Continuous location problems and Big Triangle Small Triangle. Constructing better bounds*

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Abstract

The Big Triangle Small Triangle method has shown to be a powerful global optimization procedure to address continuous location problems. In the paper published in JOGO **37** (2007) 305–319, Drezner proposes a rather general and effective approach for constructing the bounds needed. Such bounds are obtained by using the fact that the objective functions in continuous location models can usually be expressed as a difference of convex functions.

In this note we show that, exploiting further the rich structure of such objective functions, alternative bounds can be derived, yielding a significant improvement in computing times, as reported in our numerical experience.

Keywords: continuous location, big triangle small triangle, dc monotonic functions

1 Introduction

In continuous location problems, the location for one or several facilities within a subset S of the n-dimensional space $\mathbb{R}^n$ is sought so that a given function of the distances from the facilities to a set A of users is optimized. The reader is referred to [9] for an introduction to continuous location.

Many instances in single-facility continuous location can be expressed as optimization problems in the form

$$\min_{x \in S} F(x) := \sum_{a \in A} \varphi_a(\|x - a\|_a) \tag{1.1}$$

where S is a finite union of polytopes in $\mathbb{R}^n$ representing the set of possible locations for the facility, A is a finite subset of $\mathbb{R}^n$ with the coordinates of the users, $\|\cdot\|_a$ is a norm in $\mathbb{R}^n$ for each $a \in A$ which models travel distances from user a, and φ_a is a function, $\varphi_a : \mathbb{R}_+ \longrightarrow \mathbb{R}$ so

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that $\varphi_a(d)$ gives the cost associated with the interaction between user a and the facility located at distance d.

In general F is not convex, and global optimization procedures are needed to solve (1.1). The first solution method proposed in the literature was a branch-and-bound algorithm called Big Square Small Square, BSSS, [7], later generalized by Plastria, [8]. Recently, Drezner and Suzuki have introduced in [5] a variant, called the Big Triangle Small Triangle, BTST. BTST differs from its ancestor BSSS in the subdivision elements used: whereas BSSS uses hyper-rectangles, BTST uses simplices. Both may share the bounding strategies, but in the literature one finds that bounds in BSSS are mostly constructed exploiting the (piecewise) monotonicity of the functions φ_a, whereas in BTST functions φ_a are assumed to be dc, i.e., they can be decomposed as a difference of convex functions, and then standard bounding procedures for dc functions are then used. See [3, 4, 6] for some examples.

In most applications, such as all those mentioned in [2], functions φ_a are dc, and a dc decomposition of φ_a as $\varphi_a = \varphi_a^1 - \varphi_a^2$ is available. This immediately yields lower and upper bounds for φ_a. Indeed, an upper bound of a univariate finite convex function on a segment is given by the chord interpolating at the endpoints; a lower bound is obtained by taking the supporting line at an arbitrary interior point. Using this strategy, given an interval $[d_a^{\min}, d_a^{\max}] \subset \mathbb{R}_+$, one constructs coefficients $K_a^1, L_a^1, K_a^2, L_a^2$, such that, for any $d \in [d_a^{\min}, d_a^{\max}]$,

$$\begin{array}{rcl} \varphi_a^1(d) & \geq & K_a^1 d + L_a^1 \\ \varphi_a^2(d) & \leq & K_a^2 d + L_a^2, \end{array} \tag{1.2}$$

implying

$$\varphi_a(d) \geq (K_a^1 - K_a^2)d + (L_a^1 - L_a^2). \tag{1.3}$$

Hence, for any $x \in \mathbb{R}^n$ such that $d_a^{\min} \leq \|x - a\|_a \leq d_a^{\max}$, we have by (1.3) that

$$\varphi_a(\|x - a\|_a) \geq (K_a^1 - K_a^2)\|x - a\|_a + (L_a^1 - L_a^2). \tag{1.4}$$

A concave minorant $l_a(\cdot)$ of $\varphi_a(\|\cdot - a\|_a)$, i.e., a concave function with $l_a(x) \leq \varphi_a(\|x - a\|_a)\, \forall x$, is obtained as follows. If $K_a^1 - K_a^2 \geq 0$, then the function in the right term of (1.4) is convex, a concave minorant of which is obtained by linearizing below the convex function $(K_a^1 - K_a^2)\|x - a\|_a + (L_a^1 - L_a^2)$. On the other hand, if $K_a^1 - K_a^2 < 0$, then the function in the right term of (1.4) is concave, which is obviously a concave minorant of itself.

With this, we can construct a concave minorant $m_a(x)$ on a simplex (triangle if $n = 2$) of $\varphi_a(\|x - a\|_a)$, and, by summing such minorants, one obtains a concave minorant of F on the simplex. A lower bound of F is thus obtained by inspecting at the extreme points of the simplex such concave minorant.

The aim of this paper is to show that, in many cases, it is possible to obtain a dc decomposition of the functions φ_a with additional properties, namely, that the corresponding φ_a^1, φ_a^2 are not only convex, but also monotonic. Moreover, this decomposition leads to bounding procedures which may be more successful in terms of running times than the standard bounding procedures in the same branch and bound scheme.

The remainder of the paper is structured as follows. In Section 2 we introduce a subclass of dc functions. Some general properties are studied, and different examples (in the context of continuous location problems) are given. In Section 3 a bounding strategy for dcm functions is given, which is tested in a set of numerical examples in Section 4, showing that this new strategy is very competitive in running times.

