MOTIVIC POINCARÉ SERIES, TORIC SINGULARITIES AND LOGARITHMIC JACOBIAN IDEALS

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Abstract. The *geometric motivic Poincaré series* of a variety, which was introduced by Denef and Loeser, takes into account the classes in the Grothendieck ring of the sequence of jets of arcs in the variety. Denef and Loeser proved that this series has a rational form. We describe it in the case of an affine toric variety of arbitrary dimension. The result, which provides an explicit set of candidate poles, is expressed in terms of the sequence of Newton polyhedra of certain monomial ideals, which we call *logarithmic jacobian ideals*, associated to the modules of differential forms with logarithmic poles outside the torus of the toric variety.

Introduction

Let S denote an irreducible and reduced algebraic variety of dimension d defined over the field $\mathbf{C}$ of complex numbers. The set $H(S)$ of formal arcs of the form, Spec $\mathbf{C}[[t]] \to S$ can be given the structure of scheme over $\mathbf{C}$ (not necessarily of finite type). If $0 \in S$ we denote by $H(S)_0 := j_0^{-1}(0)$ the subscheme of the arc space consisting on arcs in $H(S)$ with origin at 0. The set $H_k(S)$ of k-jets of S, of the form Spec $\mathbf{C}[t]/(t^{k+1}) \to S$, has the structure of algebraic variety over $\mathbf{C}$. By a theorem of Greenberg, the image of the space of arcs $H(S)$ by the natural morphism of schemes $j_k : H(S) \to H_k(S)$ which maps any arc to its k-jet, is a constructible subset of $H_k(S)$. Notice that $j_k(H(S)) = H_k(S)$ if S is smooth but $j_k(H(S)) \neq H_k(S)$ in general. Since a constructible set W has an image $[W]$ in the *Grothendieck ring of varieties* $K_0(\mathrm{Var}_{\mathbf{C}})$ it is natural to measure the singularities of S by considering the formal power series:

$$P^S_{\mathrm{geom}}(T) := \sum_{s\geq 0} [j_s(H(S))]T^s \in K_0(\mathrm{Var}_{\mathbf{C}})[[T]], \tag{1}$$

which is called the *geometric motivic Poincaré series* of S. Similarly, the *local geometric motivic Poincaré series* of the germ $(S,0)$, denoted by $P^{(S,0)}_{\mathrm{geom}}(T)$, is defined by replacing $H(S)$ by $H(S)_0$ in the right hand side of (1). Denef and Loeser introduced these series, inspired by the Poincaré series of Serre-Oesterlé in arithmetic geometry (see [D-L5]). They proved that the image of these series in the ring $K_0(\mathrm{Var}_{\mathbf{C}})[\mathbf{L}^{-1}][[T]]$ (where $\mathbf{L} = [\mathbf{A}^1_{\mathbf{C}}]$ denotes the class of the affine line) has a rational form (see [D-L1]).

If $(S,0)$ is an analytically irreducible germ of plane curve the series $P^{(S,0)}_{\mathrm{geom}}(T)$ is determined by the multiplicity of $(S,0)$ (see [D-L3]). For a general singular variety S, the invariants of S encoded by the series $P^{(S,0)}_{\mathrm{geom}}(T)$, in particular by the denominator of its rational form, are not well understood. In comparison with other motivic series, as the motivic zeta functions of a polynomial or ideal, there is not a general formula for $P^{(S,0)}_{\mathrm{geom}}(T)$ in terms of a resolution of singularities of S (see [D-L2]). Some positive results in this direction have been obtained by Nicaise for a class of singularities defined in terms of the existence of an embedded resolution of special type; the simplest example in this class are those hypersurface singularities with embedded resolution obtained by one blowing up (see [N2]). Lejeune-Jalabert and Reguera [LJ-R] have given a formula for the local geometric motivic Poincaré series of a germ $(S,0)$ of *normal* toric surface at its distinguished point in terms of the Hirzebruch-Jung continued fraction describing the resolution of singularities of the germ $(S,0)$. The alternative

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computation of the geometric motivic Poincaré series of a normal toric surface singularity is given in [N2]. The comparison with the arithmetic and the Igusa series is contained in [N1].

In this paper we consider the case of $(S,0)$ being the germ of an affine toric variety $(Z^\Lambda,0)$ of dimension d at its 0-dimensional orbit. Here Λ denotes a semigroup of finite type of a rank d lattice M such that $Z^\Lambda := \mathrm{Spec}\mathbf{C}[\Lambda]$ (cf. Notation 1.1).

Our approach is inspired by Lejeune-Jalabert and Reguera [LJ-R], though there are substantial differences coming from the particularities of normal toric surfaces, which verify that:

(a) Every truncated arc is the jet of an arc with generic point in the torus.
(b) Any pair of consecutive integral vectors in the boundary of the Newton polygon of the maximal ideal define a basis of the lattice.

Property (a) holds more generally for normal toric singularities (see [N1]) but does not hold in general without the assumption of normality, the simplest example is the Whitney umbrella (see Remark 3.4). Property (b), which plays also an essential role in the comparison of various types of motivic series in [N1], does not generalize even for normal toric germs of dimension ≥ 3.

We deal with the failure of property (a) by characterizing combinatorially the jets of those arcs which cannot be obtained as jets of arcs factoring through proper orbit closures of the action of the torus on Z^Λ. We define the auxiliary series $P(\Lambda)$ by taking classes in the Grothendieck ring of these sets and considering the associated Poincaré series. We have that $P_{\mathrm{geom}}^{(Z^\Lambda,0)}(T) = \sum P(\Lambda\cap\tau)$, where τ runs through the faces of the cone $\mathbf{R}_{\geq 0}\Lambda$ of $M_\mathbf{R} := M \otimes_\mathbf{Z} \mathbf{R}$. The term $P(\Lambda\cap\tau)$ is the auxiliary series associated to the toric variety $Z^{\Lambda\cap\tau}$, which is an orbit closure of the torus action on Z^Λ.

The failure of (b) is overcome by the systematic use of the *logarithmic jacobian ideals* associated to the toric variety Z^Λ to study jet spaces. The logarithmic jacobian ideals $\mathcal{J}_1,\dots,\mathcal{J}_d$ of Z^Λ are defined in terms of the minimal set of generators of the semigroup Λ in Section 4. The ideal $\mathcal{J}_1$ is the maximal ideal defining the closed point of the germ $(Z^\Lambda,0)$. The ideal $\mathcal{J}_d$ appears in [LJ-R] in connection with the combinatorial description of the Nash blowing up (see also [GS, T]). If $1\leq k\leq d$, the logarithmic jacobian ideal $\mathcal{J}_k$ can be described in terms of the module of Kähler differential k-forms on Z^Λ over $\mathbf{C}$, in a way which generalizes the one given for $\mathcal{J}_d$ in the Appendix of [LJ-R] (see Section 11). Up to our knowledge, if $d\geq 3$ the ideals $\mathcal{J}_2,\dots,\mathcal{J}_{d-1}$ appear in this paper for the first time in the literature.

Ishii noticed that the arc space of the torus acts on the arc space of the toric variety Z^Λ (see [I1, I2]). The set H^* consisting of those arcs of $H(Z^\Lambda)_0$ which have their generic point in the torus is a union of orbits. These orbits are in bijection with the possible orders of the arcs, naturally identified with the elements of the dual lattice $N := M^*$, which are in the interior of the dual cone σ of $\mathbf{R}_{\geq 0}\Lambda$. For $\nu \in \overset{\circ}{\sigma} \cap N$ we denote by H_ν^* the corresponding orbit in the arc space. We show that the jets of these orbits are either disjoint or equal and we characterize the equality in combinatorial terms. We prove that the coefficient of T^m in the auxiliary series $P(\Lambda)$ expands as the sum of classes $[j_m(H_\nu^*)]$ in the Grothendieck ring, for ν running through a finite subset of $\overset{\circ}{\sigma} \cap N$. The combinatorial convexity properties of the Newton polyhedra of the logarithmic jacobian ideals allow us to determine a simple formula for the class of $j_m(H_\nu^*)$ in the Grothendieck ring (see Theorem 7.1).

The main result states that the rational form of the geometric motivic Poincaré series $P_{\mathrm{geom}}^{(Z^\Lambda,0)}(T)$ is determined by the Newton polyhedra (with integral structure) of the logarithmic jacobian ideals of the orbit closures of Z^Λ (see Theorem 4.9 and Corollary 4.10). In particular we describe explicitly a finite set of candidate poles for the rational form of $P_{\mathrm{geom}}^{(Z^\Lambda,0)}(T)$. We give a geometrical interpretation of the candidate poles in terms of the order of vanishing of certain sheaves of locally principal monomial ideals along the exceptional divisors of certain modifications, which are both defined in terms of the logarithmic jacobian ideals. The rationality of the series is deduced at this point from a purely combinatorial result: the rationality of the generating series of the projection of the set of integral points in the interior of a rational open cone (see Theorem 12.4). The appearance of these projections seems the combinatorial analogue of the quantifier elimination results used in [D-L1].

We give two applications:

- We deduce a formula for the global geometric motivic Poincaré series $P_{\mathrm{geom}}^{Z^\Lambda}(T)$ in the normal case (see Theorem 4.11).
- We prove a formula for the *motivic volume* of the arc space of the germ $(Z^\Lambda,0)$ in terms of the logarithmic jacobian ideal $\mathcal{J}_d$ (see Proposition 10.2). We have obtained this result without

using Denef and Loeser's formula for the motivic volume of a variety S in terms of a resolution of singularities (see [D-L1]).

In the normal toric surface case, property (b) allows an explicit description of the series in [LJ-R]. In this case only the terms $1-T$ and $1-\mathbf{L}T$, which appear then in the denominator of the rational form of the series in Corollary 4.10, are not actual poles. This property is a particularity of the normal toric surface case. We give an example of toric surface Z^Λ such that all terms in the denominator of the rational form of $P_{\rm geom}^{(Z^\Lambda,0)}(T)$ in Corollary 4.10, are actually poles (see Section 13).

In [C-GP] we extend the results and approach of this paper to the case of a germ of irreducible quasi-ordinary hypersurface singularity of arbitrary dimension d in terms of similar notions of logarithmic jacobian ideals. Rond states some partial results on this case in [R]. In general it is a challenge to analyse this motivic series in terms of some suitable notion of logarithmic jacobian ideals associated to a partial resolution of singularities of a given singularity. It is a natural problem to find a geometrical meaning for the logarithmic jacobian ideals in terms of limits of tangent spaces.

The results of this paper hold if the base field of complex numbers $\mathbf{C}$ is replaced by an algebraically closed field of zero characteristic. The assumption that the base field has characteristic zero is used in Section 6.

The paper is organized as follows. In Section 1 we set our notations on toric varieties. In Section 2 some results on arcs and jets spaces are recalled. We describe the orbit decomposition of the arc space of a toric variety in Section 3. In Section 4 we state the main results. In Section 5 we give some combinatorial convexity properties of the Newton polyhedra of the logarithmic jacobian ideals. Section 6 deals with the universal family of arcs in the torus. In Section 7 we analyze the jets of the orbits in the arc space. The main results on the geometric motivic Poincaré series are proved in Sections 8 and 9. A formula for the motivic volume is given in Section 10. Sections 11 and 12 can be read independently of the rest of the paper. Section 11 is dedicated to the definition of the sequence of logarithmic jacobian ideals in terms of differential forms. Section 12 deals with generating functions.

1. A reminder of toric geometry

In this Section we introduce the basic notions and notations from toric geometry (see [Ew, O, F, GKZ] for proofs). Following the convention established at the meeting "Convex and algebraic geometry", Oberwolfach (2006), we do not assume the normality in the definition of toric varieties.

If $N \cong \mathbf{Z}^d$ is a lattice we denote by $N_{\mathbf{R}} := N \otimes \mathbf{R}$ (resp. $N_{\mathbf{Q}} := N \otimes \mathbf{Q}$) the vector space spanned by N over the field $\mathbf{R}$ (resp. over $\mathbf{Q}$). In what follows a *cone* in $N_{\mathbf{R}}$ mean a *rational convex polyhedral cone*: the set of non negative linear combinations of vectors $a_1, \dots, a_r \in N$. The cone τ is *strictly convex* if it contains no line through the origin, in that case we denote by 0 the 0-dimensional face of τ; the cone τ is *simplicial* if the primitive vectors of the 1-dimensional faces are linearly independent over $\mathbf{R}$. We denote by $\mathring{\tau}$ or by $\mathrm{int}(\tau)$ the relative interior of the cone τ. We denote by $\mathbf{R}\tau$ the real vector subspace spanned by τ in $N_{\mathbf{R}}$.

We denote by M the dual lattice. The *dual* cone $\tau^\vee \subset M_{\mathbf{R}}$ (resp. *orthogonal* cone $\tau^\perp$) of τ is the set $\{w \in M_{\mathbf{R}} \mid \langle w, u\rangle \geq 0$, (resp. $\langle w, u\rangle = 0$) $\forall u \in \tau\}$.

A *fan* Σ is a family of strictly convex cones in $N_{\mathbf{R}}$ such that any face of such a cone is in the family and the intersection of any two of them is a face of each. The relation $\theta \leq \tau$ (resp. $\theta < \tau$) denotes that θ is a face of τ (resp. $\theta \neq \tau$ is a face of τ). The *support* (resp. the *k-skeleton*) of the fan Σ is the set $|\Sigma| := \bigcup_{\tau\in\Sigma} \tau \subset N_{\mathbf{R}}$ (resp. $\Sigma^{(k)} = \{\tau \in \Sigma \mid \dim \tau = k\}$). We say that a fan Σ' is a *subdivision* of the fan Σ if both fans have the same support and if every cone of Σ' is contained in a cone of Σ. If Σ_i for $i = 1, \dots, n$, are fans with the same support their intersection $\cap_{i=1}^n \Sigma_i := \{\cap_{i=1}^n \tau_i \mid \tau_i \in \Sigma_i\}$ is also a fan. The 1-skeleton of $\cap_{i=1}^n \Sigma_i$ is $\cup_{i=1}^n \Sigma_i^{(1)}$.

Notation 1.1. In this paper Λ is a sub-semigroup of finite type of a lattice M, which generates M as a group and such that the cone $\sigma^\vee = \mathbf{R}_{\geq 0}\Lambda$ is strictly convex and of dimension d. We denote by N the dual lattice of M and by $\sigma \subset N_{\mathbf{R}}$ the dual cone of $\sigma^\vee$. We denote by Z^Λ the *affine toric variety* $Z^\Lambda = \operatorname{Spec} \mathbf{C}[\Lambda]$, where $\mathbf{C}[\Lambda] = \{\sum_{\rm finite} a_\lambda X^\lambda \mid a_\lambda \in \mathbf{C}\}$ denotes the semigroup algebra of the semigroup Λ with coefficients in the field $\mathbf{C}$. The semigroup Λ has a unique minimal set of generators $e_1, \dots, e_n$ (see the proof of Chapter V, Lemma 3.5, page 155 [Ew]). We have an *embedding* of $Z^\Lambda \subset \mathbf{C}^n$ given by, $x_i := X^{e_i}$ for $i = 1, \dots n$.

If $\Lambda = \sigma^\vee \cap M$ then the variety Z^Λ, which we denote also by $Z_{\sigma,N}$ or by Z_σ when the lattice is clear from the context, is normal. If $\Lambda \neq \sigma^\vee \cap M$ the inclusion of semigroups $\Lambda \to \bar\Lambda := \sigma^\vee \cap M$ defines a toric modification $Z^{\bar\Lambda} \to Z^\Lambda$, which is the *normalization map*.

The affine varieties Z_τ corresponding to cones in a fan Σ glue up to define a *toric variety* Z_Σ. For instance, the toric variety defined by the fan formed by the faces of the cone σ coincides with the affine toric variety Z_σ. The subdivision Σ' of a fan Σ defines a *toric modification* $\pi_{\Sigma'} : Z_{\Sigma'} \to Z_\Sigma$.

