# ON THE EXISTENCE OF ALMOST CONTACT STRUCTURE AND THE CONTACT MAGNETIC FIELD

J. L. CABRERIZO[1]*, M. FERNÁNDEZ[1] and J. S. GÓMEZ[1]

[1]*Department of Geometry and Topology, University of Seville, Apdo. Correos 1160, 41080-Seville (Spain)*
*e-mails: jaraiz@us.es, mafernan@us.es, jsalvador@us.es*

(Received

**Abstract.** In this short note we give a simple proof of the existence of an almost contact metric structure on any orientable 3-dimensional Riemannian manifold $(M^3, g)$ with the prescribed metric $g$ as the adapted metric of the almost contact metric structure. By using the key formula for the structure tensor obtained in the proof of this theorem, we give an application which allows us to completely determine the magnetic flow of the contact magnetic field in any 3-dimensional Sasakian manifold.

## 1 Introduction

The existence of particular geometric structures (complex, almost complex, contact, almost contact, symplectic, etc.) on a given $n$-dimensional manifold $M^n$ is, in general, a nontrivial problem. The more interesting and well-known results correspond with the low-dimensional cases. For example, F. Hirzebruch [8] proved that the $n$-dimensional quaternionic projective space $P^n(H)$ does not admit any almost complex structure in case $n \neq 2, 3$. According to Hirzebruch's lecture at the 1958 International Congress [9], Milnor has since proved that $P^2(H)$ and $P^3(H)$ do not admit almost complex structure.

An *almost contact structure* [4] on a connected $(2n+1)$-dimensional manifold $M^{2n+1}$ is a triple $(\varphi, \xi, \eta)$, where $\varphi$ is a field of endomorphisms of the tangent spaces, $\xi$ is a vector field and $\eta$ a 1-form such that

$$\varphi^2 = -Id + \eta \otimes \xi, \;\; \eta(\xi) = 1. \tag{1}$$

Then $(M^{2n+1}, \varphi, \xi, \eta)$ is called an *almost contact manifold.* As a consequence of Eq. (1) we have also $\varphi(\xi) = 0$ and $\eta \cdot \varphi = 0$. Moreover, $\varphi$ has rank $2n$.

A Riemannian metric $g$ on the almost contact manifold $(M^{2n+1}, \varphi, \xi, \eta)$ is said to be *adapted or compatible* [4] if for all $X, Y \in \mathfrak{X}(M^{2n+1})$ the following equation is satisfied:

$$g(\varphi(X), \varphi(Y)) = g(X, Y) - \eta(X)\eta(Y). \tag{2}$$

*Corresponding Author

*Key words and phrases:* Sasakian manifold, magnetic field, magnetic curve.
*2000 Mathematics Subject Classification*: 53D10, 53C80

An immediate consequence is that $\eta$ is the covariant form of $\xi$, that is, $\eta(X) = g(\xi, X)$.

An *almost contact metric manifold* is an almost contact manifold endowed with a compatible metric $g$, which is denoted by $(M^{2n+1}, \varphi, \xi, \eta, g)$. From Eq. (2) we have that $\|\xi\|^2 = g(\xi, \xi) = 1$. Note that a conformal change of the metric, $\bar{g} = \rho^2 g$ gives an almost contact structure $(\bar{\varphi}, \bar{\xi}, \bar{\eta})$ where $\bar{\varphi} = \varphi$, $\bar{\xi} = (1/\rho)\xi$, $\bar{\eta} = \rho\eta$.

Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. The *fundamental 2-form* of the almost contact metric structure [4] is the 2-form $\omega$ on $M^{2n+1}$ given by $\omega(X, Y) = g(X, \varphi(Y))$. Notice that in general $\omega \neq \mathrm{d}\eta$. More precisely, an almost contact metric manifold with $\omega = \mathrm{d}\eta$ is called a *contact metric manifold*. In a contact metric manifold the integral curves of $\xi$ are geodesics. A contact metric manifold such that the vector field $\xi$ is a Killing vector field with respect to $g$ is called a *$K$-contact manifold*. The first basic property of a $K$-contact manifold is that

$$\nabla_X \xi = -\varphi(X). \tag{3}$$

A *Sasakian manifold* is an almost contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ such that

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X$$

for all $X, Y \in \mathfrak{X}(M^{2n+1})$. It is easy to see that any Sasakian manifold is $K$-contact. In dimension 3, the converse is true.