2 Dcm functions

2.1 General properties

Definition 1 *Given a nondegenerate interval $K \subset \mathbb{R}$, a function $\varphi : K \longrightarrow \mathbb{R}$ is said to be* difference of convex monotonic *(dcm) in K if there exist $\varphi^1, \varphi^2 : K \longrightarrow \mathbb{R}$, convex and monotonic in K such that $\varphi = \varphi^1 - \varphi^2$.*

Smooth functions are dcm, as shown in the following results.

Proposition 2 *Let φ be C^2 in a nondegenerate interval $K \subset \mathbb{R}$. Then φ is dcm in K.*

Proof
Let t_0 be interior to K. Define

$$\begin{aligned} \alpha(t) &= \int_{t_0}^{t} \left[\varphi''(s)\right]^{+} ds \\ \beta(t) &= -\int_{t_0}^{t} \left[\varphi''(s)\right]^{-} ds, \end{aligned}$$

where $[z]^+$ and $[z]^-$ denote, respectively, the positive and negative part of z, that is, $[z]^+ = \max\{z, 0\}$ and $[z]^- = \min\{z, 0\}$. We have the following decomposition for φ :

$$\varphi(t) = \int_{t_0}^{t} \alpha(s)\, ds + \big(\varphi(t_0) + \varphi'(t_0)(t - t_0)\big) - \int_{t_0}^{t} \beta(s)\, ds.$$

If $\varphi'(t_0) \geq 0$, then both $\int_{t_0}^{t} \alpha(s)\, ds + \varphi(t_0) + \varphi'(t_0)(t - t_0)$ and $\int_{t_0}^{t} \beta(s)\, ds$ are convex and non-decreasing, whereas if $\varphi'(t_0) < 0$, then $\int_{t_0}^{t} \alpha(s)\, ds$ and $-\varphi(t_0) - \varphi'(t_0)(t - t_0) + \int_{t_0}^{t} \beta(s)\, ds$ are convex and non-decreasing, giving a dcm decomposition of φ. □

Proposition 3 *Let K be a compact interval. Assume $\varphi = \varphi^1 - \varphi^2$, with both $\varphi^1, \varphi^2 \in C^1(K)$ and convex in K. Then, φ is dcm in K.*

Proof
Let $M \geq \max\{\max_{t \in K} -(\varphi^1(t))', \max_{t \in K} -(\varphi^2(t))'\}$. Then

$$\begin{aligned} \varphi(t) &= \varphi^1(t) - \varphi^2(t) \\ &= \big(\varphi^1(t) + tM\big) - \big(\varphi^2(t) + tM\big), \end{aligned}$$

giving a dcm decomposition for φ. □

Remark 4 *Although the class of dcm functions is rather broad, it is a proper subset of the class of dc functions. Indeed, let $K = [0, 1]$, and consider the function $\varphi(t) = \sqrt{t(1 - t)}$, which is concave in K, and thus dc in K. Let us show that φ is not dcm. By contradiction, suppose φ is dcm in K, and let $\varphi = \varphi^1 - \varphi^2$ be a dcm decomposition in K. Since the side derivative of φ at $t = 0$ is $+\infty$, the right derivative of φ^1 at $t = 0$ is also $+\infty$, or the right derivative of φ^2 at $t = 0$ is $-\infty$. In the former case, the convexity of φ^1 would imply that its right derivative* *would be non-decreasing, and thus equally constant to $+\infty$ in K, which is a contradiction. In the latter case, φ^2 would be non-increasing in K. Since the left derivative of φ at $t=1$ is $-\infty$, we would need that the left derivative of φ^1 is also $-\infty$ (impossible, by convexity of φ^1), or the left derivative of φ^2 at $t=1$ would be $+\infty$, which contradicts the fact that φ^2 is non-increasing in K. Hence, no dcm decomposition for φ exists.*

Remark 5 *Contrary to the case of dc functions, which enjoy a rich algebra (dc functions are closed under usual operations), the class of dcm functions is not closed by sums. This is shown with the following example: take $K=\mathbb{R}_+$, and consider the dcm functions in K*

$$\begin{aligned} \alpha(t) &= \sqrt{t} = 0 - \big(-\sqrt{t}\big) \\ \beta(t) &= -e^t = 0 - e^t. \end{aligned}$$

The function $\varphi(t)=\alpha(t)+\beta(t)$ is not dcm in K. Indeed, suppose by contradiction that it is dcm, and a dcm decomposition is given by $\varphi=\varphi^1-\varphi^2$. As in Remark 4, let us analyze the directional derivatives at the endpoints of K. Since the right derivative $\varphi'_+(0)$ of φ at $t=0$ is $+\infty$, by convexity of φ^1 and φ^2 we would have that φ^2 should have right derivative at $t=0$ equal to $-\infty$, which would imply, in particular, that φ^2 would be non-increasing in K. Now, for sufficiently large t, the derivative of φ goes to $-\infty$. Hence, since φ^2 is non-increasing, one would need φ^1 to have derivative going to $-\infty$, which is impossible by convexity of φ^1.

2.2 Examples in continuous location

Although the theory of dcm functions is general, we show that it is directly applicable, among others, to continuous location problems. In what follows we describe some of the models already presented in [2], which fit within the framework given in this paper.

