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Research Article

Generalized Asymptotic Pointwise Contractions and Nonexpansive Mappings Involving Orbits

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We give fixed point results for classes of mappings that generalize pointwise contractions, asymptotic contractions, asymptotic pointwise contractions, and nonexpansive and asymptotic nonexpansive mappings. We consider the case of metric spaces and, in particular, CAT(0) spaces. We also study the well-posedness of these fixed point problems.

1. Introduction

Four recent papers [1–4] present simple and elegant proofs of fixed point results for pointwise contractions, asymptotic pointwise contractions, and asymptotic nonexpansive mappings. Kirk and Xu [1] study these mappings in the context of weakly compact convex subsets of Banach spaces, respectively, in uniformly convex Banach spaces. Hussain and Khamsi [2] consider these problems in the framework of metric spaces and CAT(0) spaces. In [3], the authors prove coincidence results for asymptotic pointwise nonexpansive mappings. Espínola et al. [4] examine the existence of fixed points and convergence of iterates for asymptotic pointwise contractions in uniformly convex metric spaces.

In this paper we do not consider more general spaces, but instead we formulate less restrictive conditions for the mappings and show that the conclusions of the theorems still stand even in such weaker settings.

2. Preliminaries

Let (X, d) be a metric space. For $z \in X$ and $r > 0$ we denote the closed ball centered at z with radius r by $\widetilde{B}(z, r) := \{x \in X : d(x, z) \le r\}$.

Let $K \subseteq X$ and let $T : K \to K$. Throughout this paper we will denote the fixed point set of T by $\mathrm{Fix}(T)$. The mapping T is called a Picard operator if it has a unique fixed point z and $(T^n(x))_{n\in\mathbb{N}}$ converges to z for each $x \in K$.

A sequence $(x_n)_{n\in\mathbb{N}} \subseteq K$ is said to be an approximate fixed point sequence for the mapping T if $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$.

The fixed point problem for T is well-posed (see [5, 6]) if T has a unique fixed point and every approximate fixed point sequence converges to the unique fixed point of T.

A mapping $T : X \to X$ is called a pointwise contraction if there exists a function $\alpha : X \to [0,1)$ such that

$$d(T(x), T(y)) \leq \alpha(x) d(x,y) \quad \text{for every } x, y \in X. \tag{2.1}$$

Let $T : X \to X$ and for $n \in \mathbb{N}$ let $\alpha_n : X \to \mathbb{R}_+$ such that

$$d(T^n(x), T^n(y)) \leq \alpha_n(x) d(x,y) \quad \text{for every } x, y \in X. \tag{2.2}$$

If the sequence $(\alpha_n)_{n\in\mathbb{N}}$ converges pointwise to the function $\alpha : X \to [0,1)$, then T is called an asymptotic pointwise contraction.

If for every $x \in X$, $\limsup_{n\to\infty} \alpha_n(x) \leq 1$, then T is called an asymptotic pointwise nonexpansive mapping.

If there exists $0 < k < 1$ such that for every $x \in X$, $\limsup_{n\to\infty} \alpha_n(x) \leq k$, then T is called a strongly asymptotic pointwise contraction.

For a mapping $T : X \to X$ and $x \in X$ we define the orbit starting at x by

$$O_T(x) = \left\{x, T(x), T^2(x), \ldots, T^n(x), \ldots\right\}, \tag{2.3}$$

where $T^{n+1}(x) = T(T^n(x))$ for $n \geq 0$ and $T^0(x) = x$. Denote also $O_T(x,y) = O_T(x) \cup O_T(y)$.

Given $D \subseteq X$ and $x \in X$, the number $r_x(D) = \sup_{y\in D} d(x,y)$ is called the radius of D relative to x. The diameter of D is $\mathrm{diam}(D) = \sup_{x,y\in D} d(x,y)$ and the cover of D is defined as $\mathrm{cov}(D) = \bigcap\{B : B \text{ is a closed ball and } D \subseteq B\}$.

As in [2], we say that a family $\mathcal{F}$ of subsets of X defines a convexity structure on X if it contains the closed balls and is stable by intersection. A subset of X is admissible if it is a nonempty intersection of closed balls. The class of admissible subsets of X denoted by $\mathcal{A}(X)$ defines a convexity structure on X. A convexity structure $\mathcal{F}$ is called compact if any family $(A_\alpha)_{\alpha\in\Gamma}$ of elements of $\mathcal{F}$ has nonempty intersection provided $\bigcap_{\alpha\in F} A_\alpha \neq \emptyset$ for any finite subset $F \subseteq \Gamma$.

According to [2], for a convexity structure $\mathcal{F}$, a function $\varphi : X \to \mathbb{R}_+$ is called $\mathcal{F}$-convex if $\{x : \varphi(x) \leq r\} \in \mathcal{F}$ for any $r \geq 0$. A type is defined as $\varphi : X \to \mathbb{R}_+$, $\varphi(u) = \limsup_{n\to\infty} d(u, x_n)$ where $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in X. A convexity structure $\mathcal{F}$ is T-stable if all types are $\mathcal{F}$-convex.

The following lemma is mentioned in [2].

Lemma 2.1. *Let X be a metric space and $\mathcal{F}$ a compact convexity structure on X which is T-stable. Then for any type φ there is $x_0 \in X$ such that*

$$\varphi(x_0) = \inf_{x\in X} \varphi(x). \tag{2.4}$$

A metric space (X, d) is a geodesic space if every two points $x, y \in X$ can be joined by a geodesic. A geodesic from x to y is a mapping $c : [0, l] \to X$, where $[0, l] \subseteq \mathbb{R}$, such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for every $t, t' \in [0, l]$. The image $c([0, l])$ of c forms a geodesic segment which joins x and y. A geodesic triangle $\Delta(x_1, x_2, x_3)$ consists of three points x_1, x_2, and x_3 in X (the vertices of the triangle) and three geodesic segments corresponding to each pair of points (the edges of the triangle). For the geodesic traingle $\Delta = \Delta(x_1, x_2, x_3)$, a comparison triangle is the triangle $\overline{\Delta} = \Delta(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean space $\mathbb{E}^2$ such that $d(x_i, x_j) = d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic triangle Δ satisfies the CAT(0) inequality if for every comparison triangle $\overline{\Delta}$ of Δ and for every $x, y \in \Delta$ we have

$$d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}), \tag{2.5}$$

where $\overline{x}, \overline{y} \in \overline{\Delta}$ are the comparison points of x and y. A geodesic metric space is a CAT(0) space if every geodesic traingle satisfies the CAT(0) inequality. In a similar way we can define CAT(k) spaces for $k > 0$ or $k < 0$ using the model spaces M_k^2.

