. $$-\Delta\underline{u}_1 - g_1(x, \underline{u}_1, v, B_1(u), B_2(v), C_1(u, v)) \leq 0 \leq -\Delta\overline{u}_1 - g_1(x, \overline{u}_1, v, B_1(u), B_2(v), C_1(u, v))$$

   in the weak sense for all $(u, v) \in [\underline{u}_1, \overline{u}_1] \times [\underline{u}_2, \overline{u}_2]$.

3.

$$-\Delta\underline{u}_2-g_2(x,u,\underline{u}_2,B_1(u),B_2(v),C_2(u,v))\leq 0\leq -\Delta\overline{u}_2-g_2(x,u,\overline{u}_2,B_1(u),B_2(v),C_2(u,v))$$

in the weak sense for all $(u,v)\in[\underline{u}_1,\overline{u}_1]\times[\underline{u}_2,\overline{u}_2]$.

The main result in this section is:

Theorem 2.2. *Assume that there exists a pair of sub-supersolution of (2.1) in the sense of Definition 2.1. Then, there exists a solution* $(u_1,u_2)\in(H_0^1(\Omega)\cap L^\infty(\Omega))^2$ *of (2.1) such that* $u_i\in[\underline{u}_i,\overline{u}_i]$, $i=1,2$.

Proof. For $i=1,2$, define the truncation operators

$$T_iu(x):=\begin{cases}\overline{u}_i(x) & \text{if } u(x)\geq\overline{u}_i(x),\\ u(x) & \text{if } \underline{u}_i(x)\leq u(x)\leq\overline{u}_i(x),\\ \underline{u}_i(x) & \text{if } u(x)\leq\underline{u}_i(x),\end{cases}\tag{2.2}$$

and the Nemytskii operators $F_i:(L^\infty(\Omega))^2\mapsto L^\infty(\Omega)$ given by

$$F_i(u_1,u_2)(x):=g_i(x,T_1(u_1)(x),T_2(u_2)(x),B_1(T_1(u_1)),B_2(T_2(u_2)),C_i(T_1(u_1),T_2(u_2))).$$

It is clear that F_i is continuous and bounded, because there exists $M>0$ such that

$$\|F_i(u_1,u_2)\|_\infty\leq M\quad\text{for all } u_1,u_2\in L^\infty(\Omega).$$

Consider the problem

$$\begin{cases}-\Delta w_1=F_1(u_1,u_2) & \text{in } \Omega,\\ -\Delta w_2=F_2(u_1,u_2) & \text{in } \Omega,\\ w_1=w_2=0 & \text{on } \partial\Omega.\end{cases}\tag{2.3}$$

We can define the operator $\mathcal{T}$ by $(u_1,u_2)\mapsto(w_1,w_2):=\mathcal{T}(u_1,u_2)$ being (w_1,w_2) the unique solution of (2.3). It is clear that $\mathcal{T}$ is well-defined, it is a compact operator and $\mathcal{T}(B_M)\subset B_M$ for some $M>0$, where B_M denotes the ball in $(L^\infty(\Omega))^2$ centered in $(0,0)$ and radius M. Hence, by the Schauder Fixed Point Theorem there exists $(u_1,u_2)\in(L^\infty(\Omega))^2$ such that $(u_1,u_2)=\mathcal{T}(u_1,u_2)$, and then

$$\begin{cases}-\Delta u_1=F_1(u_1,u_2) & \text{in } \Omega,\\ -\Delta u_2=F_2(u_1,u_2) & \text{in } \Omega,\\ u_1=u_2=0 & \text{on } \partial\Omega.\end{cases}\tag{2.4}$$

Now, we show that $u_i\in[\underline{u}_i,\overline{u}_i]$, which implies that (u_1,u_2) is solution of (2.1). Let us show that

$$u_1\leq\overline{u}_1\quad\text{in } \Omega,$$

the other inequalities can be proved similarly. Indeed, in the definition of supersolution of $\overline{u}_1$ we can take $u=T_1(u_1)$, $v=T_2(u_2)$ and then,

$$-\Delta\overline{u}_1\geq g_1(x,\overline{u}_1,T_2(u_2),B_1(T_1(u_1)),B_2(T_2(u_2)),C_1(T_1(u_1),T_2(u_2))),$$