2.2.1 Obnoxious facility location

$$\min_x \sum_{a\in A} \frac{\omega_a}{\|x-a\|^2} \qquad (\omega_a > 0 \quad \forall a)$$

A dcm decomposition in $\mathbb{R}_{++}$ for $\varphi_a(d)=\omega_a/d^2$ is given by

$$\varphi_a^1(d) = \omega_a/d^2 \qquad\qquad \varphi_a^2(d) = 0 \tag{2.5}$$

2.2.2 Weber problem with some negative weights

$$\min_x \sum_{a\in A} \omega_a \|x-a\| \qquad (\omega_a \in \mathbb{R})$$

A dcm decomposition in $\mathbb{R}_+$ for $\varphi_a(d)=\omega_a d$ is given by

$$\varphi_a^1(d) = \max\{\omega_a, 0\} d \qquad \varphi_a^2(d) = \max\{-\omega_a, 0\} d \tag{2.6}$$

2.2.3 Huff competitive location

$$\max_x \quad \sum_{a\in A} \frac{b_a}{1+h_a\|x-a\|^\lambda} \qquad (h_a, b_a > 0 \quad \lambda \geq 1)$$

A dcm decomposition in $\mathbb{R}_+$ of $\varphi_a(d) = -\frac{b_a}{1+h_a d^\lambda}$ is given by

$$\begin{aligned} \varphi_a^1(d) &= b_a h_a d^\lambda \\ \varphi_a^2(d) &= b_a h_a d^\lambda - \varphi_a(d) \end{aligned} \tag{2.7}$$

An alternative dcm decomposition for φ_a is given by

$$\begin{aligned} \varphi_a^1(d) &= \begin{cases} -\varphi_a(\bar{d}) - \varphi_a'(\bar{d})(d-\bar{d}) + \varphi_a(d) & \text{if } d < \bar{d} \\ 0 & \text{if } d \geq \bar{d} \end{cases} \\ \varphi_a^2(d) &= \begin{cases} -\varphi_a(\bar{d}) - \varphi_a'(\bar{d})(d-\bar{d}) & \text{if } d < \bar{d} \\ -\varphi_a(d) & \text{if } d \geq \bar{d} \end{cases} \end{aligned} \tag{2.8}$$

In these expressions, $\bar{d}$ is root of equation $\varphi_a''(d) = 0$.

2.2.4 Stochastic weighted minimax

$$\max_x \quad \sum_{a\in A} \ln(\|x-a\|) - \ln(T_a - \alpha_a\|x-a\|)$$

A dcm decomposition in $\mathbb{R}_{++}$ for $\varphi_a(d) = -\ln(d) + \ln(T_a - \alpha_a d)$ is given by

$$\varphi_a^1(d) = -\ln(d), \qquad \varphi_a^2(d) = -\ln(T_a - \alpha_a d). \tag{2.9}$$

2.2.5 Unserviced demand (I)

$$\max_x \quad \sum_{a\in A} \exp(-\|x-a\|)$$

As dcm decomposition in $\mathbb{R}_+$ for $\varphi_a(d) = -\exp(-d)$ we have

$$\begin{aligned} \varphi_a^1(d) &= 0 \\ \varphi_a^2(d) &= \exp(-d). \end{aligned} \tag{2.10}$$

2.2.6 Unserviced demand (II)

$$\max_x \quad \sum_{a\in A} \frac{1}{1+\|x-a\|}$$

We have as dcm decomposition in $\mathbb{R}_+$ for $\varphi_a(d) = -1/(1+d)$

$$\begin{aligned} \varphi_a^1(d) &= 0 \\ \varphi_a^2(d) &= 1/(1+d) \end{aligned} \tag{2.11}$$

2.2.7 Inventory-location model

$$\min_x \quad \sum_{a\in A} \alpha_a\|x-a\| + \omega_a\sqrt{R_a\|x-a\|^2 + S_a\|x-a\| + T_a}$$

A dcm decomposition in $\mathbb{R}_+$ for $\varphi_a(d) = \omega_a\sqrt{R_a d^2 + S_a d + T_a}$ is given by

$$\begin{aligned} \varphi_a^1(d) &= 0 \\ \varphi_a^2(d) &= -\alpha_a d - \omega\sqrt{R_a d^2 + S_a d + T_a} \end{aligned} \tag{2.12}$$

2.2.8 Gradual covering

$$\min_x \quad \sum_{a\in A} \phi_a(x),$$

with

$$\phi_a(x) := \begin{cases} 0 & \text{if } \|x-a\| \le l_a \\ \omega_a(\|x-a\| - l_a) & \text{if } l_a < \|x-a\| \le u_a \\ \omega_a(u_a - l_a) & \text{if } \|x-a\| > u_a \end{cases}$$

A dcm decomposition is given by

$$\begin{aligned} \varphi_a^1(d) &= \begin{cases} 0 & \text{if } d < l_a \\ \omega(d - l_a) & \text{if } d \ge l_a \end{cases} \\ \varphi_a^2(d) &= \begin{cases} 0 & \text{if } d < u_a \\ \omega_a(d - u_a) & \text{if } d \ge u_a \end{cases} \end{aligned} \tag{2.13}$$

2.2.9 The acceleration-deceleration distance

$$\min_x \quad \sum_{a\in A} \phi_a(x),$$

where

$$\phi_a(x) = \begin{cases} 2\sqrt{\|x-a\| d_{0a}} & \text{if } \|x-a\| < d_{0a} \\ \|x-a\| + d_{0a} & \text{if } \|x-a\| \ge d_{0a} \end{cases}$$

A dcm decomposition is given by

$$\varphi_a^1(d) = 0 \qquad \varphi_a^2(d) = -\begin{cases} 2\sqrt{d d_{0a}} & \text{if } d < d_{0a} \\ d + d_{0a} & \text{if } d \ge d_{0a} \end{cases} \tag{2.14}$$

3 Bounds for dcm functions

Consider, for each $a \in A$, a function $\varphi_a = \varphi_a^1 - \varphi_a^2$, dcm in $\mathbb{R}_+$. Let S be a polytope in $\mathbb{R}^n$, expressed as the convex hull of a finite set of points $\{v_i : i \in I\}$. W.l.o.g. we assume that S contains at least a non-degenerate segment. Let us construct a lower bound in S for $F(x) = \sum_{a\in A} \varphi_a(\|x-a\|_a)$ using the monotonicity of the functions φ_a^1, φ_a^2.