The torus $T_N := Z^M$ is an open dense subset of Z^Λ, which acts on Z^Λ and the action extends the action of the torus on itself by multiplication. The *origin* 0 of the affine toric variety Z^Λ is the 0-dimensional orbit, which is defined by the maximal ideal $(X^\lambda)_{0\neq\lambda\in\Lambda}$ of $\mathbf{C}[\Lambda]$. There is a one to one inclusion reversing correspondence between the faces of σ and the orbit closures of the torus action on Z^Λ. If $\theta \leq \sigma$, we denote by $\mathrm{orb}_\theta^\Lambda$ the orbit corresponding to the face θ of σ. The set $\Lambda \cap \theta^\perp$ is a semigroup of finite type which generates a sublattice $M(\theta, \Lambda)$ of finite index $i(\theta, \Lambda)$ of the lattice $M \cap \theta^\perp$. We denote by $N(\theta, \Lambda)$ the dual lattice of $M(\theta, \Lambda)$. The image $\sigma/\mathbf{R}\theta$ of the cone σ in the vector space $N_\mathbf{R}/\mathbf{R}\theta$ is the dual cone of $\mathbf{R}_{\geq 0}(\Lambda \cap \theta^\perp)$. The toric variety $Z^{\Lambda\cap\theta^\perp}$, which is embedded in Z^Λ, is the closure of $\mathrm{orb}_\theta^\Lambda$. The origins of Z^Λ and $Z^{\Lambda\cap\theta^\perp}$ coincide. We say that Z^Λ is *analytically unibranched* if for any $x \in Z^\Lambda$ the germ (Z^Λ, x) is analytically irreducible, i.e., for all $\theta \leq \sigma$ we have $i(\theta, \Lambda) = 1$ (see [GKZ] Chapter 5). Notice that normal toric varieties are analytically unibranched but the converse is not true.

The ring $\mathbf{C}[[\Lambda]]$ of formal power series with coefficients in $\mathbf{C}$ and exponents in the semigroup Λ is isomorphic to the completion of the local ring of germs of holomorphic functions at $(Z^\Lambda, 0)$ with respect to its maximal ideal.

The *Newton polyhedron* of a monomial ideal corresponding to a non empty set of lattice vectors $\mathcal{I} \subset \Lambda$ is defined as the convex hull of the Minkowski sum of sets $\mathcal{I} + \sigma^\vee$. We denote this polyhedron by $\mathcal{N}(\mathcal{I})$. We denote by $\mathrm{ord}_\mathcal{I}$ the *support function* of the polyhedron $\mathcal{N}(\mathcal{I})$, which is defined by $\mathrm{ord}_\mathcal{I} : \sigma \to \mathbf{R}$, $\nu \mapsto \inf_{\omega\in\mathcal{N}(\mathcal{I})} \langle \nu, \omega\rangle$. The face of the polyhedron $\mathcal{N}(\mathcal{I})$ determined by $\nu \in \sigma$ is the set $\mathcal{F}_\nu := \{\omega \in \mathcal{N}(\mathcal{I}) \mid \langle \nu, \omega\rangle = \mathrm{ord}_\mathcal{I}(\nu)\}$. All faces of $\mathcal{N}(\mathcal{I})$ are of this form, the compact faces are defined by vectors $\nu \in \overset{\circ}{\sigma}$. The *dual fan* $\Sigma(\mathcal{I})$ associated to an integral polyhedron $\mathcal{N}(\mathcal{I})$ is a fan supported on σ which is formed by the cones $\sigma(\mathcal{F}) := \{\nu \in \sigma \mid \langle \nu, \omega\rangle = \mathrm{ord}_\mathcal{I}(\nu),\ \forall \omega \in \mathcal{F}\}$, for $\mathcal{F}$ running through the faces of $\mathcal{N}(\mathcal{I})$. Notice that if $\tau \in \Sigma(\mathcal{I})$ and if $\nu, \nu' \in \overset{\circ}{\tau}$ then $\mathcal{F}_\nu = \mathcal{F}_{\nu'}$. We denote this face of $\mathcal{N}(\mathcal{I})$ also by $\mathcal{F}_\tau$. Notice that the vertices of $\mathcal{N}(\mathcal{I})$ are elements of $\mathcal{I}$.

If $\mathcal{I}$ is a monomial ideal of Z_σ and $\Sigma = \Sigma(\mathcal{I})$, then the toric modification $\pi_\Sigma : Z_\Sigma \to Z_\sigma$ is the *normalized blowing up* of Z_σ centered at $\mathcal{I}$ (see [LJ-R] for instance).

2. Arcs, jet spaces and the geometric motivic Poincaré series

In this Section we introduce arc and jet spaces on a variety S, i.e., a reduced separated scheme of finite type over $\mathbf{C}$. For simplicity we assume that S is affine and equidimensional of dimension d. We refer to [I3, E-M] for expository papers on arc and jet schemes. See [D-L2, Lo, V], for expository papers on motivic integration on arc spaces and applications.

We have that for all integers $m \geq 0$ the functor from the category of $\mathbf{C}$-algebras to the category of sets, sending a $\mathbf{C}$-algebra R to the set of $R[t]/(t^{m+1})$-rational points of S is representable by a $\mathbf{C}$-scheme $H_m(S)$ of finite type over $\mathbf{C}$, called the m-jet scheme of S. The natural maps induced by truncation $j_m^{m+1} : H_{m+1}(S) \to H_m(S)$ are affine and hence the projective limit $H(S) := \varprojlim H_m(S)$ is a $\mathbf{C}$-scheme, not necessarily of finite type, called the arc space of S. The scheme $H(S)$ represents a functor sending a $\mathbf{C}$-algebra R to the set of $R[[t]]$-rational points of S. It is the arc space of S. We consider the schemes $H_m(S)$ and $H(S)$ with their reduced structure. If $Z \subset S$ is a closed subvariety then $H(S)_Z := j_0^{-1}(Z)$ (resp. $H_m(S)_Z := (j_0^m)^{-1}(Z)$) denotes the subscheme of $H(S)$ (resp. of $H_m(S)$) formed by arcs (resp. m-jets) in S with origin in Z.

We have natural morphisms $j_m : H(S) \to H_m(S)$. By an *arc* we mean a $\mathbf{C}$-rational point of $H(S)$, i.e., a morphism Spec $\mathbf{C}[[t]] \to S$. By an m-jet we mean a $\mathbf{C}$-rational point of $H_m(S)$, i.e., a morphism Spec $\mathbf{C}[t]/(t^{m+1}) \to S$. The origin of the arc (resp. of the m-jet) is the image of the closed point 0 of Spec $\mathbf{C}[[t]]$ (resp. of Spec $\mathbf{C}[t]/(t^{m+1})$).

If $h(t) = \sum_{i\geq 0} a_i t^i$ is a formal power series and $m \geq 0$ we set $j_m(h(t)) := h(t) \mod t^{m+1}$.

Suppose that $S \subset \mathbf{A}^n_{\mathbf{C}}$ is a closed affine subvariety with ideal $I \subset \mathbf{C}[x_1, \dots, x_n]$, for $(x_1, \dots, x_n)$ coordinates of $\mathbf{C}^n$. An arc Spec $\mathbf{C}[[t]] \to \mathbf{A}^n_{\mathbf{C}}$ is defined by n power series

$$x_i(t) = a_i^{(0)} + a_i^{(1)} t + a_i^{(2)} t^2 + \cdots + a_i^{(r)} t^r + \cdots, \quad i = 1, \dots, n. \tag{2}$$

An m-jet Spec$\mathbf{C}[t]/(t^{m+1}) \to \mathbf{A}^n_{\mathbf{C}}$ is defined by n-polynomials of the form given by (2) $\mod t^{m+1}$. If $F \in I$ we have a power series expansion

$$F(x_1(t), \dots, x_n(t)) = \alpha_F^{(0)}(\underline{a}^{(0)}) + \alpha_F^{(1)}(\underline{a}^{(0)}, \underline{a}^{(1)}) t + \alpha_F^{(2)}(\underline{a}^{(0)}, \underline{a}^{(1)}, \underline{a}^{(2)}) t^2 + \cdots \tag{3}$$

where the coefficients $\alpha_F^{(k)}$ are polynomials expressions in $(\underline{a}^{(0)}, \dots, \underline{a}^{(k)})$ where $\underline{a}^{(j)} = (a_1^{(j)}, \dots, a_n^{(j)})$, for $j \in \mathbf{Z}_{\geq 0}$. The arc (2) (resp. the m-jet of (2)) factors through S if (3) vanishes for all $F \in I$ (resp. if (3) vanishes $\mod t^{m+1}$ for all $F \in I$). The arc space $H(S)$ (resp. m-jet space $H_m(S)$) is the reduced scheme underlying the affine scheme Spec$\mathcal{A}_S$, where $\mathcal{A}_S = \mathbf{C}[\underline{a}^{(0)}, \dots, \underline{a}^{(k)}, \dots]/(\alpha_F^{(0)}, \dots, \alpha_F^{(k)}, \dots)_{F \in I}$ (resp. Spec $\mathcal{A}_{S,m}$, where $\mathcal{A}_{S,m} := \mathbf{C}[\underline{a}^{(0)}, \dots, \underline{a}^{(m)}]/(\alpha_F^{(0)}, \dots, \alpha_F^{(m)})_{F \in I}$). The *universal family of arcs* of S, which is the map Spec $\mathcal{A}_S[[t]] \to S$ defined by (2), parametrizes the arcs in $H(S)$.

We recall the definition of the *Grothendieck ring* $K_0(\mathrm{Var}_{\mathbf{C}})$ of $\mathbf{C}$-*varieties*. This ring is generated by the symbols $[X]$ for X an algebraic variety, subject to relations: $[X] = [X']$ if X is isomorphic to X', $[X] = [X - X'] + [X']$ if X' is closed in X and $[X][X'] = [X \times X']$. We denote by $\mathbf{L} := [\mathbf{A}^1_{\mathbf{C}}]$ the class of the affine line and by $\mathcal{M}$ the localization $K_0(\mathrm{Var}_{\mathbf{C}})[\mathbf{L}^{-1}]$.

If C is a constructible subset of some variety X, i.e. a disjoint union of finitely many locally closed subvarieties A_i of X , then it is easy to see that $[C] \in K_0(\mathrm{Var}_{\mathbf{C}})$ is well defined as $[C] := \sum_i [A_i]$.

A set $A \subset H(S)$ is *constructible* or *cylindric* if $A = j_m^{-1}(C)$, for some integer m and some constructible subset $C \subset H_m(S)$; the constructible set A is *stable* if, in addition, for all $p \geq m$ the projection $j_p^{p+1} : j_{p+1}(A) \to j_p(A)$ is a piece-wise trivial fibration with fiber $\mathbf{A}^d_{\mathbf{C}}$ (where $d = \dim S$). If $A \subset H(S)$ is constructible and $A \cap H(\mathrm{Sing} S) = \emptyset$ then A is stable (see [D-L1]). For a stable set it makes sense to consider the *naive motivic measure*, defined as the limit $\lim_{m \to \infty} [j_m(A)] \mathbf{L}^{-md} \in \mathcal{M}$ (by definition of stability all the terms $[j_m(A)] \mathbf{L}^{-md}$ are equal for m large enough). Kontsevich introduced a completion $\hat{\mathcal{M}} := \varprojlim \mathcal{M}/F^m$ of the ring $\mathcal{M}$, where F^m for $m \in \mathbf{Z}$, is the subgroup of $\mathcal{M}$ generated by $[X] \mathbf{L}^{-i}$ such that $\dim X + m \leq i$ and (F^m) defines a ring filtration since $F^m F^p \subset F^{m+p}$.

Theorem 2.1. (see [D-L1] Theorem 7.1) *Let A be a constructible subset of $H(S)$. Then the limit $\mu(A) := \lim_{m \to \infty} [j_m(A)] \mathbf{L}^{-md}$ exists in $\hat{\mathcal{M}}$. If $A = H(S)$ this limit is nonzero.*

If A is a constructible subset of $H(S)$, then $\mu(A)$ is called the *motivic measure* of A. Notice that if S is irreducible and $Z \subset S$ is a proper closed subset then $H(Z) \subset H(S)$ is not cylindric. There exists a class of *measurable* sets containing $H(Z)$ and the cylinders and a measure μ with values on $\hat{\mathcal{M}}$, extending the motivic measure of constructible sets. We refer to [D-L1, D-L4, Lo] for the precise definition.

Definition 2.2. The motivic measure of the arc space $H(S)_Z$ for Z a closed subvariety of S, is called the *motivic volume* of $H(S)_Z$.

Proposition 2.3. (see [D-L1, D-L4, Lo]) *If $A \subset H(S)$ is a measurable set such that $A \subset H(Z)$ for some closed subvariety $Z \subset S$ with* $\dim Z < \dim S$ *then* $\mu(A) = 0$.

By a Theorem of Greenberg [Gr], see also [E-M], $j_m(H(S))$ is a constructible subset of $H_m(S)$, hence it has an image in the Grothendieck ring $K_0(\mathrm{Var}_{\mathbf{C}})$. The same applies for $j_m(H(S)_Z)$ if $Z \subset S$ is a closed subvariety.

Definition 2.4. Let S be a variety and $Z \subset S$ a closed subvariety. The *geometric motivic Poincaré series* of S (resp. of (S, Z)) is the element of $K_0(\mathrm{Var}_{\mathbf{C}})[[T]]$ defined by

$$P^S_{\mathrm{geom}}(T) := \sum_{m \geq 0} [j_m(H(S))] T^m \quad (\text{resp. } P^{(S,Z)}_{\mathrm{geom}}(T) := \sum_{m \geq 0} [j_m(H(S)_Z)] T^m \,).$$

For instance, it is easy to see that $P^{\mathbf{C}^d}_{\mathrm{geom}}(T) = \mathbf{L}^d (1 - \mathbf{L}^d T)^{-1}$ and $P^{(\mathbf{C}^d, 0)}_{\mathrm{geom}}(T) = (1 - \mathbf{L}^d T)^{-1}$. We often call the series $P^S_{\mathrm{geom}}(T)$ (resp. $P^{(S,Z)}_{\mathrm{geom}}(T)$) the motivic Poincaré series of S (resp. of (S, Z)) for short. Denef and Loeser proved that these series have a rational form:

Theorem 2.5. (see [D-L1] Theorem 1.1) *The series $P^S_{\mathrm{geom}}(T)$ (resp. $P^{(S,Z)}_{\mathrm{geom}}(T)$), considered as an element of $\mathcal{M}[[T]]$ belongs to $\mathcal{M}(T)$, more precisely there exist $Q(T) \in \mathcal{M}[T]$, $a_i \in \mathbf{Z}$ and $b_i \in \mathbf{Z}_{\geq 1}$, for $i = 1, \dots, r$, such that the series is of the form $Q(T) \prod_{i=1}^r (1 - \mathbf{L}^{a_i} T^{b_i})^{-1}$.*

The proof of this deep result is based on quantifier elimination for semi-algebraic sets of power series, a substantial development of the theory of motivic integration introduced by Kontsevich and the existence of resolution of singularities of varieties over a field of zero characteristic. See [D-L5] for relations with other Poincaré series in arithmetic geometry.

3. Arcs and jets on a toric singularity

Let Λ be a semigroup, as in Notation 1.1. If R is a $\mathbf{C}$-algebra, a R-rational point of Z^Λ is a homomorphism of semigroups $(\Lambda, +) \to (R, \cdot)$, where $(R, \cdot)$ denotes the semigroup R for the multiplication. In particular, the closed points are obtained for $R = \mathbf{C}$. An arc h on the affine toric variety Z^Λ is given by a semigroup homomorphism $(\Lambda, +) \to (\mathbf{C}[[t]], \cdot)$. An arc in the torus T_N is defined by a semigroup homomorphisms $\Lambda \to \mathbf{C}[[t]]^*$, where $\mathbf{C}[[t]]^*$ denotes the group of units of the ring $\mathbf{C}[[t]]$.

Notation 3.1. We denote the set of arcs $H(Z^\Lambda)_0$ of Z^Λ with origin at the distinguished point 0 of Z^Λ simply by H_Λ, and by H^*_Λ the set consisting of those arcs of H_Λ with generic point in the torus T_N.

Notice that $h \in H^*_\Lambda$ if and only if for all $u \in \Lambda$ the formal power series $X^u \circ h \in \mathbf{C}[[t]]$ is non-zero. Any arc $h \in H^*_\Lambda$ defines two group homomorphisms $\nu_h : M \to \mathbf{Z}$ and $\omega_h : M \to \mathbf{C}[[t]]^*$ by: $X^m \circ h = t^{\nu_h(m)} \omega_h(m)$. If $m \in \Lambda$ then $\nu_h(m) > 0$ hence ν_h belongs to $\overset{\circ}{\sigma} \cap N$. Notice that ω_h defines an arc in the torus, i.e., $\omega_h \in H(T_N)$.