A geodesic space is a CAT(0) space if and only if it satisfies the following inequality known as the (CN) inequality of Bruhat and Tits [7]. Let x, y_1, y_2 be points of a CAT(0) space and let m be the midpoint of $[y_1, y_2]$. Then

$$d(x, m)^2 \leq \frac{1}{2} d(x, y_1)^2 + \frac{1}{2} d(x, y_2)^2 - \frac{1}{4} d(y_1, y_2)^2. \tag{2.6}$$

It is also known (see [8]) that in a complete CAT(0) space, respectively, in a closed convex subset of a complete CAT(0) space every type attains its infimum at a single point. For more details about CAT(k) spaces one can consult, for instance, the papers [9, 10].

In [2], the authors prove the following fixed point theorems.

Theorem 2.2. *Let X be a bounded metric space. Assume that the convexity structure $\mathcal{A}(X)$ is compact. Let $T : X \to X$ be a pointwise contraction. Then T is a Picard operator.*

Theorem 2.3. *Let X be a bounded metric space. Assume that the convexity structure $\mathcal{A}(X)$ is compact. Let $T : X \to X$ be a strongly asymptotic pointwise contraction. Then T is a Picard operator.*

Theorem 2.4. *Let X be a bounded metric space. Assume that there exists a convexity structure $\mathcal{F}$ that is compact and T-stable. Let $T : X \to X$ be an asymptotic pointwise contraction. Then T is a Picard operator.*

Theorem 2.5. *Let X be a complete* CAT(0) *space and let K be a nonempty, bounded, closed and convex subset of X. Then any mapping $T : K \to K$ that is asymptotic pointwise nonexpansive has a fixed point. Moreover, Fix(T) is closed and convex.*

The purpose of this paper is to present fixed point theorems for mappings that satisfy more general conditions than the ones which appear in the classical definitions of pointwise contractions, asymptotic contractions, asymptotic pointwise contractions and asymptotic nonexpansive mappings. Besides this, we show that the fixed point problems are well-posed. Some generalizations of nonexpansive mappings are also considered. We work in the context of metric spaces and CAT(0) spaces.

3. Generalizations Using the Radius of the Orbit

In the sequel we extend the results obtained by Hussain and Khamsi [2] using the radius of the orbit. We also study the well-posedness of the fixed point problem. We start by introducing a property for a mapping $T : X \to X$, where X is a metric space. Namely, we will say that T satisfies property (S) if

(S) for every approximate fixed point sequence $(x_n)_{n\in\mathbb{N}}$ and for every $m \in \mathbb{N}$, the sequence $(\mathrm{d}(x_n, T^m(x_n)))_{n\in\mathbb{N}}$ converges to 0 uniformly with respect to m.

For instance, if for every $x \in X$, $d(x, T^2(x)) \leq d(x, T(x))$ then property (S) is fulfilled.

Proposition 3.1. *Let X be a metric space and let $T : X \to X$ be a mapping which satisfies (S). If $(x_n)_{n\in\mathbb{N}}$ is an approximate fixed point sequence, then for every $m \in \mathbb{N}$ and every $x \in X$,*

$$\limsup_{n\to\infty} d(x, T^m(x_n)) = \limsup_{n\to\infty} d(x, x_n), \tag{3.1}$$

$$\limsup_{n\to\infty} r_x(O_T(x_n)) = \limsup_{n\to\infty} d(x, x_n), \tag{3.2}$$

$$\lim_{n\to\infty} \operatorname{diam} \mathrm{O_T(x_n)} = 0. \tag{3.3}$$

Proof. Since T satisfies (S) and $(x_n)_{n\in\mathbb{N}}$ is an approximate fixed point sequence, it easily follows that (3.1) holds. To prove (3.2), let $\epsilon > 0$. Then there exists $m \in \mathbb{N}$ such that

$$r_x(O_T(x_n)) \leq d(x, T^m(x_n)) + \epsilon \leq d(x, x_n) + d(x_n, T^m(x_n)) + \epsilon. \tag{3.4}$$

Taking the superior limit,

$$\limsup_{n\to\infty} r_x(O_T(x_n)) \leq \limsup_{n\to\infty} d(x, x_n) + \epsilon. \tag{3.5}$$

Hence, (3.2) holds. Now let again $\epsilon > 0$. Then there exist $m_1, m_2 \in \mathbb{N}$ such that

$$\operatorname{diam} O_T(x_n) \leq d(T^{m_1}(x_n), T^{m_2}(x_n)) + \epsilon \leq d(x_n, T^{m_1}(x_n)) + d(x_n, T^{m_2}(x_n)) + \epsilon. \tag{3.6}$$

We only need to let $n \to \infty$ in the above relation to prove (3.3). □

Theorem 3.2. *Let X be a bounded metric space such that $\mathcal{A}(X)$ is compact. Also let $T : X \to X$ for which there exists $\alpha : X \to [0, 1)$ such that*

$$d(T(x), T(y)) \leq \alpha(x) r_x(O_T(y)) \quad \text{for every } x, y \in X. \tag{3.7}$$

Then T is a Picard operator. Moreover, if additionally T satisfies (S), then the fixed point problem is well-posed.

Proof. Because $\mathcal{A}(X)$ is compact, there exists a nonempty minimal T-invariant $K \in \mathcal{A}(X)$ for which $\operatorname{cov}(T(K)) = K$. If $x, y \in K$ then $r_x(O_T(y)) \leq r_x(K)$. In a similar way as in the proof of Theorem 3.1 of [2] we show now that T has a fixed point. Let $x \in K$. Then,

$$d(T(x), T(y)) \leq \alpha(x) r_x(O_T(y)) \leq \alpha(x) r_x(K) \quad \text{for every } y \in X. \tag{3.8}$$

This means that $T(K) \subseteq \widetilde{B}(T(x), \alpha(x) r_x(K))$, so $K = \operatorname{cov}(T(K)) \subseteq \widetilde{B}(T(x), \alpha(x) r_x(K))$. Therefore,

$$r_{T(x)}(K) \leq \alpha(x) r_x(K). \tag{3.9}$$

Denote

$$K_x = \{ y \in K : r_y(K) \leq r_x(K) \}. \tag{3.10}$$

$K_x \in \mathcal{A}(X)$ since it is nonempty and $K_x = \bigcap_{y \in K} \widetilde{B}(y, r_x(K)) \cap K$.