For $x \in S$, we express x as $x = \sum_i \lambda_i v_i$, $\lambda_i \ge 0\,\forall i$, $\sum_i \lambda_i = 1$. We first obtain a concave minorant of φ_a^1 as follows:

1. If φ_a^1 is non-decreasing, then $\varphi_a^1(\|\cdot - a\|_a)$ is the composition of a non-decreasing convex function with a convex function. Hence, it is also convex. Let $x_0 \in S$, $x_0 \neq a$, and let p_a be a subgradient at x_0 of the convex function $\varphi_a^1(\|\cdot - a\|_a)$. By definition of subgradient we have that

$$\varphi_a^1(\|\sum_i \lambda_i v_i - a\|_a) \geq \varphi_a^1(\|x_0 - a\|_a) + p_a^\top(\sum_i \lambda_i v_i - x_0).$$

   Observe that the minorant found is an affine function.

2. If φ_a^1 is non-increasing, then given $x_0 \in S$, $x_0 \neq a$, for any p_a, subgradient at $d_0 := \|x_0 - a\|_a$ of φ^1, by definition of subgradient, one has

$$\varphi_a^1(d) \geq \varphi_a^1(d_0) + p_a\,(d - d_0)\,,$$

   and then,

$$\varphi_a^1(\|\sum_i \lambda_i v_i - a\|_a) \geq \varphi_a^1(\|x_0 - a\|_a) + p_a(\|\sum_i \lambda_i v_i - a\|_a - \|x_0 - a\|_a).$$

   Since φ^1 is assumed to be non-increasing, one has that $p_a \leq 0$, and hence the minorant found is concave.

Now we obtain a convex majorant of $\varphi_a^2(\|\cdot - a\|_a)$, i.e., a convex function u_a with $u_a(x) \geq \varphi_a^2(\|x - a\|_a)\,\forall x:$

1. If φ_a^2 is non-decreasing, then $\varphi_a^2(\|\cdot - a\|_a)$ is the composition of a non-decreasing convex function with a convex function. Hence, it is also convex, and the very same function $\varphi_a^2(\|\cdot - a\|_a)$ is taken as convex majorant of itself.

2. If φ_a^2 is non-increasing, then given $x_0 \in S$, $x_0 \neq a$, let p_a be a subgradient of $\|\cdot - a\|_a$ at x_0. Then, by definition of subgradient, one has

$$\|\sum_i \lambda_i v_i - a\|_a \geq \|x_0 - a\|_a + p_a^\top \left(\sum_i \lambda_i v_i - x_0\right),$$

   and, since φ^2 is non-increasing,

$$\varphi_a^2(\|\sum_i \lambda_i v_i - a\|_a) \leq \varphi_a^2\left(\|x_0 - a\|_a + p_a^\top(\sum_i \lambda_i v_i - x_0)\right).$$

   We have then found a convex majorant of $\varphi_a^2(\|\cdot - a\|_a)$.

The procedure above yields a concave minorant $l_a(x)$ of $\varphi_a^1(\|x - a\|_a)$ and a convex majorant $u_a(x)$ of $\varphi_a^2(\|x - a\|_a)$. This implies that the function $l_a(x) - u_a(x)$ is a concave minorant of $\varphi_a(\|\cdot - a\|_a) = \varphi_a^1(\|\cdot - a\|_a) - \varphi_a^2(\|\cdot - a\|_a)$, and hence the concave function $\sum_{a \in A}(l_a(x) - u_a(x))$ is a concave minorant of $F(x)$. This implies that

$$\min_{x \in S} F(x) \geq \min_{x \in S} \sum_{a \in A}(l_a(x) - u_a(x)) = \min_{i \in I} \sum_{a \in A}(l_a(v_i) - u_a(v_i))\,,$$

and this is the bound we propose.

4 Computational experience

In order to show empirically that the bounding strategy described in the paper is competitive compared with the approach suggested by Drezner, we have implemented the branch and bound method BTST using the two bounding procedures and run the algorithm on a set of instances of the 2-dimensional problems described in Section 2.2. The algorithm was implemented in a Fortran program compiled by Intel Fortran 10.1, and run on a 2.4GHz computer under Windows XP. The solutions were found to a relative accuracy of 10^{-10}.

Two issues must be taken into account, namely, the dcm decomposition and the bounding process. In Table 1 we show the problems that have been considered in the numerical experience, as well as the dcm decompositions used in the two bounding strategies: the bounding procedure described in [2], summarized in Section 1, and the new procedure, detailed in Section 3. The numbers in the last two columns of Table 1 are the labels of the corresponding dc decompositions given in Section 2.2. Note that in experiment D (Huff competitive location) two different dc decompositions (namely 2.7 and 2.8) have been used, whereas in the remaining experiments the two bounding methods are compared with respect the same decomposition.

Every problem was solved, using the two bounding procedures, for a different numbers of demand points N, ranging from 10 to 10000, randomly generated in the unit square $[0,1] \times [0,1]$.

