SEMI–SLANT SUBMANIFOLDS OF A SASAKIAN MANIFOLD

J.L. CABRERIZO, A. CARRIAZO, L. M. FERNÁNDEZ

and M. FERNÁNDEZ (Sevilla)[1]

Abstract

In this paper, we define and study both bi–slant and semi–slant submanifolds of an almost contact metric manifold and, in particular, of a Sasakian manifold.

Key words: Sasakian manifold, slant submanifold, slant distribution, bi–slant submanifold, semi–slant submanifold.

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0 Introduction.

The differential geometry of slant submanifolds have shown an increasing development since B.-Y. Chen defined slant immersions in complex geometry as a natural generalization of both holomorphic immersions and totally real immersions (see [6]). Many authors have studied such slant immersions in almost Hermitian manifolds. In [8], A. Lotta has introduced the notion of slant immersion of a Riemannian manifold into an almost contact metric manifold. In [4], we have studied and characterized slant submanifolds of K–contact and Sasakian manifolds and we have given several examples of such immersions. We have also studied other properties of slant submanifolds in [5]. Moreover, in [3], we have presented existence and uniqueness theorems for slant immersions into Sasakian-space-forms, which are similar to that of B.-Y. Chen and L. Vrancken in complex geometry [7].

Recently, in [9] N. Papaghiuc has introduced a class of submanifolds in an almost Hermitian manifold, called the semi–slant submanifolds, such that the class of proper CR–submanifolds and the class of slant submanifolds appear as particular cases in the class of semi–slant submanifolds. The purpose of the present paper is to define and study a contact version of semi–slant submanifolds, so that both semi–invariant [1] and contact slant submanifolds to appear as particular cases of the introduced notion.

In Section 1, we review basic formulas and definitions for almost contact metric manifolds and their submanifolds, which we shall use later. In Section 2, we recall the definitions and some properties given in [4, 8, 9]. We also study some relations

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between slant submanifolds of an almost contact metric manifold and semi–slant submanifolds of an almost Hermitian manifold. In Section 3, we introduce the notion of slant distribution in contact geometry and we present a more general class of submanifolds: bi–slant submanifolds. Finally, we define and study semi–slant submanifolds in Section 4.

1 Preliminaries.

Let $(\widetilde{M}, g)$ be an odd–dimensional Riemannian manifold and denote by $T\widetilde{M}$ the Lie algebra of vector fields in $\widetilde{M}$. Then $\widetilde{M}$ is said to be an *almost contact metric manifold* [2] if there exist on $\widetilde{M}$ a tensor ϕ of type $(1,1)$ and a global vector field ξ (*structure vector field*) such that, if η is the dual 1–form of ξ, then

$$\phi^2 X = -X + \eta(X)\xi, \quad g(X,\xi) = \eta(X),$$

$$g(\phi X, \phi Y) \ = \ g(X,Y) - \eta(X)\eta(Y),$$

for any $X, Y \in T\widetilde{M}$. Let Φ denote the 2–form in $\widetilde{M}$ given by $\Phi(X,Y) \ = \ g(X, \phi Y)$ for all $X, Y \in T\widetilde{M}$. The 2–form Φ is called *the fundamental 2–form* in $\widetilde{M}$ and the manifold is said to be a *contact metric manifold* if $\Phi = \mathrm{d}\eta$. If ξ is a Killing vector field with respect to g, the contact metric structure is called a *K–contact structure*. It is known that a contact metric manifold is K–contact if and only if $\widetilde{\nabla}_X \xi = -\phi X$, for any $X \in T\widetilde{M}$, where $\widetilde{\nabla}$ denotes the Levi–Civita connection of $\widetilde{M}$.

The almost contact structure of $\widetilde{M}$ is said to be *normal* if $[\phi, \phi] + 2\mathrm{d}\eta \otimes \xi \ = \ 0$, where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ. A *Sasakian manifold* is a normal contact metric manifold. Every Sasakian manifold is a K–contact manifold. It is easy to show that an almost contact metric manifold is a Sasakian manifold if and only if

$$(\widetilde{\nabla}_X \phi) Y \ = \ g(X,Y)\xi - \eta(Y)X, \tag{1.1}$$

for any $X, Y \in T\widetilde{M}$.

Now, let M be a submanifold immersed in $\widetilde{M}$. We also denote by g the induced metric on M. Let TM be the Lie algebra of vector fields in M and $T^{\perp}M$ the set of all vector fields normal to M. Denote by ∇ the Levi–Civita connection of M. Then, the Gauss – Weingarten formulas are given by

$$\widetilde{\nabla}_X Y \ = \ \nabla_X Y + \sigma(X,Y), \quad \widetilde{\nabla}_X V \ = \ -A_V X + D_X V,$$

for any $X, Y \in TM$ and any $V \in T^{\perp}M$, where D is the connection in the normal bundle, σ is the second fundamental form of M and A_V is the Weingarten endomorphism associated with V. The second fundamental form σ and the shape operator A are related by $g(A_V X, Y) \ = \ g(\sigma(X,Y), V)$.

For any $X \in TM$, we write $\phi X = TX + NX$, where TX is the tangential component of ϕX and NX is the normal component of ϕX. Similarly, for any $V \in T^{\perp}M$, we have $\phi V = tV + nV$, where tV (resp. nV) is the tangential component (resp. normal component) of ϕV.

The submanifold M is said to be *invariant* if N is identically zero, that is, $\phi X \in TM$, for any $X \in TM$. On the other hand, M is said to be an *anti–invariant* submanifold if T is identically zero, that is, $\phi X \in T^{\perp}M$, for any $X \in TM$.

2 Slant Immersions.

Let M be a Riemannian manifold, isometrically immersed in an almost contact metric manifold $(\widetilde{M}, \phi, \xi, \eta, g)$. From now on, we suppose that the structure vector field ξ is tangent to M. Hence, if we denote by $\mathcal{D}$ the orthogonal distribution to ξ in TM, we can consider the orthogonal direct decomposition $TM = \mathcal{D} \oplus < \xi >$.

For each nonzero vector X tangent to M at x, such that X is not proportional to ξ_x, we denote by $\theta(X)$ the angle between ϕX and T_xM.