and so, denoting $z := \overline{u}_1 - u_1$ we get

$$\begin{aligned} -\Delta z &\geq g_1(x, \overline{u}_1, T_2(u_2), B_1(T_1(u_1)), B_2(T_2(u_2)), C_1(T_1(u_1), T_2(u_2))) - F(u_1, u_1) \\ &= g_1(x, \overline{u}_1, T_2(u_2), B_1(T_1(u_1)), B_2(T_2(u_2)), C_1(T_1(u_1), T_2(u_2))) \\ &\quad - g_1(x, T_1(u_1)(x), T_2(u_2)(x), B_1(T_1(u_1)), B_2(T_2(u_2)), C_1(T_1(u_1), T_2(u_2))). \end{aligned}$$

Now, multiplying by $(\overline{u}_1 - u_1)^-$ we obtain

$$\int_\Omega |\nabla(\overline{u}_1 - u_1)^-|^2 \leq 0,$$

whence we conclude the result. □

3 The sub-supersolution for Kirchhoff systems

First, we are going to transform (1.1) into a nonlocal system as (2.1). Indeed, define

$$N_i(t) := M_i(t)t$$

and assume that N_i is invertible, and so define

$$G_i(t) = N_i^{-1}(t).$$

Finally, define the non-local operators $\mathcal{R}_i : (L^\infty(\Omega))^2 \mapsto \mathbb{R}$ by

$$\mathcal{R}_i(u_1, u_2) = M_i\left(G_i\left(\int_\Omega f_i(x, u_1, u_2)u_i\right)\right).$$

Lemma 3.1. *Assume that*

(*N*) $\quad N_i,\ i = 1, 2$ *are invertible.*

Then, (1.1) is equivalent to

$$\begin{cases} -\Delta u_1 = F_1(x, u_1, u_2, C_1(u_1, u_2)) & \text{in } \Omega, \\ -\Delta u_2 = F_2(x, u_1, u_2, C_2(u_1, u_2)) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where

$$C_i(u_1, u_2) = \mathcal{R}_i(u_1, u_2), \quad F_i(x, t_1, t_2, r) = \frac{f_i(x, t_1, t_2)}{r},\ i = 1, 2.$$

Proof. Assume that (u_1, u_2) is solution of (1.1). Multiplying (1.1) by u_i and integrating, we get

$$M_i(\|u_i\|^2)\|u_i\|^2 = \int_\Omega f_i(x, u_1, u_2)u_i,$$

and then,

$$\|u_i\|^2 = G_i\left(\int_\Omega f_i(x, u_1, u_2)u_i\right) \Longrightarrow M_i(\|u_i\|^2) = \mathcal{R}_i(u_1, u_2).$$

By (M), $\mathcal{R}_i(u_1,u_2)\geq m_0$ and then we can divide by $\mathcal{R}_i(u_1,u_2)$. Hence, we conclude that (u_1,u_2) is solution of (3.1).

Reciprocally, if (u_1,u_2) is solution of (3.1), then multiplying by u_i we obtain

$$\|u_i\|^2=\frac{\int_\Omega f_i(x,u_1,u_2)u_i}{\mathcal{R}_i(u_1,u_2)}=\frac{\int_\Omega f_i(x,u_1,u_2)u_i}{M_i(G_i(\int_\Omega f_i(x,u_1,u_2)u_i)}=G_i\left(\int_\Omega f_i(x,u_1,u_2)u_i\right),$$

where we have used that $N_i\circ G_i(t)=t$, that is $M_i(G_i(t))G_i(t)=t$. Applying M_i in that above equality we get

$$M_i(\|u_i\|^2)=\mathcal{R}_i(u_1,u_2),$$

and so (u_1,u_2) is solution of (1.1). This completes the proof. □

As consequence of this result and Theorem 2.2, we have the following results.

Definition 3.2. *We say that the pair $(\underline{u}_1,\overline{u}_1)$, $(\underline{u}_2,\overline{u}_2)$, with $\underline{u}_i,\overline{u}_i\in H^1(\Omega)\cap L^\infty(\Omega)$, is a pair of sub-supersolution of (1.1) if*

1. $\underline{u}_i\leq\overline{u}_i$ *in* Ω *and* $\underline{u}_i\leq 0\leq\overline{u}_i$ *on* $\partial\Omega$ *for* $i=1,2$,

2. $$-\mathcal{R}_1(u,v)\Delta\underline{u}_1-f_1(x,\underline{u}_1,v)\leq 0\leq -\mathcal{R}_1(u,v)\Delta\overline{u}_1-f_1(x,\overline{u}_1,v)$$
   in the weak sense for all $(u,v)\in[\underline{u}_1,\overline{u}_1]\times[\underline{u}_2,\overline{u}_2]$.