Let $y \in K_x$. As above we have $K \subseteq \widetilde{B}(T(y), \alpha(y) r_y(K)) \subseteq \widetilde{B}(T(y), \alpha(y) r_x(K))$ and hence $T(y) \in K_x$. Because K is minimal T-invariant it follows that $K_x = K$. This yields $r_y(K) = r_x(K)$ for every $x, y \in K$. In particular, $r_{T(x)}(K) = r_x(K)$ and using (3.9) we obtain $r_x(K) = 0$ which implies that K consists of exactly one point which will be fixed under T.

Now suppose $x, y \in X$, $x \neq y$ are fixed points of T. Then

$$d(x, y) \leq \alpha(x) r_x(O_T(y)) = \alpha(x) d(x, y). \tag{3.11}$$

This means that $\alpha(x) \geq 1$ which is impossible.

Let z denote the unique fixed point of T, let $x \in X$ and $l_x = \limsup_{n\to\infty} d(z, T^n(x))$. Observe that the sequence $(r_z(O_T(T^n(x))))_{n\in\mathbb{N}}$ is decreasing and bounded below by 0 so its limit exists and is precisely l_x. Then

$$l_x \leq \alpha(z) \lim_{n\to\infty} r_z\left(O_T\left(T^{n-1}(x)\right)\right) = \alpha(z) l_x. \tag{3.12}$$

This implies that $l_x = 0$ and hence $\lim_{n\to\infty} T^n(x) = z$.

Next we prove that the problem is well-posed. Let $(x_n)_{n\in\mathbb{N}}$ be an approximate fixed point sequence. We know that

$$d(z, x_n) \leq d(x_n, T(x_n)) + d(T(x_n), T(z)) \leq d(x_n, T(x_n)) + \alpha(z) r_z(O_T(x_n)). \tag{3.13}$$

Taking the superior limit and applying (3.2) of Proposition 3.1 for z,

$$\limsup_{n\to\infty} d(z, x_n) \leq \alpha(z) \limsup_{n\to\infty} d(z, x_n), \tag{3.14}$$

which implies $\lim_{n\to\infty} d(z, x_n) = 0$. □

We remark that if in the above result T is, in particular, a pointwise contraction then the fixed point problem is well-posed without additional assumptions for T.

Next we give an example of a mapping which is not a pointwise contraction, but fulfills (3.7).

Example 3.3. Let $T : [0,1] \to [0,1]$,

$$T(x) = \begin{cases} \dfrac{1-x}{2}, & \text{if } x \geq \dfrac{1}{2}, \\ \dfrac{3}{4}x, & \text{if } x < \dfrac{1}{2}, \end{cases} \tag{3.15}$$

and let $\alpha : [0,1] \to [0,1)$,

$$\alpha(x) = \begin{cases} \dfrac{1}{2}, & \text{if } x \geq \dfrac{1}{2}, \\ \dfrac{3}{4} + x^2, & \text{if } x < \dfrac{1}{2}. \end{cases} \tag{3.16}$$

Then T is not a pointwise contraction, but (3.7) is verified.

Proof. T is not continuous, so it is not nonexpansive and hence it cannot be a pointwise contraction. If $x, y \geq 1/2$ or $x, y < 1/2$ the conclusion is immediate. Suppose $x \geq 1/2$ and $y < 1/2$. Then

$$r_x(O_T(y)) = x, \qquad r_y(O_T(x)) = \max\{x - y, y\}. \tag{3.17}$$

(i) If $T(x) - T(y) \geq 0$, then

$$\begin{aligned} \frac{1-x}{2} - \frac{3}{4}y &\leq \frac{x}{2} = \alpha(x) r_x(O_T(y)), \\ \frac{1-x}{2} - \frac{3}{4}y &\leq \left(\frac{3}{4} + y^2\right)(x - y) \leq \alpha(y) r_y(O_T(x)). \end{aligned} \tag{3.18}$$

The above is true because $1/2 - 5/4x < 0 \leq y^2(x - y)$.

(ii) If $T(x) - T(y) < 0$, then

$$\begin{aligned} \frac{3}{4}y - \frac{1-x}{2} &\leq -\frac{1}{8} + \frac{x}{2} < \frac{x}{2} = \alpha(x) r_x(O_T(y)), \\ \frac{3}{4}y - \frac{1-x}{2} &\leq \left(\frac{3}{4} + y^2\right) y \leq \alpha(y) r_y(O_T(x)). \end{aligned} \tag{3.19}$$

□

Theorem 3.4. *Let X be a bounded metric space, $T : X \to X$, and suppose there exists a convexity structure $\mathcal{F}$ which is compact and T-stable. Assume*

$$d(T^n(x), T^n(y)) \le \alpha_n(x) r_x(O_T(y)) \quad \text{for every } x, y \in X, \tag{3.20}$$

where for each $n \in \mathbb{N}$, $\alpha_n : X \to \mathbb{R}_+$, and the sequence $(\alpha_n)_{n\in\mathbb{N}}$ converges pointwise to a function $\alpha : X \to [0,1)$. Then T is a Picard operator. Moreover, if additionally T satisfies (S), then the fixed point problem is well-posed.

Proof. Assume T has two fixed points $x, y \in X$, $x \ne y$. Then for each $n \in \mathbb{N}$,

$$d(x,y) \le \alpha_n(x) d(x,y). \tag{3.21}$$

When $n \to \infty$ we obtain $\alpha(x) \ge 1$ which is false. Hence, T has at most one fixed point.

Let $x \in X$. We consider $\varphi : X \to \mathbb{R}_+$,

$$\varphi(u) = \limsup_{n\to\infty} d(u, T^n(x)) \quad \text{for } u \in X. \tag{3.22}$$

Because $\mathcal{F}$ is compact and T-stable there exists $z \in X$ such that

$$\varphi(z) = \inf_{u\in X} \varphi(u). \tag{3.23}$$

For $p \in \mathbb{N}$,

$$\varphi(z) \le \varphi(T^p(z)) \le \alpha_p(z) \lim_{n\to\infty} r_z(O_T(T^n(x))) = \alpha_p(z)\varphi(z). \tag{3.24}$$

Letting $p \to \infty$ in the above relation yields $\varphi(z) = 0$ so $(T^n(x))_{n\in\mathbb{N}}$ converges to z which will be the unique fixed point of T because $d(T(z), T^{n+1}(x)) \le \alpha_1(z) r_z(O_T(T^n(x)))$ and $\lim_{n\to\infty} r_z(O_T(T^n(x))) = 0$. Thus, all the Picard iterates will converge to z.