Obviously, the first nontrivial case for $M^{2n+1}$ to admit an almost contact metric structure is when $n = 1$, i.e., the 3-dimensional case. In 1971 J. Martinet proved that every compact orientable 3-dimensional manifold carries a contact structure [11]. The requirement on $M^{2n+1}$ of compactness is unnecessary for our purpose, therefore in this note we first give in Section 2 a simple proof of the following existence theorem.

**THEOREM 1.1** *Let $(M^3, g)$ be an oriented 3-dimensional Riemannian manifold. Then, there exists on $M^3$ an almost contact metric structure with $g$ as the adapted metric.*

As a direct application of the key formula obtained in the proof of this theorem, in Section 3 we completely determine the normal magnetic flow of the contact magnetic field on any 3-dimensional Sasakian manifold. It is a well-known fact that a charged particle in a static uniform magnetic field in Euclidean space $\mathbb{R}^3$ moves along a circular helix (i.e., a curve of constant curvature and torsion) around the line flow of the magnetic fields (the charged particle has a circular component of motion in the plane normal to the magnetic field, the constant angular frequency of revolution is the cyclotron frequency, but also drifts at constant speed in the direction of the field). As a generalization of this fact, we shall prove in Section 3 that on any (not necessarily compact) 3-dimensional Sasakian manifold, for the contact magnetic field $F = \mathrm{d}\eta$, charged particles move along helices around the orbits of the global Reeb vector field $\xi$ of the Sasakian manifold (a helix in a Riemannian manifold is an arclength parametrized curve such that all its curvatures are constants).

## 2 Proof of the theorem

Let us first recall the following well-known standard topological result (see, for example, Refs. [13] p. 149 or [14] p. 11-30 and p. 11-51).

**LEMMA 2.1** *For a connected orientable manifold $M^m$ the following assertions are equivalent.*

1. *There is a nonvanishing vector field on $M^m$.*
2. *Either $M^m$ is noncompact, or $M^m$ is compact and has Euler number $\chi(M^m) = 0$.*

PROOF OF THEOREM 1.1. From Lemma 2.1, if $m = 3$ there is a nonvanishing vector field $Z$ on $M^3$. Define on $M^3$ the unit vector field $\xi = Z / \parallel Z \parallel$ .

Let $\Omega$ denote the volume element on $(M^3, g)$. For any vector fields $U, V \in \mathfrak{X}(M^3)$ define their *cross product* $U \times V$ as the vector field on $M^3$ such that

$$g(U \times V, W) = \Omega(U, V, W), \tag{4}$$

for all $W \in \mathfrak{X}(M^3)$. Now, let $\varphi : \mathfrak{X}(M^3) \longrightarrow \mathfrak{X}(M^3)$ be the 2-rank endomorphism defined by

$$\varphi(U) = \xi \times U, \tag{5}$$

for any $U \in \mathfrak{X}(M^3)$. Then, Eq. (5) shows that $\varphi(\xi) = 0$, and if $\eta = \xi^\flat$ is the $g$-dual 1-form of $\xi$ then we have also that $\eta(\xi) = 1$. It is not difficult to see that if $X, Y, Z, W \in \mathfrak{X}(M^3)$, then the identities

$$X \times (Y \times Z) = g(X, Z)Y - g(X, Y)Z,$$

$$g(X \times Y, Z \times W) = g(X, Z)g(Y, W) - g(X, W)g(Y, Z)$$

are fulfilled. Therefore, from Eq. (5) we have also that

$$\varphi^2(U) = \varphi(\xi \times U) = \xi \times (\xi \times U) = -g(\xi, \xi)U + g(\xi, U)\xi = -U + \eta(U)\xi,$$

that is, $\varphi^2 = -Id + \eta \otimes \xi$.

What is left to show is that the metric $g$ is adapted to the almost contact structure $(\varphi, \xi, \eta)$. In fact, if $U, V \in \mathfrak{X}(M^3)$, then

$$g(\varphi U, \varphi V) = g(\xi \times U, \xi \times V) = g(U, V) - \eta(U)\eta(V),$$