The computational results obtained for these problems are shown in Tables 2 - 11. Each table shows some statistics (minimum, maximum and average) for three indicators of the algorithm performance: number of iterations, maximum number of triangles in the branch-and-bound list (we remind that in the BTST method simplices, and thus triangles when the dimension $n = 2$, are used as partition elements) and running time. Every problem was run ten times for each value of N in order to obtain the above-mentioned measurements.

Experiment	Problem name	dcm decomposition	
		JOGO(2007) bounding method	dcm-based bounding method
A	Obnoxious facility location	2.5	2.5
B	Weber problem with some negative weights	2.6	2.6
C	Huff competitive location	2.7	2.7
D	Huff competitive location	2.7	2.8
E	Huff competitive location	2.8	2.8
F	Stochastic weighted minimax	2.9	2.9
G	Unserviced demand (I)	2.10	2.10
H	Unserviced demand (II)	2.11	2.11
I	Inventory-location model	2.12	2.12
J	Gradual covering	2.13	2.13
K	The accelaration-deceleration distance	2.14	2.14

Table 1: Problems, dcm decomposition and bounding strategies considered

The Huff competitive location problem is analyzed in experiments C,D and E. When the decomposition 2.7 is used, Drezner's method outperforms the dcm-based method. However, when one uses the dc decomposition 2.8, which exploits more the structure of the functions φ_a, the gains in time and memory use of the dcm-based method are very important. Moreover, as shown in Table 5, the decomposition 2.8 combined with our dcm-based method, clearly outperforms the decomposition 2.7 combined with Drezner's method.

In the remaining problems, when the same dc decomposition is used, the two bounding methods yield roughly the same number of iterations and memory use (measured as the maximum number of triangles to be inspected), but our dcm-based method tends to run in much less time (Experiments F, G, H, I, J and K) or the same time (experiments A and B).

To sum up, it is evident that in most cases the new bounding procedure suggested in this paper reduces considerably the running times for the same dcm decomposition. The choice of the dcm decomposition may have important consequences, as shown in experiments C-E. An adequate choice of the dc decomposition, following [1], deserves further study.

JOGO(2007) bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	37	77	65,50	10	34	17,30	0,00	0,00	0,00
20	71	141	88,00	15	40	27,80	0,00	0,00	0,00
50	76	186	105,40	23	108	50,00	0,00	0,01	0,00
100	98	387	170,10	38	184	80,40	0,00	0,03	0,01
200	103	496	202,00	57	211	133,40	0,01	0,06	0,03
500	115	215	164,30	104	199	157,10	0,09	0,10	0,10
1000	158	542	317,90	196	614	334,80	0,31	0,54	0,40
2000	207	1898	567,20	272	781	527,30	1,12	3,07	1,54
5000	312	1984	609,10	399	1768	710,80	6,42	11,31	7,28
10000	368	1703	647,90	623	1511	986,80	24,10	31,93	25,75

dcm-based bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	37	77	65,50	10	34	17,30	0,00	0,00	0,00
20	71	141	88,00	15	40	27,80	0,00	0,01	0,00
50	76	186	105,40	23	108	50,00	0,00	0,01	0,00
100	98	387	170,10	38	184	80,40	0,00	0,03	0,01
200	103	496	202,00	57	211	133,40	0,01	0,06	0,03
500	115	215	164,30	104	199	157,10	0,09	0,12	0,10
1000	158	542	317,90	196	614	334,80	0,31	0,54	0,41
2000	207	1898	567,20	272	781	527,30	1,15	3,09	1,56
5000	312	1984	609,10	399	1768	710,80	6,53	11,43	7,41
10000	368	1703	647,90	623	1511	986,80	24,68	32,43	26,30

Table 2: Computational results for Experiment A in Table 1

References

[1] Blanquero, R., E. Carrizosa and E. Conde, "Finding GM-Estimators With Global Optimization Techniques", *Journal of Global Optimization* **21** (2001) 223–237.

[2] Drezner, Z., "A General Global Optimization Approach for Solving Location Problems in the Plane". *Journal of Global Optimization* **37** (2007) 305–319.

JOGO(2007) bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	115	1712	393,10	12	169	39,90	0,00	0,01	0,00
20	120	825	292,60	12	127	44,20	0,00	0,01	0,00
50	132	5689	769,00	16	479	88,30	0,00	0,10	0,01
100	124	481	259,00	19	119	41,20	0,00	0,01	0,01
200	181	633	327,50	25	99	49,30	0,01	0,06	0,03
500	269	1762	598,30	32	775	174,10	0,09	0,40	0,16
1000	248	760	406,20	27	221	79,80	0,26	0,46	0,32
2000	371	4233	888,30	62	1428	237,80	0,93	4,10	1,36
5000	389	19638	4040,70	74	7115	1335,60	4,75	44,25	12,22
10000	362	13761	4218,40	90	4308	1382,60	17,28	72,14	33,06

dcm-based bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	115	1712	393,10	12	169	39,90	0,00	0,01	0,00
20	120	825	292,60	12	127	44,20	0,00	0,01	0,00
50	132	5689	769,00	16	479	88,30	0,00	0,10	0,01
100	124	481	259,00	19	119	41,20	0,00	0,03	0,01
200	181	633	327,50	25	99	49,30	0,01	0,06	0,03
500	269	1762	598,30	32	775	174,10	0,09	0,39	0,16
1000	248	760	406,20	27	221	79,80	0,25	0,45	0,32
2000	371	4233	888,30	62	1428	237,80	0,92	4,06	1,34
5000	389	19638	4040,70	74	7115	1335,60	4,68	43,64	12,07
10000	362	13761	4218,40	90	4308	1382,60	17,01	71,32	32,63