Then, M is said to be *slant* ([8]) if the angle $\theta(X)$ is a constant, which is independent of the choice of $x \in M$ and $X \in T_xM - < \xi_x >$. The angle θ of a slant immersion is called the *slant angle* of the immersion. Invariant and anti–invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \pi/2$ respectively. A slant immersion which is not invariant nor anti–invariant is called a *proper* slant immersion.

An useful characterization of slant submanifolds in almost contact metric manifolds is given by the following theorem:

Theorem 2.1.– [4] *Let M be a submanifold of an almost contact metric manifold $\widetilde{M}$ such that $\xi \in TM$. Then, M is slant if and only if there exists a constant $\lambda \in [0,1]$ such that $T^2 = -\lambda I + \lambda \eta \otimes \xi$. Furthermore, in such case, if θ is the slant angle of M, then, $\lambda = \cos^2 \theta$.*

Another important notion about slant immersions is that of *semi–slant* submanifolds of an almost Hermitian manifold, introduced by N. Papaghiuc in [9]. Given a submanifold S, isometrically immersed in an almost Hermitian manifold $(\widetilde{S}, J, g_1)$, Papaghiuc says a differentiable distribution D on S to be a *slant distribution* if for any nonzero vector $X \in D_x$; $x \in S$, the angle between JX and the vector space D_x is constant, i.e., it is independent of the choice of $x \in S$ and of $X \in D_x$. This constant angle is called the *slant angle* of the slant distribution D.

Now, S is said to be a *semi–slant* submanifold if there exist on S two differentiable orthogonal distributions D_1 and D_2 such that $TM = D_1 \oplus D_2$, D_1 is a complex distribution (i.e. $J(D_1) = D_1$) and D_2 is a slant distribution with the slant angle $\theta \neq 0$. In particular, if $\dim D_1 = 0$ and $\theta \neq \pi/2$, then Papaghiuc obtain the proper slant submanifolds of almost Hermitian manifolds, introduced by B.Y. Chen (see [6]).

In fact, by studying two classic examples, we can find some relations between slant submanifolds of almost contact metric manifolds and semi–slant submanifolds of almost Hermitian manifolds.

Let $(\widetilde{M}, \phi, \xi, \eta, g)$ be an almost contact metric manifold. We consider the manifold $\widetilde{M} \times \mathbf{R}$. We denote by $(X, f\frac{d}{dt})$ a vectorial field of $\widetilde{M} \times \mathbf{R}$, where X is tangent to $\widetilde{M}$, t is the coordinate of $\mathbf{R}$ and f is a differentiable function on $\widetilde{M} \times \mathbf{R}$. We define on this manifold the almost complex structure J given by:

$$J(X, f\frac{d}{dt}) \;=\; (\phi X - f\xi, \eta(X)\frac{d}{dt}). \tag{2.1}$$

Then, it is well known that $(\widetilde{M} \times \mathbf{R}, J, g_1)$ is an almost Hermitian manifold [2], where g_1 denotes the product metric:

$$g_1((X, f\frac{d}{dt}), (Y, h\frac{d}{dt})) \;=\; g(X, Y) + fh.$$

The following result show how to obtain semi–slant submanifolds of $\widetilde{M} \times \mathbf{R}$ from slant submanifolds of $\widetilde{M}$.

Theorem 2.2.– *Let M be a non-invariant slant submanifold of an almost contact metric manifold $\widetilde{M}$. Then, $M \times \mathbf{R}$ is a semi–slant submanifold of $\widetilde{M} \times \mathbf{R}$, with complex distribution $D_1 =< (\xi, 0), (0, d/dt) >$ and slant distribution $D_2 = \{(X, 0) \;/\; X \in \mathcal{D}\}$, respectively.*

Proof: It is clear that distributions D_1 and D_2 are perpendicular and that $T(M \times \mathbf{R}) = D_1 \oplus D_2$. Moreover, D_1 is a complex distribution, given that, by virtue of (2.1), $J(\xi, 0) = (0, d/dt)$ and so $J(0, d/dt) = -(\xi, 0)$.

Finally, it is easy to see that D_2 is a slant distribution, in the sense of [9]. □

Remark 2.3.– We have excluded the invariant case in the above theorem since, in [9], the slant distribution of a semi–slant submanifold must have a non-zero angle. Nevertheless, it is easy to prove that, if M is an invariant submanifold of $\widetilde{M}$, then $M \times \mathbf{R}$ is an invariant submanifold of $\widetilde{M} \times \mathbf{R}$.

Remark 2.4.– In [8, p. 193], Lotta has proved that, if $M \times \mathbf{R}$ is slant in $\widetilde{M} \times \mathbf{R}$, then M must be an invariant submanifold of $\widetilde{M}$ and so, $M \times \mathbf{R}$ is a complex submanifold of $\widetilde{M} \times \mathbf{R}$. Note that there is not a contradiction between this result and Theorem 2.2, because the semi–slant submanifold $M \times \mathbf{R}$ is slant if and only if it is a complex submanifold, since $J(\xi, 0) = (0, d/dt)$.

Note also that, by using the notion of semi–slant submanifold of an almost Hermitian manifold, we have obtained a more general result.

The second classic example was given by Tashiro in [10]. Let $\widetilde{S}$ be an almost Hermitian manifold with almost complex structure J. Let $\widetilde{M} \hookrightarrow \widetilde{S}$ be an orientable hypersurface, isometrically immersed in $\widetilde{S}$. Denote by g both the metric of $\widetilde{S}$ and the induced one in $\widetilde{M}$.

Let C be the unit normal to $\widetilde{M}$. Then, $\xi = -JC$ is tangent to M. We define ϕ and η by $JX = \phi X + \eta(X)C$, for any X tangent to $\widetilde{M}$. It is easy to see that (ϕ, ξ, η, g) is an almost contact metric structure on $\widetilde{M}$.

