Rigidity of pseudo-isotropic immersions

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Abstract

Several notions of isotropy of a (pseudo)Riemannian manifold have been introduced in the literature, in particular, the concept of pseudo-isotropic immersion. The aim of this paper is to look more closely at this notion of pseudo-isotropy and then to study the rigidity of this class of immersion into the pseudo-Euclidean space. It is worth pointing out that we first obtain a characterization of the pseudo-isotropy condition by using tangent vectors of any causal character. Then, rigidity theorems for pseudo-isotropic immersions are proved, and in particular, some well known results for the Riemannian case arise. Later, we bring together the notions of pseudo-isotropy, intrinsically and extrinsically isotropic manifolds, and prove interesting relations among them. Finally, we pay special attention to the case of codimension two Lorentz surfaces.

2000 *Mathematics Subject Classification*: Primary: 53C40; Secondary: 53C42.
Key words and phrases: Pseudo-Riemannian manifold, isotropic immersion, pseudo-isotropic submanifold, rigid immersion.

1 Introduction

The concept of *isotropic submanifold* of a Riemannian manifold was introduced by B. O' Neill [17], who studied the general properties of such a class of submanifolds. These submanifolds can be considered as a generalization of the totally umbilical submanifolds, and constitute a distinguished family in Submanifold Theory. On the other hand, Y. H. Kim defined the notion of *pseudo-isotropic submanifold* [11] by extending the O'Neill's notion to the case of pseudo-Riemannian submanifolds of the pseudo-Euclidean space. Nevertheless, some remarkable differences will arise between both environments along this paper.

Let N^n_s and M^m_ν be pseudo-Riemannian manifolds of dimension n, m and index s, ν, respectively. Let $\mathcal{I}$ be the space of isometric immersions $\phi : N^n_s \longrightarrow M^m_\nu$, and let us denote by $\mathcal{I}(\mathcal{P})$ the class of these immersions which satisfy a property $\mathcal{P}$. The class $\mathcal{I}(\mathcal{P})$ is said to be *rigid* if any two immersions $\phi, \phi' \in \mathcal{I}(\mathcal{P})$ are *congruent*, that is, there exists an isometry A of M^m_ν such that $A\circ\phi = \phi'$. In general, it is worth studying rigidity for a class $\mathcal{I}(\mathcal{P})$ in order to obtain their classification. For instance, T. Itoh y K. Ogiue [10] showed the rigidity of certain class of isotropic immersions between

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Riemannian manifolds of constant curvature. Later, K. Sakamoto [21] extended these results by proving that any isotropic parallel submanifold of a Riemannian manifold of constant curvature is rigid, which provided classification results for this class.

In this paper we first review in Section 2 some of the standard facts on isometric immersions of a pseudo-Riemannian manifold N^n_s into the pseudo-Euclidean space $\mathbb{R}^{n+d}_\nu$. Section 3 provides some simple characterizations of pseudo-isotropic immersions by using tangent vectors of any causal character. Section 4 is devoted to the study of rigidity of pseudo-isotropic immersions. In particular, we prove the rigidity for Riemannian isotropic immersions with parallel second fundamental form. This allows us to extend some rigidity results of T. Itoh, K. Ogiue [10] and T. Sakamoto [21]. In Section 5 we first recall that besides Y. H. Kim's definition of pseudo-isotropic submanifold, there are other two notions of isotropy in the literature closely related to that one: the J. A. Wolf's concept of *intrinsically isotropic manifold* [23], and the *extrinsic version* in terms of the immersion. More precisely, a pseudo-Riemannian manifold N^n_s is said to be intrinsically isotropic if, given $p \in N^n_s$ and a real number r, the subgroup of isometries preserving p is transitive on the set of all nonzero tangent vector v at p for which $g(v, v) = r$. If besides, N^n_s is isometrically immersed into a pseudo-Riemannian manifold M^{n+d}_ν and these isometries come from rigid motions of M^{n+d}_ν, then the submanifold N^n_s is called *extrinsically isotropic*. For instance, any hyperquadric of the pseudo-Euclidean space is extrinsically isotropic [18]. On the other hand, note that Wolf's notion of isotropy is related with the physical notion of spacelike isotropy for time-orientable spacetimes [20].

Since we prove that an extrinsically isotropic submanifold is intrinsically isotropic as a manifold, and pseudo-isotropic in the sense of Y. H. Kim, this fact may raise the following question:

> *Given a pseudo-isotropic immersion of an intrinsically isotropic manifold into the pseudo-Euclidean space, is it also extrinsically isotropic?*

We shall give an affirmative answer to this question in Section 5 under some additional conditions.

Finally, Section 6 will look more closely at the case of pseudo-isotropic immersions of a Lorentz surface N^2_1 into the pseudo-Euclidean space $\mathbb{R}^4_2$. We first classify constant pseudo-isotropic immersions. Moreover, if the surface has non-vanishing mean curvature vector, then it is *marginally trapped* [15, 19] and 0-pseudo-isotropic. A rigidity theorem for non-totally umbilical pseudo-isotropic immersions of a Lorentz surface with non-vanishing Gauss curvature is also proved.

2 Preliminaries and basic results

Let $\mathbb{R}^{n+d}_\nu$ be the $(n+d)$-dimensional pseudo-Euclidean space with metric tensor $\langle \cdot\,,\cdot\rangle$ of index ν given by

$$\langle\cdot,\cdot\rangle = -\sum_{i=1}^{\nu} dx_i^2 + \sum_{i=\nu+1}^{n+d} dx_i^2$$

in terms of the natural coordinate system $(x_1,\ldots,x_{n+d})$ of the Euclidean space $\mathbb{R}^{n+d}$.