3. $$-\mathcal{R}_2(u,v)\Delta\underline{u}_2-f_2(x,u,\underline{u}_2)\leq 0\leq -\mathcal{R}_2(u,v)\Delta\overline{u}_2-f_2(x,u,\overline{u}_2)$$
   in the weak sense for all $(u,v)\in[\underline{u}_1,\overline{u}_1]\times[\underline{u}_2,\overline{u}_2]$.

Theorem 3.3. *Assume (M) and (N). If there exists a pair of sub-supersolution of (3.1) in the sense of Definition 3.2, then there exists a solution (u_1,u_2) of (1.1) such that $(u_1,u_2)\in[\underline{u}_1,\overline{u}_1]\times[\underline{u}_2,\overline{u}_2]$.*

Remark 3.4. *Observe that if M_i is increasing, then it verifies (N).*

4 Applications

4.1 Non-local Lotka-Volterra models

Consider the classical diffusive Lotka-Volterra model with non-local interaction

$$\begin{cases} -\Delta u_1=u_1(\lambda-u_1-b\int_\Omega u_2) & \text{in } \Omega,\\ -\Delta u_2=u_2(\mu-u_2-c\int_\Omega u_1) & \text{in } \Omega,\\ u_1=u_2=0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\lambda,\mu\in\mathbb{R}$ and $b,c\in\mathbb{R}$. Here, u_1 and u_2 denote two species inhabiting in Ω, the habitat, which is surrounded by inhospitable areas. Here, λ and μ represent the intrinsic growth rates of each species, and b, c the interaction rates between the species: if both b and c are positive numbers the species compete, if both are negative they cooperate and finally in the case $b>0$ and $c<0$, u_1 denotes the prey and u_2 the predator. The main novelty in (4.1) is that this interaction is non-local, that is, the interaction between both species at the point $x \in \Omega$ not only depends on the value at x but the value to the entire domain Ω, see [7].

In order to enunciate the main result, we need introduce some notation. Denote by $\varphi > 0$ the eigenfunction associated to λ_1, the principal eigenvalue of the $-\Delta$ under Dirichlet boundary conditions, such that $\|\varphi\|_\infty = 1$. It is well-known that the classical logistic equation

$$\begin{cases} -\Delta w = w(\gamma - w) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

possesses a unique positive solution if and only if $\gamma > \lambda_1$. In such case, the positive solution is unique. We denote it by θ_γ. We prolong the definition of $\theta_\gamma \equiv 0$ when $\gamma \le \lambda_1$. It is well-known that $\gamma \mapsto \theta_\gamma$ is increasing in γ and that $\theta_\gamma \le \gamma$.

Theorem 4.1. *1. Assume that $b, c > 0$. Then, (4.1) possesses at least a positive solution if*

$$\lambda - b\int_\Omega \theta_\mu > \lambda_1 \quad \text{and} \quad \mu - c\int_\Omega \theta_\lambda > \lambda_1. \tag{4.3}$$

2. Assume that $b, c < 0$ and $bc|\Omega|^2 < 1$. Then, (4.1) possesses at least a positive solution if (λ, μ) verifies condition (4.3).

3. Assume $b > 0$, $c < 0$ and

$$\lambda - b|\Omega|(\mu + c\int_\Omega \theta_\lambda) > \lambda_1 \quad \text{and} \quad \mu > \lambda_1. \tag{4.4}$$

Proof. 1. We can take as pair of sub-supersolution

$$(\underline{u}_1, \overline{u}_1) = (\theta_{\lambda - b\int_\Omega \theta_\mu}, \theta_\lambda), \quad (\underline{u}_2, \overline{u}_2) = (\theta_{\mu - c\int_\Omega \theta_\lambda}, \theta_\mu).$$