Let $S \hookrightarrow \widetilde{S}$ be an immersion such that C and ξ are tangent to S. Denote by D_1 the distribution in TS spanned by C and ξ, $D_1 \ = < C, \xi >$, and by D_2, the orthogonal distribution in TS. Suppose that there are an orientable hypersurface $M \hookrightarrow S$, normal to C, and an immersion $M \hookrightarrow \widetilde{M}$ such that the following diagram is conmutative:

$$\begin{array}{ccc} \widetilde{M} & \hookrightarrow & \widetilde{S} \\ \uparrow & & \uparrow \\ M & \hookrightarrow & S \end{array} .$$

We can state the following result:

Theorem 2.5.– *In the above conditions, M is a slant submanifold of $\widetilde{M}$, with slant angle $\theta \neq 0$, if and only if S is a semi–slant submanifold of $\widetilde{S}$, with complex distribution D_1 and θ–slant distribution D_2.*

PROOF: Given that $\xi = -JC$, it is clear that D_1 is a complex distribution. Thus, it is enough to prove that M is slant if and only if D_2 is a slant distribution on S, but this is easy to see since ϕ and J are equal on D_2. □

Remark 2.6.– It is clear that M is invariant if and only if S is a complex submanifold.

We can easily obtain an example of the above situation:

Example 2.7.– Let $\mathbf{R}^6$ be the Euclidean space of dimension 6, with the standard metric and the almost complex structure given by $J(\partial/\partial x^i) = \partial/\partial y^i$, for any $i = 1, 2, 3$, where (x^i, y^i) denote the cartesian coordinates.

Let $\mathbf{R}^5 \hookrightarrow \mathbf{R}^6$ be the usual immersion. Then, $C = \partial/\partial y^3$ is the unit normal to $\mathbf{R}^5$ and so, $\xi = -JC = \partial/\partial x^3$.

Now, for any $\theta \neq 0$, we can consider the immersions:

$$\varphi_1 : \mathbf{R}^4 \to \mathbf{R}^6 : (u, v, t, s) \mapsto (u\cos\theta, u\sin\theta, t, v, 0, s),$$

$$\varphi_2 : \mathbf{R}^3 \to \mathbf{R}^5 : (u, v, t) \mapsto (u\cos\theta, u\sin\theta, t, v, 0).$$

Then, it is easy to show that all conditions of Theorem 2.5 are satisfied, where the immersion $\mathbf{R}^3 \hookrightarrow \mathbf{R}^4$ is the usual one. In fact, we can directly prove that φ_1

is a semi–slant immersion, with complex distribution $D_1 =< \partial/\partial x^3, \partial/\partial y^3 >$ and θ–slant distribution $D_2 =< \cos\theta\ \partial/\partial x^1 + \sin\theta\ \partial/\partial x^2, \partial/\partial y^1 >$. On the other hand, φ_2 is a θ–slant immersion, where $\mathbf{R}^5$ has the almost contact metric structure induced by the described almost Hermitian structure on $\mathbf{R}^6$.

Hence, it is clear that semi–slant submanifolds are interesting from a geometrical point of view. In the following sections, we are going to introduce the contact versions of slant distributions and semi–slant immersions.

3 Slant Distributions.

In this section, we are going to give the notion of *slant distribution* on a submanifold of an almost contact metric manifold. By using this notion, we will define *bi–slant* submanifolds, which appear as a natural generalization of slant submanifolds.

From now on, let M be a Riemannian manifold, isometrically immersed in an almost contact metric manifold $(\widetilde{M}, \phi, \xi, \eta, g)$, such that $\xi \in TM$. We call a differentiable distribution $\mathcal{V}$ on M a *slant distribution* if for each $x \in M$ and each nonzero vector $X \in \mathcal{V}_x$, the angle $\theta'(X)$ between ϕX and the vectorial subspace $\mathcal{V}_x$ is a constant, which is independent of the choice of $x \in M$ and $X \in \mathcal{V}_x$. In this case, the constant angle θ' is called the *slant angle* of the distribution $\mathcal{V}$ (compare with the case of almost Hermitian manifolds).

If we consider the distribution $\mathcal{D}$, then we can state the following lemma:

Lemma 3.1.– *Let x be a point of M and $X \in T_xM$. Then, $\theta(X) = \theta'(X)$, where $\theta'(X)$ denotes the angle between ϕX and $\mathcal{D}_x$.*

Proof: It is enough to prove that $TX = \Pi_{\mathcal{D}} X$, for any $X \in TM$, where $\Pi_{\mathcal{D}}$ denotes the orthogonal proyection of ϕ on $\mathcal{D}$, but this is clear, since $\phi\xi = 0$. □

Hence, we can give a new notion of slant submanifold, equivalent to Lotta's definition:

Proposition 3.2.– *The submanifold M is slant if and only if $\mathcal{D}$ is a slant distribution, with the same slant angle.*

Proof: By virtue of Lemma 3.1, it is obvious that if M is a slant submanifold, then $\mathcal{D}$ is a slant distribution with the same slant angle. Conversely, given $X \in TM - < \xi >$, the angle $\theta(X)$ satisfies:

$$\cos\theta(X) = \frac{g(TX, \phi X)}{|TX||\phi X|} = \frac{|TX|}{\sqrt{|X|^2 - \eta^2(X)}}. \tag{3.1}$$

On the other hand, if we consider $X-\eta(X)\xi \in \mathcal{D}$, then we have:

$$\cos\theta'(X-\eta(X)\xi) \;=\; \frac{|\Pi_{\mathcal{D}}(X-\eta(X)\xi)|}{|X-\eta(X)\xi|}. \tag{3.2}$$

But, $\sqrt{|X|^2-\eta^2(X)} = |X-\eta(X)\xi|$ and $TX = \Pi_{\mathcal{D}}(X-\eta(X)\xi)$. Thus, (3.1) is equal to (3.2), which is a constant since $\mathcal{D}$ is a slant distribution. $\square$

Now, we can define bi–slant submanifolds. We say that M is a *bi–slant* submanifold of $\widetilde{M}$ if there exist two orthogonal distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on M such that:

(i) TM admits the orthogonal direct decomposition: $TM \;=\; \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus <\xi>$.

(ii) For any $i=1,2$, $\mathcal{D}_i$ is a slant distribution with slant angle θ_i.