Throughout this paper we shall denote N^n_s a connected pseudo-Riemannian manifold of dimension $n \geq 2$ and signature $(s, n-s)$. Let $\phi : N^n_s \to \mathbb{R}^{n+d}_\nu$ be an isometric immersion of N^n_s into the pseudo-Euclidean space $\mathbb{R}^{n+d}_\nu$. For all local formulae and computations we may assume ϕ is an imbedding and thus we shall often identify $p \in N^n_s$ with $\phi(p) \in \mathbb{R}^{n+d}_\nu$. The tangent space $T_pN^n_s$ is identified with the subspace $\phi_*(T_pN^n_s)$ of $T_p\mathbb{R}^{n+d}_\nu$, and the normal space is denoted by $T^\perp_pN^n_s$. We will use letters X, Y, Z (resp. ξ, η, ζ) to denote vectors fields tangent (resp. normal) to N^n_s. Let $\widetilde{\nabla}$ and ∇ be the Levi-Civita connections of $\mathbb{R}^{n+d}_\nu$ and N^n_s, respectively. Then, the Gauss-Weingarten equations are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X\xi, \tag{2.2}$$

where h denotes the second fundamental form of ϕ, A_ξ the shape operator, and D is the normal connection. A is related to h by $\langle A_\xi X, Y\rangle = \langle h(X,Y), \xi\rangle$.

A point $p \in N^n_s$ is *umbilic* [18] provided there exists a vector $\xi_p \in T^\perp_pN^n_s$ such that for all $u, v \in T_pN^n_s$ then $h(u,v) = \langle u,v\rangle\xi_p$. When every point of N^n_s is umbilic the immersion ϕ is called *totally umbilical*. In such case, it is well known that $h(X,Y) = \langle X,Y\rangle\mathbf{H}$ where $\mathbf{H} = (1/n)\,trace\,(h)$ is the *normal curvature vector* of N^n_s.

The covariant derivative $\bar\nabla h$ of h is defined by

$$(\bar\nabla_X h)(Y,Z) \quad = \quad D_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

Sometimes we shall write $(\bar\nabla_X h)(Y,Z)$ as $(\bar\nabla h)(Y,Z,X)$. The second fundamental form h is said to be *parallel* if h is covariantly constant, that is, $\bar\nabla h = 0$. If R denotes the curvature tensor of N^n_s, then the equations of Gauss and Codazzi are given respectively by

$$\langle R(X,Y)Z,W\rangle = \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle, \tag{2.3}$$

$$(\bar\nabla h)(Z,Y,X) = (\bar\nabla h)(Z,X,Y). \tag{2.4}$$

We define the *first normal space at* p by $(\mathrm{Im}\ h)_p = \{h(x,y) : x,y \in T_pN^n_s\}$. The second fundamental form h is said to be *surjective* if $(\mathrm{Im}\ h)_p = T^\perp_pN^n_s$ at each point p of N^n_s. As in the Riemannian case [16], we have the following.

Lemma 2.1 *Let $\phi : N_s^n \to \mathbb{R}_\nu^{n+d}$ be an isometric immersion with surjective second fundamental form h. Then, the normal connection D is the only connection in $T^\perp N_s^n$ which is compatible with the metric and satisfies Codazzi's equation.*

The following result is essential in this paper (the proof is omitted because it is basically the same as in the Riemannian case, see Theorem 1.1 and remark in [5]).

Lemma 2.2 (Uniqueness Theorem for Immersions) *Assume $\phi, \phi' : N_s^n \to \mathbb{R}_\nu^m$ are isometric immersions of a connected manifold N_s^n into $\mathbb{R}_\nu^m$. Let us denote by E, h and D (resp. E', h' and D') the normal bundle, the second fundamental form and the normal connection of ϕ (resp. ϕ'). Suppose that there exists also an isometry $f : N_s^n \to N_s^n$ which can be covered by a fibered isomorphism $\widetilde{f} : E \to E'$ that preserve the fibered metric, the second fundamental form and the normal connection. Then, there exists a unique rigid motion A of $\mathbb{R}_\nu^{n+d}$ such that $A \circ \phi = \phi' \circ f$ and $A_*|_E = \widetilde{f}$.*

3 Pseudo-isotropic immersion

We first recall [11] that an isometric immersion $\phi : N_s^n \to \mathbb{R}_\nu^{n+d}$ is called *pseudo-isotropic at $p \in N_s^n$* if

$$\langle h(u,u), h(u,u)\rangle = \lambda(p) \in \mathbb{R} \tag{3.1}$$

does not depends on the choice of the *unit* tangent vector $u \in \Sigma_p = \{u \in T_pN_s^n : \langle u,u\rangle = \pm 1\}$, and ϕ is said to be *pseudo-isotropic* if ϕ is pseudo-isotropic at each point of N_s^n. In such a case, the smooth function $\lambda : N_s^n \to \mathbb{R}$ defined by equation (3.1) is called the *pseudo-isotropy function*, and the isometric immersion ϕ is said to be λ-*pseudo-isotropic*. In particular, if λ is a constant function, the immersion is called *constant pseudo-isotropic*.

Remark 3.1 It is clear that every totally umbilical immersion $\phi : N_s^n \to \mathbb{R}_\nu^{n+d}$ is a constant $\langle \mathbf{H}, \mathbf{H}\rangle$-pseudo-isotropic immersion. For example, pseudo-Riemannian spheres $\mathbb{S}_s^n$, $\mathbb{H}_s^n$ and (non-degenerate) n-planes into a pseudo-Euclidean spaces are constant pseudo-isotropic submanifolds.

We need the following result for later use.

Lemma 3.2 *Let $\phi : N_s^n \to \mathbb{R}_\nu^{n+d}$ be an isometric immersion. Then, the following conditions are equivalent:*

(1) ϕ is pseudo-isotropic at p.

(2) $\langle h(x,x), h(x,x)\rangle = \lambda(p)\langle x,x\rangle^2$ for all $x \in T_pN_s^n$.

(3) $\langle h(x,y), h(z,w)\rangle + \langle h(y,z), h(x,w)\rangle + \langle h(x,z), h(y,w)\rangle = \lambda(p)\{\langle x,y\rangle\langle z,w\rangle + \langle y,z\rangle\langle x,w\rangle + \langle x,z\rangle\langle y,w\rangle\}$ for all $x,y,z,w \in T_pN_s^n$.

Proof. Assume ϕ is pseudo-isotropic at p. Then, the identity $\langle h(u,u), h(u,u)\rangle = \lambda(p)\langle u,u\rangle^2$ holds for any timelike (or spacelike) tangent vector u at p, and by continuity, we get

$$\langle h(z,z), h(z,z)\rangle = \lambda(p)\langle z,z\rangle^2 = 0, \tag{3.2}$$

for any ligthlike tangent vector z at p. Thus, we have

$$\langle h(x,x), h(x,x)\rangle = \lambda(p)\langle x,x\rangle^2$$

for all $x \in T_pN^n_s$ and, in consequence, condition *(3)* is satisfied (see B. O'Neill [17]). □

Next we prove that it suffices to check the pseudo-isotropy condition (3.1) for only timelike or spacelike tangent vectors $u \in T_pN^n_s$.