Table 3: Computational results for Experiment B in Table 1

JOGO(2007) bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	1546	10272	3455,40	353	4384	1210,60	0,04	0,25	0,08
20	3261	23784	10546,30	942	10401	4202,50	0,15	1,17	0,51
50	3424	26321	9037,90	902	3033	1787,10	0,42	3,21	1,10
100	8631	38740	20839,70	1440	15614	5968,30	2,10	9,40	5,06
200	11374	31149	18604,70	2561	7331	4418,20	5,56	15,15	9,05
500	25686	225775	60419,40	2974	56600	12151,50	31,34	272,73	73,28
1000	16119	121410	49726,00	4060	36009	9533,30	40,03	294,57	121,40
2000	25003	91161	44350,30	3547	12011	6788,30	125,26	447,03	219,12
5000	23615	76870	43315,50	4086	18449	7244,70	312,31	955,73	550,42
10000	25322	113933	57053,70	4098	25665	9383,30	717,07	2852,37	1482,95

dcm-based bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	4348	38008	11682,80	928	16534	3903,50	0,07	0,67	0,20
20	10334	135627	42372,90	3518	50889	14189,70	0,35	4,70	1,46
50	10009	63998	28842,10	2706	10653	6057,80	0,85	5,43	2,44
100	26202	126039	69249,00	5218	46734	19825,60	4,43	21,35	11,72
200	39972	141281	69207,90	9098	24792	15131,00	13,50	47,62	23,35
500	84410	879358	258658,30	9687	229944	46271,50	71,15	740,73	217,86
1000	51124	588205	185917,70	11504	142366	32350,20	86,70	991,01	313,44
2000	92079	243985	145699,50	12715	34554	22735,80	312,32	822,62	492,63
5000	83212	273440	157018,10	14070	57043	25126,70	716,48	2315,28	1336,59
10000	88832	420086	202536,40	16210	87786	31930,20	1561,79	7138,21	3472,83

Table 4: Computational results for Experiment C in Table 1

JOGO(2007) bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	1546	10272	3455,40	353	4384	1210,60	0,04	0,25	0,08
20	3261	23784	10546,30	942	10401	4202,50	0,15	1,17	0,51
50	3424	26321	9037,90	902	3033	1787,10	0,42	3,21	1,10
100	8631	38740	20839,70	1440	15614	5968,30	2,10	9,40	5,06
200	11374	31149	18604,70	2561	7331	4418,20	5,56	15,15	9,05
500	25686	225775	60419,40	2974	56600	12151,50	31,34	272,73	73,28
1000	16119	121410	49726,00	4060	36009	9533,30	40,03	294,57	121,40
2000	25003	91161	44350,30	3547	12011	6788,30	125,26	447,03	219,12
5000	23615	76870	43315,50	4086	18449	7244,70	312,31	955,73	550,42
10000	25322	113933	57053,70	4098	25665	9383,30	717,07	2852,37	1482,95

dcm-based bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	187	981	409,00	27	93	52,40	0,00	0,03	0,00
20	200	598	346,60	29	88	54,90	0,00	0,03	0,01
50	169	495	308,60	24	77	42,40	0,01	0,06	0,03
100	200	557	311,30	32	70	51,10	0,04	0,12	0,07
200	208	361	279,20	31	71	47,00	0,12	0,18	0,15
500	316	642	457,90	51	104	71,70	0,56	0,89	0,70
1000	264	866	490,10	45	149	73,00	1,45	2,71	1,93
2000	221	554	375,60	34	79	58,70	4,50	5,93	5,17
5000	214	579	356,70	37	81	54,20	24,65	28,54	26,18
10000	209	685	420,70	33	123	70,90	94,00	104,15	98,51

Table 5: Computational results for Experiment D in Table 1

JOGO(2007) bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	7590	191606	90873,30	3138	78502	37112,90	0,20	5,28	2,49
20	670	511323	130306,00	214	251744	60708,40	0,04	27,46	7,02
50	418	104870	31748,90	58	41895	12497,20	0,06	14,00	4,23
100	163	271540	62316,10	33	113022	25752,30	0,04	72,81	16,58
200	275	94917	26174,30	75	47849	11692,20	0,18	50,78	13,96
500	853	333568	81006,20	135	143653	34609,90	1,42	440,35	107,47
1000	3546	133319	39303,60	1132	75639	18829,20	10,42	355,70	105,47
2000	691	93329	24589,30	220	45317	10874,40	8,31	500,03	135,11
5000	5744	36640	13439,10	2334	16416	6069,40	105,25	516,76	207,22
10000	285	67258	11445,70	67	32693	5335,10	123,50	1895,34	418,85

dcm-based bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	187	981	409,00	27	93	52,40	0,00	0,03	0,00
20	200	598	346,60	29	88	54,90	0,00	0,03	0,01
50	169	495	308,60	24	77	42,40	0,01	0,06	0,03
100	200	557	311,30	32	70	51,10	0,04	0,12	0,07
200	208	361	279,20	31	71	47,00	0,12	0,18	0,15
500	316	642	457,90	51	104	71,70	0,56	0,89	0,70
1000	264	866	490,10	45	149	73,00	1,45	2,71	1,93
2000	221	554	375,60	34	79	58,70	4,50	5,93	5,17
5000	214	579	356,70	37	81	54,20	24,65	28,54	26,18
10000	209	685	420,70	33	123	70,90	94,00	104,15	98,51