Given a bi–slant submanifold M, we can write, for any $X \in TM$,

$$X \;=\; P_1X + P_2X + \eta(X)\xi, \tag{3.3}$$

where P_iX denotes the component of X in $\mathcal{D}_i$, for any $i=1,2$. In particular, if $X \in \mathcal{D}_i$, then we obtain $X = P_iX$. If we define $T_i = P_i \circ T$, then we have

$$\phi X \;=\; T_1X + T_2X + NX, \tag{3.4}$$

for any $X \in TM$, given that $\eta(\phi X) = 0$.

Let d_1 (resp. d_2) denote the dimension of the distribution $\mathcal{D}_1$ (resp. $\mathcal{D}_2$). By virtue of Proposition 3.2, if either d_1 or d_2 vanishes, the bi–slant submanifold is a slant submanifold. Thus, slant submanifolds (and, therefore, invariant and anti-invariant submanifolds) are particular cases of bi–slant submanifolds. Moreover, we easily find examples of non-trivial bi–slant submanifolds.

Example 3.3.– For any $\theta_1, \theta_2 \in [0, \pi/2]$,

$$x(u,v,w,s,t) \;=\; 2(u,0,w,0,v\cos\theta_1, v\sin\theta_1, s\cos\theta_2, s\sin\theta_2, t)$$

defines a 5–dimensional bi–slant submanifold M, with slant angles θ_1 and θ_2, in $\mathbf{R}^9$ with its usual Sasakian structure (ϕ_0, ξ, η, g) (see [4]).

In fact, it is easy to see that

$$\begin{gathered} e_1 = 2(\frac{\partial}{\partial x^1} + y^1\frac{\partial}{\partial z}), \quad e_2 = \cos\theta_1(2\frac{\partial}{\partial y^1}) + \sin\theta_1(2\frac{\partial}{\partial y^2}), \\ e_3 = 2(\frac{\partial}{\partial x^3} + y^3\frac{\partial}{\partial z}), \quad e_4 = \cos\theta_2(2\frac{\partial}{\partial y^3}) + \sin\theta_2(2\frac{\partial}{\partial y^4}), \quad e_5 = 2\frac{\partial}{\partial z} = \xi, \end{gathered} \tag{3.5}$$

form a local orthonormal frame of TM. We define the distributions $\mathcal{D}_1 =< e_1, e_2 >$ and $\mathcal{D}_2 =< e_3, e_4 >$.

Then, it is clear that $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus < \xi >$. It can be easily proved that $\mathcal{D}_i$ is a slant distribution with slant angle θ_i, for any $i = 1, 2$.

In particular, if we consider $\theta_1 = \theta_2 = \theta$ in Example 3.3, it results that M is a θ–slant submanifold. Nevertheless, this is not a general fact, as we can see in the following example:

Example 3.4.– For any $\theta_1 \in [0, \pi/2]$, we choose $\theta_2 \in (0, \pi/2]$ such that $\cos\theta_2 = (\cos\theta_1)/\sqrt{2}$. Then,

$$x(u, v, w, s, t) \;=\; 2(u, 0, w, u, v\cos\theta_1, v\sin\theta_1, s\cos\theta_2, s\sin\theta_2, t)$$

defines a 5–dimensional bi–slant submanifold M in $(\mathbf{R}^9, \phi_0, \xi, \eta, g)$, with both slant angles equal to θ_2, such that it is not a slant submanifold.

In fact, we can choose a local orthonormal frame $\{e_1, \ldots, e_5\}$ of TM such that $e_1 = 1/\sqrt{2}\,\{2(\partial/\partial x^1 + y^1\partial/\partial z) + 2(\partial/\partial x^4 + y^4\partial/\partial z)\}$ and $e_2, \ldots, e_5$ are given by (3.5). We define $\mathcal{D}_1 =< e_1, e_2 >$ and $\mathcal{D}_2 =< e_3, e_4 >$. It is easy to see that both $\mathcal{D}_1$ and $\mathcal{D}_2$ are slant distributions with the same slant angle θ_2. Nevertheless, we can obtain that M is not slant since $\theta_2 \neq 0$.

However, we can prove, by a direct calculation, the following result:

Proposition 3.5.– *Let M be a bi–slant submanifold with angles $\theta_1 = \theta_2 = \theta$. If $g(\phi X, Y) = 0$, for any $X \in \mathcal{D}_1$ and any $Y \in \mathcal{D}_2$, then M is slant with angle θ.*

Remark 3.6.– Note that Example 3.3 satisfies the aditional condition of Proposition 3.5.

Bi–slant submanifolds are also a generalization of semi–invariant submanifolds, introduced by Bejancu and Papaghiuc in [1]. In that paper, M is said to be a semi–invariant submanifold if there exist two orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on M, such that:

(i) $TM \;=\; \mathcal{D} \oplus \mathcal{D}^\perp \oplus < \xi >$.

(ii) The distribution $\mathcal{D}$ is invariant, i.e., $\phi\mathcal{D} = \mathcal{D}$.

(iii) The distribution $\mathcal{D}^\perp$ is anti-invariant, i.e., $\phi\mathcal{D}^\perp \subset T^\perp M$.

We are now going to show that semi–invariant submanifolds are bi–slant submanifolds, with angles $\theta_1 = 0$ and $\theta_2 = \pi/2$. We first need the following two lemmas:

Lemma 3.7.– *Suppose that there exist two orthogonal distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on M such that $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus < \xi >$. Then, $\mathcal{D}_1$ is invariant if and only if it is slant with angle $\theta_1 = 0$.*

Proof: It is clear that, if $\mathcal{D}_1$ is invariant, then it is slant with zero angle. The converse is easy to prove, by taking into account (3.4) and that, if $\mathcal{D}_1$ is a slant distribution with zero slant angle, then $|T_1X| = |X|$, for any $X \in \mathcal{D}_1$. $\square$

Lemma 3.8.– *In the above conditions, if $\mathcal{D}_1$ is invariant, then $TX = T_2X$ for any $X \in \mathcal{D}_2$.*

PROOF: For any $X \in \mathcal{D}_1$ and $Y \in \mathcal{D}_2$, we have $g(TX, Y) = -g(X, \phi Y) = 0$, since $\mathcal{D}_1$ is invariant. Thus, $T_1X = 0$ and the result holds. $\square$