First, observe that $\underline{u}_1 \le \overline{u}_1$ and $\underline{u}_2 \le \overline{u}_2$ in Ω. Now, we have to verify four inequalities. Let us only check two of them:

$$-\Delta \underline{u}_1 \le \underline{u}_1(\lambda - \underline{u}_1 - b\int_\Omega \overline{u}_2), \quad -\Delta \overline{u}_1 \ge \overline{u}_1(\lambda - \overline{u}_1 - b\int_\Omega \underline{u}_2).$$

Observe that

$$-\Delta \underline{u}_1 = -\Delta \theta_{\lambda - b\int_\Omega \theta_\mu} = \theta_{\lambda - b\int_\Omega \theta_\mu}(\lambda - b\int_\Omega \theta_\mu - \theta_{\lambda - b\int_\Omega \theta_\mu}) = \underline{u}_1(\lambda - \underline{u}_1 - b\int_\Omega \overline{u}_2).$$

On the other hand,

$$-\Delta \overline{u}_1 = -\Delta \theta_\lambda = \theta_\lambda(\lambda - \theta_\lambda) \ge \theta_\lambda(\lambda - \theta_\lambda - b\int_\Omega \underline{u}_2) = \overline{u}_1(\lambda - \overline{u}_1 - b\int_\Omega \underline{u}_2).$$

This completes the first paragraph.

2. In this case, take

$$(\underline{u}_1, \overline{u}_1) = (\theta_{\lambda - b\int_\Omega \theta_\mu}, M), \quad (\underline{u}_2, \overline{u}_2) = (\theta_{\mu - c\int_\Omega \theta_\lambda}, N),$$

where M, N are positive constants verifying

$$M \geq \lambda - bN|\Omega| \quad \text{and} \quad N \geq \mu - cM|\Omega|,$$

which exist because $bc|\Omega|^2 < 1$.

We prove now that they are sub-supersolutions. Again we only show two inequalities:

$$-\Delta \underline{u}_1 \leq \underline{u}_1(\lambda - \underline{u}_1 - b\int_\Omega \underline{u}_2), \quad -\Delta \overline{u}_1 \geq \overline{u}_1(\lambda - \overline{u}_1 - b\int_\Omega \overline{u}_2).$$

The first inequality is equivalent to

$$\theta_\mu \leq \theta_{\mu - c\int_\Omega \theta_\lambda},$$

and the second one to

$$0 \geq \lambda - M - bN|\Omega|.$$

Taking M and N large we get both inequalities and $\underline{u}_1 \leq \overline{u}_1$ and $\underline{u}_2 \leq \overline{u}_2$.

3. Take in this case

$$(\underline{u}_1, \overline{u}_1) = (\varepsilon\varphi, \theta_\lambda), \quad (\underline{u}_2, \overline{u}_2) = (\theta_\mu, N),$$

with $\varepsilon, N > 0$ to choose. Observe that N has to verify that $N \geq \mu - c\int_\Omega \overline{u}_1$, and so, we can take

$$N = \mu - c\int_\Omega \theta_\lambda.$$

It is clear that $\overline{u}_1$ and $\underline{u}_2$ verify the inequalities. Finally, we consider $\underline{u}_1$. It has to verify that

$$\lambda_1 \leq \lambda - \varepsilon\varphi - bN|\Omega|,$$

so, if $\lambda - bN|\Omega| > \lambda_1$ we can take ε small enough that the above inequality holds and $\underline{u}_1 \leq \overline{u}_1$. Finally, observe that since $\theta_\mu \leq \mu < N$ we get that $\underline{u}_2 \leq \overline{u}_2$.

□

4.2 Kirchhoff systems

Along this section, we assume that M_i verifies (M) and (N). We present different applications of Theorem 3.3. First, we study a system with concave nonlinearities

$$\begin{cases} -M_1(\|u_1\|^2)\Delta u_1 = \lambda u_1^{q_1} + u_2^{q_2} & \text{in } \Omega, \\ -M_2(\|u_2\|^2)\Delta u_2 = \mu u_2^{p_2} + u_1^{p_1} & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.5}$$

where $\lambda, \mu \in \mathbb{R}$ and $0 < q_i, p_i < 1$.