Lemma 3.3 *Let $\phi : N^n_s \to \mathbb{R}^{n+d}_\nu$ be an isometric immersion. Then, the following conditions are equivalent,*

(1) ϕ is pseudo-isotropic at p.

(2) $\langle h(u,u), h(u,u)\rangle = \lambda(p)\langle u,u\rangle^2$ for any timelike $u \in T_pN^n_s$.

(3) $\langle h(u,u), h(u,u)\rangle = \lambda(p)\langle u,u\rangle^2$ for any spacelike $u \in T_pN^n_s$.

Proof. Assume $\langle h(u,u), h(u,u)\rangle = \lambda(p)\langle u,u\rangle^2$ holds for any timelike tangent vector u at a point $p \in N^n_s$ and let us prove that ϕ is pseudo-isotropic at p. In fact, by applying the Hawking-Ellis useful argument [8, p. 61], take $v \in T_pN^n_s$ of any causal character, and define the curve $u_t = u + tv$, where $u \in T_pN^n_s$ is a (fixed) timelike vector. By continuity, there exists $\delta > 0$ such that $\langle u_t, u_t\rangle < 0$ for any $t \in (-\delta, \delta)$, and therefore

$$\langle h(u_t,u_t), h(u_t,u_t)\rangle = \lambda(p)\langle u_t,u_t\rangle^2.$$

In particular, this identity gives

$$\langle h(v,v), h(v,v)\rangle = \lambda(p)\langle v,v\rangle^2$$

for any $v \in T_pN^n_s$. Thus *(1)* and *(2)* are equivalents. The case *(1)* $\Longleftrightarrow$ *(3)* is completely similar. □

Remark 3.4 There exists no general result as *(2)* or *(3)* in Lemma 3.3 for lightlike vectors. Notice that any indefinite pseudo-isotropic immersion satisfies equation (3.2). Nevertheless, the isometric immersion $\phi : \mathbb{R}^2_1 \to \mathbb{R}^4_2$ defined by

$$\begin{aligned}\sqrt{2}\,\phi(x,y) &= (\sin(x), \cos(x), \sin(x), \cos(x)) \\ &\quad + y(\cos(x), -\sin(x), -\cos(x), \sin(x)),\end{aligned}$$

is a non-pseudo-isotropic immersion which satisfies the ligthlike pseudo-isotropic property (equation (3.2)). In fact, $\{\partial_x, \partial_y\}$ is a pseudo-orthonormal basis, i.e.,

$\langle \partial_x, \partial_x \rangle = 0$, $\langle \partial_y, \partial_y \rangle = 0$ and $\langle \partial_x, \partial_y \rangle = -1$. Furthermore, the second fundamental form h of ϕ satisfies

$$\begin{aligned} h\left(\partial_x, \partial_x\right) &= -\phi(x,y), \\ h\left(\partial_y, \partial_y\right) &= 0, \\ h\left(\partial_x, \partial_y\right) &= (-\sin(x), -\cos(x), \sin(x), \cos(x))/\sqrt{2}. \end{aligned}$$

But $h(\partial_x, \partial_x)$ and $h(\partial_x, \partial_y)$ are two ligthlike normal vectors, and

$$\langle h\left(\partial_x, \partial_x\right), h\left(\partial_x, \partial_y\right) \rangle = -1.$$

Consequently, ϕ is a non-pseudo-isotropic immersion (because the condition *(3)* in Lemma 3.2 is no satisfied) and $\langle h(z,z), h(z,z) \rangle = 0$ holds for any ligthlike tangent vector z at each point.

Next, we give a characterization of pseudo-isotropic immersions in terms of ligthlike vectors.

Lemma 3.5 *Let* $\phi : N_s^n \to \mathbb{R}_\nu^{n+d}$ *be an isometric immersion with* $0 < s < n$. *Then, the following conditions are equivalent:*

(1) ϕ *is pseudo-isotropic at* p.

(2) $\langle h(x,x), h(x,y) \rangle = 0$ *for any ligthlike vectors* $x, y \in T_pN_s^n$.

(3) $\langle h(u,u), h(u,v) \rangle = 0$ *for any orthonormal vectors* $u, v \in T_pN_s^n$.

Proof.

(1) $\Rightarrow$ *(2)*. This follows from *(3)* in Lemma 3.2.

(2) $\Rightarrow$ *(3)*. For each vector $w \in T_pN_s^n$ define $f(w) = \langle h(w,w), h(w,w) \rangle$ and choose $\{e_1, e_2\}$ two orthonormal vectors with opposite causal character, say e_1 spacelike and e_2 timelike. Since $f(e_1 + e_2) - f(e_1 - e_2) = 0$, we have

$$\langle h(e_1,e_1), h(e_1,e_2) \rangle + \langle h(e_2,e_2), h(e_1,e_2) \rangle = 0. \tag{3.3}$$

But on the other hand,

$$\langle h(e_1+e_2, e_1+e_2), h(e_1+e_2, e_1-e_2) \rangle = 0,$$

$$\langle h(e_1-e_2, e_1-e_2), h(e_1-e_2, e_1+e_2) \rangle = 0,$$

and therefore we have

$$\langle h(e_1,e_1), h(e_1,e_2) \rangle = \langle h(e_2,e_2), h(e_1,e_2) \rangle = 0. \tag{3.4}$$

It is clear that $\langle h(u,u), h(u,v) \rangle = 0$ for any u, v orthogonal tangent vectors at p of opposite causal character. If $n = 2$, the proof is complete. For $n > 2$, we consider the curve $w_t = w + tu$, where w, u are orthogonal tangent vectors at p having opposite causal character. Then, by continuity, there exists $\delta > 0$, such that the product $\langle w_t, w_t\rangle \cdot \langle w, w\rangle > 0$ for any $t \in (-\delta, \delta)$, that is, w_t and w have the same causal character. Since $n > 2$, take a unit tangent vector v at p having the same causal character as u and orthogonal to both, w and u. Since the polynomial in t, $P(t) = \langle h(w_t, w_t), h(w_t, v)\rangle = 0$ for any $t \in (-\delta, \delta)$, we obtain $\langle h(u, u), h(u, v)\rangle = 0$, and this gives *(3)*.