Proposition 3.9.– *The submanifold M is semi–invariant if and only if M is bi–slant with angles $\theta_1 = 0$ and $\theta_2 = \pi/2$.*

PROOF: The direct implication is immediate. The converse follows directly from Lemmas 3.7 and 3.8. $\square$

Remark 3.10.– In Proposition 3.9, we have proved that, if $\mathcal{D}_2$ is an anti-invariant distribution, then it is slant with angle $\pi/2$. This is not a general fact, as we can show by putting $\theta_1 = \theta_2 = \pi/2$ in Example 3.4. Thus, if we write $\mathcal{D}_1 =< e_3, e_4 >$ and $\mathcal{D}_2 =< e_1, e_2 >$, we obtain a submanifold M endowed with two orthogonal distributions such that $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus < \xi >$, $\mathcal{D}_2$ is slant with angle $\pi/2$ but it is not anti-invariant, because $\phi_0 e_1$ is not normal to M.

We finish this section by stablishing a general theorem for bi–slant submanifolds, which we shall use later. Note that this result is an obvious generalization of the direct implication of Theorem 2.1, and it can be proved by following the same steps.

Theorem 3.11.– *Let M be a bi–slant submanifold of an almost contact metric manifold $\widetilde{M}$. Denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the slant distributions of M, with slant angles θ_1 and θ_2, respectively. Then, given $i = 1, 2$, for any $X \in \mathcal{D}_i$:*

$$T_i^2 X = -\cos^2\theta_i \ X. \tag{3.6}$$

4 Semi–Slant Submanifolds.

Now, we can introduce the notion of semi–slant submanifolds in contact geometry. We say that M is a *semi–slant* submanifold of $\widetilde{M}$ if there exist two orthogonal distribution $\mathcal{D}_1$ and $\mathcal{D}_2$ on M such that:

(i) TM admits the orthogonal direct decomposition $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus < \xi >$.

(ii) The distribution $\mathcal{D}_1$ is an invariant distribution, i.e., $\phi(\mathcal{D}_1) = \mathcal{D}_1$.

(iii) The distribution $\mathcal{D}_2$ is slant with angle $\theta \neq 0$.

In this case, we call the angle θ the *slant angle* of submanifold M. By virtue of Lemma 3.7, the invariant distribution of a semi–slant submanifold is a slant distribution with zero angle. Thus, it is obvious that, in fact, semi–slant submanifolds are particular cases of bi–slant submanifolds. Moreover, by virtue of Proposition 3.9, it is clear that, if $\theta = \pi/2$, then the semi–slant submanifold is a semi–invariant submanifold. On the other hand, if we denote the dimension of $\mathcal{D}_i$ by d_i, for $i = 1, 2$, then we find the following cases:

(a) If $d_2 = 0$, then M is an invariant submanifold.

(b) If $d_1 = 0$ and $\theta = \pi/2$, then M is an anti-invariant submanifold.

(c) If $d_1 = 0$ and $\theta \neq \pi/2$, then M is a proper slant submanifold, with slant angle θ.

On the other hand, we say that a semi–slant submanifold is *proper* if $d_1 d_2 \neq 0$ and $\theta \neq \pi/2$. It is easy to show that there are proper semi–slant submanifolds. For example, it is enough to put $\theta_1 = 0$ and $\theta_2 = \theta \in (0, \pi/2)$ in Example 3.3, in order to obtain proper semi–slant submanifolds, with slant angle θ, in $\mathbf{R}^9$ with its usual Sasakian structure (ϕ_0, ξ, η, g).

Given a semi–slant submanifold M, we denote by P_i the projection on the distribution $\mathcal{D}_i$, for any $i = 1, 2$. We also put $T_i = P_i \circ T$. Hence, equations (3.3) and (3.4) are still right. On the other hand, by applying ϕ on (3.3), we obtain

$$\phi X = \phi P_1 X + T P_2 X + N P_2 X, \tag{4.1}$$

for any $X \in TM$. By a direct calculation, we can state the following result, which we shall use later:

Lemma 4.1.– *If M is a semi–slant submanifold, then, for any $X \in TM$:*
(i) $\phi P_1 X = T P_1 X$ and $N P_1 X = 0$.
(ii) $T P_2 X \in \mathcal{D}_2$.

In particular, (4.1) and statement (i) of Lemma 4.1 imply, for any $X \in TM$:

$$TX = \phi P_1 X + T P_2 X. \tag{4.2}$$

From (3.6) and Lemma 4.1, we obtain the following generalization of [4, Corollary 2.3]:

Lemma 4.2.– *Let M be a semi–slant submanifold, with angle θ, of a K–contact manifold $\widetilde{M}$. Then, for any $X, Y \in TM$:*

$$g(TX, TP_2 Y) = \cos^2\theta g(X, P_2 Y), \qquad g(NX, NP_2 Y) = \sin^2\theta g(X, P_2 Y). \tag{4.3}$$

In the following, we are dealing with semi–slant submanifolds of a Sasakian manifold. Our goal is to study the integrability of distributions $\mathcal{D}_1$ and $\mathcal{D}_2$, as well as to find the generalization of some interesting slant submanifolds studied in [4].