and this proves the theorem. □

**REMARK 2.2**

(a) For a given unit vector field $\xi$ on $M^3$, the particular dimension of this manifold allowed us to define the tensor field $\varphi$ by means of equation (5). But if we start with any given almost contact metric manifold of dimension 3 with the suitable orientation, then this equation (5) is always satisfied. In fact, assume that $(M^3, \varphi, \xi, \eta, g)$ is a 3-dimensional almost contact metric manifold. Let $G$ be a coordinate neighborhood and take $U$ a unit vector field on $G$ orthogonal to $\xi$. Then $\{\xi, U, \varphi(U)\}$ is a local orthonormal frame which is called a $\varphi$-basis [4]. Define the orientation in such a way that the volume element $\Omega$ satisfies $\Omega(\xi, U, \varphi(U)) = 1$. As $\varphi(U)$ is an unit vector field orthogonal to $\xi$ and $U$, then $\varphi(U) = \pm \xi \times U$. But $g(\xi \times U, \varphi(U)) = \Omega(\xi, U, \varphi(U)) = 1$, and hence $\varphi(U) = \xi \times U$. Therefore $\varphi(X) = \xi \times X$ for any $X \in \mathfrak{X}(M^3)$.

(b) A similar geometrical construction to the one showed in the proof of Theorem 1.1 has been used to define special almost contact structures on 7-dimensional manifolds endowed with a 2-fold vector cross product [12]. On the other hand, the cross product defined by Eq. (4) is an example of the $r$-fold cross product on manifolds introduced by Brown and Gray (see Refs. [5] and [7]).

(c) The topology of 3-dimensional Sasakian manifolds is well-known in the compact case. In fact, any compact Sasakian manifold is a Seifert fibration but the Sasakian structures can be explicitly described [3].

# 3 An application to magnetic fields

Let $(M^m, g)$ be a Riemannian manifold and denote by $\nabla$ its Levi-Civita connection. A *magnetic field* on $(M^m, g)$ is a closed 2-form $F$ on $M^m$ ([1, 2, 6, 10]). The *Lorentz force* of $F$ is the skew-symmetric tensor field $\Phi$ given by

$$g(\Phi(X), Y) = F(X, Y). \tag{6}$$

Let us remark that $\Phi$ is metrically equivalent to $F$, so that no information is lost when $\Phi$ is considered instead of $F$. In classical terminology, it is said that $\Phi$ is obtained from $F$ by raising its second index, and $\Phi$ an $F$ are then said to be physically equivalent.

A smooth parametrized curve $\gamma(t)$ in $M^m$ is called a *magnetic curve* or a *flowline* of the magnetic field $F$ if it satisfies the Lorentz force equation

$$\nabla_{\gamma'}\gamma' = \Phi(\gamma'). \tag{7}$$

Since the Lorentz force is skew-symmetric

$$\frac{d}{\mathrm{d}t} g(\gamma', \gamma') = 2g(\nabla_{\gamma'}\gamma', \gamma') = 0,$$

that is, magnetic curves have constant speed $v(t) = \| \gamma'(t) \| = v_0$. When the magnetic curve $\gamma(t)$ is arc-length parametrized ($v_0 = 1$), then it is called a *normal magnetic curve.*

For the trivial magnetic field, $F = 0$, Eq. (7) says that normal magnetic curves are the geodesics of $(M^m, g)$. As it is well-known, they represent trajectories of free fall charged particles travelling under the influence of only gravity. Moreover, for each point $p \in M^m$ and for any unit direction $u \in T_pM^m$ there exits a unique geodesic $\gamma(t)$ such that $\gamma(0) = p$ and $\gamma'(0) = u$. When $F \neq 0$, the same existence and uniqueness property can be stated for normal magnetic curves [2]. Nevertheless, it is worth pointing out that the well-known homogeneity result for geodesics is no longer true for magnetic curves. More precisely, if $\gamma$ is the inextendible magnetic curve of $(M^m, g, F)$ determined from the initial data $(p, u)$, the curve $\beta$, defined by $\beta(t) = \gamma(\lambda t)$, $\lambda \in \mathbb{R}\backslash\{0\}$, is a magnetic trajectory of $(M^m, g, \lambda F)$ and also, when $\lambda > 0$, of $(M^m, \frac{1}{\lambda}\, g, F)$, in both cases determined from initial data $(p, \lambda u)$. Furthermore, for any constant $\lambda > 0$ the whole families of magnetic curves of $(M^m, g, F)$ and $(M^m, \lambda g, \lambda F)$ coincide. Consequently, we see that *for a nontrivial magnetic field $F$ on $(M^m, g)$ there exists no affine connection on $M^m$ whose geodesics are magnetic curves of $(M^m, g, F)$* [2].

Note that we are dealing with time-independent magnetic fields, so that in physical terminology our approach belongs to the *classical magnetostatic theory* [16].