(3) ⇒ *(1)*. If $p \in N^n_s$ and $\varepsilon \in \{1, -1\}$, define the spaces $\Sigma_p(\varepsilon)$ by

$$\Sigma_p(\varepsilon) = \{u \in T_pN^n_s : \langle u, u\rangle = \varepsilon\} \subseteq \Sigma_p = \{u \in T_pN^n_s : \langle u, u\rangle = \pm 1\}.$$

Then it suffices to prove that the function $f : \Sigma_p \to \mathbb{R}$, $f(w) = \langle h(w, w), h(w, w)\rangle$ is a constant mapping. For this end, take $u \in \Sigma_p(\varepsilon)$ and a unit vector $v \in T_u(\Sigma_p(\varepsilon)) = \{v \in T_pN^n_s : \langle u, v\rangle = 0\}$. Next, we consider a curve $\alpha\colon (-\delta, \delta) \to \Sigma_p(\varepsilon)$ such that $\alpha(0) = u$, $\alpha'(0) = v$. Since

$$v(f) \quad = \quad \left.\frac{\mathrm{d}(f \circ \alpha)}{\mathrm{d}t}\right|_{t=0} = 2\langle h(u, u), h(u, v)\rangle = 0,$$

this means that f is a constant mapping on each connected component of $\Sigma_p(\varepsilon)$. Let C be a connected component of $\Sigma_p(\varepsilon)$ and put $\lambda = \lambda(p)$ for the corresponding constant value of f in C. Then, equation $\langle h(u, u), h(u, u)\rangle = \lambda\langle u, u\rangle^2$ is satisfied for any non-lightlike tangent vector u such that $u/||u|| \in C$. Fix one of these u and define the curve $u_t = u + tv$ where v is a tangent vector at p of any causal character. There exist $\delta > 0$ such that $\langle u_t, u_t\rangle \cdot \langle u, u\rangle > 0$ and $u_t/||u_t|| \in C$ for all $t \in (-\delta, \delta)$. Therefore,

$$\langle h(u_t, u_t), h(u_t, u_t)\rangle = \lambda\langle u_t, u_t\rangle^2.$$

In particular, this identity gives $\langle h(v, v), h(v, v)\rangle = \lambda\langle v, v\rangle^2$. As v has arbitrary causal character, f is the constant function $f \equiv \lambda$ on Σ_p.

□

As for constant isotropic submanifolds in the Euclidean space [4], by using *(3)* in Lemma 3.5 we have the following characterization of the constant pseudo-isotropic immersions in the pseudo-Euclidean space.

Lemma 3.6 *Let $\phi : N^n_s \to \mathbb{R}^{n+d}_\nu$ be a pseudo-isotropic immersion. Then, the following assertions are equivalent:*

(1) ϕ is constant pseudo-isotropic.

(2) $\langle(\bar\nabla h)(u, u, u), h(u, v)\rangle = 0$ for any orthonormal vectors $u, v \in T_pN^n_s$ at each $p \in N^n_s$.

(3) $A_{(\bar\nabla h)(u,u,u)}u = 0$ for any $u \in T_pN^n_s$ at each $p \in N^n_s$.

As a consequence of Lemma 3.6 we have the following.

Corollary 3.7 *A pseudo-isotropic immersion with parallel second fundamental form is constant pseudo-isotropic.*

We recall [1] that an isometric immersion $\phi : N_s^n \to \mathbb{R}_\nu^{n+d}$ is said to be *planar geodesic* if, locally, the image of each geodesic of N_s^n lies in a 2-plane of $\mathbb{R}_\nu^{n+d}$. Notice that if timelike and spacelike geodesics are planar, then lightlike geodesics will be planar by continuity. As in the Riemannian case [9, 13], it can be easily seen that a planar geodesic immersion is constant λ-pseudo-isotropic, and if $\lambda \neq 0$, then the second fundamental form is parallel. We have also the following.

Lemma 3.8 *Let $\phi : N_s^n \to \mathbb{R}_\nu^{n+d}$ be a non-vanishing λ-pseudo-isotropic immersion with parallel second fundamental form. Then, ϕ is planar geodesic.*

Proof. Let $\phi \circ \gamma$ be a non-lightlike geodesic of $\phi(N_s^n)$ with unit tangent vector X_t and signature ε. Putting $Y_t = \varepsilon|\lambda|^{-1/2}\, h(X_t, X_t)$, then, from the equations of Gauss (2.1), Weingarten (2.2), and *(3)* in Lemma 3.5, we obtain

$$\begin{cases} \dfrac{dX_t}{dt} = \varepsilon|\lambda|^{1/2} Y_t, \\[2ex] \dfrac{dY_t}{dt} = -(\mathrm{sgn}\lambda)|\lambda|^{1/2} X_t. \end{cases} \tag{3.5}$$

Thus, the differential equations (3.5) yield the desired result (see Proposition 2.1 in [1]).

□

4 The rigidity results

Theorem 4.1 *Let $\phi, \phi' : N_s^n \to \mathbb{R}_\nu^{n+d}$ be constant λ-pseudo-isotropic immersions and assume that their respective second fundamental forms h and h' are both surjective mappings. Then, any isometry $f : N_s^n \to N_s^n$ is the restriction of a rigid motion A of $\mathbb{R}_\nu^{n+d}$, that is, $A \circ \phi = \phi' \circ f$.*

Proof. It follows from the Gauss equation (2.3) and *(3)* in Lemma 3.2 that

$$\begin{aligned} \langle h(X,Y), h(Z,W)\rangle &= \frac{\lambda}{3}\{\langle X,Y\rangle\langle Z,W\rangle + \langle Y,Z\rangle\langle X,W\rangle + \langle X,Z\rangle\langle Y,W\rangle\} \\ &\quad + \frac{1}{3}\{\langle R(Z,X)Y,W\rangle + \langle R(Z,Y)X,W\rangle\}, \end{aligned} \tag{4.1}$$

for any vector fields X, Y, Z, W tangent to N_s^n. Since f is an isometry and the immersions are constant λ-pseudo-isotropic, (4.1) gives

$$\langle h(X,Y), h(Z,W)\rangle = \big\langle h'(f_*X, f_*Y), h'(f_*Z, f_*W)\big\rangle \circ f. \tag{4.2}$$