Let $(x_n)_{n\in\mathbb{N}}$ be an approximate fixed point sequence and let $m \in \mathbb{N}$. Then

$$d(z, x_n) \le d(x_n, T^m(x_n)) + d(T^m(x_n), T^m(z)) \le d(x_n, T^m(x_n)) + \alpha_m(z) r_z(O_T(x_n)). \tag{3.25}$$

Taking the superior limit and applying (3.2) of Proposition 3.1,

$$\limsup_{n\to\infty} d(z, x_n) \le \alpha_m(z) \limsup_{n\to\infty} d(z, x_n). \tag{3.26}$$

Letting $m \to \infty$ we have $\lim_{n\to\infty} d(z, x_n) = 0$. □

Theorem 3.5. *Let X be a complete* CAT(0) *space and let $K \subseteq X$ be nonempty, bounded, closed, and convex. Let $T : K \to K$ and for $n \in \mathbb{N}$, let $\alpha_n : K \to \mathbb{R}_+$ be such that* $\limsup_{n\to\infty}\alpha_n(x) \leq 1$ *for all $x \in K$. If for all $n \in \mathbb{N}$,*

$$d(T^n(x), T^n(y)) \leq \alpha_n(x) r_x(O_T(y)) \quad \text{for every } x, y \in K, \tag{3.27}$$

then T has a fixed point. Moreover, Fix(T) is closed and convex.

Proof. The idea of the proof follows to a certain extend the proof of Theorem 5.1 in [2]. Let $x \in K$. Denote $\varphi : K \to \mathbb{R}_+$,

$$\varphi(u) = \limsup_{n\to\infty} d(u, T^n(x)) \quad \text{for } u \in K. \tag{3.28}$$

Since K is a nonempty, closed, and convex subset of a complete CAT(0) space there exists a unique $z \in K$ such that

$$\varphi(z) = \inf_{u\in K} \varphi(u). \tag{3.29}$$

For $p \in \mathbb{N}$,

$$\varphi(T^p(z)) \leq \alpha_p(z) \lim_{n\to\infty} r_z(O_T(T^n(x))) = \alpha_p(z)\varphi(z). \tag{3.30}$$

Let $p, q \in \mathbb{N}$ and let m denote the midpoint of the segment $[T^p(z), T^q(z)]$. Using the (CN) inequality, we have

$$d(m, T^n(x))^2 \leq \frac{1}{2} d(T^p(z), T^n(x))^2 + \frac{1}{2} d(T^q(z), T^n(x))^2 - \frac{1}{4} d(T^p(z), T^q(z))^2. \tag{3.31}$$

Letting $n \to \infty$ and considering $\varphi(z) \leq \varphi(m)$, we have

$$\begin{aligned} \varphi(z)^2 &\leq \frac{1}{2}\varphi(T^p(z))^2 + \frac{1}{2}\varphi(T^q(z))^2 - \frac{1}{4} d(T^p(z), T^q(z))^2 \\ &\leq \frac{1}{2}\alpha_p(z)^2\varphi(z)^2 + \frac{1}{2}\alpha_q(z)^2\varphi(z)^2 - \frac{1}{4} d(T^p(z), T^q(z))^2. \end{aligned} \tag{3.32}$$

Letting $p, q \to \infty$ we obtain that $(T^n(z))_{n\in\mathbb{N}}$ is a Cauchy sequence which converges to $\omega \in K$. As in the proof of Theorem 3.4 we can show that ω is a fixed point for T. To prove that $\mathrm{Fix}(T)$ is closed take $(x_n)_{n\in\mathbb{N}}$ a sequence of fixed points which converges to $x^* \in K$. Then

$$d(T(x^*), T(x_n)) \leq \alpha_1(x^*) d(x^*, x_n), \tag{3.33}$$

which shows that x^* is a fixed point of T.

The fact that Fix(T) is convex follows from the (CN) inequality. Let $x, y \in$ Fix(T) and let m be the midpoint of $[x, y]$. For $n \in \mathbb{N}$ we have

$$\begin{aligned} d(m, T^n(m))^2 &\leq \frac{1}{2}d(x, T^n(m))^2 + \frac{1}{2}d(y, T^n(m))^2 - \frac{1}{4}d(x, y)^2 \\ &\leq \frac{1}{2}\alpha_n(m)^2 r_m(O_T(x))^2 + \frac{1}{2}\alpha_n(m)^2 r_m(O_T(y))^2 - \frac{1}{4}d(x, y)^2 \\ &= \frac{1}{2}\alpha_n(m)^2\left(d(m, x)^2 + d(m, y)^2\right) - \frac{1}{4}d(x, y)^2 \\ &= \frac{1}{4}\left(\alpha_n(m)^2 - 1\right)d(x, y)^2. \end{aligned} \tag{3.34}$$

Letting $n \to \infty$ we obtain $\lim_{n\to\infty} T^n(m) = m$. This yields m which is a fixed point since

$$\limsup_{n\to\infty} d\left(T(m), T^{n+1}(m)\right) \leq \alpha_1(m) \limsup_{n\to\infty} d(m, T^n(m)). \tag{3.35}$$

Hence, Fix(T) is convex. □

We conclude this section by proving a demi-closed principle similarly to [2, Proposition 1]. To this end, for $K \subseteq X$, K closed and convex and $\varphi : K \to \mathbb{R}_+$, $\varphi(x) = \limsup_{n\to\infty} d(x, x_n)$, as in [2], we introduce the following notation:

$$x_n \overset{\varphi}{\rightharpoonup} \omega \quad \text{iff } \varphi(\omega) = \inf_{x\in K} \varphi(x), \tag{3.36}$$

where the bounded sequence $(x_n)_{n\in\mathbb{N}}$ is contained in K.

Theorem 3.6. *Let X be a* CAT(0) *space and let $K \subseteq X$, K bounded, closed, and convex. Let $T : K \to K$ satisfy (S) and for $n \in \mathbb{N}$, let $\alpha_n : K \to \mathbb{R}_+$ be such that $\limsup_{n\to\infty} \alpha_n(x) \leq 1$ for all $x \in K$. Suppose that for $n \in \mathbb{N}$,*

$$d(T^n(x), T^n(y)) \leq \alpha_n(x) r_x(O_T(y)) \quad \textit{for every } x, y \in K. \tag{3.37}$$

Let also $(x_n)_{n\in\mathbb{N}} \subseteq K$ be an approximate fixed point sequence such that $x_n \overset{\varphi}{\rightharpoonup} \omega$. Then $\omega \in$ Fix(T).