Lemma 4.3.– *Let M be a semi–slant submanifold of a Sasakian manifold $\widetilde{M}$. Then, for any $X, Y \in TM$, we have:*

$$P_1(\nabla_X \phi P_1 Y) + P_1(\nabla_X T P_2 Y) = \phi P_1(\nabla_X Y) + P_1 A_{NP_2 Y} X - \eta(Y) P_1 X, \tag{4.4}$$

$$
\begin{aligned}
P_2(\nabla_X \phi P_1 Y) + P_2(\nabla_X TP_2 Y) &= \\
= TP_2(\nabla_X Y) + P_2 A_{NP_2Y} X + t\sigma(X,Y) - \eta(Y) P_2 X, && (4.5)\\
\eta(\nabla_X \phi P_1 Y) + \eta(\nabla_X TP_2 Y) &= \eta(A_{NP_2Y}X) + g(\phi X, \phi Y), && (4.6)\\
\sigma(\phi P_1 Y, X) + \sigma(TP_2 Y, X) + D_X NP_2 Y &= NP_2(\nabla_X Y) + n\sigma(X,Y). && (4.7)
\end{aligned}
$$

PROOF: By using Gauss – Weingarten formulas, (1.1), (3.3), (4.1) and Lemma 4.1, we obtain

$$
\begin{aligned}
&\nabla_X \phi P_1 Y + \sigma(\phi P_1 Y, X) + \nabla_X TP_2 Y + \sigma(TP_2 Y, X) - A_{NP_2Y}X + \\
&+ D_X NP_2 Y \;=\; \phi P_1 \nabla_X Y + TP_2 \nabla_X Y + NP_2 \nabla_X Y + t\sigma(X,Y) + \\
&+ n\sigma(X,Y) + g(X,Y)\xi - \eta(Y) P_1 X - \eta(Y) P_2 X - \eta(Y)\eta(X)\xi, \qquad (4.8)
\end{aligned}
$$

for any $X, Y \in TM$.

Hence, (4.4)–(4.7) follow from (4.8), by identifying the components on $\mathcal{D}_1$, $\mathcal{D}_2$, $< \xi >$ and $T^{\perp}M$ respectively. $\square$

Remark 4.4.– Note that it is not necessary for $\mathcal{D}_2$ to be a slant distribution in proofs of Lemmas 4.1 and 4.3. Hence, these results hold for a submanifold satisfying conditions (i) and (ii) of semi–slant submanifold definition.

Note also that, to obtain (4.6), it is not necessary for $\widetilde{M}$ to be a Sasakian manifold. From (4.3), we have the following proposition, in which we do need for $\mathcal{D}_2$ to be a slant distribution.

Proposition 4.5.– *Let M be a semi–slant submanifold, with angle θ, of a K–contact manifold $\widetilde{M}$. Then, for any $X, Y \in TM$, we have:*

$$
\eta(\nabla_X \phi P_1 Y) = g(X, P_1 Y), \qquad (4.9)
$$

$$
\eta(\nabla_X TP_2 Y) = \cos^2\theta g(X, P_2 Y), \qquad \eta(A_{NP_2Y}X) = -\sin^2\theta g(X, P_2 Y). \qquad (4.10)
$$

In particular, (4.6) *holds.*

Now we can study the integrability conditions of distributions $\mathcal{D}_1$ and $\mathcal{D}_2$, which are involved in the definition of a semi–slant submanifold.

Proposition 4.6.– *Let M be a semi–slant submanifold of a Sasakian manifold $\widetilde{M}$ such that $d_1 \neq 0$. Then, the invariant distribution $\mathcal{D}_1$ is not integrable.*

PROOF: It is easy to see that $g([X, \phi X], \xi) = 0$, for any $X \in \mathcal{D}_1$. $\square$

Proposition 4.7.– *Let M be a semi–slant submanifold of a Sasakian manifold $\widetilde{M}$. Then the slant distribution $\mathcal{D}_2$ is integrable if and only if M is a semi–invariant submanifold.*

Proof: It is easy to see that $g([X,Y],\xi) = 2g(Y,T_2X)$, for any $X,Y \in \mathcal{D}_2$. Hence, if $\mathcal{D}_2$ is integrable, then $T_2 \equiv 0$ and so, $\theta = \pi/2$, by virtue of (3.6). In this case, we know that M is a semi–invariant submanifold. The converse is given by [1, Theorem 2.1]. □

Remark 4.8.– Note that, in Proposition 4.6, it is enough for $\widetilde{M}$ to be a contact metric manifold, since, in this case, $g([X,Y],\xi) = -2g(X,\phi Y)$, for any $X,Y \in TM$. This fact also works for the necessary condition in Proposition 4.7. Nevertheless, Theorem 2.1 of [1] uses a Sasakian structure on $\widetilde{M}$.

On the other hand, we can also note that, if $d_1 = 0$ in Proposition 4.7, then we obtain Proposition 3.2 of [4] in a Sasakian manifold.

It is more interesting to study the integrability of distributions $\mathcal{D}_1 \oplus <\xi>$ and $\mathcal{D}_2 \oplus <\xi>$:

Proposition 4.9.– *Let M be a semi–slant submanifold of a Sasakian manifold $\widetilde{M}$. Then, we have:*
(i) The distribution $\mathcal{D}_1 \oplus <\xi>$ is integrable if and only if

$$\sigma(X,\phi Y) \;=\; \sigma(Y,\phi X), \tag{4.11}$$

for any $X,Y \in \mathcal{D}_1$.
(ii) The distribution $\mathcal{D}_2 \oplus <\xi>$ is integrable if and only if

$$P_1(\nabla_X TY - \nabla_Y TX) \;=\; P_1(A_{NY}X - A_{NX}Y), \tag{4.12}$$

for any $X,Y \in \mathcal{D}_2 \oplus <\xi>$.

Proof: By using (4.7), we see that

$$\sigma(X,\phi Y) - \sigma(Y,\phi X) \;=\; NP_2[X,Y], \tag{4.13}$$

for any $X,Y \in \mathcal{D}_1 \oplus <\xi>$. Hence, if $\mathcal{D}_1 \oplus <\xi>$ is integrable, then (4.11) holds directly from (4.13). Conversely, let $X,Y \in \mathcal{D}_1 \oplus <\xi>$ be. It is easy to prove that

$$\sigma(X,\phi Y) - \sigma(Y,\phi X) \;=\; \sigma(P_1X,\phi P_1Y) - \sigma(P_1Y,\phi P_1X) \;=\; 0,$$

by virtue of (4.11). Thus, by applying (4.13) it follows $NP_2[X,Y] = 0$. So, we can easily deduce that $P_2[X,Y]$ must vanish, since $\mathcal{D}_2$ is a slant distribution with non-zero slant angle. Therefore, $[X,Y] \in \mathcal{D}_1 \oplus <\xi>$. This ends the proof of statement (i).