Ishii noticed that the space of arcs in the torus acts on the arc space of a toric variety (see [I1, I2]).

Lemma 3.2. (Theorem 4.1 of [I1], and Lemma 5.6 of [I2]). *The map $\overset{\circ}{\sigma} \cap N \times H(T_N) \to H^*_\Lambda$ which sends a pair (ν, ω) to the arc h defined by $X^u \circ h = t^{\langle \nu, u \rangle} \omega(u)$, for $u \in \Lambda$, is a one to one correspondence. The sets $H^*_{\Lambda,\nu} := \{h \in H^*_\Lambda \mid \nu_h = \nu\}$ for $\nu \in \overset{\circ}{\sigma} \cap N$ are orbits for the action of H_{T_N} on H^*_Λ and we have that $H^*_\Lambda = \bigsqcup_{\nu \in \overset{\circ}{\sigma} \cap N} H^*_{\Lambda,\nu}$.*

The sets defining these orbits were also considered by Lejeune-Jalabert and Reguera in the normal toric surface case (Proposition 3.3 of [LJ-R]).

Remark 3.3. We often denote the set H^*_Λ (resp. the orbit $H^*_{\Lambda,\nu}$) by H^* (resp. by H^*_ν) if Λ is clear from the context.

An arc $h \in H_\Lambda$ has its generic point η contained in exactly one orbit of the torus action on Z^Λ. If $h(\eta) \in \mathrm{orb}^\Lambda_\theta$, for some $\theta \leq \sigma$, then h factors through the orbit closure $Z^{\Lambda \cap \theta^\perp}$ and $h \in H^*_{\Lambda \cap \theta^\perp}$, i.e., h is an arc through $(Z^{\Lambda \cap \theta^\perp}, 0)$ with generic point in $T_{N(\Lambda,\theta)} = \mathrm{orb}^\Lambda_\theta$. We can apply Lemma 3.2 to describe the set $H^*_{\Lambda \cap \theta^\perp}$, just replacing Λ, σ, M and N by $\Lambda \cap \theta^\perp$, $\sigma/\mathbf{R}\theta$, $M(\Lambda, \theta)$ and $N(\Lambda, \theta)$ respectively (cf. with notations in Section 1). In particular, if $\theta = 0$ then $h \in H^*_\Lambda$; if $\theta = \sigma$ then $\Lambda \cap \theta^\perp = 0$ and h is the constant arc at the distinguished point $0 \in Z^\Lambda$. We have a partition $H_\Lambda = \sqcup_{\theta \leq \sigma} H^*_{\Lambda \cap \theta^\perp}$.

Remark 3.4. In the normal case the equality $j_m(H_\Lambda) = j_m(H^*_\Lambda)$ holds for all $m \geq 0$, see [N1]. This property fails in general, for instance, the arc $h(t) = (0, t, 0)$ of the *Whitney umbrella*, $\{(x_1, x_2, x_2) \mid x_1^2 x_2 - x_3^2 = 0\}$, is contained in the singular locus but its 1-jet is not obtained as the jet of an arc h' with generic point in the torus.

4. Statement of the main results on the geometric motivic Poincaré series

In this Section we state the main results of the paper. The proofs are given in Section 9.

We consider the following auxiliary Poincaré series:

$$P(\Lambda) := \sum_{s \geq 0} \big[j_s(H^*_\Lambda) \setminus \bigcup_{0 \neq \theta \leq \sigma} j_s(H_{\Lambda \cap \theta^\perp}) \big] T^s \in K_0(\mathrm{Var}_{\mathbf{C}})[[T]]. \tag{4}$$

Notice that the Poincaré series $P(\Lambda)$ measures the classes in the Grothendieck ring of the jets of arcs in H^*_Λ which are not jets of arcs in $H_{\Lambda \cap \theta^\perp}$, for any $0 \neq \theta \leq \sigma$, i.e., jets of arcs with origin in 0 which are not jets of arcs factoring through proper orbit closures of the toric variety Z^Λ. It follows that:

Proposition 4.1.

$$P_{\text{geom}}^{(Z^\Lambda,0)}(T) = \sum_{\theta \leq \sigma} P(\Lambda \cap \theta^\perp).$$

Example 4.2. The series $P(\Lambda \cap \sigma^\perp)$ takes into account those jets of arcs in H_Λ which coincide with the jet of the constant arc. We have that $P(\Lambda \cap \sigma^\perp) = \sum_{s \geq 0} [\{0\}] T^s = \sum_{s \geq 0} T^s$ hence $P(\Lambda \cap \sigma^\perp) = (1-T)^{-1}$.

Proposition 4.3. *If $d = 1$ and the multiplicity of the monomial curve Z^Λ at the origin is equal to m then the series $P(\Lambda)$ is equal to $(\mathbf{L}-1)T^m(1-\mathbf{L}T)^{-1}(1-T^m)^{-1}$.*

Example 4.4. Let Λ be a semigroup, as in Notation 1.1, defining a toric variety of arbitrary dimension. For any $\theta \leq \sigma$ of codimension 1 we denote by m_θ the multiplicity of the monomial curve $(Z^{\Lambda \cap \theta^\perp}, 0)$. Then we have $P(\Lambda \cap \theta^\perp) = (\mathbf{L}-1)T^{m_\theta}(1-\mathbf{L}T)^{-1}(1-T^{m_\theta})^{-1}$.

Definition 4.5. Recall that $e_1 \ldots, e_n$ denote the minimal system of generators of the semigroup Λ. The *k^{th}-logarithmic jacobian ideal* of Z^Λ is the monomial ideal $\mathcal{J}_k$ of $\mathbf{C}[\Lambda]$ corresponding to the following subset of Λ,

$$\{e_{i_1} + \cdots + e_{i_k} \mid e_{i_1} \wedge \cdots \wedge e_{i_k} \neq 0, \text{ for } 1 \leq i_1 < \cdots < i_k \leq n\}. \tag{5}$$

We abuse of notation by denoting also by $\mathcal{J}_k$ the set (5).

Remark 4.6. The motivation for this terminology is inspired by the Appendix in [LJ-R] and by the fact that these ideals are defined geometrically in terms of differential forms on Z^Λ with logarithmic poles outside the torus (see Section 11).

Notation 4.7. We denote by Σ_k (resp. by $\operatorname{ord}_{\mathcal{J}_k}$) the dual subdivision of σ (resp. the support function) of the Newton polyhedron of the k^{th}-logarithmic jacobian ideal $\mathcal{J}_k$, for $k = 1, \ldots, d$. The maps

$$\begin{cases} \phi_1 := \operatorname{ord}_{\mathcal{J}_1} & \text{and} \quad \phi_k := \operatorname{ord}_{\mathcal{J}_k} - \operatorname{ord}_{\mathcal{J}_{k-1}} & \text{for } k = 2, \ldots, d, \\ \Psi_1 := 0 & \text{and} \quad \Psi_k := (k-1)\operatorname{ord}_{\mathcal{J}_k} - k \operatorname{ord}_{\mathcal{J}_{k-1}} & \text{for } k = 2, \ldots, d, \end{cases}$$

are piece-wise linear functions defined on the cone σ. If $\nu \in \sigma$ we put $\phi_0(\nu) := 0$ and $\phi_{d+1}(\nu) := +\infty$. If $\rho \subset \sigma$ is a cone of dimension one, we denote by ν_ρ the generator of the semigroup $\rho \cap N$. We define the finite set:

$$B(\Lambda) := \{(d,1)\} \cup \bigcup_{k=1}^{d} \left\{ (\Psi_k(\nu_\rho), \phi_k(\nu_\rho)) \mid \rho \in \cup_{i=1}^k \Sigma_i^{(1)}, \text{ and } \mathring{\rho} \cap \mathring{\sigma} \neq \emptyset \text{ if } k < d \right\}. \tag{6}$$

Remark 4.8. Notice that the set $B(\Lambda)$ depends only on the Newton polyhedra (with integral structure) of the logarithmic jacobian ideals of Z^Λ. In particular, we apply this observation to the sets $B(\Lambda \cap \theta^\perp)$ for $\theta < \sigma$. For $\theta = \sigma$ we convey that $B(\Lambda \cap \sigma^\perp) := \{(0,1)\}$.

Theorem 4.9. *The series $P(\Lambda)$ is of the form*

$$P(\Lambda) = Q_\Lambda \prod_{(a,b) \in B(\Lambda)} (1 - \mathbf{L}^a T^b)^{-1}, \text{ where } Q_\Lambda \in \mathbf{Z}[\mathbf{L}, T]$$

is determined by the lattice M and the Newton polyhedra of the logarithmic jacobian ideals of Z^Λ.

Corollary 4.10. *With notations of Theorem 4.9 the local geometric motivic Poincaré series of $(Z^\Lambda, 0)$,*

$$P_{\text{geom}}^{(Z^\Lambda,0)}(T) = \sum_{\theta \leq \sigma} Q_{\Lambda \cap \theta^\perp} \prod_{(a,b) \in B(\Lambda \cap \theta^\perp)} (1 - \mathbf{L}^a T^b)^{-1}, \tag{7}$$

is determined by the sequences of Newton polyhedra of the logarithmic jacobian ideals of $Z^{\Lambda \cap \theta^\perp}$ and lattices $M(\theta, \Lambda)$, for $\theta \leq \sigma$.

Corollary 4.11. *Suppose that the affine toric variety Z^Λ is normal. If $\theta \leq \sigma$ we denote by $\sigma_\theta^\vee$ the image of the cone $\sigma^\vee$ in $(M_\theta)_{\mathbf{R}}$, where $M_\theta := M/\theta^\perp \cap M$ and by $\Lambda(\theta)$ the semigroup $\Lambda(\theta) := (\sigma_\theta^\vee \cap M_\theta) \times \mathbf{Z}_{\geq 0}^{\mathrm{codim}\theta}$. With this notation we have*

$$P_{\mathrm{geom}}^{Z^\Lambda}(T) = \sum_{\theta \leq \sigma} (\mathbf{L}-1)^{\mathrm{codim}\theta} P_{\mathrm{geom}}^{(Z^{\Lambda(\theta)},0)}(T).$$

Remark 4.12. See also [C] for a generalization of this result to the class of affine toric varieties which are locally analytically unibranched.

We make more explicit the result for surfaces:

Corollary 4.13. *Let Z^Λ be an affine toric surface (case $d = 2$ in Notation 1.1). We denote by θ_1 and θ_2 the one dimensional faces of the cone σ. The terms which appear in the denominator of the rational expression (7) of $P_{\mathrm{geom}}^{(Z^\Lambda,0)}(T)$ are $1-T$, $1-\mathbf{L}T$, $1-\mathbf{L}^2T$, $1-T^{m_{\theta_i}}$ and $1-\mathbf{L}^{\Psi_2(\nu_\rho)}T^{\phi_2(\nu_\rho)}$, where the integer m_{θ_i} is the multiplicity of the curve $Z^{\Lambda\cap\theta_i^\perp}$, for $i = 1, 2$, and ρ runs through the rays of $\Sigma_1 \cap \Sigma_2$.*

Remark 4.14. Suppose that $\mathbf{C}$ denotes the field of complex numbers. If V is a variety the map $V \mapsto HD(V) \in \mathbf{Z}[u,v]$, where $HD(V)$ denotes the Hodge-Deligne polynomial, factors through $K_0(\mathrm{Var}_{\mathbf{C}})$ inducing a ring morphism $HD : K_0(\mathrm{Var}_{\mathbf{C}}) \to \mathbf{Z}[u,v]$ which maps $\mathbf{L} \mapsto uv$ (see [D-L1]). It follows that $\mathbf{Z}[\mathbf{L}] \cong \mathbf{Z}[X]$ where X is an indeterminate. By Corollary 4.10 the geometric motivic Poincaré series of a toric singularity is an element of $\mathbf{Z}[\mathbf{L}](T)$, a ring in which the notion of the pole in T of a non zero element is well defined since $\mathbf{Z}[\mathbf{L}]$ is an integral domain.

Remark 4.15. If Z^Λ is a normal toric surface then $T = 1$ and $T = \mathbf{L}^{-1}$ are not poles of $P_{\mathrm{geom}}^{(Z^\Lambda,0)}$ (see [LJ-R]). In general it may happen that all candidate poles mentioned in the statement of Corollary 4.13 are actual poles (see an example in Section 13).

We give a geometrical interpretation of the set of candidate poles of the series $P_{\mathrm{geom}}^{(Z^\Lambda,0)}(T)$.

Definition 4.16. For $1 \leq k \leq d$ we denote by π_k the composite of the normalization map $Z_\sigma \to Z^\Lambda$ with the toric modification of $Z_k \to Z_\sigma$ defined by the subdivision $\cap_{i=1}^k \Sigma_i$ of σ.

The modification π_k is the minimal toric modification which factors through the normalization of Z^Λ and the normalized blowing up of Z^Λ with center $\mathcal{J}_i$, for $i = 1, \ldots, k$. The rays ρ in the fan $\cap_{i=1}^k \Sigma_i$ correspond bijectively to orbit closures of Z_k which are of codimension one. If ν_ρ is the generator of the semigroup $\rho \cap N$ we denote by E_{ν_ρ} the irreducible component corresponding to ρ. We denote by val_{ν_ρ} the divisorial valuation of the field of fractions of $\mathbf{C}[\Lambda]$, which is associated to the divisor E_{ν_ρ}. If $m \in M$ then we have that

$$\mathrm{val}_{\nu_\rho}(X^m) = \langle \nu_\rho, m \rangle. \tag{8}$$

We have that $\mathring{\rho} \subset \mathring{\sigma}$ if and only if E_{ν_ρ} is a codimension one irreducible component of the exceptional fiber of $\pi_k^{-1}(0)$. If $1 \leq i \leq k \leq d$, the pull-back $\pi_k^*(\mathcal{J}_i)$ of $\mathcal{J}_i$ by π_k defines a sheaf of locally principal monomial ideals on the toric variety Z_k and by (8) we deduce

$$\mathrm{val}_{\nu_\rho}(\pi_k^*(\mathcal{J}_i)) = \mathrm{ord}_{\mathcal{J}_i}(\nu_\rho). \tag{9}$$

Proposition 4.17. *For $1 \leq k \leq d$,*

$$\mathcal{L}_k := (\pi_k^*(\mathcal{J}_k))^{k-1} / (\pi_k^*(\mathcal{J}_{k-1}))^k \textit{ and } \mathcal{Q}_k := \pi_k^*(\mathcal{J}_k)/\pi_k^*(\mathcal{J}_{k-1})$$

are sheaves of locally principal monomial ideals on Z_k such that

$$\begin{array}{rcl} B(\Lambda) & = & \{(d,1)\} \cup \bigcup_{k=1}^{d-1} \{(val_{\nu_\rho}(\mathcal{L}_k), val_{\nu_\rho}(\mathcal{Q}_k)) \mid E_{\nu_\rho} \subset \pi_k^{-1}(0)\} \\ & & \cup \{(val_{\nu_\rho}(\mathcal{L}_d), val_{\nu_\rho}(\mathcal{Q}_d)) \mid \rho \in \cap_{i=1}^d \Sigma_i, \dim \rho = 1\}. \end{array}$$

Remark 4.18. If $(S,0)$ is equidimensional of dimension d then the term $1-\mathbf{L}^dT$ appears always in the denominator of the rational form of $P_{\mathrm{geom}}^{(S,0)}(T)$. This is consequence of Theorem 7.1 of [D-L1].

5. Combinatorial convexity properties of Newton polyhedra of $\mathcal{J}_k$

We study the combinatorial convexity properties of the support functions of the Newton polyhedra of the monomial ideals $\mathcal{J}_k \subset \mathbf{C}[\Lambda]$, for Λ as in Notation 1.1.

If $\nu \in \sigma$ then the relation $\leq_\nu$ defined by

$$v \leq_\nu v' \Leftrightarrow \langle \nu, v \rangle \leq \langle \nu, v' \rangle, \tag{10}$$

is a preorder on the set $\sigma^\vee \cap M$. We give an algorithm to determine a vector $w_k \in \mathcal{J}_k$ such that $\mathrm{ord}_{\mathcal{J}_k}(\nu) = \langle \nu, w_k \rangle$.