Proof. Using (3.1) of Proposition 3.1 we obtain that for every $x \in K$ and every $p \in \mathbb{N}$,

$$\varphi(x) = \limsup_{n\to\infty} d(x, T^p(x_n)). \tag{3.38}$$

Applying (3.2) of Proposition 3.1 for ω, we have

$$\varphi(T^p(\omega)) = \limsup_{n\to\infty} d(T^p(\omega), T^p(x_n)) \leq \alpha_p(\omega) \limsup_{n\to\infty} r_\omega(O_T(x_n)) = \alpha_p(\omega)\varphi(\omega). \tag{3.39}$$

Let $p \in \mathbb{N}$ and let m be the midpoint of $[\omega, T^p(\omega)]$. As in the above proof, using the (CN) inequality we have

$$\varphi(m)^2 \leq \frac{1}{2}\varphi(\omega)^2 + \frac{1}{2}\varphi(T^p(\omega))^2 - \frac{1}{4}d(\omega, T^p(\omega))^2. \tag{3.40}$$

Since $\varphi(\omega) \leq \varphi(m)$,

$$\varphi(\omega)^2 \leq \frac{1}{2}\varphi(\omega)^2 + \frac{1}{2}\alpha_p(\omega)^2\varphi(\omega)^2 - \frac{1}{4}d(\omega, T^p(\omega))^2. \tag{3.41}$$

Letting $p \to \infty$, we have $\lim_{p\to\infty} T^p(\omega) = \omega$. This means $\omega \in \mathrm{Fix}(T)$ because

$$\limsup_{p\to\infty} d\Big(T(\omega), T^{p+1}(\omega)\Big) \leq \alpha_1(\omega) \limsup_{p\to\infty} d(\omega, T^p(\omega)). \tag{3.42}$$

□

4. Generalized Strongly Asymptotic Pointwise Contractions

In this section we generalize the strongly asymptotic pointwise contraction condition, by using the diameter of the orbit. We begin with a fixed point result that holds in a complete metric space.

Theorem 4.1. *Let X be a complete metric space and let $T : X \to X$ be a mapping with bounded orbits that is orbitally continuous. Also, for $n \in \mathbb{N}$, let $\alpha_n : X \to \mathbb{R}_+$ for which there exists $0 < k < 1$ such that for every $x \in X$, $\limsup_{n\to\infty} \alpha_n(x) \leq k$. If for each $n \in \mathbb{N}$,*

$$d(T^n(x), T^n(y)) \leq \alpha_n(x) \operatorname{diam} \mathrm{O_T}(\mathrm{x}, \mathrm{y}) \quad \textit{for every } \mathrm{x}, \mathrm{y} \in \mathrm{X}, \tag{4.1}$$

then T is a Picard operator. Moreover, if additionally T satisfies (S), then the fixed point problem is well-posed.

Proof. First, suppose that T has two fixed points $x, y \in X, x \neq y$. Then for each $n \in \mathbb{N}$,

$$d(x, y) \leq \alpha_n(x) d(x, y). \tag{4.2}$$

Letting $n \to \infty$ we obtain that $k \geq 1$ which is impossible. Hence, T has at most one fixed point. Let $x \in X$. Notice that the sequence $(\operatorname{diam} \mathrm{O_T}(\mathrm{T^n}(\mathrm{x})))_{n\in\mathbb{N}}$ is decreasing and bounded below by 0 so it converges to $l_x \geq 0$. For $n, p_1, p_2 \in \mathbb{N},\ p_1 \leq p_2$ we have

$$d(T^{n+p_1}(x), T^{n+p_2}(x)) \leq \alpha_{n+p_1}(x) \operatorname{diam} O_T(x). \tag{4.3}$$

Taking the supremum with respect to p_1 and p_2 and then letting $n \to \infty$ we obtain

$$l_x \leq k \operatorname{diam} O_T(x). \tag{4.4}$$

For $p \in \mathbb{N}$,

$$l_x = \lim_{n\to\infty} \operatorname{diam} O_T(T^n(T^p(x))) \le k \operatorname{diam} O_T(T^p(x)). \tag{4.5}$$

Letting $p \to \infty$ in the above relation we have $l_x \le kl_x$ which implies that $l_x = 0$. This means that the sequence $(T^n(x))_{n\in\mathbb{N}}$ is Cauchy so it converges to a point $z \in X$. Because T is orbitally continuous it follows that z is a fixed point, which is unique. Therefore, all Picard iterates converge to z.

Next we prove that the problem is well-posed. Let $(x_n)_{n\in\mathbb{N}}$ be an approximate fixed point sequence. Taking into account (3.2) applied for z and (3.3) of Proposition 3.1,

$$\limsup_{n\to\infty} \operatorname{diam} O_T(z, x_n) = \limsup_{n\to\infty} \operatorname{diam}(\{z\} \cup O_T(x_n)) = \limsup_{n\to\infty} d(z, x_n). \tag{4.6}$$

Knowing that

$$d(z, x_n) \le d(x_n, T^m(x_n)) + d(T^m(x_n), T^m(z)) \le d(x_n, T^m(x_n)) + \alpha_m(z) \operatorname{diam} O_T(z, x_n), \tag{4.7}$$

and taking the superior limit we obtain

$$\limsup_{n\to\infty} d(z, x_n) \le \alpha_m(z) \limsup_{n\to\infty} d(z, x_n). \tag{4.8}$$

If we let here $m \to \infty$ it is clear that $(x_n)_{n\in\mathbb{N}}$ converges to z. □

A similar result can be given in a bounded metric space where the convexity structure defined by the class of admissible subsets is compact.

Theorem 4.2. *Let X be a bounded metric space such that $\mathcal{A}(X)$ is compact and let $T : X \to X$ be an orbitally continuous mapping. Also, for $n \in \mathbb{N}$, let $\alpha_n : X \to \mathbb{R}_+$ for which there exists $0 < k < 1$ such that for every $x \in X$, $\limsup_{n\to\infty} \alpha_n(x) \le k$. If for each $n \in \mathbb{N}$,*

$$d(T^n(x), T^n(y)) \le \alpha_n(x) \operatorname{diam} \mathrm{O}_\mathrm{T}(\mathrm{x}, \mathrm{y}) \quad \text{for every } \mathrm{x}, \mathrm{y} \in \mathrm{X}, \tag{4.9}$$

then T is a Picard operator. Moreover, if additionally T satisfies (S), then the fixed point problem is well-posed.