Let $\{v_1, \dots, v_n\}$ a reference frame at $p \in N^n_s$ and set $v'_i = f_*(v_i)$, $i = 1, \dots, n$. Then, from equation (4.2), we have

$$\langle h(v_i, v_j), h(v_k, v_\ell)\rangle = \left\langle h'(v'_i, v'_j), h'(v'_k, v'_\ell)\right\rangle .$$

On the other hand, since h and h' are both surjective mappings, a linear isometry $h(v_i, v_j) \longmapsto h'(v'_i, v'_j)$ can be defined between the respective normal spaces of ϕ and ϕ' at p. In this way, we obtain a fibered isomorphism $\widetilde{f} : E \to E'$, where E and E' are the normal bundles of ϕ and ϕ', respectively. Clearly $\widetilde{f}$ covers f and preserve metrics and second fundamental forms. But $\widetilde{f}$ preserves also the respective normal connections D and D'. In fact, any normal field along ϕ' is written as $\widetilde{f}(\xi)$, whereas any tangent vector field of N^n_s as f_*X, and hence we can define on E' a connection δ by

$$\delta_{f_*X}(\widetilde{f}(\xi)) = \widetilde{f}(D_X\xi).$$

It is easy to see that δ is compatible with the metric and satisfies Codazzi's equation. But Lemma 2.1 asserts that $\delta = D'$ and then we that the mapping $\widetilde{f} : E \to E'$ covers an isometry, preserves bundle metrics, bundle connections, and second fundamental forms. By the Uniqueness Theorem for Immersions (Lemma 2.2), there is a rigid motion from $\phi(N^n_s)$ onto $\phi'(N^n_s)$.

□

Now, for a given pseudo-isotropy function λ we prove the following rigidity result.

Theorem 4.2 *Let $\phi : N^n_s \to \mathbb{R}^{n+d}_\nu$ be a λ-pseudo-isotropic immersion and suppose that the second fundamental forms h is a surjective mapping. Then, ϕ is rigid.*

Proof. Let $\phi, \phi' : N^n_s \to \mathbb{R}^{n+d}_\nu$ be λ-pseudo-isotropic immersions with second fundamental forms h and h' both surjective at any point. Then, from equation (4.1), we obtain

$$\langle h(X, Y), h(Z, W)\rangle = \left\langle h'(X, Y), h'(Z, W)\right\rangle . \tag{4.3}$$

Now, if we take the identity map as f in the proof of Theorem 4.1, then ϕ and ϕ' are congruent.

□

Next we show that the assumption of surjectivity on the second fundamental form in Theorem 4.2 is essential. In fact, we have the following example.

Example 4.3 [1] *Expansions of $\mathbb{R}^n_s$ into $\mathbb{R}^{n+\ell}_{s,\ell}$.* Let $f_1, \dots, f_\ell : \mathbb{R}^n_s \to \mathbb{R}$ be smooth functions. Define the space $\mathbb{R}^{n+\ell}_{s,\ell}$ as $\mathbb{R}^{n+\ell}$ equipped with the degenerate metric tensor given by the matrix

$$\begin{pmatrix} -I_s & & \\ & I_{n-s} & \\ & & 0_\ell \end{pmatrix} .$$

The isometric immersion

$$\phi : \mathbb{R}^n_s \to \mathbb{R}^{n+2\ell}_{s+\ell}, \quad \phi(x) = (f_1(x), \dots, f_\ell(x), x, f_1(x), \dots, f_\ell(x)) ,$$

is 0-pseudo-isotropic. In fact, the second fundamental form h is given by

$$h\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)=\left(\frac{\partial^2 f_1}{\partial x_i\partial x_j},\ldots,\frac{\partial^2 f_\ell}{\partial x_i\partial x_j},0,\ldots,0,\frac{\partial^2 f_1}{\partial x_i\partial x_j},\ldots,\frac{\partial^2 f_\ell}{\partial x_i\partial x_j}\right),$$

where $(x_1,\ldots,x_n)$ denotes the canonical coordinates of $\mathbb{R}^n$. Notice that $\phi(\mathbb{R}^n_s)$ is contained in the degenerate $(n+\ell)$-plane $\mathbb{R}^{n+\ell}_{s,\ell}$ of $\mathbb{R}^{n+2\ell}_{s+\ell}$. Thus, the second fundamental form h is not surjective, and, clearly, two immersions of this type need not be congruent.

It is easy to verify that the mean curvature vector field of ϕ satisfies $\langle \mathbf{H},\mathbf{H}\rangle=0$, and $\mathbf{H}=0$ if and only if

$$\frac{\partial^2 f_j}{\partial x_1^2}+\cdots+\frac{\partial^2 f_j}{\partial x_s^2}=\frac{\partial^2 f_j}{\partial x_{s+1}^2}+\cdots+\frac{\partial^2 f_j}{\partial x_n^2}$$

for any $j=1,\ldots,\ell$. Moreover, ϕ is totally umbilical if and only if f_j is given by

$$f_j(x_1,\ldots,x_n)=a_j\left(-\sum_{i=1}^{s}x_i^2+\sum_{i=s+1}^{n}x_i^2\right)+\sum_{i=1}^{n}b_j^i x_i+c_j,$$

with $a_j,b_j^1,\ldots,b_j^n,c_j\in\mathbb{R}$, $j=1,\ldots\ell$.

Corollary 4.4 *Let $\phi : N^n_s \to \mathbb{R}^{n+d}_\nu$ be a pseudo-isotropic immersion with parallel second fundamental form. Assume that at a point p the first normal space of ϕ is a non-degenerate subspace of $\mathbb{R}^{n+d}_\nu$. Then, ϕ is rigid.*

Proof. Note that since the second fundamental form h is parallel, opportune changes in Lemma 4.2 of [21] allows us to prove that the dimension k and index μ of the first normal space of ϕ are constant on N^n_s. Therefore the codimension can be reduced [1, 7] in such a way that N^n_s is contained in a non-degenerate $(n+k)$-plane Π of $\mathbb{R}^{n+d}_\nu$. Consequently, we have defined an isotropic immersion $\phi : N^n_s \to \mathbb{R}^{n+k}_{s+\mu}$ whose second fundamental form is surjective and parallel. Now, Theorem 4.2 gives the proof.