Lemma 5.1. *Let ν be an element of $\overset{\circ}{\sigma} \cap N$ such that*

$$e_1 \leq_\nu e_2 \leq_\nu \cdots \leq_\nu e_n, \tag{11}$$

for the preorder $\leq_\nu$ defined by (10). Define the sequence $i_1 < i_2 < \cdots < i_d \leq n$ of $1, \ldots, n$ in the following inductive form: set $i_1 := 1$, suppose that $i_2, \ldots, i_k$ have already been defined and set:

$$i_{k+1} := \min\{1 \leq i \leq n \mid e_{i_1} \wedge \cdots \wedge e_{i_k} \wedge e_i \neq 0\}. \tag{12}$$

Set $w_k := e_{i_1} + \cdots + e_{i_k}$ for $k = 1, \ldots, d$. Then we have:

$$\mathrm{ord}_{\mathcal{J}_k}(\nu) = \langle \nu, w_k \rangle \quad \text{and} \quad \phi_k(\nu) = \langle \nu, e_{i_k} \rangle. \tag{13}$$

Proof. We deduce from (12) that:

$$\langle \nu, e_{i_{k+1}} \rangle = \min\{\langle \nu, e_i \rangle \mid 1 \leq i \leq n, \exists 1 \leq j_1 < \cdots < j_k \leq n,\ e_{j_1} \wedge \cdots \wedge e_{j_k} \wedge e_i \neq 0\}. \tag{14}$$

The statement is obvious for $k = 1$. Suppose the result for $1 < k < d$. We have then that:

$$\mathrm{ord}_{\mathcal{J}_{k+1}}(\nu) \overset{\text{Def.}}{\leq} \langle \nu, e_{i_1} + \cdots + e_{i_{k+1}} \rangle \overset{\text{by induction}}{=} \mathrm{ord}_{\mathcal{J}_k}(\nu) + \langle \nu, e_{i_{k+1}} \rangle.$$

The other inequality follows from Formula (14) since

$$\begin{array}{rcl} \mathrm{ord}_{\mathcal{J}_{k+1}}(\nu) & = & \min\{\langle \nu, e_{j_1} + \cdots + e_{j_{k+1}} \rangle\}_{0 \neq e_{j_1} \wedge \cdots \wedge e_{j_{k+1}}} \\ & \geq & \min\{\langle \nu, e_{j_1} + \cdots + e_{j_k} \rangle\}_{0 \neq e_{j_1} \wedge \cdots \wedge e_{j_k}} + \min \langle \nu, e_{j_{k+1}} \rangle \\ & = & \mathrm{ord}\mathcal{J}_k(\nu) + \langle \nu, e_{i_{k+1}} \rangle. \end{array}$$

□

Proposition 5.2. *For every ν in $\overset{\circ}{\sigma} \cap N$ there exist $1 \leq i_1, \ldots, i_d \leq n$ such that $\phi_k(\nu) = \langle \nu, e_{i_k} \rangle$, $\sum_{r=1}^k e_{i_r} \in \mathcal{J}_k$ and $\mathrm{ord}_{\mathcal{J}_k}(\nu) = \langle \nu, \sum_{r=1}^k e_{i_r} \rangle$, for $k = 1, \ldots, d$.*

Proof. It follows immediately from Lemma 5.1. □

Corollary 5.3. *If $\nu \in \overset{\circ}{\sigma} \cap N$ we have that:*

$$0 = \phi_0(\nu) < \phi_1(\nu) \leq \phi_2(\nu) \leq \cdots \leq \phi_d(\nu) < \phi_{d+1}(\nu) = +\infty. \tag{15}$$

Definition 5.4. For $0 \leq k \leq d$ we set $A_k := \{(\nu, s) \mid \nu \in \overset{\circ}{\sigma} \cap N,\ \phi_k(\nu) \leq s < \phi_{k+1}(\nu)\}$.

Proposition 5.5. *The sets $A_0, \ldots, A_d$ define a partition of $(\overset{\circ}{\sigma} \cap N) \times \mathbf{Z}_{>0}$.*

Proof. It follows from Corollary 5.3. □

Definition 5.6. If $(\nu, s) \in (\overset{\circ}{\sigma} \cap N) \times \mathbf{Z}_{>0}$ we denote by ℓ_ν^s the linear subspace of $M_{\mathbf{Q}}$ given by

$$\ell_\nu^s := \mathrm{span}_{\mathbf{Q}}\{e_i \mid 1 \leq i \leq n \text{ and } \langle \nu, e_i \rangle \leq s\}.$$

Lemma 5.7. *Let (ν, s) belong to A_k for some $1 \leq k \leq d$. Let $w_k \in \mathcal{J}_k$ verify that $\mathrm{ord}_{\mathcal{J}_k}(\nu) = \langle \nu, w_k \rangle$. If $w_k = e_{j_1} + \cdots + e_{j_k}$ is an expansion as a sum of k linearly independent vectors in $\{e_1, \ldots, e_n\}$ then $\{e_{j_1}, \ldots, e_{j_k}\}$ is a basis of the vector space ℓ_ν^s. If $e_i \in \ell_\nu^s$ and $e_i = \sum_{r=1}^k \alpha_r e_{j_r}$ then $\alpha_r \neq 0$ implies that $\langle \nu, e_{j_r} \rangle \leq \langle \nu, e_i \rangle$, for $r = 1, \ldots, k$.*

Proof. Let $e_{i_1}, \dots, e_{i_d}$ be the vectors defined by Proposition 5.2. We set $w'_r := e_{i_1} + \cdots + e_{i_r}$ for $r = 1, \dots, d$. By Proposition 5.2 and Corollary 5.3 if $1 \leq r \leq k$ then we deduce $\phi_r(\nu) = \langle \nu, e_{i_r} \rangle \leq s$. This implies that $\mathrm{span}_{\mathbf{Q}}\{e_{i_1}, \dots, e_{i_k}\} \subset \ell^s_\nu$. If for some $1 \leq i \leq n$, $\langle \nu, e_i \rangle \leq s$ and $e_i \notin \mathrm{span}_{\mathbf{Q}}\{e_{i_1}, \dots, e_{i_k}\}$ then the vector $\bar{w}_{k+1} := e_{i_1} + \cdots + e_{i_k} + e_i$ belongs to $\mathcal{J}_{k+1}$ hence $\mathrm{ord}_{\mathcal{J}_{k+1}}(\nu) \leq \langle \nu, \bar{w}_{k+1} \rangle = \mathrm{ord}_{\mathcal{J}_k}(\nu) + \langle \nu, e_i \rangle$. This implies that $\phi_{k+1}(\nu) \leq \langle \nu, e_i \rangle \leq s$, a contradiction with the fact that $(\nu, s) \in A_k$. We have shown that $\ell^s_\nu = \mathrm{span}_{\mathbf{Q}}\{e_{i_1}, \dots, e_{i_k}\}$.

Suppose that there exists a vector $w_k = e_{j_1} + \cdots + e_{j_k} \in \mathcal{J}_k$ such that:

$$\mathrm{ord}_{\mathcal{J}_k}(\nu) = \langle \nu, w'_k \rangle = \langle \nu, w_k \rangle \quad \text{and} \quad \mathrm{span}_{\mathbf{Q}}\{e_{i_1}, \dots, e_{i_k}\} \neq \mathrm{span}_{\mathbf{Q}}\{e_{j_1}, \dots, e_{j_k}\}. \tag{16}$$

Then there exists $1 \leq k_0 \leq k$ such that $e_{j_{k_0}} \notin \mathrm{span}_{\mathbf{Q}}\{e_{i_1}, \dots, e_{i_k}\}$. If $\langle \nu, e_{j_{k_0}} \rangle < \langle \nu, e_{i_k} \rangle$ then the vector $\hat{w}_k := e_{i_1} + \cdots + e_{i_{k-1}} + e_{j_{k_0}}$ belongs to $\mathcal{J}_k$ and $\langle \nu, \hat{w}_k \rangle < \mathrm{ord}_{\mathcal{J}_k}(\nu)$ by (16), a contradiction. If $\langle \nu, e_{i_k} \rangle < \langle \nu, e_{j_{k_0}} \rangle$, then the vector $w'_k - e_{j_{k_0}}$ belongs to $\mathcal{J}_{k-1}$ and we have $\langle \nu, w'_k - e_{j_{k_0}} \rangle < \langle \nu, w'_{k-1} \rangle$, which contradicts the formula $\mathrm{ord}_{\mathcal{J}_{k-1}}(\nu) = \langle \nu, w'_{k-1} \rangle$. Thus, the equality

$$\langle \nu, e_{j_{k_0}} \rangle = \langle \nu, e_{i_k} \rangle \tag{17}$$

holds. The equality $\phi_k(\nu) = \phi_{k+1}(\nu)$ follows from (17) and the inequalities:

$$\langle \nu, e_{i_k} \rangle \overset{(13)}{=} \phi_k(\nu) \overset{(15)}{\leq} \phi_{k+1}(\nu) \overset{\text{Def.}}{=} \mathrm{ord}_{\mathcal{J}_{k+1}}(\nu) - \mathrm{ord}_{\mathcal{J}_k}(\nu),$$

$$\mathrm{ord}_{\mathcal{J}_{k+1}}(\nu) \overset{\text{Def.}}{\leq} \langle \nu, e_{i_1} + \cdots + e_{i_k} + e_{j_{k_0}} \rangle \overset{(13)}{=} \mathrm{ord}_{\mathcal{J}_k}(\nu) + \langle \nu, e_{j_{k_0}} \rangle.$$

Finally, if we have an expansion $e_i = \sum_{r=1}^k \alpha_r e_{j_r}$ with $\alpha_{r_0} \neq 0$ and $\langle \nu, e_i \rangle < \langle \nu, e_{j_{r_0}} \rangle$ then the vector $\tilde{w}_k := e_i + \sum_{r=1, r \neq r_0}^k e_{j_r}$ belongs to $\mathcal{J}_k$. The inequality $\mathrm{ord}_{\mathcal{J}_k}(\nu) = \langle \nu, e_{j_1} + \cdots + e_{j_k} \rangle > \langle \nu, \tilde{w}_k \rangle$, is a contradiction with the definition of the support function. □

6. The universal family of arcs in the torus

We describe the universal family of arcs in the torus T_N of a rank d lattice N and some properties of its functions which are useful to deal with jets of arcs in toric varieties. In this section we use that the characteristic of the base field $\mathbf{C}$ is zero.

Let us fix a basis $m_1, \dots, m_d$ of M. We set

$$\mathcal{A} := \mathbf{C}[c(m_i)^{\pm 1}] \otimes_{\mathbf{C}} \mathbf{C}[u_j(m_i)]_{i=1,\dots,d}^{j \geq 1},$$

where $\{c(m_1), \dots, c(m_d)\} \cup \{u_j(m_i)\}_{i=1,\dots,d}^{j \geq 1}$ are algebraically independent over $\mathbf{C}$. Then there is one homomorphism of semigroups $h^* : (M, +) \to (\mathcal{A}[[t]]^*, \cdot)$ such that $m_i \mapsto c(m_i)(1 + \sum_{j \geq 1} u_j(m_i) t^j)$ for $i = 1, \dots, d$. We have that the image of $m \in M$ by this homomorphism is to $c(m)u(m)$ where $u(m) \in \mathcal{A}[[t]]$ is a series of the form $u(m) = 1 + \sum_{j \geq 1} u_j(m) t^j$. Notice that if $m, m' \in M$ then we have $c(m + m') = c(m)c(m')$ and $u(m + m') = u(m)u(m')$. By the description of Section 2 we check that $\mathcal{A} = \mathcal{A}_{T_N}$ and the map $h : \mathrm{Spec}\mathcal{A}[[t]] \to T_N$ corresponding to h^*, is the universal family of arcs in the torus. The following two lemmas show some relations among the elements $u_i(m) \in \mathcal{A}_{T_N}$, when we vary i and $m \in M$, in terms of linear dependency relations among the $m \in M$.

Lemma 6.1. *Let $m_1, \dots, m_k$ be a set of linearly independent vectors of M. If $0 \neq m = \sum_{j=1}^k a_j m_j$ with $a_j \in \mathbf{Z}$ then we have:*

$$u_i(m) = \sum_{r=1}^k a_r \, u_i(m_r) + R_{i-1}^{(m)}(u_j(m_r))_{r=1,\dots,k}^{j=1,\dots,i-1}, \tag{18}$$

for all $i \geq 1$, where $R_{i-1}^{(m)}$ is a quasi-homogeneous polynomial of weight i, with rational coefficients in $(u_j(m_r))_{r=1,\dots,k}^{j=1,\dots,i-1}$, where $u_j(m_r)$ is given weight equal to j, for $r = 1, \dots, k$ and $j = 1, \dots, i-1$.

Proof. We have that $u(m) = \prod_{j=1}^k u(m_j)^{a_j} = \prod_{j=1}^k \left(\sum_{i \geq 0} u_i(m_j) t^i \right)^{a_j}$. Remark that if $\phi = \sum u_i t^i \in \mathbf{C}[[t]]$ is a series with constant term equal to one and if $n \in \mathbf{Z}$ then the series ϕ^n is of the form: $\phi^n = \sum P_i^{(n)} t^i$ where $P_i^{(n)} = n u_i + R_i^{(n)}(u_1, \dots, u_{i-1})$ is a quasi-homogeneous polynomial in $u_1, \dots, u_i$, where u_l has weight equal to l; notice that the coefficient n of u_i does not vanish since $\mathbf{C}$ is a field of characteristic zero. We use this observation to compute the expansion of $\prod_j \left(\sum_{i\geq 0} u_i(m_j)t^i\right)^{a_j}$ as a series in t. □

Lemma 6.2. *If $m'_1, \dots, m'_k$ and $m_1, \dots, m_k$ are linearly independent vectors in the lattice M spanning the same linear subspace ℓ of $M_{\mathbf{Q}}$ then for any $s \geq 1$ we have the equality of $\mathbf{Q}$-algebras:*

$$\mathbf{Q}[u_1(m'_j), \dots, u_s(m'_j)]_{j=1}^k = \mathbf{Q}[u_1(m_j), \dots, u_s(m_j)]_{j=1}^k. \tag{19}$$

In particular, if $m = \sum_{j=1}^k a_j m_j$, with $a_j \in \mathbf{Q}$ then $u_i(m)$ belongs to the $\mathbf{Q}$-algebra (19) for $i = 1, \dots, s$.

Proof. It is sufficient to prove it in the case that $m_1, \dots, m_k$ are a basis of the rank k lattice $\ell \cap M$. We show the result by induction on s. Since $m_1, \dots, m_k$ form a basis of $\ell \cap M$ and $m'_r \in \ell \cap M$ we have expansions: $m'_r = a_{r,1}m_1 + \cdots + a_{r,k}m_k$ with $a_{r,j} \in \mathbf{Z}$, for $j = 1, \dots, k$, and $r = 1, \dots, k$. Since $m'_1, \dots, m'_k$ are linearly independent we have expansions: $m_r = b_{r,1}m'_1 + \cdots + b_{r,k}m'_k$ $b_{r,j} \in \mathbf{Q}$, for $j = 1, \dots, k$ and $r = 1, \dots, k$. For $s = 1$ the term $R_0^{(m)}$ appearing in formula (18) is equal to zero thus by Lemma 6.1 we obtain that $u_1(m'_r) = a_{r,1}u_1(m_1) + \cdots + a_{r,k}u_1(m_k)$ and $u_1(m_r) = b_{r,1}u_1(m'_1) + \cdots + b_{r,k}u_1(m'_k)$, for $r = 1, \dots, k$. Using the induction hypothesis for all $1 \leq s' < s$ and the triangular form of formula (18) for $u_s(m'_r)$ we deduce that $u_s(m_r)$ is of the form:

$$u_s(m_r) = b_{r,1}u_s(m'_1) + \cdots + b_{r,k}u_s(m'_k) + P_{r,s},$$

where $P_{r,s}$ belongs to $\mathbf{Q}[u_1(m'_j), \dots u_{s-1}(m'_j)]_{j=1}^k$. □

Proposition 6.3. *If the vectors $m_1, \dots, m_k$ in M are linearly independent then the following elements of $\mathcal{A}_{T_N}$ are algebraically independent over $\mathbf{C}$:*

$$c(m_1), \dots, c(m_k), \text{ and } u_i(m_1), \dots, u_i(m_k), \quad \forall i \geq 1 \tag{20}$$

7. The image in the Grothendieck ring of the jets of the orbits

Let ν belong to the set $\overset{\circ}{\sigma} \cap N$. We consider the orbit H_ν^* of the action of the arc space of the torus on H_Λ^*. The universal family of arcs in the torus parametrizes the arcs in H_ν^* by the morphism $\Psi_\nu : \mathrm{Spec}\mathcal{A}_{T_N}[[t]] \to Z^\Lambda$ given by:

$$X^{e_i} \circ \Psi_\nu = t^{\langle \nu, e_i \rangle} c(e_i) u(e_i), \text{ for } i = 1, \dots, n.$$

Recall that $e_1, \dots, e_n$ is the minimal system of generators of Λ (cf. Notations 1.1).