Now, let $\omega$ be the fundamental 2-form of the contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$. In such a background we have the distinguished magnetic field $F = \mathrm{d}\eta = \omega$, which is naturally called the *contact magnetic field*. In fact, in Ref. [10] if $(M^m, J, g)$ is a Kähler manifold with Kähler form $\omega$, the *Kähler magnetic field* is $F = \omega$.

**LEMMA 3.1** *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a Sasakian manifold. Then*

1. *The Lorentz force $\Phi$ of the contact magnetic field $F = \mathrm{d}\eta = \omega$ satisfies $\Phi = -\varphi$.*
2. *If $\gamma(t)$ is a normal magnetic curve of $F$ and $\theta(t)$ denotes the angle between $\gamma'(t)$ and $\xi_{\gamma(t)}$, then $\theta(t)$ is a constant $\theta(t) = \theta_0$.*

PROOF. For the Lorentz force $\Phi$ of $F$ we have that

$$g(\Phi(X), Y) = F(X, Y) = \omega(X, Y) = g(X, \varphi(Y)) = -g(\varphi(X), Y)$$

for any $X, Y \in \mathfrak{X}(M^{2n+1})$, and hence $\Phi = -\varphi$.

Since every Sasakian manifold $M^{2n+1}$ is $K$-contact, equation $\nabla_X \xi = -\varphi(X)$ is satisfied for any $X \in \mathfrak{X}(M^{2n+1})$. Covariant derivation of $\cos\theta(t) = g(\gamma'(t), \xi_{\gamma(t)})$ gives

$$\frac{d}{\mathrm{d}t}\cos\theta(t) = g(\nabla_{\gamma'(t)}\gamma'(t), \xi_{\gamma(t)}) + g(\gamma'(t), \nabla_{\gamma'(t)}\xi_{\gamma(t)}) =$$

$$= g(\Phi(\gamma'(t)), \xi_{\gamma(t)}) + g(\gamma'(t), -\varphi(\gamma'(t)) = 0,$$

because $\Phi = -\varphi$ are both skew-symmetric tensors and $\varphi(\xi) = 0$. Thus, $\theta(t)$ is a constant $\theta(t) = \theta_0$. In particular, if $\theta_0 = 0$ or $\theta_0 = \pi$, the normal magnetic curve $\gamma(t)$ is an integral curve of $\xi$, and therefore $\gamma(t)$ is simultaneously geodesic and magnetic curve.

□

From now on, we will assume $n = 1$, therefore our background is an oriented 3-dimensional Riemannian manifold $(M^3, g)$. The theory of magnetic fields in dimension three is quite special. In particular, 2-forms are in bijective correspondence with vector fields. In fact, given a 2-form, $F \in \Lambda_2(M^3)$, we consider its star 1-form $\star F \in \Lambda_1(M^3)$ and the $g$-equivalent vector field $(\star F)^\sharp \in \mathfrak{X}(M^3)$. Thus we have defined a one-to-one map between 2-forms and vector fields. The converse trip is described as follows. For a given vector field $V \in \mathfrak{X}(M^3)$, consider its $g$-equivalent 1-form $V^\flat$ and then compute its star, $\star V^\flat$. Then we obtain a 2-form which can be also written, using the interior contraction $i_V$, as $\star V^\flat = i_V\,\Omega$, where $\Omega$ is the volume form of $(M^3, g)$.

On the other hand, it is well-known that the Lie derivative of the volume form satisfies

$$\mathcal{L}_V \Omega = d\,(i_V\,\Omega) = \mathrm{div}(V)\Omega$$

and therefore the 2-form $\star V^\flat = i_V\,\Omega$ is closed if and only if $\mathrm{div}(V) = 0$, i.e., the volume element is invariant by the local flows of $V$. This allows us to regard the magnetic fields in dimension three as divergence free vector fields. In particular, if $V$ is Killing, then $\mathrm{div}(V) = 0$.

Let $(M^3, \varphi, \xi, \eta, g)$ be a 3-dimensional (not necessarily compact) Sasakian manifold. The vector field $\xi$ is Killing, and therefore, it has associated a magnetic field $F_\xi$. The normal magnetic flow of this magnetic field $F_\xi$ is completely determined as follows.