□

Let us denote by $Q^{n+d}(c)$ the Riemannian model space form of constant curvature c. It is well known (see [3, p. 49]) the uniqueness theorem for immersions into $Q^{n+d}(c)$. Analogously, for Riemannian λ-isotropic immersions we have the following.

Theorem 4.5 *Let $\phi : N^n \to Q^{n+d}(c)$ be a λ-isotropic immersion. If the second fundamental form h is a surjective mapping, then ϕ is rigid.*

Corollary 4.6 *Let $\phi : N^n \to Q^{n+d}(c)$ be a λ-isotropic immersion. Suppose that the second fundamental forms h is parallel. Then, ϕ is rigid.*

Remark 4.7 We have obtained the rigidity of the isotropic immersions with parallel second fundamental form in the Riemannian case, but in a different way than K. Sakamoto [21]. Locally, this kind of immersions is completely determined up to congruences.

5 Extrinsic isotropic immersions

We first recall that a pseudo-Riemannian manifold N_s^n is called *intrinsically isotropic at* $p \in N_s^n$ [23] if, given a real number r, the action of the isotropy group $\mathrm{Iso}_p(N_s^n)$ on the set $\Sigma_p^*(r) = \{v \in T_pN_s^n : g(v,v) = r, v \neq 0\}$ is transitive, that is, if for any nonzero vectors $u, v \in T_pN_s^n$ with $g(u,u) = g(v,v)$, there exists an isometry f of N_s^n such that $f(p) = p$ and $f_*(u) = v$. A manifold is said to be *intrinsically isotropic* when it is intrinsically isotropic at each point.

Definition 5.1 *An isometric immersion $\phi : N_s^n \to \mathbb{R}_\nu^{n+d}$ is called extrinsically isotropic at $p \in N_s^n$ if, given nonzero $u, v \in T_pN_s^n$ with $g(u,u) = g(v,v)$, there exists a rigid motion A of $\mathbb{R}_\nu^{n+d}$ such that*

(i) $A(\phi(N_s^n)) = \phi(N_s^n)$,

(ii) $A(\phi(p)) = \phi(p)$ and $(A \circ \phi)_*(u) = \phi_*(v)$.

An isometric immersion $\phi : N_s^n \to \mathbb{R}_\nu^{n+d}$ is said to be extrinsically isotropic if ϕ is extrinsically isotropic at each point.

Proposition 5.1 *Let $\phi : N_s^n \to \mathbb{R}_\nu^{n+d}$ be an extrinsically isotropic immersion. Then, we have*

(a) Locally, N_s^n is intrinsically isotropic and, in particular, a locally symmetric space.

(b) ϕ is a pseudo-isotropic immersion.

Proof. *(a)* Take $p \in N_s^n$ and nonzero tangent vectors $u, v \in T_pN_s^n$ with $g(u,u) = g(v,v)$, and let A be an isometry of $\mathbb{R}_\nu^{n+d}$ such that $A(\phi(N_s^n)) = \phi(N_s^n)$, $A(\phi(p)) = \phi(p)$ and $(A \circ \phi)_*(u) = \phi_*(v)$. Let $\mathcal{U}$ be an open neighborhood of p in N_s^n such that the restriction $\phi|_{\mathcal{U}}$ is an isometric imbedding. Then, $f = \phi^{-1} \circ A \circ \phi|_{\mathcal{U}}$ is a local isometry around p such that $f(p) = p$ and $f_*(u) = v$. Thus, locally N_s^n is intrinsically isotropic. In particular [23, Theorem 12.3.1], N_s^n is locally symmetric. *(b)* Just bear in mind that the second fundamental form is an extrinsic invariant [18] and then apply Lemma 3.3.

□

Note that by equation (2.3), if the second fundamental form h of an isometric immersion $\phi : N_s^n \to \mathbb{R}_\nu^{n+d}$ is parallel, then N_s^n is locally symmetric. Next we shall prove that for constant isotropic immersions the converse holds, provided an additional condition is fulfilled. Thus, we obtain an extension of the corresponding Riemannian result [2].

Lemma 5.2 *Let $\phi : N_s^n \to \mathbb{R}_\nu^{n+d}$ be a constant λ-pseudo-isotropic immersion. If N_s^n is locally symmetric and the second fundamental form h of ϕ is a surjective mapping, then h is parallel.*

Proof. Since λ is a constant and N_s^n is locally symmetric, a simple differentiation of equation (4.1) with respect to any tangent vector field T on N_s^n gives

$$\langle(\bar{\nabla}_T h)(X,Y), h(Z,W)\rangle = -\langle(h(X,Y), (\bar{\nabla}_T h)(Z,W)\rangle. \tag{5.1}$$

But combining repeatedly Codazzi's equation (2.4) and equation (5.1), we find

$$\langle(\bar{\nabla}_T h)(X,Y), h(Z,W)\rangle = 0,$$

which, joined to the hypothesis that the first normal space spans the normal space at any point of N_s^n, shows that the second fundamental form of ϕ is parallel.

□

Theorem 5.3 *Let $\phi : N_s^n \to \mathbb{R}_\nu^{n+d}$ be a constant pseudo-isotropic immersion. If N_s^n is a intrinsically isotropic manifold and the second fundamental form h is a surjective mapping, then ϕ is extrinsically isotropic and h is parallel.*

Proof. This follows immediately from Theorem 4.1 and Lemma 5.2.

□

Remark 5.4 (a) It is well known that intrinsically isotropic pseudo-Riemannian manifolds N_s^n has been classified by J. A. Wolf [23, Theorem 12.4.5]. (b) If $s = 1$ and the isotropy group at each point is transitive on the set of unit timelike tangent vectors of $T_p N_1^n$ then N_1^n has constant sectional curvature [12].

6 Codimension two Lorentz surfaces

In this section we study the rigidity of codimension two pseudo-isotropic Lorentz surfaces N_1^2 of the pseudo-Euclidean space $\mathbb{R}_\nu^m$. First, we have the following simple criterion for studying the pseudo-isotropy of N_1^2.