We prove that the set $j_s(H_\nu^*)$ is a locally closed subset of $j_s(\mathbf{A}^n_{\mathbf{C}})_0 \cong \mathbf{A}^{ns}_{\mathbf{C}}$ and we determine its class in the Grothendieck ring of varieties.

Theorem 7.1. *If $(\nu, s) \in A_k$ for some $0 \leq k \leq d$, then the jet space $j_s(H_\nu^*)$ is a locally closed subset of $H_s(Z^\Lambda)_0$ isomorphic to $\{0\}$ if $k = 0$ or to $(\mathbf{C}^*)^k \times \mathbf{A}_{\mathbf{C}}^{ks - \mathrm{ord}_{\mathcal{J}_k}(\nu)}$ if $k > 0$.*

Proof. If $h \in H_\nu^*$ the equality $\mathrm{ord}_t(X^{e_i} \circ h) = \langle \nu, e_i \rangle$ holds for $1 \leq i \leq n$. By Definition 5.6 those vectors e_i such that $j_s(X^{e_i} \circ h) \neq 0$ span the $\mathbf{Q}$-vector space ℓ_ν^s since $\langle \nu, e_i \rangle \leq s$. If $k = 0$ this vector space is empty, the jet space $j_s(H_\nu^*)$ consists of the constant 0-jet and the conclusion follows.

Suppose then that $k > 0$. We denote by $\mathcal{O}_\nu^s$ (resp. by C_ν^s) the $\mathbf{C}$-algebra of $\mathcal{A}_{T_N}$ generated by:

$$c(e_i)^{\pm 1}, u_1(e_i), \dots, u_{s - \langle \nu, e_i \rangle}(e_i) \text{ for those } 1 \leq i \leq n \text{ such that } \langle \nu, e_i \rangle \leq s, \tag{21}$$

$$(\text{resp. } \; c(e_i)^{\pm 1} \text{ for those } 1 \leq i \leq n \text{ such that } \langle \nu, e_i \rangle \leq s \;).$$

By Proposition 5.2 the vector ν determines integers $1 \leq i_1, \dots, i_k \leq n$ such that $\mathrm{ord}_{\mathcal{J}_k}(\nu) = \sum_{r=1}^k \langle \nu, e_{i_r} \rangle$.

Assertion. *We have the following properties*:

(i) *Denote by $\underline{U}$ the variables $(U_1, \dots, U_{ks - \mathrm{ord}_{\mathcal{J}_k(\nu)}})$. For any $1 \leq i \leq n$ and l such that $1 \leq l \leq s - \langle \nu, e_i \rangle$ there exists a polynomial $P_{l,i} \in \mathbf{Q}[\underline{U}]$ such that*

$$u_l(e_i) = P_{l,i}\left(u_1(e_{i_1}), \dots, u_{s - \langle \nu, e_{i_1} \rangle}(e_{i_1}), \dots, u_1(e_{i_k}), \dots, u_{s - \langle \nu, e_{i_k} \rangle}(e_{i_k})\right).$$

(ii) *The ring $\mathcal{O}_\nu^s$ is generated as a C_ν^s-algebra by*

$$u_1(e_{i_r}), \dots, u_{s - \langle \nu, e_{i_r} \rangle}(e_{i_r}), \text{ for } r = 1, \dots, k. \tag{22}$$

(iii) *The lattice M_ν^s spanned by $\{e_i \mid 1 \leq i \leq n, \; \langle \nu, e_i \rangle \leq s\}$ is of rank k. The map $C_\nu^s \to \mathbf{C}[M_\nu^s]$ given by $c(e_i) \mapsto X^{e_i}$ is an isomorphism.*

(iv) *The variety* $\mathrm{Spec}\mathcal{O}_\nu^s$ *is isomorphic to* $(\mathbf{C}^*)^k \times \mathbf{A}_{\mathbf{C}}^{ks-\mathrm{ord}_{\mathcal{J}_k}(\nu)}$.

Proof of the Assertion. By Lemma 5.7 the vectors $e_{i_1}, \dots, e_{i_k}$ define a basis of ℓ_ν^s. If e_i is a vector in ℓ_ν^s with $\langle \nu, e_i \rangle \leq s$, and $e_i = \sum_{r=1}^k \alpha_r e_{i_r} \in \ell_\nu^s$, then $\alpha_r \neq 0$ implies that $\langle \nu, e_i \rangle \geq \langle \nu, e_{i_r} \rangle$. We deduce from Lemma 6.2 that the elements $u_1(e_i), \dots, u_{s-\langle \nu, e_i \rangle}(e_i)$ belong to the $\mathbf{Q}$-algebra generated by (22). This implies that (i) and (ii) hold.

By Lemma 6.3 and the definitions the ring C_ν^s is isomorphic to the $\mathbf{C}$-algebra of the lattice M_ν^s. The assertion (iii) follows, since by definition M_ν^s is a sublattice of finite index of the rank k lattice $\ell_\nu^s \cap M$.

Finally, (iv) follows from these observations and Lemma 6.3. This ends the proof of the Assertion.

By the Assertion the morphism

$$\psi : \mathrm{Spec}\ C_\nu^s \otimes_{\mathbf{C}} \mathbf{C}[\underline{U}] \longrightarrow j_s(\mathbf{A}_{\mathbf{C}}^n)_0,$$

given by

$$x_i(t) := \left\{ \begin{array}{ccc} c(e_i)\, t^{\langle \nu, e_i \rangle} \left(1 + P_{1,i}(\underline{U})t + \cdots P_{s-\langle \nu, e_i \rangle, i}(\underline{U}) t^{s-\langle \nu, e_i \rangle}\right) & \text{if} & \langle \nu, e_i \rangle \leq s, \\ 0 & \text{if} & \langle \nu, e_i \rangle > s, \end{array} \right.$$

for $1 \leq i \leq n$, is an inmersion. The image $\mathrm{Im}(\psi)$ of ψ is a locally closed subset.

Finally, if h belongs to H_ν^*, then $j_s(h)$ belongs to $\mathrm{Im}(\psi)$ by the Assertion. Conversely, if $\xi \in \mathrm{Im}(\psi)$ we define an arc $h \in H_\nu^*$ such that $j_s(h) = \xi$ by specialization from the universal family. First, for $1 \leq i \leq n$ and $l \geq 1$ we set $u_l(e_i) = 0$ if $l > s - \langle \nu, e_i \rangle$. By the Assertion the coefficients $u_l(e_i)$ associated to h, for $1 \leq s - \langle \nu, e_i \rangle \leq l$, are complex numbers determined by ξ. In order to complete the definition of h we have to give values for the coefficients $c(e_i)$ corresponding to h. We have an injection of $\mathbf{C}$-algebras $\mathbf{C}[M_\nu^s] \subset \mathbf{C}[M] = \mathbf{C}[c(e_i)^{\pm 1}]_{i=1}^n$ which corresponds to a surjective map of torus. The inicial coefficients associated to ξ define a closed point $p(\xi)$ of the torus $\mathrm{Spec}\mathbf{C}[M_\nu^s]$. Any closed point in the fiber of $p(\xi)$ by this map provides suitable initial coefficients $c(e_i) \in \mathbf{C}^*$, in such a way that the resulting arc h verifies that $j_s(h) = \xi$.

Notice that $\mathcal{O}_\nu^s$ is the coordinate ring of the locally closed subset $j_s(H_\nu^*)$. □

8. Description of the series $P(\Lambda)$

We describe the coefficients of the auxiliary series $P(\Lambda)$. We study in which cases the intersections $j_s(H_\nu^*) \cap j_s(H_{\nu'}^*)$ and $j_s(H_{\Lambda,\nu}^*) \cap j_s(H_{\Lambda \cap \theta^\perp})$ are non-empty, for $(\nu, s), (\nu', s) \in A_k$ and $\theta \leq \sigma$.

Definition 8.1. Define an equivalence relation in the set A_k for any $1 \leq k \leq d$:

$$(\nu, s) \sim (\nu', s') \in A_k \Leftrightarrow \left\{ \begin{array}{c} s = s',\ \nu \text{ and } \nu' \text{ define the same face of } \mathcal{N}(\mathcal{J}_j) \\ \text{and } \mathrm{ord}_{\mathcal{J}_j}(\nu) = \mathrm{ord}_{\mathcal{J}_j}(\nu'), \text{ for } 1 \leq j \leq k. \end{array} \right. \tag{23}$$

We denote by $[(\nu, s)]$ the equivalence class of (ν, s) in A_k by this relation.

Remarks 8.2.

(i) For any fixed integer $s_0 > 0$ the set $\{[(\nu, s_0)] \mid (\nu, s_0) \in A_k\}$ is finite for $1 \leq k \leq d$.
(ii) If $k = d$ the equivalence relation defined in the set A_d is the equality.

Proposition 8.3. *If* $(\nu, s), (\nu', s) \in A_k$ *the following relations are equivalent:*

(i) $(\nu, s) \sim (\nu', s)$,
(ii) $\ell_\nu^s = \ell_{\nu'}^s$ *and* $\nu_{|\ell_\nu^s} = \nu'_{|\ell_{\nu'}^s}$,
(iii) $j_s(H_\nu^*) = j_s(H_{\nu'}^*)$,
(iv) $j_s(H_\nu^*) \cap j_s(H_{\nu'}^*) \neq \emptyset$.

Proof. The condition $(\nu, s) \sim (\nu', s)$ implies that $\ell_\nu^s = \ell_{\nu'}^s$ by Lemma 5.7. The condition $\mathrm{ord}_{\mathcal{J}_j}(\nu) = \mathrm{ord}_{\mathcal{J}_j}(\nu')$ for $j = 1, \dots, k$ is equivalent to $\nu_{|\ell_\nu^s} = \nu'_{|\ell_{\nu'}^s}$ by Lemma 5.1. It follows that the conditions (i) and (ii) are equivalent.

If (ii) holds then the basis $e_{j_1}, \dots, e_{j_k}$ of the vector space introduced in Lemma 5.1, coincides for the vectors ν and ν' and $\langle \nu, e_{j_r} \rangle = \langle \nu', e_{j_r} \rangle$, for $r = 1, \dots, k$. This implies that the inmersion ψ of $j_s(H_\nu^*)$ defined for (ν, s) in the proof of Theorem 7.1 is the same map as the one defined for (ν', s), hence $j_s(H_\nu^*) = j_s(H_{\nu'}^*)$.

Suppose that (iii) or (iv) holds. If $h \in H^*_\nu$, $h' \in H^*_{\nu'}$ verify that $0 \neq j_s(X^{e_i} \circ h) = j_s(X^{e_i} \circ h')$ for some $1 \leq i \leq n$, then $X^{e_i} \circ h$ and $X^{e_i} \circ h'$ have the same order $\langle \nu, e_i \rangle = \langle \nu', e_i \rangle \leq s$ and those vectors e_i generate the linear subspace $\ell^s_\nu = \ell^s_{\nu'}$, hence (ii) holds. □

Notation 8.4. The cone $\hat{\sigma} := \sigma \times \mathbf{R}_{\geq 0}$ is rational for the lattice $\hat{N} := N \times \mathbf{Z}$.

(i) If $\tau \subset \sigma$ and $1 \leq k \leq d$ we set $\tau(k) := \{(\nu, s) \mid \nu \in \overset{\circ}{\tau} \cap \overset{\circ}{\sigma} \text{ and } \phi_k(\nu) \leq s < \phi_{k+1}(\nu)\}$.
(ii) If $\tau \in \cap_{i=1}^k \Sigma_i$ then we set $A_{k,\tau} := \tau(k) \cap \hat{N}$.

Remark 8.5. If τ is a cone contained in a cone of the fan $\cap_{i=1}^k \Sigma_i$ and if $\tau(k) \neq \emptyset$, then the closure of $\tau(k)$ in $\hat{\sigma}$ is a convex polyhedral cone, rational for the lattice $\hat{N}$ (since in this case the functions $\mathrm{ord}_{\mathcal{J}_1}, \dots . \mathrm{ord}_{\mathcal{J}_k}$, hence also $\phi_1, \dots, \phi_k$, are linear on τ and the function $\mathrm{ord}_{\mathcal{J}_{k+1}}$, hence also ϕ_{k+1}, is piece-wise linear and convex on τ). In particular, the set $A_{k,\tau}$ may be empty, for instance, if τ is contained in the boundary of σ or if for all ν in the interior of τ we have that $\phi_k(\nu) = \phi_{k+1}(\nu)$.

Remark 8.6. We deduce the following:

(i) $A_k = \sqcup_{\tau \in \cap_{i=1}^k \Sigma_i} A_{k,\tau}$ for $1 \leq k \leq d$.
(ii) The vectors $(\nu, s), (\nu', s) \in A_k$ are equivalent by the relation $\sim$ in (23) if and only if there exists a cone $\tau \in \cap_{i=1}^k \Sigma_i$ such that ν and ν' belong to the relative interior of τ and $\phi_i(\nu) = \phi_i(\nu')$ for $i = 1, \dots, k$.
(iii) It follows that $A_k/_\sim = \sqcup_{\tau \in \cap_{i=1}^k \Sigma_i} A_{k,\tau}/_\sim$, where $A_{k,\tau}/_\sim$ is the set of equivalent classes of elements in the set $A_{k,\tau}$ by the relation (23).

Proposition 8.7. *If $\nu \in \overset{\circ}{\sigma}$, $s \geq 1$ and $\theta \leq \sigma$ then the following relations are equivalent:*

(i) $j_s(H^*_{\Lambda,\nu}) \cap j_s(H_{\Lambda \cap \theta^\perp}) \neq \emptyset$,
(ii) $j_s(H^*_{\Lambda,\nu}) \subset j_s(H_{\Lambda \cap \theta^\perp})$,
(iii) $\ell^s_\nu \subset \theta^\perp$.

Proof. Suppose that (i) holds. Then there is an arc $h \in H^*_\nu$ whose s-jet belongs to $j_s(H_{\Lambda \cap \theta^\perp})$. If the truncation $j_s(X^{e_i} \circ h)$ does not vanish then the vector e_i belongs to $\Lambda \cap \theta^\perp$. By Definition 5.6, those vectors e_i for which $j_s(X^{e_i} \circ h) \neq 0$ span the linear subspace ℓ^s_ν. This proves the inclusion (iii).

Assume that (iii) holds. Let $h \in H^*_{\Lambda,\nu}$. Define an arc $h' \in H_{\Lambda \cap \theta^\perp}$ by the semigroup homomorphism $\Lambda \cap \theta^\perp \to \mathbf{C}[[t]]$, given by $e \mapsto X^e \circ h$, for $e \in \Lambda \cap \theta^\perp$. We have that $h' \in H^*_{\Lambda \cap \theta^\perp, \nu'}$ where ν' is the restriction of ν to $M(\theta, \Lambda)$. Since ℓ^s_ν is contained in $\theta^\perp$ by hypothesis, the vector space $\ell^s_{\nu'}$ associated to the pair (ν', s) with respect to $\Lambda \cap \theta^\perp$ is equal to ℓ^s_ν and the restrictions of ν and ν' to this subspace coincide. The inclusion (ii) holds by the argument in the proof of (ii) ⇒ (iii) in Proposition 8.3. □

Proposition 8.8. *If $1 \leq k \leq d$ and $(\nu, s) \in A_k$ then the following assertions are equivalent:*

(i) *The intersection $j_s(H^*_{\nu,\Lambda}) \cap (\bigcup_{0 \neq \theta \leq \sigma} j_s(H_{\Lambda \cap \theta^\perp}))$ is empty.*
(ii) *The face $\mathcal{F}_\nu$ of the polyhedron $\mathcal{N}(\mathcal{J}_k)$ determined by ν is contained in the interior of $\sigma^\vee$.*

Proof. By Proposition 8.7 we have that (i) holds if and only if for any face θ of σ the inclusion $\ell^s_\nu \subset \theta^\perp$ implies that $\theta = 0$, or equivalently if and only if $\ell^s_\nu \cap \mathrm{int}(\sigma^\vee) \neq \emptyset$.