Table 6: Computational results for Experiment E in Table 1

JOGO(2007) bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	91	154	122,30	9	16	13,40	0,00	0,01	0,00
20	89	166	131,70	10	26	20,00	0,00	0,01	0,00
50	116	182	156,70	22	39	28,40	0,01	0,03	0,01
100	138	235	182,10	23	35	29,10	0,03	0,04	0,04
200	164	237	197,40	33	41	37,60	0,09	0,12	0,10
500	181	297	247,40	34	49	41,00	0,35	0,46	0,41
1000	177	327	277,20	35	55	46,60	1,09	1,35	1,26
2000	199	343	292,00	30	57	44,90	3,79	4,31	4,13
5000	248	350	293,50	38	63	51,90	21,51	22,42	21,91
10000	255	475	343,50	42	67	55,30	81,64	85,60	83,23

dcm-based bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	96	159	127,30	11	26	19,40	0,00	0,01	0,00
20	97	201	139,70	15	38	27,00	0,00	0,01	0,00
50	128	224	172,60	25	44	33,90	0,00	0,01	0,01
100	147	228	191,80	27	42	36,10	0,01	0,03	0,02
200	187	248	214,10	37	53	43,90	0,04	0,06	0,06
500	202	318	264,50	38	54	43,50	0,21	0,28	0,24
1000	214	355	287,10	42	57	48,80	0,64	0,79	0,72
2000	213	340	304,60	28	65	47,60	2,07	2,35	2,28
5000	272	384	314,90	41	63	52,90	11,57	12,18	11,80
10000	261	480	352,80	43	71	57,00	43,15	45,57	44,16

Table 7: Computational results for Experiment F in Table 1

JOGO(2007) bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	136	402	220,90	20	53	30,00	0,00	0,01	0,00
20	132	601	209,50	18	62	27,80	0,00	0,03	0,00
50	133	410	181,50	20	55	28,30	0,01	0,03	0,01
100	116	230	165,60	20	35	25,40	0,03	0,04	0,03
200	117	235	161,60	19	33	24,60	0,06	0,09	0,07
500	116	186	135,80	17	30	22,60	0,26	0,31	0,27
1000	94	173	125,20	19	29	21,60	0,81	0,93	0,86
2000	102	125	111,30	18	22	19,90	2,98	3,06	3,01
5000	103	150	117,70	18	33	23,10	17,46	17,82	17,58
10000	83	183	115,80	18	35	23,30	67,81	69,40	68,36

dcm-based bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	132	401	215,20	18	53	29,30	0,00	0,01	0,00
20	125	601	202,50	18	61	26,70	0,00	0,01	0,00
50	125	411	174,00	18	56	26,80	0,00	0,01	0,00
100	109	222	159,30	18	34	24,00	0,00	0,03	0,01
200	113	228	154,80	17	32	23,40	0,03	0,04	0,03
500	111	184	129,90	17	28	21,40	0,09	0,12	0,10
1000	88	164	118,40	17	28	20,90	0,29	0,34	0,31
2000	92	121	106,10	16	21	18,70	1,07	1,10	1,09
5000	99	147	112,50	18	30	21,50	6,29	6,46	6,34
10000	80	175	110,50	16	33	21,70	24,40	25,06	24,61

Table 8: Computational results for Experiment G in Table 1

JOGO(2007) bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	158	940	346,30	19	109	45,00	0,00	0,01	0,00
20	170	663	265,10	24	85	35,70	0,00	0,01	0,00
50	142	705	271,50	19	107	39,90	0,00	0,06	0,02
100	132	277	185,20	21	41	27,80	0,01	0,04	0,03
200	137	303	191,50	24	46	30,70	0,06	0,12	0,08
500	133	219	154,40	20	33	25,90	0,26	0,32	0,27
1000	111	192	139,80	21	34	24,90	0,82	0,93	0,86
2000	107	137	126,70	20	27	23,50	2,93	3,03	2,99
5000	107	195	127,90	20	34	24,70	17,14	17,79	17,29
10000	104	199	129,10	21	35	24,90	66,89	68,28	67,25

dcm-based bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	151	940	335,50	18	109	42,40	0,00	0,01	0,00
20	160	662	256,50	23	86	34,40	0,00	0,01	0,00
50	129	701	262,40	19	106	38,70	0,00	0,01	0,00
100	124	267	175,60	21	41	26,50	0,00	0,01	0,01
200	129	297	180,80	21	43	28,30	0,01	0,03	0,02
500	122	214	145,20	20	33	24,10	0,07	0,09	0,07
1000	103	187	132,90	19	33	23,50	0,23	0,26	0,24
2000	101	129	116,80	18	24	20,60	0,81	0,84	0,82
5000	101	189	120,10	20	35	23,00	4,68	4,90	4,73
10000	94	192	122,10	19	34	23,90	18,20	18,68	18,33