**THEOREM 3.2** *Let $(M^3, \varphi, \xi, \eta, g)$ be a 3-dimensional Sasakian manifold. The normal flowlines $\gamma(t)$ of the contact magnetic field $F_\xi$ are the helices of axis $\xi$ with constant curvature $\kappa_0 = \sin\theta_0$ and torsion $\tau_0 = 1 - \cos\theta_0$, where $\theta_0$ is the (constant) angle between $\gamma'(t)$ and $\xi_{\gamma(t)}$.*

Proof. Suppose $\gamma(t)$ is a normal magnetic curve of $F_\xi$. Then, Lemma 3.1 says that $g(\gamma'(t), \xi_{\gamma(t)}) = \cos\theta_0$, and it was also noticed that if $\theta_0$ satisfies $\theta_0 = 0, \pi$ then $\gamma(t)$ can be regarded as a degenerate helix of axis $\xi$. Therefore from now on we shall assume that $0 < \theta_0 < \pi$. On the other hand, the key formula (5), Eq. (7) and $\Phi = -\varphi$ yield

$$\nabla_{\gamma'}\gamma' = \Phi(\gamma') = -\varphi(\gamma') = -\xi \times \gamma'. \tag{8}$$

Let $\{\gamma'(t), N(t), B(t)\}$, $\kappa(t)$, $\tau(t)$ be the Frenet frame, the (geodesic) curvature and the torsion of $\gamma(t)$, respectively. The first Frenet equation for $\gamma$ reads

$$\nabla_{\gamma'}\gamma' = \kappa N, \tag{9}$$

Then equations (8) and (9) give $\kappa N = -\xi \times \gamma'$, and hence

$$\kappa^2 = g(\xi \times \gamma', \xi \times \gamma') = 1 - \cos^2\theta_0 = \sin^2\theta_0. \tag{10}$$

Thus $\kappa(t) = \kappa_0 = \sin\theta_0$ is a constant. Now, the binormal vector of $\gamma$ is defined by

$$B = \gamma' \times N = -\frac{1}{\kappa_0}\,\gamma' \times (\xi \times \gamma') = -\frac{1}{\kappa_0}\,(\xi - \cos\theta_0\gamma'). \tag{11}$$

From Eq. (11) and the third Frenet equation $\nabla_{\gamma'}B = -\tau N$ we have

$$-\frac{1}{\kappa_0}\,(\nabla_{\gamma'}\xi - \cos\theta_0\nabla_{\gamma'}\gamma') = \tau\left(\frac{1}{\kappa_0}\,\xi \times \gamma'\right), \tag{12}$$

and hence

$$\varphi(\gamma') - \cos\theta_0\,\varphi(\gamma') = \tau\,\xi \times \gamma' = \tau\,\varphi(\gamma'),$$

which gives $\tau = \tau_0 = 1 - \cos\theta_0$. Therefore $\gamma$ is a helix (curvature and torsion are constant) with axis $\xi$.

Conversely, assume that $\gamma(t)$ is an arc-length parametrized helix with axis $\xi$, constant curvature $\kappa_0 = \sin\theta_0 > 0$ and constant torsion $\tau_0 = 1 - \cos\theta_0$, $0 < \theta_0 < \pi$, where $\theta_0$ is the angle between $\gamma'(t)$ and $\xi_{\gamma(t)}$. Then, the covariant derivative of $\cos\theta_0 = g(\gamma', \xi)$ along $\gamma$ gives

$$0 = g(\nabla_{\gamma'}\gamma', \xi) + g(\gamma', \nabla_{\gamma'}\xi) = g(\kappa_0 N, \xi) + g(\gamma', -\varphi(\gamma')) = \kappa_0 g(N, \xi),$$

where we have used the Frenet equation (9) and the fundamental equation on Sasakian manifolds $\nabla_X\xi = -\varphi(X)$. Thus, $N$ is orthogonal to $\xi$, and therefore $N = \lambda\,\xi \times \gamma'$, where $\lambda(t)$ is a nonvanishing function. Computing modules on both sides of this equation we obtain $1 = |\lambda(t)| \sin\theta_0$ and hence we conclude that $\lambda(t) = \lambda_0 \neq 0$ is a constant. Thus, we have that

$$B = \gamma' \times N = \lambda_0 \gamma' \times (\xi \times \gamma') = \lambda_0 (\xi - \cos\theta_0 \, \gamma') \,.$$

A substitution of this formula for $B$ in the third Frenet equation $\nabla_{\gamma'} B = -\tau_0 \, N$ yields

$$\lambda_0 \left( \nabla_{\gamma'} \xi - \cos\theta_0 \nabla_{\gamma'} \gamma' \right) = -\tau_0 \, \lambda_0 \, \xi \times \gamma' = -\tau_0 \, \lambda_0 \, \varphi(\gamma').$$