Lemma 6.1 *Let $\phi : N_1^2 \to \mathbb{R}_\nu^m$ be an isometric immersion. Then, ϕ is pseudo-isotropic at p if and only if there exists a pseudo-orthonormal basis $\{x, y\}$ of $T_p N_1^2$ such that*

$$\langle h(x,x), h(x,x)\rangle = 0,\ \langle h(y,y), h(y,y)\rangle = 0,\ \langle h(x,x), \mathbf{H}\rangle = 0,\ \langle h(y,y), \mathbf{H}\rangle = 0,$$

where $\mathbf{H}$ denotes the mean curvature vector field of ϕ.

Proof. A pseudo-orthonormal basis $\{x, y\}$ of $T_p N_1^2$ satisfies $\langle x, x\rangle = \langle y, y\rangle = 0$ and $\langle x, y\rangle = -1$. If $u = (x - y)/\sqrt{2}$ and $v = (x + y)/\sqrt{2}$, then $\{u, v\}$ is an orthonormal basis at p, and the mean curvature vector is given by $\mathbf{H} = (h(u,u) - h(v,v))/2 = -h(x,y)$. Since any lightlike vector is proportional to x or y, by Lemma 3.5 the result follows.

□

Lemma 6.2 *Let ϕ be a λ-pseudo-isotropic immersion of a surface N_s^2 into the pseudo-Euclidean space $\mathbb{R}_\nu^m$. Then, the Gauss curvature $\mathcal{K}$ of N_s^n satisfies*

$$\mathcal{K} = -2\lambda + 3\langle \mathbf{H}, \mathbf{H} \rangle.$$

Proof. This follows from the equation of Gauss and *(3)* in Lemma 3.2. □

Corollary 6.3 *Let $\phi : N_1^2 \to \mathbb{R}_2^4$ be a non-totally umbilical λ-pseudo-isotropic immersion. Then, $\langle \mathbf{H}, \mathbf{H} \rangle = 0$ and $\mathcal{K} = -2\lambda$.*

Proof. Notice that if p is a non-totally umbilical point, there exist a ligthlike vector $x \in T_pN_1^2$ such that $h(x, x) \neq 0$ (see [6]). Since the normal space is a Lorentzian plane, by Lemma 6.1 $\langle \mathbf{H}, \mathbf{H} \rangle = 0$ at p. Then, by continuity, the mean curvature vector $\mathbf{H}$ satisfies $\langle \mathbf{H}, \mathbf{H} \rangle = 0$ at each point and then, Lemma 6.2 gives $\mathcal{K} = -2\lambda$. □

For constant pseudo-isotropic immersions we have the following.

Theorem 6.4 *Let ϕ be a non-totally umbilical constant pseudo-isotropic immersion of a Lorentz surface N_1^2 into the pseudo-Euclidean space $\mathbb{R}_\nu^4$. Then, ϕ is 0-pseudo-isotropic. Besides, if N_1^2 is complete, then it is also congruent to an expansion of $\mathbb{R}_1^2$ into $\mathbb{R}_{1,1}^3$.*

Proof. Notice that the index $\nu = 2$. In fact, if $\nu = 1$ or 3, for any lightlike $x \in T_pN_1^2$ the lightlike pseudo-isotropic condition (3.2) says that $h(x, x) = 0$ because each normal space has a definite (positive or negative) induced metric. This means [6] that ϕ is totally umbilical, which is a contradiction. Let $\lambda \in \mathbb{R}$ be the pseudo-isotropy constant, and denote by $\mathcal{U}$ the set of non umbilical points of ϕ. Assume $\lambda \neq 0$. Note that at any $p \in \mathcal{U}$ we can take orthonormal tangent vectors u, v such that $h(u, v) \neq 0$. Then, Definition 3.1, Lemma 3.5 and Lemma 3.6 yield

$$\langle h(u,u), h(u,u) \rangle = \langle h(v,v), h(v,v) \rangle = \lambda \neq 0 \tag{6.1}$$

$$\langle h(u,u), h(u,v) \rangle = \langle h(v,v), h(u,v) \rangle = 0, \tag{6.2}$$

$$\langle (\bar{\nabla} h)(u,u,u), h(u,v) \rangle = 0, \tag{6.3}$$

$$\langle (\bar{\nabla} h)(u,u,u), h(u,u) \rangle = 0. \tag{6.4}$$

Since any normal space to ϕ is isometric to the Lorentz plane $\mathbb{L}^2$, equations (6.1)—(6.4) give $(\bar{\nabla} h)(u, u, u) = 0$. Therefore, by Codazzi's equation (2.4), the second fundamental form of ϕ is parallel. Thus, by Lemma 3.8, ϕ is a planar geodesic immersion. But after the paper by C. Blomstrom [1], this is a contradiction with $\lambda \neq 0$. Consequently, ϕ is a 0-pseudo-isotropic immersion.

Now, equation (6.4) says that $(\bar{\nabla} h)(u, u, u) \in Im(h)$ and $h(u, u)$ are linearly dependent vectors. This joined to Codazzi's equation (2.4) can be used to show that $(\bar{\nabla} h)(u, v, w) \in Im(h)$ for any u, v, w. Thus [1, 7], the codimension can be reduced in such a way that N_1^2 is contained in a degenerate hyperplane of $\mathbb{R}_2^4$. Since this hyperplane is isometric to $\mathbb{R}^3_{1,1}$, then N^2_1 is imbedded in $\mathbb{R}^3_{1,1}$. Let $\pi\colon \mathbb{R}^3_{1,1} \to \mathbb{R}^4_2$ be the projection map on the first 2 coordinates. Then $\pi(N^2_1)$ is an open subset of the pseudo-Euclidean space $\mathbb{R}^2_1$ [14]. By the completeness hypotesis, there exists a smooth function $f\colon \mathbb{R}^2_1 \to \mathbb{R}$ such that N^2_1 (viewed in $\mathbb{R}^3_{1,1}$) can be realized as the set of points $(x, f(x))$. Finally, it suffices to note that the map $\mathbb{R}^3_{1,1} \hookrightarrow \mathbb{R}^4_2$ given by $(y_1, y_2, y_3) \mapsto (y_3, y_1, y_2, y_3)$ is an isometric embedding.