With regards to statement (ii), we first compute

$$\phi P_1[X,Y] \;=\; P_1(\nabla_X TY - \nabla_Y TX) - P_1(A_{NY}X - A_{NX}Y),$$

for any $X, Y \in \mathcal{D}_2 \oplus < \xi >$, by virtue of (4.4). Hence, (4.12) holds if and only if

$$\phi P_1[X,Y] = 0, \tag{4.14}$$

for any $X, Y \in \mathcal{D}_2 \oplus < \xi >$. But it can be showed that (4.14) is equivalent to $\mathcal{D}_2 \oplus < \xi >$ being an integrable distribution. □

Remark 4.10.– Statement (i) of Proposition 4.9 is a clear generalization of Theorem 2.4 of [1]. It can be easily proved that statement (ii) generalizes Theorem 2.2 of the same paper.

In [4], we have paid special attention to proper θ–slant submanifolds satisfying

$$(\nabla_X T)Y \;=\; \cos^2\theta(g(X,Y)\xi - \eta(Y)X), \tag{4.15}$$

for any $X, Y \in TM$. In fact, in [5] we have pointed out that these submanifolds are the contact version of Kaehlerian slant submanifolds (see [6]). Now, we want to find a similar condition for semi–slant submanifolds.

If we compute ∇T in Example 3.3, with $\theta_1 = 0$ and $\theta_2 = \theta \in (0, \pi/2)$, then, we obtain

$$(\nabla_X T)Y \;=\; g(P_1X,Y)\xi - \eta(Y)P_1X + \cos^2\theta(g(P_2X,Y)\xi - \eta(Y)P_2X), \tag{4.16}$$

for any $X, Y \in TM$. If we put $X = P_2X + \eta(X)\xi, Y = P_2Y + \eta(Y)\xi \in \mathcal{D}_2 \oplus < \xi >$, then (4.16) implies:

$$(\nabla_X T)Y \;=\; \cos^2\theta(g(X,Y)\xi - \eta(Y)X).$$

Thus, with respect to ∇T, the slant distribution $\mathcal{D}_2 \oplus < \xi >$ works as the tangent bundle of a proper slant submanifold satisfying (4.15). On the other hand, if $X, Y \in \mathcal{D}_1 \oplus < \xi >$, then it follows from (4.16):

$$(\nabla_X T)Y \;=\; g(X,Y) - \eta(Y)X.$$

Invariant submanifolds satisfy this equation. Moreover, we are going to show that (4.16) is a "natural" condition. We first need the following lemma:

Lemma 4.11.– *Let M be a proper semi–slant submanifold, with angle θ, of a Sasakian manifold $\widetilde{M}$. For any $X, Y \in TM$, we have:*

$$(\nabla_X T)Y \;=\; A_{NP_2Y}X + t\sigma(X,Y) + g(X,Y)\xi - \eta(Y)X. \tag{4.17}$$

Hence, M satisfies (4.16) *if and only if*

$$A_{NP_2Y}X \;=\; A_{NP_2X}Y - \sin^2\theta(\eta(X)P_2Y - \eta(Y)P_2X), \tag{4.18}$$

for any $X, Y \in TM$.

Proof: Equation (4.17) can be obtained by using (3.3), (4.2) and (4.4)–(4.6).

Now, suppose that M is a proper semi–slant submanifold satisfying (4.16). Then, by applying (3.3), (4.18) follows directly from (4.16) and (4.17).

Conversely, suppose that we have (4.18) for any $X, Y \in TM$. Thus, it is easy to see that

$$g(A_{NP_2Y}Z, X) \;=\; -g(t\sigma(Y,Z), X) - \sin^2\theta g(g(P_2Y, Z)\xi - \eta(Y)P_2Z, X),$$

for any $X, Y, Z \in TM$. Then, by applying (4.17), this implies

$$\begin{aligned}(\nabla_Z T)Y \;&=\; g(Z,Y)\xi - \eta(Y)Z - \sin^2\theta(g(P_2Y, Z)\xi - \eta(Y)P_2Z) \;=\\ &=\; g(P_1Z, Y)\xi - \eta(Y)P_1Z + \cos^2\theta(g(P_2Z, Y)\xi - \eta(Y)P_2Z),\end{aligned}$$

for any $Y, Z \in TM$ and the proof concludes. □

The following Theorem shows that (4.16) must be the expected generalization of (4.15).

Theorem 4.12.– *Let M be a proper semi–slant submanifold, with angle θ, of a Sasakian manifold $\widetilde{M}$. The following statements are equivalent:*

(i) *M satisfies* (4.16)*.*

(ii) $(\nabla_X TP_2)Y \;=\; \cos^2\theta(g(P_2X, Y)\xi - \eta(Y)P_2X)$*, for any $X, Y \in TM$.*

Proof: Suppose that M satisfies (4.16). Then, by proceeding as in Lemma 4.11, we have

$$t\sigma(X,Y) + A_{NP_2Y}X + \sin^2\theta(g(P_2X, Y)\xi - \eta(Y)P_2X) \;=\; 0, \tag{4.19}$$

for any $X, Y \in TM$. By taking P_1 in (4.19), it is easy to see that

$$P_1A_{NP_2Y}X \;=\; 0, \tag{4.20}$$

for any $X, Y \in TM$.

If we write (4.19) with $Y \in \mathcal{D}_1 \oplus < \xi >$, then we obtain

$$t\sigma(X,Y) \;=\; \sin^2\theta\eta(Y)P_2X, \tag{4.21}$$

for any $X \in TM$ and any $Y \in \mathcal{D}_1 \oplus <\xi>$. On the other hand, if we write (4.7) in the same case, it results

$$\sigma(\phi Y, X) \;=\; NP_2\nabla_X Y + n\sigma(X,Y), \tag{4.22}$$

for any $X \in TM$ and any $Y \in \mathcal{D}_1 \oplus <\xi>$. Then, it follows from (4.21) and (4.22):

$$\sigma(\phi Y, X) \;=\; NP_2\nabla_X Y + \phi\sigma(X,Y) - \sin^2\theta\eta(Y)P_2X. \tag{4.23}$$