If (ii) holds then $\ell^s_\nu \cap \mathrm{int}(\sigma^\vee) \neq \emptyset$ by Lemma 5.7 hence (i) holds.

Suppose that (ii) does not hold, that is there exists a vertex w of $\mathcal{F}_\nu$ which belongs to a proper face $\sigma^\vee \cap \theta^\perp$ of the cone $\sigma^\vee$, for some $0 \neq \theta \leq \sigma$. Such w belongs to $\mathcal{J}_k$, hence it is of the form $w = e_{j_1} + \cdots + e_{j_k}$. Since e_{j_r} belongs to $\sigma^\vee$ it follows that e_{j_r} must belong to $\theta^\perp$, for $r = 1, \dots, k$. It follows that $\ell^s_\nu \subset \theta^\perp$ by Lemma 5.7, hence (i) does not hold. □

Definition 8.9. If $1 \leq k \leq d$ we define the set $\mathcal{D}_k$ as the subset of cones $\tau \in \bigcap_{i=1}^k \Sigma_i$ such that the face $\mathcal{F}_\tau$ of $\mathcal{N}(\mathcal{J}_k)$ is contained in the interior of $\sigma^\vee$.

Remark 8.10. Notice that $\mathcal{D}_d = \bigcap_{i=1}^k \Sigma_i$. If $\tau \in \mathcal{D}_d$, the set $\tau(d)$ is non-empty if and only if $\overset{\circ}{\tau} \subset \overset{\circ}{\sigma}$.

As a consequence of the results of this Section we have the following Propositions:

Proposition 8.11. *Let us fix an integer $s_0 \geq 1$. The set $j_{s_0}(H^*_\Lambda) \setminus \bigcup_{0\neq\theta\leq\sigma} j_{s_0}(H_{\Lambda\cap\theta^\perp})$ expresses as a finite disjoint union of locally closed subsets, as follows:*

$$j_{s_0}(H^*_\Lambda) \setminus \bigcup_{0\neq\theta\leq\sigma} j_{s_0}(H_{\Lambda\cap\theta^\perp}) = \bigsqcup_{k=1}^{d} \bigsqcup_{\tau\in\mathcal{D}_k} \bigsqcup_{[(\nu,s_0)]\in A_{k,\tau}/\sim} j_{s_0}(H^*_{\Lambda,\nu}). \tag{24}$$

□

If $s_0 \geq 1$ the coefficient of T^{s_0} in the auxiliary series $P(\Lambda)$ is obtained by taking classes in the Grothendieck ring in (24), and then using Theorem 7.1.

For each cone $\tau \in \mathcal{D}_k$ we define the auxiliary series:

$$P_{k,\tau}(\Lambda) = (\mathbf{L}-1)^k \sum_{s\geq 1} \sum_{[(\nu,s)]\in A_{k,\tau}/\sim} \mathbf{L}^{sk-\mathrm{ord}_{\mathcal{J}_k}(\nu)} T^s. \tag{25}$$

Proposition 8.12. *We have that*

$$P(\Lambda) = \sum_{k=1}^{d} \sum_{\tau\in\mathcal{D}_k} P_{k,\tau}(\Lambda). \tag{26}$$

□

9. The proofs of the main results

In this Section we fix a cone $\tau \in \mathcal{D}_k$ such that $A_{k,\tau} \neq \emptyset$ and we describe the rational form of the series $P_{k,\tau}(\Lambda)$. For convenience, we do not stress the dependency on the cone τ in the notations introduced in this Section.

We denote the closure of $\tau(k)$ by $\hat{\tau}$. By Remark 8.5 the functions $\phi_1, \ldots, \phi_k$ are linear on τ. More precisely, if $\nu_0 \in \overset{\circ}{\tau}$ we consider the vectors $e_{i_1}, \ldots, e_{i_d}$ introduced in Proposition 5.2. Then we deduce that $\phi_r(\nu) = \langle \nu, e_{i_r}\rangle$ for $1 \leq r \leq k$ and for all $\nu \in \tau$, since τ is a cone in the fan $\cap_{r=1}^k \Sigma_r$.

Notation 9.1. Let us define the lattice homomorphisms

$$\begin{array}{lll} \pi : \hat{N} \to \mathbf{Z}^{k+1} & \text{by} & (\nu, s) \mapsto (\langle \nu, e_{i_1}\rangle, \ldots, \langle \nu, e_{i_k}\rangle, s), \\ \zeta : \mathbf{Z}^{k+1} \to \mathbf{Z}^2 & \text{by} & a = (a_1, \ldots, a_{k+1}) \mapsto (k a_{k+1} - \sum_{r=1}^k a_r, a_{k+1}), \end{array}$$

We set also $\xi := \zeta \circ \pi : \hat{N} \to \mathbf{Z}^2$. We abuse of notation by denoting by the same letter the linear extension of these maps to the corresponding real vector spaces.

Notice that the intersection of the kernel of π (and also of ξ) with the cone $\hat{\tau}$ is $\{0\}$.

Remark 9.2. If $(\nu, s) \neq (0,0)$ belongs to a ray in $\hat{\tau}$ then $\xi(\nu, s) \neq (0,0)$ and

$$\xi(\nu, s) = \begin{cases} (\Psi_k(\nu), \phi_k(\nu)) & \text{if} \quad s = \phi_k(\nu), \\ (\Psi_{k+1}(\nu), \phi_{k+1}(\nu)) & \text{if} \quad k \neq d \text{ and } s = \phi_{k+1}(\nu), \\ (ds, s) & \text{if} \quad k = d \text{ and } \nu = 0. \end{cases}$$

It follows from Remark 9.2 and Corollary 5.3 that $\tilde{\tau} := \xi(\hat{\tau}) \subset \mathbf{R}^2_{\geq 0}$ and $\bar{\tau} := \pi(\hat{\tau}) \subset \mathbf{R}^{k+1}_{\geq 0}$ are strictly convex and rational for the lattices $\mathbf{Z}^2$ and $\mathbf{Z}^{k+1}$ respectively. Hence the map of $\mathbf{C}$-algebras

$$\zeta_* : \mathbf{C}[[\bar{\tau} \cap \mathbf{Z}^{k+1}]] \to \mathbf{C}[[\mathbf{Z}^2_{\geq 0}]] = \mathbf{C}[[\mathbf{L}, T]] \quad \text{given by } X^a \mapsto \mathbf{L}^{k a_{k+1} - \sum_{r=1}^k a_r} T^{a_{k+1}},$$

for $a = (a_1, \ldots, a_{k+1})$ is well defined. If $B \subset \bar{\tau} \cap \mathbf{Z}^{k+1}$ the generating function $F_B := \sum_{a\in B} X^a$ belongs to the ring $\mathbf{C}[[\bar{\tau} \cap \mathbf{Z}^{k+1}]]$.

Lemma 9.3. *The sets $A_{k,\tau}$, $\bar{A}_{k,\tau} := \pi(A_{k,\tau})$ and $\tilde{A}_{k,\tau} := \xi(A_{k,\tau})$ are subsemigroups, not necessarily of finite type of $\hat{\tau} \cap \hat{N}$, $\mathbf{Z}^{k+1}_{\geq 0}$ and $\mathbf{Z}^2_{\geq 0}$ respectively. The restriction $\pi_{|A_{k,\tau}} : A_{k,\tau} \to \bar{A}_{k,\tau}$ induces a bijection $A_{k,\tau}/\sim \to \bar{A}_{k,\tau}$, by $[(\nu, s)] \mapsto \pi(\nu, s)$ (see Definition 8.1). We have $P_{k,\tau}(\Lambda) = \zeta_*(F_{\bar{A}_{k,\tau}})$.*

Proof. The result follows from Definition 8.1, Remark 8.6 and the previous discussion. □

Notation 9.4. If $\rho \subset \tau$ is a one-dimensional cone rational for the lattice N we denote by ν_ρ the primitive integral vector on ρ, that is, the generator of the semigroup $\rho \cap N$.

Proposition 9.5. *If $1 \leq k \leq d-1$ there exists $R_{k,\tau} \in \mathbf{Z}[\mathbf{L}, T]$ such that:*

$$P_{k,\tau}(\Lambda) = R_{k,\tau} \prod_{\rho\leq\tau}^{\dim\rho=1} (1-\mathbf{L}^{\Psi_k(\nu_\rho)}T^{\phi_k(\nu_\rho)})^{-1} \prod_{\rho\in\Sigma_{k+1},\rho\subset\tau}^{\dim\rho=1,\phi_{k+1}(\nu_\rho)\neq\phi_k(\nu_\rho)} (1-\mathbf{L}^{\Psi_{k+1}(\nu_\rho)}T^{\phi_{k+1}(\nu_\rho)})^{-1}. \tag{27}$$

If $k = d$ then (27) holds by replacing the term $\prod_{\rho\in\Sigma_{k+1},\rho\subset\tau}^{\dim\rho=1,\phi_{k+1}(\nu_\rho)\neq\phi_k(\nu_\rho)}(1-\mathbf{L}^{\Psi_{k+1}(\nu_\rho)}T^{\phi_{k+1}(\nu_\rho)})$ by $(1-\mathbf{L}^dT)$. Both numerator and denominator in (27) are determined by the lattice M and the Newton polyhedra of the logarithmic jacobian ideals.

Proof. We call the set $\partial_-\hat{\tau} = \{(\nu, \phi_k(\nu)) \mid \nu \in \tau\}$ the lower boundary of $\hat{\tau}$. The set $\partial_-\hat{\tau}$ is a convex polyhedral cone of dimension d. We deduce that $A_{k,\tau} = (A_{k,\tau} \cap \partial_-\hat{\tau}) \sqcup (A_{k,\tau} \cap \mathrm{int}(\hat{\tau}))$. The sets $A_{k,\tau} \cap \partial_-\hat{\tau}$ and $A_{k,\tau} \cap \mathrm{int}(\hat{\tau})$ consist of the integral points for the lattice $\hat{N}$ in the cones $\partial_-\hat{\tau}$ and $\hat{\tau}$, respectively. It is easy to see that if $(\nu, s) \in A_{k,\tau} \cap \partial_-\hat{\tau}$ and if $(\nu', s') \in A_{k,\tau} \cap \mathrm{int}(\hat{\tau})$ then $[(\nu, s)] \neq [(\nu', s')]$ (see Notation 8.4 and (23)). It follows that $\bar{A}_{k,\tau} = \pi(A_{k,\tau} \cap \partial_-\hat{\tau}) \sqcup \pi(A_{k,\tau} \cap \mathrm{int}(\hat{\tau}))$. We set $\bar{A}^{\circ}_{k,\tau} := \pi(A_{k,\tau} \cap \mathrm{int}(\hat{\tau}))$ and $\bar{A}^{-}_{k,\tau} := \pi(A_{k,\tau} \cap \partial_-\hat{\tau})$. It follows that

$$P_{k,\tau}(\Lambda) = (\mathbf{L}-1)^k\,(\zeta_*(F_{\bar{A}^{\circ}_{k,\tau}}) + \zeta_*(F_{\bar{A}^{-}_{k,\tau}})). \tag{28}$$

The semigroups $\bar{A}^{\circ}_{k,\tau}$ and $\bar{A}^{-}_{k,\tau}$ are the images by π of the semigroups of integral points in the relative interiors of the cones $\hat{\tau}$ and $\partial_-\hat{\tau}$, respectively. We apply Theorem 12.4 (see Section 12) using that the kernel of π intersects the cone $\hat{\tau}$ only at 0.

It follows that the denominator of the rational form of $F_{\bar{A}^{\circ}_{k,\tau}}$ (resp. of $F_{\bar{A}^{-}_{k,\tau}}$) is the product of terms $1 - X^{\pi(b)}$, for b running through the primitive integral vectors in the edges of $\hat{\tau}$ (resp. of $\partial_-\hat{\tau}$) while the numerator is a polynomial in $\mathbf{Z}[\bar{\tau} \cap \mathbf{Z}^{k+1}]$. The rational form of $P_{k,\tau}(\Lambda)$ is the image of the rational form of $F_{\bar{A}^{\circ}_{k,\tau}} + F_{\bar{A}^{-}_{k,\tau}}$ by the homomorphism ζ_* since the image by ζ_* of the denominator does not vanish by Remark 9.2.

If $1 \leq k \leq d-1$ and if ρ is an edge of $\hat{\tau}$ which is not contained in $\partial_-\hat{\tau}$ then it is necessarily of the form $\rho = (\nu, \phi_{k+1}(\nu))$ for $\nu \in \overset{\circ}{\tau}$ in some edge of Σ_{k+1}. If $k = d$ the only edge of $\hat{\tau}$ which is not contained in $\partial_-\hat{\tau}$ is $(0,1)\mathbf{R}_{\geq 0}$. Finally by this discussion and Remark 9.2 the denominator of this rational form is as indicated in (27). □

Remarks 9.6.

(i) For $k = 1, \ldots, d-1$ and $\tau \in \mathcal{D}_k$, the factor $1 - \mathbf{L}^dT$ does not appear in the denominator of $P_{k,\tau}(\Lambda)$.

(ii) The factors in the denominator of the rational form (27) of $P_{k,\tau}(\Lambda)$ are of the form $1 - \mathbf{L}^aT^b$ with $(a, b) \in B(\Lambda)$. We use that $\cup_{i=1}^k \Sigma_i^{(1)}$ is the set of rays in the fan $\cap_{i=1}^k \Sigma_i$ and Definition 8.9.

(iii) The term $1 - \mathbf{L}^dT$ appears in the denominator of $P_{d,\tau}(\Lambda)$ with multiplicity one.

Proof of Proposition 4.3. If $d = 1$ then the toric variety Z^Λ is a monomial curve. Let ν_0 be the generator of the semigroup $\sigma \cap N \cong \mathbf{Z}_{\geq 0}$. The monomial curve Z^Λ is parametrized by $x_i = t^{m_i}$, where $m_i := \langle \nu_0, e_i\rangle$, for $i = 1, \ldots, n$. The multiplicity of Z^Λ at 0 is $m = \min_{i=1,\ldots,n}\{\langle \nu_0, e_i\rangle\} = \mathrm{ord}_{\mathcal{J}_1}(\nu_0)$.

By Definition 5.4, the set A_1 is $A_1 = \{(\nu, s) \mid \nu = r\nu_0,\ 0 < mr \leq s\}$. By Theorem 7.1 it follows that $P(\Lambda) = \sum_{0<rm\leq s}(\mathbf{L}-1)\mathbf{L}^{s-rm}T^s = \frac{\mathbf{L}-1}{1-\mathbf{L}T}\frac{T^m}{1-T^m}$. □

Proof of Theorem 4.9. The lattice M and the Newton polyhedra of the ideals $\mathcal{J}_k$ determine and are determined by duality by N and the functions $\mathrm{ord}_{\mathcal{J}_k}$, for $k = 1, \ldots, d$. The proof follows from Propositions 9.5, Formula (26) and Remark 9.6. □

Proof of Corollary 4.10. It is a consequence of Theorem 4.9, Proposition 4.1 and Example 4.2. □

Proof of Corollary 4.11. Nicaise observed in [N1] that the motivic Poincaré series of an affine normal toric variety Z^Λ has an expansion in terms of the local motivic Poincaré series at the distinguished points of the orbits, namely:

$$P^{Z^\Lambda}_{\mathrm{geom}}(T) = \sum_{\theta\leq\sigma}(\mathbf{L}-1)^{\mathrm{codim}\theta}P^{(Z^\Lambda,o_\theta)}_{\mathrm{geom}}(T).$$

For each $\theta \leq \sigma$ there exists an open set of Z^Λ containing the distinguished point o_θ of the orbit $\mathrm{orb}_\theta^\Lambda$, which is isomorphic to $\mathrm{orb}_\theta^\Lambda \times Z^{\Lambda_\theta}$, where $\Lambda_\theta = \sigma_\theta^\vee \cap M_\theta$ is the image of Λ by the canonical map $M \to M/M \cap \theta^\perp$ (see [F] page 29). It follows that the germ (Z^Λ, o_θ) is analytically isomorphic to $(Z^{\Lambda(\theta)}, 0)$. □

Proof of Corollary 4.13. It follows from Theorem 4.9 and Example 4.4 by taking into consideration that if ρ is a ray of Σ_1 and if $\nu_\rho \in \overset{\circ}{\sigma}$ then we get $\phi_1(\nu_\rho) = \phi_2(\nu_\rho)$, thus $\Psi_1(\nu_\rho) = \Psi_2(\nu_\rho) = 0$. □

Proof of Proposition 4.17. It follows from the definitions by using (15) and (9). □

10. Motivic volume of a toric variety

We give a formula for the motivic volume $\mu(H_\Lambda)$ of the space of arcs H_Λ of the toric variety Z^Λ in terms of the support function $\mathrm{ord}_{\mathcal{J}_d}$. This formula generalizes the one given in [LJ-R].