□

Remark 6.5 As a consequence of this theorem, any pseudo-isotropic surface N^2_1 in the Lorentz-Minkowski space $\mathbb{L}^4 = \mathbb{R}^4_1$ (or in the space $\mathbb{R}^4_3$) is totally umbilical. Thus, in order to completely determine the rigidity of the pseudo-isotropic surfaces of codimension two, it suffices to study the rigidity of pseudo-isotropic and non-constant pseudo-isotropic Lorentz surfaces in $\mathbb{R}^4_2$.

Corollary 6.6 *Let $\phi : N^2_1 \to \mathbb{R}^4_2$ be a non-totally umbilical pseudo-isotropic immersion. Then, the following assertion are equivalent.*

(1) ϕ is constant pseudo-isotropic.

(2) The Gaussian curvature $\mathcal{K}$ of N^2_1 is constant.

(3) ϕ is 0-pseudo-isotropic.

(4) N^2_1 is flat.

(5) The first normal space $Im(h)$ at each non-totally geodesic point is entirely constituted by lightlike vectors.

Moreover, if N^2_1 is complete, ϕ is (up to a rigid motion) an expansion of $\mathbb{R}^2_1$ into $\mathbb{R}^3_{1,1}$.

Proof. This follows immediately from Theorem 6.4 and Corollary 6.3.

□

By Magid's classification theorem for totally umbilical immersions into the pseudo-Euclidean space [14, Theorem 1.4], and Corollary 6.6 we have the following.

Corollary 6.7 *Let ϕ be a pseudo-isotropic immersion of a Lorentz surface N^2_1 with constant curvature $c \neq 0$ into the pseudo-Euclidean space $\mathbb{R}^4_2$. Then, ϕ is totally-umbilical. In particular, $\phi(N^2_1)$ is an open portion of the De Sitter space $\mathbb{S}^2_1$ or the pseudo-hyperbolic space $\mathbb{H}^2_1$.*

For the non-constant curvature case we have also the following result.

Proposition 6.8 *Let ϕ be a non-totally umbilical pseudo-isotropic immersion of a Lorentz surface N^2_1 into the pseudo-Euclidean space $\mathbb{R}^4_2$. Suppose that the pseudo-isotropy function $\lambda \neq 0$ everywhere, or equivalently, the Gauss curvature of N^2_1 is non-zero everywhere, then* *(a) the second fundamental form h is a surjective mapping.*

(b) ϕ is minimal.

Proof. First note that since $\lambda \neq 0$ everywhere, ϕ has no umbilical points. In fact, if $p \in N_1^2$ is a umbilical point, Corollary 6.3 gives $\lambda(p) = \langle \mathbf{H}, \mathbf{H} \rangle(p) = 0$, which is a contradiction. Now, let (x, y) be a local lightlike parametrization, that is, the Lorentz metric g of the surface is locally gives as $g = 2B dx dy$ for some real function $B > 0$ (see [22, p.13]). Since the coordinate fields ∂_x, ∂_y are lightlike vector fields, we may assume that $h(\partial_x, \partial_x) \neq 0$. On the other hand, from *(3)* in Lemma 3.2, we obtain

$$\langle h(\partial_x, \partial_x), h(\partial_y, \partial_y) \rangle + 2 \langle h(\partial_x, \partial_y), h(\partial_x, \partial_y) \rangle = 2\lambda B^2.$$

But by Corollary 6.3 we have $\langle h(\partial_x, \partial_y), h(\partial_x, \partial_y) \rangle = 0$ and then

$$\langle h(\partial_x, \partial_x), h(\partial_y, \partial_y) \rangle = 2\lambda B^2. \tag{6.5}$$

Thus, by the ligthlike pseudo-isotropic property (3.2), $h(\partial_x, \partial_x)$ and $h(\partial_y, \partial_y)$ are linearly independent ligthlike normal fields [18], and the second fundamental form h is a surjective map. Now, from *(2)* in Lemma 3.5 and equation (6.5), $h(\partial_x, \partial_y) = 0$. But this means that our immersion ϕ is minimal.

□

In particular, we have:

Corollary 6.9 *Let $\phi : N_1^2 \to \mathbb{R}_2^4$ be a non-totally umbilical pseudo-isotropic immersion with non-vanishing mean curvature vector field $\mathbf{H}$. Then,*

(a) ϕ is marginally trapped, i.e., the mean curvature vector $\mathbf{H}$ is ligthlike.

(b) ϕ is 0-pseudo-isotropic.

If, in addiction, N_1^2 is complete, then ϕ is (up to rigid motion) an expansion of $\mathbb{R}_1^2$ into $\mathbb{R}_{1,1}^3$.

Proof. *(a)* This follow immediately from Corollary 6.3. *(b)* With a similar reasoning as in the proof of Proposition 6.8, we obtain $\lambda(p) = 0$ for any non-umbilical point p. Now, if $p \in N_1^2$ is an umbilical point, Corollary 6.3 yields $\lambda(p) = 0$. Thus, ϕ is 0-pseudo-isotropic. Finally, if N_1^2 is complete, Theorem 6.4 says that ϕ is congruent to an expansion.

□

Theorem 6.10 *Let ϕ be a non-totally umbilical pseudo-isotropic immersion of a Lorentz surface N_1^2 with non-vanishing Gauss curvature into the pseudo-Euclidean space $\mathbb{R}_2^4$. Then, ϕ is rigid.*

Proof. Assume $\phi, \phi' : N_1^2 \to \mathbb{R}_2^4$ are non-totally umbilical pseudo-isotropic immersions. Then, by Corollary 6.3 and Proposition 6.8, the pseudo-isotropy functions of ϕ and ϕ' are both equals to $-\mathcal{K}/2$, and the respective second fundamental forms are surjective mappings. Now, by Theorem 4.2 ϕ and ϕ' are congruent, as desired. □

ACKNOWLEDGMENTS

The authors would like to express their many thanks to Professor Alfonso Romero for his valuable suggestions. Research partially supported by MCYT FEDER Grant MTM2004-04934-C04, MCYT Grant MTM2007-61248, and by Grupo de Investigación FQM-327, Junta de Andalucía.

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