Now then, given $X \in TM$ and $Y \in \mathcal{D}_1 \oplus <\xi>$,

$$g(NP_2\nabla_X Y, \sigma(\phi Y, X)) \;=\; g(A_{NP_2\nabla_X Y}X, \phi Y) \;=\; 0, \tag{4.24}$$

since $\phi Y \in \mathcal{D}_1$ and $P_1 A_{NP_2\nabla_X Y}X = 0$, by virtue of (4.20). Moreover, by using (4.20) and Lemma 4.1, it is easy to see that

$$g(NP_2\nabla_X Y, \phi\sigma(X,Y)) \;=\; g(A_{NP_2TP_2\nabla_X Y}X, Y) \;=\; 0, \tag{4.25}$$

if we put $Y \in \mathcal{D}_1$. Therefore, from (4.23)–(4.25), it results $NP_2\nabla_X Y = 0$, for any $X \in TM$ and any $Y \in \mathcal{D}_1$. Since M is a proper semi–slant submanifold, it follows from this equation that $P_2\nabla_X Y$ must vanish. Hence,

$$\nabla_X Y \in \mathcal{D}_1 \oplus <\xi>, \tag{4.26}$$

for any $X \in TM$ and any $Y \in \mathcal{D}_1$. In particular, this implies $\nabla_X Z \in \mathcal{D}_2 \oplus <\xi>$, for any $X \in TM$ and any $Z \in \mathcal{D}_2$. Then, by applying Lemma 4.1,

$$P_1(\nabla_X TP_2 Y) \;=\; 0, \tag{4.27}$$

for any $X, Y \in TM$. From (4.4), (4.9), (4.20), (4.26) and (4.27), we have

$$(\nabla_X \phi P_1)Y \;=\; g(P_1X, Y)\xi - \eta(Y)P_1X, \tag{4.28}$$

for any $X, Y \in TM$. Now then, it follows from (4.2) that $TP_2 = T - \phi P_1$. Hence,

$$(\nabla_X TP_2)Y \;=\; (\nabla_X T)Y - (\nabla_X \phi P_1)Y,$$

for any $X, Y \in TM$, and so, by virtue of (4.16) and (4.28), statement (ii) holds.

Conversely, suppose that M satisfies (ii). By virtue of (3.6) and Lemma 4.1, it is easy to see that

$$P_1(\nabla_X Z) \;=\; -\eta(Z)T_1X, \tag{4.29}$$

for any $X \in TM$ and any $Z \in \mathcal{D}_2 \oplus <\xi>$. Hence,

$$\nabla_X Z \in \mathcal{D}_2 \oplus <\xi>, \tag{4.30}$$

for any $X \in TM$ and any $Z \in \mathcal{D}_2$. Thus, we can deduce from (4.30) that $\nabla_X Y \in \mathcal{D}_1 \oplus < \xi >$, for any $X \in TM$ and any $Y \in \mathcal{D}_1$. Therefore, by applying Lemma 4.1, (4.4), (4.9) and (4.29), we can compute

$$(\nabla_X \phi P_1)Y \;=\; P_1 A_{NP_2Y} X + g(P_1 X, Y) - \eta(Y) P_1 X, \tag{4.31}$$

for any $X, Y \in TM$. On the other hand, condition (ii), (4.5), (4.10) and (4.30) imply

$$P_1 A_{NP_2Y} X \;=\; 0, \tag{4.32}$$

for any $X, Y \in TM$. Finally, equation (4.16) follows from (4.2), (4.31), (4.32) and condition (ii). □

The following corollary appears directly from equation (4.28):

Corollary 4.13.– *If M is a proper semi–slant submanifold of a Sasakian manifold $\widetilde{M}$, satisfaying* (4.16)*, then, for any $X, Y \in TM$:*

$$(\nabla_X \phi P_1)Y \;=\; g(P_1 X, Y)\xi - \eta(Y) P_1 X.$$

On the other hand, we can also emphasize the integrability conditions obtained in the proof of Theorem 4.12:

Corollary 4.14.– *If M is a proper semi–slant submanifold of a Sasakian manifolf $\widetilde{M}$ satisfaying* (4.16)*, then*

$$\nabla_X Y \in \mathcal{D}_1 \oplus < \xi >, \qquad \nabla_X Z \in \mathcal{D}_2 \oplus < \xi >, \tag{4.33}$$

for any $X \in TM$, $Y \in \mathcal{D}_1$ and $Z \in \mathcal{D}_2$. In particular, distributions $\mathcal{D}_1 \oplus < \xi >$ and $\mathcal{D}_2 \oplus < \xi >$ are integrable.

Remark 4.15.– Given that distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ are orthogonal, we can prove that both equations in (4.33) are equivalent. We have implicitly used this fact in the proof of Theorem 4.12. Moreover, if $\dim \mathcal{D}_2 = 2$, these equations are equivalent to the conditions of above theorem, as we are going to show in Theorem 4.16. This last theorem is a clear generalization of Theorem 5.1 of [4].

Theorem 4.16.– *Let M be a proper semi–slant submanifold, with angle θ, of a Sasakian manifold $\widetilde{M}$. Suppose* $\dim \mathcal{D}_2 = 2$*. Then, the following statements are equivalent:*
(i) M satisfies (4.16)*.*
(ii) $(\nabla_X T P_2)Y \;=\; \cos^2\theta(g(P_2 X, Y)\xi - \eta(Y) P_2 X)$*, for any $X, Y \in TM$.*
(iii) $\nabla_X Y \in \mathcal{D}_1 \oplus < \xi >$*, for any $X \in TM$ and any $Y \in \mathcal{D}_1$.*
(iv) $\nabla_X Y \in \mathcal{D}_2 \oplus < \xi >$*, for any $X \in TM$ and any $Y \in \mathcal{D}_2$.*

Proof: By virtue of Theorem 4.12, Corollary 4.14 and Remark 4.15, it is enough to prove that (iv) implies (ii). This proof is similar to that of [4, Theorem 5.1]. □

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José L. Cabrerizo, Alfonso Carriazo, Luis M. Fernández and Manuel Fernández:
Departamento de Geometría y Topología.
Facultad de Matemáticas. Universidad de Sevilla.
Apartado de Correos 1160.
41080–Sevilla.
Spain.
e–mail: acarri@cica.es