Theorem 4.2. *Assume that $\lambda, \mu > 0$. Then, there exists a positive solution of (4.5).*

Proof. We are going to build again a pair of sub-supersolution. Denote also by e the unique positive solution of

$$\begin{cases} -\Delta e = 1 & \text{in } \Omega, \\ e = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

We show that $(\underline{u}_1, \overline{u}_1) = (\varepsilon_1\varphi, K_1 e)$ and $(\underline{u}_2, \overline{u}_2) = (\varepsilon_2\varphi, K_2 e)$ is a pair of sub-supersolution of (4.5) taking the positive constants $\varepsilon_1, \varepsilon_2, K_1$ and K_2 in an appropriate way. We start with $\overline{u}_1$. We need to verify that

$$-\mathcal{R}_1(u,v)\Delta\overline{u}_1 \geq \lambda\overline{u}_1^{q_1} + \overline{u}_2^{q_2}, \quad \forall (u,v) \in [\underline{u}_1, \overline{u}_1] \times [\underline{u}_2, \overline{u}_2].$$

Using (M), it suffices to show that

$$K_1 m_0 \geq \lambda K_1^{q_1}\|e\|_\infty^{q_1} + K_2^{q_2}\|e\|_\infty^{q_2}.$$

Similarly for $\overline{u}_2$,

$$K_2 m_0 \geq \mu K_2^{p_2}\|e\|_\infty^{p_2} + K_1^{p_1}\|e\|_\infty^{p_1}.$$

Fix, K_1 and K_2 verifying above inequalities. Now, we study $\underline{u}_1$ and $\underline{u}_2$. They have to verify

$$\mathcal{R}_1(u,v)\lambda_1\varepsilon_1\varphi \leq \lambda(\varepsilon_1\varphi)^{q_1} + (\varepsilon_2\varphi)^{q_2}, \quad \forall (u,v) \in [\underline{u}_1, \overline{u}_1] \times [\underline{u}_2, \overline{u}_2],$$
$$\mathcal{R}_2(u,v)\lambda_1\varepsilon_2\varphi \leq \mu(\varepsilon_2\varphi)^{p_2} + (\varepsilon_1\varphi)^{p_1}, \quad \forall (u,v) \in [\underline{u}_1, \overline{u}_1] \times [\underline{u}_2, \overline{u}_2].$$

Since $\mathcal{R}_i$ is bounded in $[0, \overline{u}_1] \times [0, \overline{u}_2]$, it is clear that we can take ε_1 and ε_2 small enough, and we conclude the result. □

Finally, we consider the competition Kirchhoff model with local nonlinearities

$$\begin{cases} -M_1(\|u_1\|^2)\Delta u_1 = u_1(\lambda - u_1 - bu_2) & \text{in } \Omega, \\ -M_2(\|u_2\|^2)\Delta u_2 = u_2(\mu - u_2 - cu_1) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.7}$$

where $\lambda, \mu \in \mathbb{R}$ and $0 < b, c$. The meaning of the parameters were given at the beginning of this Section.

Theorem 4.3. *Assume that there exist positive constants m_i^∞, $i = 1, 2$, such that $M_i \leq m_i^\infty$, and*

$$\lambda > b\mu + \lambda_1 m_1^\infty \quad \text{and} \quad \mu > c\lambda + \lambda_1 m_2^\infty.$$

Then, there exists a positive solution of (4.7).

Proof. We show that

$$(\underline{u}_1, \overline{u}_1) = (\varepsilon_1\varphi, M_1) \quad \text{and} \quad (\underline{u}_2, \overline{u}_2) = (\varepsilon_2\varphi, M_2)$$

is a pair of sub-supersolution of (4.7) taking positive constants $\varepsilon_1, \varepsilon_2$ small $M_1 = \lambda$, $M_2 = \mu$. Indeed, $\overline{u}_1$ is supersolution if

$$0 \geq \lambda - M_1 - b\varepsilon_2\varphi,$$

which is true for $M_1 = \lambda$.

Consider now $\underline{u}_1$. The function $\underline{u}_1 = \varepsilon_1\varphi$ is subsolution provided of

$$\mathcal{R}_1(u,v)\lambda_1\varepsilon_1\varphi \leq (\varepsilon_1\varphi)(\lambda - \varepsilon_1\varphi - b\mu), \quad \forall (u,v) \in [\underline{u}_1, \overline{u}_1] \times [\underline{u}_2, \overline{u}_2],$$

for which it suffices $\lambda > b\mu + \lambda_1 m_1^\infty$. Analogously for $\underline{u}_2$ and $\overline{u}_2$. □

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