If $\tau \in \Sigma_d$ we denote by $\eta_\tau : \mathbf{Z}[\tau \cap N] \longrightarrow \mathbf{Z}[\mathbf{L}^{\pm 1}]$ the ring homomorphism defined by $\eta_\tau(x^\nu) = \mathbf{L}^{-\mathrm{ord}_{\mathcal{J}_d}(\nu)}$. The generating function $F_{\overset{\circ}{\tau}\cap N}$ has a rational form $F_{\overset{\circ}{\tau}\cap N} = R_{\overset{\circ}{\tau}\cap N} \prod_{\rho \leq \tau}^{\dim \rho = 1} (1 - x^{\nu_\rho})^{-1}$, for some $R_{\overset{\circ}{\tau}\cap N} \in \mathbf{Z}[\tau \cap N]$ (see Proposition 12.2).

Proposition 10.1.

$$\mu(H_\Lambda) = (\mathbf{L} - 1)^d \sum_{\overset{\circ}{\tau}\cap\overset{\circ}{\sigma}\neq\emptyset}^{\tau \in \Sigma_d} \eta_\tau(R_{\overset{\circ}{\tau}\cap N}) \prod_{\rho \leq \tau}^{\dim \rho = 1} (1 - \mathbf{L}^{-\mathrm{ord}_{\mathcal{J}_d}(\nu_\rho)})^{-1}$$

Proof. By Theorem 2.1 (see [D-L1]) we have that the limit $\mu(H_\Lambda) = \lim_{m\to\infty} [j_m(H_\Lambda)]\mathbf{L}^{-md}$ converges in $\hat{\mathcal{M}}$. We deduce that $\mu(H_\Lambda) = ((1 - \mathbf{L}^d T) P_{\mathrm{geom}}^{(Z^\Lambda,0)}(T))_{|T = \mathbf{L}^{-d}}$, by comparing with the definition of the series $P_{\mathrm{geom}}^{(Z^\Lambda,0)}$, taking into account that $1 - \mathbf{L}^d T$ is a simple pole of $P_{\mathrm{geom}}^{(Z^\Lambda,0)}$ by Propositions 4.1 and Remark 9.6. By Proposition 2.3 the equality $\mu(H_\Lambda) = \mu(H_\Lambda^*)$ holds. By Remark 9.6 the term $1 - \mathbf{L}^d T$ does not define a pole of $P_k(\Lambda)$ for $k = 1, \ldots, d-1$, hence $\mu(H_\Lambda) = ((1 - \mathbf{L}^d T) P_d(\Lambda))_{|_{T = \mathbf{L}^{-d}}}$ and

$$\mu(H_\Lambda) = \sum_{\overset{\circ}{\tau}\cap\overset{\circ}{\sigma}\neq\emptyset}^{\tau\in\Sigma_d} \sum_{\nu \in \overset{\circ}{\tau}\cap N} (\mathbf{L} - 1)^d \mathbf{L}^{-\mathrm{ord}_{\mathcal{J}_d}(\nu)} = (\mathbf{L} - 1)^d \sum_{\overset{\circ}{\tau}\cap\overset{\circ}{\sigma}\neq\emptyset}^{\tau\in\Sigma_d} \eta_\tau(F_{\overset{\circ}{\tau}\cap N}(x)).$$

Notice that $\eta_\tau(x^{\nu_\rho}) \neq 1$ if ν_ρ is a primitive vector in a ray of $\rho \in \Sigma_d$ (see Remark 8.10). We deduce that $\eta_\tau(F_{\overset{\circ}{\tau}\cap N}) = \eta_\tau(R_{\overset{\circ}{\tau}\cap N}) \prod_{\rho\leq\tau}^{\dim\rho=1} (1 - \mathbf{L}^{-\mathrm{ord}_{\mathcal{J}_d}(\nu_\rho)})^{-1}$. □

We deduce a formula for the motivic volume of the space of arcs $H(Z^\Lambda)$ of Z^Λ (without fixing the origin of the arcs) in terms of the local data, as a consequence of Proposition 10.1 and Corollary 4.11. The same formula also holds if Z^Λ is locally analytically unibranched (see [C]).

Proposition 10.2. *If Z^Λ is an affine normal toric variety then we have that*

$$\mu(H(Z^\Lambda)) = \sum_{\theta \leq \sigma} (\mathbf{L} - 1)^{\mathrm{codim}\theta} \mu(H_{\Lambda(\theta)}).$$

□

11. Geometrical definition of the logarithmic jacobian ideals

We introduce the geometrical definition of the k^{th}-*logarithmic jacobian ideal* of an affine toric variety Z^Λ of dimension d for $1 \leq k \leq d$, following [O] Chapter 3, and [LJ-R] Appendix. We denote by D the equivariant Weil divisor defined by the orbit closures of codimension one in Z^Λ. We denote by Ω_Λ the $\mathbf{C}[\Lambda]$-module of Kähler differential forms of Z^Λ (over $\mathbf{C}$). The module $\Omega_\Lambda(\log D)$ of one forms on Z^Λ with logarithmic poles along D is identified with $\mathbf{C}[\Lambda] \otimes_{\mathbf{Z}} M$ and we have a map of $\mathbf{C}[\Lambda]$ modules:

$$\varphi : \Omega_\Lambda \to \Omega_\Lambda(\log D), \quad dX^\gamma \mapsto X^\gamma \otimes \gamma, \text{ for } \gamma \in \Lambda.$$

If $1 \leq k \leq d$ we set $\wedge^k \varphi : \Omega_\Lambda^k \to \Omega_\Lambda^k(\log D)$, $\quad dX^{\gamma_1} \wedge \cdots \wedge dX^{\gamma_k} \mapsto X^{\gamma_1 + \cdots + \gamma_k} \otimes (\gamma_1 + \cdots + \gamma_k)$, for γ_i in Λ, where $\Omega_\Lambda^k(\log D)$ is identified with $\mathbf{C}[\Lambda] \otimes_{\mathbf{Z}} \bigwedge^k M$. For $k = d$, fixing a basis $u_1, \ldots, u_d$ of the lattice M provides an isomorphism

$$\phi : \wedge^d M \to \mathbf{Z}, \quad u_1 \wedge \ldots \wedge u_d \mapsto 1,$$

which is, up to sign, independent of the choice of the basis.

Definition 11.1. The k^{th}*-logarithmic jacobian ideal* of Z^Λ is the ideal of $\mathbf{C}[\Lambda]$ generated by the set $\phi(\wedge^k \varphi(\Omega^k_\Lambda) \wedge \bigwedge^{d-k} M)$.

Proposition 11.2. *The* k^{th}*-logarithmic jacobian ideal of* Z^Λ *is the monomial ideal* $\mathcal{J}_k$ *defined by (5), for* $k = 1, \dots, d$.

Proof. The proof follows from the definitions since Ω_Λ is generated by dX^{e_i}, for $e_1, \dots, e_n$ generators of the semigroup Λ. □

The *Nash blowing up* $v : N_S \to S$ of an algebraic variety S is the minimal proper birational map such that $v^*\Omega^1_S$ has a locally free quotient of rank $\dim S$. The fibers of v at a point $x \in S$ are the limiting positions of tangent spaces at smooth points of S tending to the point x.

Proposition 11.3. ([GS, LJ-R, T]) *The blowing up of* Z^Λ *with center its* d^{th}*-logarithmic jacobian ideal* $\mathcal{J}_d$ *is the Nash blowing up of* Z^Λ.

12. Generating functions of projections of subset of cones

In this Section we state some auxiliary results on the generating function of certain subsets of integral points in a rational polyhedral cone. See [Br, B-P, 1] for an expository papers on this and related subjects. The content of this section is independent of the rest of the paper.

Let $N \subset \mathbf{R}^d$ be a rank d lattice and τ strictly convex cone rational for the lattice N.

Definition 12.1. The *generating function* of a set $B \subset \tau \cap N$ is the series $F_B(x) = \sum_{a \in B} x^a \in \mathbf{Z}[[\tau \cap N]]$. The series $F_B(x)$ is *rational* if there exist $p(x), q(x) \in \mathbf{Z}[\tau \cap N]$ such that $q(x)F_B(x) = p(x)$. In that case the ratio $p(x)/q(x)$ is well-defined and it is called the *sum* of the series $F_B(x)$.

We denote by ν_ρ the generator of the semigroup $\rho \cap N$ for each edge ρ of τ. The following Proposition is well-known (see [1] Section 4.6).

Proposition 12.2. *The generating function* $F_{\overset{\circ}{\tau} \cap N}(x)$ *is of the form:*

$$F_{\overset{\circ}{\tau} \cap N}(x) = R_{\overset{\circ}{\tau} \cap N} \prod_{\rho \leq \tau, \dim \rho = 1} (1 - x^{\nu_\rho})^{-1}, \text{ with } R_{\overset{\circ}{\tau} \cap N} \in \mathbf{Z}[\tau \cap N]. \tag{29}$$

Remark 12.3. The statement of Proposition 12.2 remains true if we replace the vector ν_ρ by a non-zero vector in $\rho \cap N$ for each edge ρ of τ.

Let $\pi : N \to \mathbf{Z}^r$ be a map of lattices for some $1 \leq r \leq d$. We abuse of notation by denoting with the same letter the extension of π to a map of real vector spaces $N_{\mathbf{R}} \to \mathbf{R}^r$. We suppose that

$$\tau \cap \ker \pi = \{0\}. \tag{30}$$

This condition implies that the cone $\bar{\tau} := \pi(\tau)$ is strictly convex. For simplicity we set $A := \overset{\circ}{\tau} \cap N$ and $\bar{A} := \pi(A)$. The sets A and $\bar{A}$ are subsemigroups, not necessarily of finite type, of $\tau \cap N$ and $\bar{\tau} \cap \mathbf{Z}^r$ respectively. Notice that for each edge $\bar{\rho}$ of $\bar{\tau}$ there exists at least one edge ρ of τ such that $\pi(\rho) = \bar{\rho}$, hence we have that

$$0 \neq \pi(\nu_\rho) \in \bar{\rho} \cap \mathbf{Z}^r. \tag{31}$$

Theorem 12.4. *The generating function* $F_{\bar{A}}(x)$ *of* $\bar{A}$ *is of the form:*

$$F_{\bar{A}}(x) = R_{\bar{A}} \prod_{\rho \leq \tau, \, \dim \rho = 1} (1 - x^{\pi(\nu_\rho)})^{-1}, \text{ with some } R_{\bar{A}} \in \mathbf{Z}[\bar{\tau} \cap \mathbf{Z}^r]. \tag{32}$$

We introduce some notations and results before proving Theorem 12.4.

If $a \neq b \in N_{\mathbf{Q}}$ we define the *length with respect to* N of the segment joining a and b by $\lg(a, b) := r$ if $a - b = rc$ where $r \in \mathbf{Q}_{>0}$ and $c \in N$ is a primitive vector.

We denote by $\mathcal{A}$ and $\mathcal{B}$ the following sets:

$$\mathcal{A} = \{\bar{\rho} \text{ edge of } \bar{\tau} \mid \dim \pi_\ell^{-1}(\bar{\rho}) \cap \tau = 1\} \text{ and } \mathcal{B} = \{\bar{\rho} \text{ edge of } \bar{\tau} \mid \dim \pi_\ell^{-1}(\bar{\rho}) \cap \tau > 1\}.$$

Lemma 12.5. *If* $r < d$ *and* $\bar{\rho} \in \mathcal{B}$ *then there exists* $\bar{u}_0 \in \bar{\rho} \cap \mathbf{Z}^r$ *such that* $(\bar{u}_0 + \mathrm{int}(\bar{\tau})) \cap \bar{N} \subset \bar{A}$.

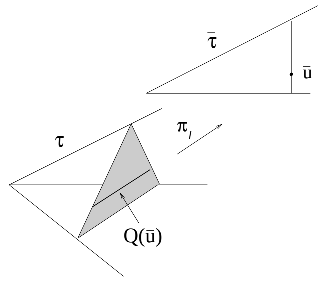


FIGURE 1. The shaded region is the preimage by π_ℓ of the segment containing $\bar{u}$ in the Figure.

Proof. If $\bar{u} \in \bar{\tau}$ we denote by $Q(\bar{u})$ the set $Q(\bar{u}) := \pi_\ell^{-1}(\bar{u}) \cap \tau$ (see Figure 1). If $\bar{u} \in \bar{\tau} \cap \mathbf{Z}^r$ then $Q(\bar{u})$ is a rational polytope for the lattice N. We denote by $\rho(\bar{u})$ the ray spanned by the sum of all the vertices of the polytope $Q(\bar{u})$, by $b(\bar{u})$ the vector $Q(\bar{u}) \cap \rho(\bar{u})$ and by $\delta(\bar{u})$ the number $\delta(\bar{u}) := \max\{\lg(v, b(\bar{u}))\}$, for v running through the vertices of $Q(\bar{u})$. Notice that if $t \in \mathbf{R}_{\geq 0}$ then we have that $\rho(t\bar{u}) = \rho(\bar{u})$ and

$$Q(t\bar{u}) = tQ(\bar{u}), \quad b(t\bar{u}) = tb(\bar{u}), \quad \delta(t\bar{u}) = t\delta(\bar{u}). \tag{33}$$

If $\bar{\rho} \in \mathcal{B}$ and $\bar{u} \in \bar{\rho} \cap \mathbf{Z}^r$ then the polytope $Q(\bar{u})$ is of dimension ≥ 1 by definition of $\mathcal{B}$, hence $\delta(\bar{u}) > 0$. Let $\bar{u}_0$ be a vector in $\bar{\rho} \cap \mathbf{Z}^r$ such that $\delta(\bar{u}_0) > 1$.

A vector $\bar{w} \in (\bar{u}_0 + \operatorname{int}(\bar{\tau})) \cap \mathbf{Z}^r$ is of the form $\bar{w} := \bar{u}_0 + \bar{v}$, with $\bar{v} \in \operatorname{int}(\bar{\tau}) \cap \mathbf{Z}^r$. By linearity of π we have the inclusion

$$Q(\bar{u}_0) + Q(\bar{v}) \subset Q(\bar{w}). \tag{34}$$

Since π maps $\operatorname{int}(\tau)$ onto $\operatorname{int}(\bar{\tau})$ the polytopes $Q(\bar{v})$ and $Q(\bar{w})$ are of dimension $d - r \geq 1$. Thus, there exists a vector $v \in N_{\mathbf{Q}}$ such that $v \in \operatorname{int}(Q(\bar{v}))$, where int denotes the relative interior. By (34) we obtain $v + \operatorname{int}(Q(\bar{u}_0)) \subset \operatorname{int}(Q(\bar{w}))$. Since $\delta(\bar{u}_0) > 1$ it follows that $Q(\bar{w})$ contains a rational segment $v + I$ of integral length > 1. Hence $Q(\bar{w})$ contains a point w of the lattice N in the set $\operatorname{int}(Q(\bar{w})) \subset \operatorname{int}(\tau)$. It follows that $\bar{w} = \pi(w)$ belongs to $\bar{A}$. □

Proof of the Theorem 12.4. We deal first with the case of a simplicial cone $\bar{\tau}$. In this case we denote $\mathcal{A} = \{\bar{\rho}_1, ..., \bar{\rho}_a\}$ and $\mathcal{B} = \{\bar{\rho}_{a+1}, ..., \bar{\rho}_{a+b}\}$, where $a, b \geq 0$ and $a + b = \dim \bar{\tau}$.