But since $\nabla_{\gamma'} \xi = -\varphi(\gamma')$, the last equation then reads

$$-\varphi(\gamma') - \cos\theta_0 \, \nabla_{\gamma'} \gamma' = -\tau_0 \, \varphi(\gamma'),$$

or equivalently,

$$\nabla_{\gamma'} \gamma' = \frac{\tau_0 - 1}{\cos\theta_0} \, \varphi(\gamma') = -\varphi(\gamma') = \Phi(\gamma').$$

Therefore $\nabla_{\gamma'} \gamma' = \Phi(\gamma')$, and this proves that $\gamma$ is a normal flowline of the contact magnetic field $F_\xi$.

□

**REMARK 3.3**

(a): As we noticed, the limit cases $\theta_0 = 0, \pi$ mean that $\gamma$ is an integral curve of $\xi$. But the trajectories of $\xi$ are then geodesics ($\nabla_\xi \, \xi = 0$), which fit with our formula $\kappa = \sin\theta_0 = 0$ for the geodesic curvature in Theorem 3.2.

(b): It is a well-known conjecture of Weinstein [17] that on a *compact* contact manifold satisfying $H^1(M^{2n+1}, R) = 0$, the vector field $\xi$ must have a closed orbit. In a recent paper Taubes [15] proved that the conjeture is true but the second hypothesis is superfluos, that is, on any compact oriented 3-dimensional contact manifold the vector field $\xi$ has a closed orbit. But even in such case, to the authors knowledge, the existence of *closed* magnetic curves (closed helices around a closed or not orbit of $\xi$) is an open problem.

# 4 ACKNOWLEDGEMENTS

Research partially supported by MCYT FEDER Grant MTM 2007-61284 and by Grupo de Investigación FQM-327, Junta de Andalucía

# References

[1] T. Adachi, Curvature bound and trajectories for magnetic fields on a Hadamard surface, *Tsukuba J. Math.*, **20** (1996), 225-230.

[2] M. Barros, A. Romero, J. L. Cabrerizo and M. Fernández, The Gauss-Landau-Hall problem on Riemannian surfaces, *J. Math. Phys.*, **46** (2005), 112905-112920.

[3] F. A. Belgun, Normal CR structures on compact 3-manifolds, *Math. Z.*, **238** (2001), 441-460.

[4] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Mathematics 203, Birkhäuser (Boston, 2002).

[5] R. Brown and A. Gray, Normal CR structures on compact 3-manifolds, *Comment. Math. Helv.*, **42** (1967), 222-236.

[6] A. Comtet, On the Landau levels on the hyperbolic plane, *Ann. Phys.*, **173** (1987), 185-209.

[7] A. Gray, Vector cross products on manifolds, *Trans. Amer. Math. Soc.*, **141** (1969), 465-504.

[8] F. Hirzebruch, *Über die quaternionalen projektiven Räume*, S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1953 (1953), 301–312.

[9] F. Hirzebruch, *Komplexe Mannigfaltigkeiten*, Proc. Int. Congr. Math. 1958. Cambridge Univ. Press (New York, (1958), 119–136).

[10] D. Kalinin, Trajectories of charged particles in Kähler magnetic fields, *Rep. Math. Phys.*, **39** (1997), 299-309.

[11] J. Martinet, Formes de contact sur les variétés de dimension 3, *Proceedings of Liverpool Singularities Symposium, II* (1969/1970), Lecture Notes in Math., **209** (1971), 142–163, Springer, Berlin.

[12] P. Mazteu and D. Munteanu, Vector cross products and almost contact structures, *Rend. Mat., Serie VII* **22** (2002), 359-376.

[13] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity,* Academic Press (New York, 1983).

[14] M. Spivak, *A comprehensive Introduction to Differential Geometry Vol I*, Publish or Perish, Inc. (Boston, 1970).

[15] C. H. Taubes, The Seiberg-Witten equations and the Weinstein conjecture, *Geom. Topol.*, **11** (2007), 2117-2202.

[16] B. Tidé, *Electromagnetic Field Theory*, Upsilon Books (Uppsala, Sweden, 2006).

[17] A. Weinstein, On the hypotheses of Rabinowitz' periodic orbit theorems, *J. Diff. Eq.*, **33** (1979), 353-358.