Table 9: Computational results for Experiment H in Table 1

JOGO(2007) bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	104	333	182,30	16	38	22,40	0,00	0,01	0,00
20	102	238	147,50	14	31	19,60	0,00	0,01	0,00
50	105	212	135,30	14	29	18,50	0,00	0,01	0,01
100	97	231	131,40	14	32	19,50	0,01	0,04	0,02
200	94	129	111,00	14	21	16,60	0,04	0,07	0,06
500	94	194	136,50	15	28	20,50	0,26	0,34	0,29
1000	86	162	110,60	14	25	17,60	0,85	0,98	0,90
2000	85	133	100,50	14	23	17,50	3,14	3,31	3,20
5000	77	209	107,10	15	38	19,80	18,53	19,60	18,77
10000	70	216	112,00	14	44	22,80	72,67	75,03	73,35

dcm-based bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	98	332	180,90	15	38	22,20	0,00	0,01	0,00
20	101	235	145,70	13	30	19,00	0,00	0,01	0,00
50	103	212	133,70	14	29	18,00	0,00	0,01	0,00
100	95	231	128,80	14	32	18,80	0,00	0,01	0,01
200	87	129	109,10	14	21	16,30	0,01	0,03	0,02
500	87	194	134,20	15	27	20,10	0,09	0,14	0,11
1000	85	161	109,40	13	25	17,50	0,32	0,37	0,34
2000	84	130	99,20	14	21	17,30	1,18	1,25	1,20
5000	74	209	106,30	15	38	19,70	6,90	7,40	7,02
10000	69	216	110,70	13	44	22,60	27,04	28,10	27,35

Table 10: Computational results for Experiment I in Table 1

JOGO(2007) bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	120	7159	3926,20	22	1483	860,10	0,00	0,10	0,06
20	130	32216	4923,20	27	4962	872,00	0,00	1,01	0,15
50	90	1357	418,90	24	304	80,40	0,00	0,10	0,03
100	138	763	469,80	46	191	109,00	0,03	0,10	0,07
200	85	614	209,30	25	135	51,50	0,06	0,21	0,09
500	100	624	234,30	29	96	56,40	0,25	0,65	0,35
1000	164	256	198,30	41	86	57,00	0,95	1,09	1,01
2000	122	387	195,80	34	85	53,20	3,18	4,01	3,42
5000	146	310	217,50	43	105	70,90	18,67	19,98	19,24
10000	162	391	249,90	48	149	87,70	72,67	76,23	74,03

dcm-based bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	165	1298546	134099,90	35	599174	60936,20	0,00	10,26	1,05
20	175	33342	5118,10	43	5498	949,60	0,00	0,48	0,07
50	157	1447	519,40	56	290	105,70	0,00	0,06	0,02
100	239	836	572,80	96	198	124,80	0,01	0,06	0,04
200	174	681	317,40	53	157	101,20	0,04	0,10	0,05
500	187	843	362,00	69	173	101,20	0,14	0,39	0,20
1000	264	452	344,20	89	152	116,10	0,48	0,60	0,54
2000	218	673	353,00	79	199	131,10	1,50	2,14	1,68
5000	246	601	395,00	77	223	141,80	8,23	9,53	8,77
10000	292	742	472,80	91	249	178,60	31,50	34,75	32,80

Table 11: Computational results for Experiment J in Table 1

JOGO(2007) bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	146	334	214,50	18	49	30,50	0,00	0,01	0,00
20	133	311	232,60	23	43	32,10	0,00	0,01	0,00
50	171	308	232,50	25	67	34,60	0,01	0,03	0,01
100	151	310	209,40	22	42	31,50	0,03	0,06	0,04
200	169	314	237,80	28	50	35,00	0,07	0,12	0,10
500	172	245	205,10	25	43	34,30	0,31	0,37	0,33
1000	164	257	193,30	24	36	29,80	0,95	1,10	1,00
2000	156	257	193,90	22	38	29,90	3,26	3,57	3,38
5000	146	247	183,80	25	39	31,30	18,37	19,15	18,66
10000	106	234	163,70	24	37	31,30	70,40	72,45	71,32

dcm-based bounding method

N	Iterations			Max triangles			Time (seconds)		
	Min	Max	Ave	Min	Max	Ave	Min	Max	Ave
10	113	302	179,10	16	50	30,20	0,00	0,01	0,00
20	144	368	197,70	22	48	31,90	0,00	0,01	0,00
50	147	660	225,90	24	78	33,40	0,00	0,03	0,00
100	127	211	177,10	23	41	30,60	0,00	0,03	0,01
200	139	264	198,70	25	60	34,00	0,03	0,04	0,03
500	163	214	182,50	24	48	32,90	0,12	0,14	0,13
1000	143	236	170,60	21	36	27,80	0,34	0,42	0,37
2000	142	243	169,30	19	33	25,80	1,20	1,35	1,25
5000	127	236	171,50	23	41	29,90	6,76	7,14	6,91
10000	127	232	159,30	21	32	27,30	26,14	26,89	26,36

Table 12: Computational results for Experiment K in Table 1

[3] Drezner, T. and Z. Drezner, "Equity models in planar location". *Computational Management Science* **4** (2007) 1–16.

[4] Drezner, T. and Z. Drezner, "Finding the optimal solution to the Huff competitive location model". *Computational Management Science* **1** (2004) 193–208.

[5] Drezner, Z. and A. Suzuki, "The Big Triangle Small Triangle Method for the Solution of Nonconvex Facility Location Problems". *Operations Research* **52** (2004) 128 – 135.

[6] Drezner, Z., G.O. Wesolowsky and T. Drezner, "The gradual covering problem". *Naval Research Logistics* **51** (2004) 841–855.

[7] Hansen, P., D. Peeters, D. Richard and J.-F.Thisse, "The Minisum and Minimax Location Problems Revisited". *Operations Research* **33** (1985) 1251–1265.

[8] Plastria, F., 'GBSSS, the generalized big square small square method for planar single facility location". *European Journal of Operational Research* **62** (1992) 163–174.

[9] Plastria, F., "Continuous location problems", in Drezner, Z. (ed.), *Facility location*, Springer Verlag, Berlin (1995) 225–262.