Proof. Let $x \in X$. Denote $\varphi : X \to \mathbb{R}_+$,

$$\varphi(u) = \limsup_{n\to\infty} d(u, T^n(x)) \quad \text{for } u \in X. \tag{4.10}$$

As in the proof of Theorem 4.1 one can show that T has at most one fixed point and for each $x \in X$, the sequence $(T^n(x))_{n\in\mathbb{N}}$ is Cauchy. This means that $\lim_{n\to\infty} \varphi(T^n(x)) = 0$ for each $x \in X$. Because $\mathcal{A}(X)$ is compact we can choose

$$\omega \in \bigcap_{n\ge1} \operatorname{cov}\left(\left\{T^k(x) : k \ge n\right\}\right). \tag{4.11}$$

Following the argument of [2, Theorem 4.1] we can show that $\varphi(\omega) = 0$. For the sake of completeness we also include this part of the proof. The definition of φ yields that for $u \in X$ and every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$,

$$d(u, T^n(x)) \leq \varphi(u) + \epsilon. \tag{4.12}$$

Hence, $T^n(x) \in \widetilde{B}(u, \varphi(u) + \epsilon)$ for every $n \geq n_0$ and so

$$\operatorname{cov}(\{T^n(x) : n \geq n_0\}) \subseteq \widetilde{B}(u, \varphi(u) + \epsilon). \tag{4.13}$$

Therefore, $\omega \in \widetilde{B}(u, \varphi(u) + \epsilon)$ for each $\epsilon > 0$. This implies $d(\omega, u) \leq \varphi(u)$ which holds for every $u \in X$. Thus,

$$\varphi(\omega) = \limsup_{n \to \infty} d(\omega, T^n(x)) \leq \limsup_{n \to \infty} \varphi(T^n(x)) = 0. \tag{4.14}$$

Now it is clear that $(T^n(x))_{n\in\mathbb{N}}$ converges to ω. Because T is orbitally continuous, ω will be the unique fixed point and all the Picard iterates will converge to this unique fixed point.

The fact that every approximate fixed point sequence $(x_n)_{n\in\mathbb{N}}$ converges to ω can be proved identically as in Theorem 4.1. □

In connection with the use of the diameter of the orbit, Walter [11] obtained a fixed point theorem that may be stated as follows.

Theorem 4.3 (Walter [11]). *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping with bounded orbits. If there exists a continuous, increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ for which $\varphi(r) < r$ for every $r > 0$ and*

$$d(T(x), T(y)) \leq \varphi(\operatorname{diam}(\mathrm{O_T(x, y)})) \quad \text{for every } x, y \in X, \tag{4.15}$$

then T is a Picard operator.

We conclude this section by proving an asymptotic version of this result. In this way we extend the notion of asymptotic contraction introduced by Kirk in [12].

Theorem 4.4. *Let (X, d) be a complete metric space and let $T : X \to X$ be an orbitally continuous mapping with bounded orbits. Suppose there exist a continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\varphi(t) < t$ for all $t > 0$ and the functions $\varphi_n : \mathbb{R}_+ \to \mathbb{R}_+$ such that the sequence $(\varphi_n)_{n\in\mathbb{N}}$ converges pointwise to φ and for each $n \in \mathbb{N}$,*

$$d(T^n(x), T^n(y)) \leq \varphi_n(\operatorname{diam}\mathrm{O_T(x, y)}) \quad \text{for all } x, y \in X, \tag{4.16}$$

then T is a Picard operator. Moreover, if additionally T satisfies (S) and φ_n is continuous for each $n \in \mathbb{N}$, then the fixed point problem is well-posed.

Proof. The proof follows closely the ideas presented in the proof of Theorem 4.1.

We begin by supposing that T has two fixed points $x, y \in X, x \neq y$. Then for each $n \in \mathbb{N}$,

$$d(x,y) \leq \varphi_n(d(x,y)). \tag{4.17}$$

Letting $n \to \infty$ we obtain that $d(x,y) \leq \varphi(d(x,y))$ which is impossible. Hence, T has at most one fixed point.

Notice that for $x \in X$ the sequence $(\operatorname{diam} O_T(T^n(x)))_{n\in\mathbb{N}}$ is decreasing and bounded below by 0 so it converges to $l_x \geq 0$. For $n, p_1, p_2 \in \mathbb{N}$, $p_1 \leq p_2$ we have

$$d(T^{n+p_1}(x), T^{n+p_2}(x)) \leq \varphi_{n+p_1}(\operatorname{diam} O_T(x)). \tag{4.18}$$

Thus, $l_x \leq \varphi(\operatorname{diam} O_T(x))$.
For $p \in \mathbb{N}$,

$$l_x = \lim_{n\to\infty} \operatorname{diam} O_T(T^n(T^p(x))) \leq \varphi(\operatorname{diam} O_T(T^p(x))). \tag{4.19}$$

Hence, $l_x \leq \varphi(l_x)$ which implies that $l_x = 0$ and the proof may be continued as in Theorem 4.1 in order to conclude that T is a Picard operator.

Let $z \in X$ be the unique fixed point of T and let $(x_n)_{n\in\mathbb{N}}$ be an approximate fixed point sequence. To show that the problem is well-posed, take $(x_{n_p})_{p\in\mathbb{N}}$ a subsequence of $(x_n)_{n\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty} d(z, x_n) = \lim_{p\to\infty} d\left(z, x_{n_p}\right). \tag{4.20}$$

Because every subsequence of $(x_n)_{n\in\mathbb{N}}$ is also an approximate fixed point sequence, the conclusions of Proposition 3.1 still stand for $(x_{n_p})_{p\in\mathbb{N}}$. This yields

$$\limsup_{p\to\infty} \operatorname{diam} O_T\left(z, x_{n_p}\right) = \limsup_{p\to\infty} \operatorname{diam}\left(\{z\} \cup O_T\left(x_{n_p}\right)\right) = \lim_{p\to\infty} d\left(z, x_{n_p}\right). \tag{4.21}$$

But since

$$d\left(z, x_{n_p}\right) \leq \operatorname{diam} O_T\left(z, x_{n_p}\right), \tag{4.22}$$

by passing to the inferior limit follows $\lim_{p\to\infty} \operatorname{diam} O_T(z, x_{n_p}) = \lim_{p\to\infty} d(z, x_{n_p})$.
For $m \in \mathbb{N}$,

$$\begin{aligned} d\left(z, x_{n_p}\right) &\leq d\left(x_{n_p}, T^m\left(x_{n_p}\right)\right) + d\left(T^m\left(x_{n_p}\right), T^m(z)\right) \\ &\leq d\left(x_{n_p}, T^m\left(x_{n_p}\right)\right) + \varphi_m\left(\operatorname{diam} O_T\left(z, x_{n_p}\right)\right). \end{aligned} \tag{4.23}$$

If we let here $p \to \infty$, we have $\lim_{p\to\infty} d(z, x_{n_p}) \leq \varphi_m(\lim_{p\to\infty} d(z, x_{n_p}))$. Passing here to the limit with respect to m implies $\lim_{p\to\infty} d(z, x_{n_p}) \leq \varphi(\lim_{p\to\infty} d(z, x_{n_p}))$ and this means $\lim_{p\to\infty} d(z, x_{n_p}) = 0$. Because of (4.20) it follows that $(x_n)_{n\in\mathbb{N}}$ converges to z. □

5. Some Generalized Nonexpansive Mappings in CAT(0) Spaces

In this section we give fixed point results in CAT(0) spaces for two classes of mappings which are more general than the nonexpansive ones.