If $\bar{\rho}_j \in \mathcal{B}$ we denote by $\bar{u}_j$ the vector $\bar{u}_j \in \bar{\rho}_j \cap \mathbf{Z}^r$ such that $(\bar{u}_j + \operatorname{int}(\bar{\tau})) \cap \mathbf{Z}^r \subset \bar{A}$ (see Lemma 12.5). The set $S := \bigcup_{\rho_j \in \mathcal{B}} (\bar{u}_j + \operatorname{int}(\bar{\tau})) \cap \mathbf{Z}^r$ is contained in $\bar{A}$. If $S' := \bar{A} \setminus S$ then $F_{\bar{A}}(x) = F_S(x) + F_{S'}(x)$.

We deal first with the rational form of $F_S(x)$. If $\emptyset \neq J \subset \mathcal{B}$ we set $R_J := \bigcap_{\rho_j \in J} (\bar{u}_j + \operatorname{int}(\bar{\tau})) \cap \mathbf{Z}^r$. Notice that $R_J = (\bar{u}_J + \operatorname{int}(\bar{\tau})) \cap \mathbf{Z}^r$, where $\bar{u}_J := \sum_{\rho_j \in J} \bar{u}_j$. By Proposition 12.2 the series $F_{R_J}(x) = x^{\bar{u}_J} F_{\operatorname{int}(\bar{\tau}) \cap \mathbf{Z}^r}(x)$ is of rational form. By Remark 12.3 and Formula (31) its denominator can be taken as in (32). By the inclusion-exclusion principle we deduce that $F_S(x)$ has a rational form as indicated in the statement of Theorem 12.4.

If $\bar{\rho}_j \in \mathcal{A}$ we denote by ρ_j the edge of τ such that $\pi(\rho_j) = \bar{\rho}_j$, for $j = 1, \ldots, a$. To study the rational form of $F_{S'}(x)$ we set

$$G = \{\sum_{i=1}^{a} \lambda_i \pi(\nu_{\rho_i}) + \sum_{j=1}^{b} \mu_j \bar{u}_{a+j} \mid 0 < \lambda_i, \mu_j \leq 1, \text{ for } 1 \leq i \leq a, \ 1 \leq j \leq b\}.$$

If $\vec{n} = (n_1, \ldots, n_a) \in \mathbf{Z}^a_{\geq 0}$ we denote by $C_{\vec{n}}$ the set $C_{\vec{n}} := n_1 \pi(\nu_{\rho_1}) + \cdots + n_a \pi(\nu_{\rho_a}) + G$ and by $k_{\vec{n}}$ the integer $k_{\vec{n}} := \#(C_{\vec{n}} \cap \bar{A})$, where $\#$ denotes cardinal. Then we have a partition

$$S' = \bigsqcup_{\vec{n} \in \mathbf{Z}^a_{\geq 0}} C_{\vec{n}} \cap \bar{A}. \tag{35}$$

If $\vec{n} \in \mathbf{Z}^a_{\geq 0}$ we have the bound:

$$k_{\vec{n}} \leq \#(G \cap \mathbf{Z}^r). \tag{36}$$

Denote by $\{\vec{e}_1, \ldots, \vec{e}_a\}$ the canonical basis of $\mathbf{Z}^a$. We have that

$$k_{\vec{n}} \leq k_{\vec{n} + \vec{e}_j} \quad \text{for } 1 \leq j \leq a \text{ and } \vec{n} \in \mathbf{Z}^a_{\geq 0}. \tag{37}$$

We deduce from (36) and (37) that there exists $m \in \mathbf{Z}_{>0}$ such that $k_{\vec{n}} = k_0$ for all $\vec{n} \in \mathbf{Z}^a_{\geq 0}$ such that $n_j \geq m$ for some $1 \leq j \leq a$. If $\vec{n}$ verifies this condition and if $1 \leq i \leq a$ we have the equality:

$$C_{\vec{n}} \cap \bar{A} + \pi(\nu_{\rho_i}) = (C_{\vec{n}+\vec{e}_i}) \cap \bar{A} \text{ for } 1 \leq i \leq a. \tag{38}$$

We deduce from these observations and (35) that

$$F_{S'}(x) = \sum_{\forall 1 \leq i \leq a : 0 \leq n_i < m} \sum_{\bar{\nu} \in C_{\vec{n}} \cap \bar{A}} x^{\bar{\nu}} + \sum_{\exists 1 \leq i \leq a : m \leq n_i} \sum_{\bar{\nu} \in C_{\vec{n}} \cap \bar{A}} x^{\bar{\nu}}.$$

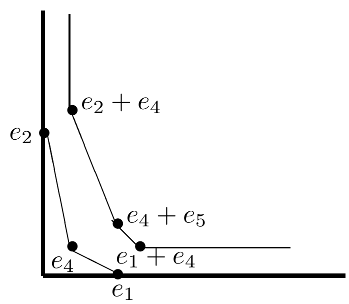
The first term is a finite sum, while the second is of the form $R(x) \prod_{j=1}^a (1 - x^{\pi(\nu_{\rho_j})})^{-1}$ for some $R(x) \in \mathbf{Z}[x]$, by (38) and a similar argument as the one used for $F_S(x)$.

If $\bar{\tau}$ is not simplicial, let $\bar{\Sigma}$ be a simplicial subdivision of $\bar{\tau}$ such that every edge of $\bar{\Sigma}$ is an edge of $\bar{\tau}$ (see Chapter V, Theorem 4.2, page 158 [Ew]). If $\bar{\theta} \in \bar{\Sigma}$ the set $\theta := \pi^{-1}(\bar{\theta}) \cap \tau$ is a rational cone for the lattice N and $\theta \cap \ker(\pi) = (0)$. We have that $\pi(\mathrm{int}(\theta)) = \mathrm{int}(\bar{\theta})$ and $\pi(\mathrm{int}(\theta) \cap N) = \bar{A} \cap \mathrm{int}(\bar{\theta})$.

By the assertion in the simplicial case $F_{\bar{A} \cap \mathrm{int}(\bar{\theta})}(x)$ has a rational form as in the statement of the Theorem. The result follows since $F_{\bar{A}}(x) = \sum_{\bar{\theta} \in \bar{\Sigma}} F_{\bar{A} \cap \mathrm{int}(\bar{\theta})}(x)$. □

13. An Example

We consider the semigroup Λ generated by $e_1 = (3,0), e_2 = (0,6), e_3 = (5,0), e_4 = (1,1), e_5 = (2,1)$ and $e_6 = (1,4)$. With notations of Section 1 the cone σ is $\mathbf{R}^2_{\geq 0}$, the lattice M is $\mathbf{Z}^2$ and the semigroups $\Lambda \cap \theta_i^{\perp}$ for $i = 1,2$ are $\Lambda \cap \theta_1^{\perp} = (0,6)\mathbf{Z}_{\geq 0}$ and $\Lambda \cap \theta_2^{\perp} = (3,0)\mathbf{Z}_{\geq 0} + (5,0)\mathbf{Z}_{\geq 0}$. The Newton polyhedron of $\mathcal{J}_1$ (resp. of $\mathcal{J}_2$) has vertices e_1, e_4 and e_2 (resp. $e_1 + e_4$, $e_4 + e_5$ and $e_2 + e_4$), see Figure 2.



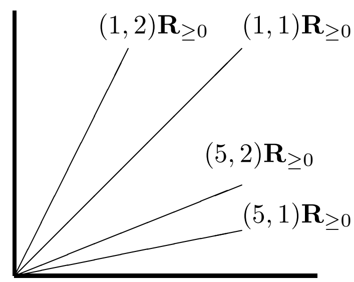


Figure 2. The Newton polygons of $\mathcal{J}_1$ and $\mathcal{J}_2$ and the subdivision $\Sigma_1 \cap \Sigma_2$

The surface Z^{Λ} is defined by binomial equations in $\mathbf{A}^6_{\mathbf{C}}$. The normalization of the germ $(Z^{\Lambda}, 0)$ is smooth. Notice that in this case there is no Hirzebruch-Jung data from the minimal resolution (compare with the results of [LJ-R] in the case of a normal toric surface singularity).

By Proposition 4.1 $P^{(Z^{\Lambda},0)}_{\mathrm{geom}}(T)$ is equal to

$$P^{(Z^{\Lambda},0)}_{\mathrm{geom}}(T) = (1-T)^{-1} + P(\Lambda \cap \theta_1^{\perp}) + P(\Lambda \cap \theta_2^{\perp}) + P(\Lambda).$$

By Proposition 4.3 we get $P(\Lambda \cap \theta_1^{\perp}) = \frac{\mathbf{L}-1}{1-\mathbf{L}T} \frac{T}{1-T}$ and $P(\Lambda \cap \theta_2^{\perp}) = \frac{\mathbf{L}-1}{1-\mathbf{L}T} \frac{T^3}{1-T^3}$. The sum of $P(\Lambda)$ is of the form $Q_{\Lambda} \prod_{(a,b) \in B(\Lambda)} (1 - \mathbf{L}^a T^b)^{-1}$ where $Q_{\Lambda} \in \mathbf{Z}[\mathbf{L}, T]$. The set $B(\Lambda) = \{(2,1), (1,1), (0,3), (0,6), (1,3), (5,12)\}$ is determined easily from the table below in which we give the values of the functions ϕ_1, ϕ_2 and Ψ_2 for the primitive vectors in the rays ρ of $\Sigma_1 \cap \Sigma_2$.

	$(1,0)$	$(0,1)$	$(1,2)$	$(5,1)$	$(1,1)$	$(5,2)$
ϕ_1	0	0	3	6	2	7
ϕ_2	1	1	3	6	3	12
Ψ_2	1	1	0	0	1	5

We have that $\mathcal{D}_1 = \{\tau\}$ where $\tau = (1,2)\mathbf{R}_{\geq 0} + (5,1)\mathbf{R}_{\geq 0} \in \Sigma_1$ (see Definition 8.9). We have that $\mathcal{D}_2 = \Sigma_1 \cap \Sigma_2$ and $P(\Lambda) = P_{1,\tau}(\Lambda) + P_2(\Lambda)$, where $P_2(\Lambda) := \sum_{\theta \in \Sigma_1 \cap \Sigma_2, \mathring{\theta} \subset \mathring{\sigma}} P_{2,\theta}$. We determine the rational form of $P_2(\Lambda)$ by computing first the rational form of the generating series $F_{\mathring{\theta} \cap N}$, for $\theta \in \Sigma_1 \cap \Sigma_2$ with $\mathring{\theta} \cap \mathring{\sigma} \neq \emptyset$. Then we apply to each term $F_{\mathring{\theta} \cap N}$ a suitable monomial transformation (see the proof of Propositions 9.5 and 12.2). We check that

$$P_2(\Lambda) = \frac{(\mathbf{L}-1)^2}{1-\mathbf{L}^2T}\Big(\frac{\mathbf{L}T^4}{(1-\mathbf{L}T)(1-T^3)} + \frac{T^3}{1-T^3} + \frac{\mathbf{L}T^6}{(1-T^3)(1-\mathbf{L}T^3)} + \frac{\mathbf{L}T^3}{1-\mathbf{L}T^3} +$$
$$+ \frac{\mathbf{L}^2T^5+\mathbf{L}^4T^{10}+\mathbf{L}^6T^{15}}{(1-\mathbf{L}T^3)(1-\mathbf{L}^5T^{12})} + \frac{\mathbf{L}^5T^{12}}{1-\mathbf{L}^5T^{12}} + \frac{\mathbf{L}^2T^6+\mathbf{L}T^6+\mathbf{L}^4T^{12}+\mathbf{L}^3T^{12}+\mathbf{L}^5T^{18}}{(1-\mathbf{L}^5T^{12})(1-T^6)} + \frac{T^6}{1-T^6} + \frac{\mathbf{L}T^7}{(1-T^6)(1-\mathbf{L}T)}\Big).$$

We determine the term $P_{1,\tau}(\Lambda)$ (see Proposition 9.5). The cone τ is associated to the vertex e_4 of $\mathcal{N}(\mathcal{J}_1)$ and it is subdivided by Σ_2 with the rays $\rho_1 = (5,2)\mathbf{R}_{\geq 0}$ and $\rho_2 = (1,1)\mathbf{R}_{\geq 0}$. We describe first the generating function $F_{\bar{A}}(x)$ of the semigroup $\bar{A} = \{(\phi_1(\nu), s) \in \mathbf{Z}^2 \mid \nu \in \overset{\circ}{\tau} \cap N,\ \phi_1(\nu) \leq s < \phi_2(\nu)\}$ (see Figure 3).

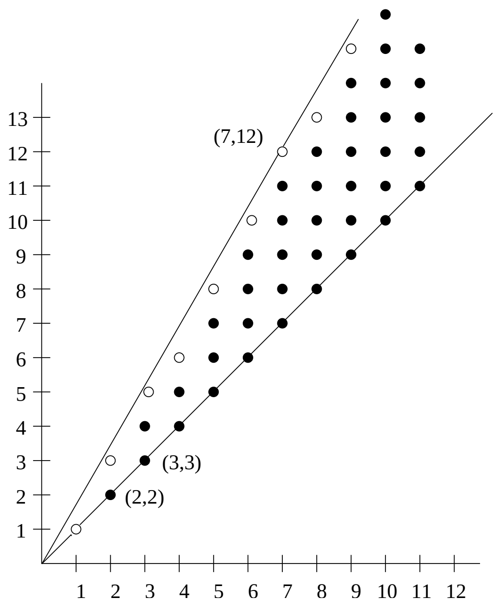


FIGURE 3. The black (resp. white) circles denote elements of $\bar{A}$ (resp. of $(\bar{\tau} \cap \mathbf{Z}^2) \setminus \bar{A}$).

We have that $\bar{A}$ is a subsemigroup of $\bar{\tau} \cap \mathbf{Z}^2$ where $\bar{\tau} = \mathbf{R}_{\geq 0}(1,1) + \mathbf{R}_{\geq 0}(7,12)$. We set $G' := \{(0,0),(2,3),(3,5),(5,8),(6,10)\}$ and $G := G' \cup \{(1,1),(4,6)\}$. We have the partitions:

$$(\bar{\tau} \cap \mathbf{Z}^2) \setminus \bar{A} = \bigsqcup_{p \geq 0} G + p(7,12) \text{ and } \bar{\tau} \cap \mathbf{Z}^2 = \bigsqcup_{(p,q) \in \mathbf{Z}^2_{\geq 0}} G' + p(1,1) + q(7,12).$$

We deduce that $F_{\bar{A}} = F_{\bar{\tau} \cap \mathbf{Z}^2} - F_{(\bar{\tau} \cap \mathbf{Z}^2) \setminus \bar{A}}$ hence

$$F_{\bar{A}} = (1 - x_1^7 x_2^{12})^{-1} \Big(\sum_{(i,j) \in G} x_1^i x_2^j + (\sum_{(i,j) \in G'} x_1^i x_2^j)(1 - x_1 x_2)^{-1} \Big).$$

To get the series $P_{1,\tau}(\Lambda)$ we apply to $F_{\bar{A}}$ the ring homomorphism which maps $x_1^i x_2^j \mapsto \mathbf{L}^{j-i} T^j$ and then we multiply the result by $\mathbf{L} - 1$.

We check that none of the candidate poles of $P_{\mathrm{geom}}^{(Z^\Lambda, 0)}(T)$ cancels. The motivic volume is

$$\mu(H_\Lambda) = (\mathbf{L} - 1)^2 \Big(\frac{1}{(1-\mathbf{L})(1-\mathbf{L}^{19})} + \frac{-1}{1-\mathbf{L}^{19}} + \frac{1+\mathbf{L}^8+\mathbf{L}^{16}}{(1-\mathbf{L}^{19})(1-\mathbf{L}^5)} + \frac{-1}{1-\mathbf{L}^5} + \frac{1}{(1-\mathbf{L}^5)(1-\mathbf{L})} \Big).$$

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