Theorem 5.1. *Let* X *be a bounded complete* CAT(0) *space and let* $T : X \to X$ *be such that for every* $x, y \in X$,

$$d\big(T(x), T(y)\big) \le r_x\big(O_T(y)\big). \tag{5.1}$$

Then T *has a fixed point. Moreover, Fix(*T*) is closed and convex.*

Proof. Let $x \in X$. Denote $\varphi : X \to \mathbb{R}_+$,

$$\varphi(u) = \limsup_{n\to\infty} d(u, T^n(x)) \quad \text{for } u \in X. \tag{5.2}$$

Since X is a complete CAT(0) space there exists a unique $z \in X$ such that

$$\varphi(z) = \inf_{u\in X} \varphi(u). \tag{5.3}$$

Supposing z is not a fixed point of T, we have

$$\varphi(z) < \varphi(T(z)) \le \lim_{n\to\infty} r_z\Big(O_T\Big(T^{n-1}(x)\Big)\Big) = \varphi(z). \tag{5.4}$$

This is a contradiction and thus $z \in \mathrm{Fix}(T)$.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of fixed points which converges to $x^* \in X$. Then,

$$d(T(x^*), T(x_n)) \le d(x^*, x_n) \tag{5.5}$$

which proves that x^* is a fixed point of T so $\mathrm{Fix}(T)$ is closed.

Now take $x, y \in \mathrm{Fix}(T)$. We show that the midpoint of $[x, y]$ denoted by m is a fixed point of T using the (CN) inequality. More precisely we have

$$\begin{aligned} d(m, T(m))^2 &\le \frac{1}{2} d(T(m), T(x))^2 + \frac{1}{2} d\big(T(m), T(y)\big)^2 - \frac{1}{4} d(x, y)^2 \\ &\le \frac{1}{2} d(m, x)^2 + \frac{1}{2} d(m, y)^2 - \frac{1}{4} d(x, y)^2 = 0. \end{aligned} \tag{5.6}$$

Hence, $\mathrm{Fix}(T)$ is convex. □

A simple example of a mapping which is not nonexpansive, but satisfies (5.1), is the following.

Example 5.2. Let $T : [0,1] \longrightarrow [0,1]$,

$$T(x) = \begin{cases} \dfrac{x}{2}, & \text{if } x \geq \dfrac{1}{2}, \\ \dfrac{x}{4}, & \text{if } x < \dfrac{1}{2}. \end{cases} \tag{5.7}$$

Then T is not nonexpansive but (5.1) is verified.

Proof. T is not continuous, so it cannot be nonexpansive. To show that (5.1) holds, we only consider the situation when $x \geq 1/2$ and $y < 1/2$ because in all other the condition is clearly satisfied. Then $|T(x) - T(y)| = x/2 - y/4$. We can easily observe that

$$\begin{aligned} r_x(O_T(y)) &= x \geq \frac{x}{2} - \frac{y}{4}, \\ r_y(O_T(x)) &= \max\{x - y, y\}. \end{aligned} \tag{5.8}$$

If $(3/4)y \leq x/2$ then $x/2 - y/4 \leq x - y$. Otherwise, $x/2 - y/4 \leq (3/4)y - y/4 = y/2 \leq y$. In this way we have shown that (5.1) is accomplished. □

Theorem 5.3. *Let X be a bounded complete* CAT(0) *space and let $T : X \to X$ be such that for every $x, y \in X$,*

$$d(T(x), T(y)) \leq diam(\{x\} \cup O_T(y)), \tag{5.9}$$

$$d(T(x), T(y)) \leq r_x(O_T(y)) + \sup_{k,p\in\mathbb{N}} \left(diam\left(\left\{T^k(x)\right\} \cup O_T\left(T^{k+p}(y)\right)\right) - diam O_T\left(T^{k+p}(y)\right)\right). \tag{5.10}$$

Then T has a fixed point. Moreover, Fix(T) is closed and convex.

Proof. Let $x \in X$. Denote $\varphi : X \to \mathbb{R}_+$,

$$\varphi(u) = \limsup_{n\to\infty} d(u, T^n(x)) \quad \text{for } u \in X. \tag{5.11}$$

Since X is a complete CAT(0) space there exists a unique $z \in X$ such that

$$\varphi(z) = \inf_{u\in X} \varphi(u). \tag{5.12}$$

Let $l_x = \lim_{n\to\infty} \operatorname{diam} O_T(T^n(x))$. This limit exists since the sequence is decreasing and bounded below by 0.

Suppose z is not a fixed point of T. Then

$$\limsup_{n\to\infty} d(z,T^n(x)) = \varphi(z) < \varphi(T(z)) \leq \lim_{n\to\infty} \operatorname{diam}(\{z\} \cup O_T(T^n(x))). \tag{5.13}$$

This means that

$$\lim_{n\to\infty} \operatorname{diam}(\{z\} \cup O_T(T^n(x))) = l_x, \tag{5.14}$$

$$\begin{gathered}\limsup_{n\to\infty} d(T(z),T^n(x)) \leq l_x,\\ \lim_{n\to\infty} \operatorname{diam}(\{T(z)\} \cup O_T(T^n(x))) = l_x.\end{gathered} \tag{5.15}$$

Inductively, it follows that for $k \geq 0$,

$$\limsup_{n\to\infty} d\Big(T^k(z),T^n(x)\Big) \leq l_x. \tag{5.16}$$

Let $k,p,n \in \mathbb{N}$ and let $d_{k,n} = \operatorname{diam}(\{T^k(z)\} \cup O_T(T^{n+k}(x)))$. Obviously,

$$\operatorname{diam}\Big(\Big\{T^k(z)\Big\} \cup O_T\Big(T^{n+p+k}(x)\Big)\Big) \leq d_{k,n}, \tag{5.17}$$

since $O_T(T^{n+p+k}(x)) \subseteq O_T(T^{n+k}(x))$.
Because of (5.9) we have

$$r_{T^k(z)}\Big(O_T\Big(T^{n+k}(x)\Big)\Big) \leq \operatorname{diam}\Big(\Big\{T^{k-1}(z)\Big\} \cup O_T\Big(T^{n+k-1}(x)\Big)\Big). \tag{5.18}$$

Since diam $(O_T(T^{n+k}(x))) \leq \operatorname{diam}(O_T(T^{n+k-1}(x)))$, it is clear that $d_{k,n} \leq d_{k-1,n}$.
Hence,

$$\sup_{k\in\mathbb{N}} d_{k,n} = \operatorname{diam}(\{z\} \cup O_T(T^n(x))). \tag{5.19}$$

Let $s_n = \sup_{k,p\in\mathbb{N}}(\operatorname{diam}(\{T^k(z)\} \cup O_T(T^{n+p+k}(x))) - \operatorname{diam} O_T(T^{n+p+k}(x)))$.
Then,

$$s_n \leq \sup_{k\in\mathbb{N}} d_{k,n} - \inf_{k,p\in\mathbb{N}} \operatorname{diam} O_T\Big(T^{n+p+k}(x)\Big) \leq \operatorname{diam}(\{z\} \cup O_T(T^n(x))) - l_x. \tag{5.20}$$

Taking into account (5.14), $\lim_{n\to\infty} s_n = 0$. Now,

$$\begin{aligned}\varphi(T(z)) &= \limsup_{n\to\infty} d(T(z), T^n(x)) \le \lim_{n\to\infty} r_z\Big(O_T\Big(T^{n-1}(x)\Big)\Big) + \lim_{n\to\infty} s_{n-1} \\ &= \limsup_{n\to\infty} d\Big(z, T^{n-1}(x)\Big) = \varphi(z),\end{aligned} \tag{5.21}$$

which is a contradiction. Hence, $T(z) = z$.

The fact that $\mathrm{Fix}(T)$ is closed and convex follows as in the previous proof. □

Remark 5.4. It is clear that nonexpansive mappings and mappings for which (5.1) holds satisfy (5.9) and (5.10). However, there are mappings which satisfy these two conditions without verifying (5.1) as the following example shows.

Example 5.5. The set $[0,1]$ with the usual metric is a CAT(0) space. Let us take $T : [0,1] \to [0,1]$,

$$T(x) = \begin{cases} \dfrac{2}{3}x, & \text{if } x \ge \dfrac{1}{2}, \\ \dfrac{x}{4}, & \text{if } x < \dfrac{1}{2}. \end{cases} \tag{5.22}$$

Then T does not satisfy (5.1) but conditions (5.9), (5.10) hold.

Proof. To prove that T does not verify (5.1) we take $x = 1/2$ and $y = 1/4$. Then $|T(x) - T(y)| = 1/3 - 1/16 = 13/48$. However,

$$r_{1/4}\left(O_T\left(\frac{1}{2}\right)\right) = \frac{1}{4} < \frac{13}{48}. \tag{5.23}$$

Next we show that (5.9) and (5.10) hold. We only need to consider the case when $x \ge 1/2$ and $y < 1/2$ because in all the other situations this is evident. Then $|T(x) - T(y)| = (2/3)x - y/4$. Since

$$\mathrm{diam}(\{x\} \cup O_T(y)) = \mathrm{diam}(\{y\} \cup O_T(x)) = x \ge \frac{2}{3}x - \frac{y}{4}, \tag{5.24}$$

relation (5.9) is satisfied.

Also,

$$\begin{aligned} r_x(O_T(y)) &\ge x - \frac{y}{4} \ge \frac{2}{3}x - \frac{y}{4}, \\ r_y(O_T(x)) &\ge x - y. \end{aligned} \tag{5.25}$$

Since $\sup_{p\in\mathbb{N}}(\mathrm{diam}(\{y\} \cup O_T(T^p(x))) - \mathrm{diam}\, O_T(T^p(x))) \ge (3/4)y$, we obtain $x - y + (3/4)y \ge 2/3x - y/4$. Hence, relation (5.10) is also accomplished. □

Remark 5.6. If we replace condition (5.9) of Theorem 5.3 with

$$d(T(x),T(y)) \leq \alpha(x)\operatorname{diam}(\{x\} \cup O_T(y)) \quad \text{for every } x,y \in X, \tag{5.26}$$

where $\alpha : X \to [0,1)$, then we may conclude that T has s unique fixed point.

It is also clear that a pointwise contraction satisfies these conditions so we can apply this result to prove that it has a unique fixed point.

We next prove a demi-closed principle. We will use the notations introduced at the end of Section 3.

Theorem 5.7. *Let X be a* CAT(0) *space, $K \subseteq X$, K bounded, closed, and convex. Let $T : K \to K$ be a mapping that safisfies (S) and* (5.9) *for each $x, y \in K$ and let $(x_n)_{n\in\mathbb{N}} \subseteq K$ be an approximate fixed point sequence such that $x_n \overset{\varphi}{\rightharpoonup} \omega$. Then $\omega \in Fix(T)$.*

Proof. Using (3.1) of Proposition 3.1 we have $\varphi(x) = \limsup_{n\to\infty} d(x,T(x_n))$. Applying (3.2) and (3.3) of Proposition 3.1 for ω,

$$\limsup_{n\to\infty} \operatorname{diam}(\{\omega\} \cup O_T(x_n)) = \limsup_{n\to\infty} d(\omega, x_n). \tag{5.27}$$

Then,

$$\varphi(T(\omega)) = \limsup_{n\to\infty} d(T(\omega),T(x_n)) \leq \limsup_{n\to\infty} \operatorname{diam}(\{\omega\} \cup O_T(x_n)) = \varphi(\omega). \tag{5.28}$$

Let m denote the midpoint of $[\omega, T(\omega)]$. The (CN) inequality yields

$$d(m,x_n)^2 \leq \frac{1}{2}d(\omega,x_n)^2 + \frac{1}{2}d(T(\omega),x_n)^2 - \frac{1}{4}d(\omega,T(\omega))^2. \tag{5.29}$$

Taking the superior limit, we have

$$\varphi(m)^2 \leq \frac{1}{2}\varphi(\omega)^2 + \frac{1}{2}\varphi(T(\omega))^2 - \frac{1}{4}d(\omega,T(\omega))^2. \tag{5.30}$$

But since $x_n \overset{\varphi}{\rightharpoonup} \omega$,

$$\frac{1}{4}d(\omega,T(\omega))^2 \leq \frac{1}{2}\varphi(\omega)^2 + \frac{1}{2}\varphi(\omega)^2 - \varphi(\omega)^2 = 0. \tag{5.31}$$

Hence, $\omega \in \operatorname{Fix}(T)$. □

We conclude this paper with the following remarks.

Remark 5.8. All the above results obtained in the context of CAT(0) spaces also hold in the more general setting used in [4] of uniformly convex metric spaces with monotone modulus of convexity.

Remark 5.9. In a similar way as for nonexpansive mappings, one can develop a theory for the classes of mappings introduced in this section. An interesting idea would be to study the approximate fixed point property of such mappings. A nice synthesis in the case of nonexpansive mappings can be found in the recent paper of Kirk [13].

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